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Uniqueness of some differential polynomials of meromorphic functions

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Abstract. In this paper, we prove some uniqueness results which improve and generalize several earlier works. Also, we prove a value distribution result concerning $f^{(k)}$ which is related to a conjecture of Fang and Wang [A note on the conjectures of Hayman, Mues and Gol'dberg, *Comp. Methods, Funct. Theory* (2013) **13**, 533–543].

1. Introduction

Throughout, by a meromorphic function we always mean a non-constant meromorphic function in the complex plane $\mathbb{C}$.

We use the notations of Nevanlinna value distribution theory [2] such as $m(r,f)$, $N(r,f)$, $T(r,f)$ and $S(r,f)$ defined as follows:

$$m(r,f) = m(r,\infty) := \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta,$$

where $r > 0$ and $\log^+ x = \max\{\log x, 0\}$;

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t}\,dt + n(0,f)\log r,$$

where $n(t,f)$ denotes the number of poles of f in $\{z : |z| \le t\}$, each pole is counted according to its multiplicity;

$$T(r,f) = m(r,f) + N(r,f);$$

and $S(r,f)$ is any quantity satisfying

$$\lim_{r\to\infty} \frac{S(r,f)}{T(r,f)} = 0,$$

possibly outside a set of finite linear measure.

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By $E(a, f)$, we denote the set of zeros of $f - a$ counting multiplicities (CM) and by $\bar{E}(a, f)$, the set of zeros of $f - a$ ignoring multiplicities (IM). Two meromorphic functions f and g are said to share the value a CM if $E(a, f) = E(a, g)$ and to share the value a IM if $\bar{E}(a, f) = \bar{E}(a, g)$. Further, by $E_{k)}(a, f)$, we denote the set of zeros of $f - a$ with multiplicities at most k in which each zero is counted according to its multiplicity. Also, by $\bar{E}_{k)}(a, f)$, we denote the set of zeros of $f - a$ with multiplicity at most k, counted once.

We denote by $\mathscr{A}$, the class of meromorphic functions f satisfying

$$\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) = S(r, f).$$

Clearly, each member of class $\mathscr{A}$ is a transcendental meromorphic function. Also for any $a \in \mathbf{C}$, we define

$$N_1\left(r, \frac{1}{f - a}\right) = N\left(r, \frac{1}{f - a}\right) - \overline{N}\left(r, \frac{1}{f - a}\right)$$

and

$$N_2\left(r, \frac{1}{f - a}\right) = \overline{N}\left(r, \frac{1}{f - a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f - a}\right),$$

where $N_{(k}(r, 1/(f - a))$ is the counting function of those zeros of $f - a$ whose multiplicity is at least k, and $\overline{N}_{(k}(r, 1/(f - a))$ is the one corresponding to ignoring multiplicity. Finally, by $S(f)$, we denote the set of small functions of f; that is,

$$S(f) := \{a \,|\, a \text{ is meromorphic and } T(r, a) = S(r, f) \text{ as } r \to \infty\}.$$

The uniqueness theory of meromorphic functions has perfected the value distribution theory of Nevanlinna and has a vast range of applications in complex analysis. For recent developments in the uniqueness theory of meromorphic functions (sharing, weighted sharing and q-difference sharing of polynomials), one may refer to [6, 8, 11].

In the present paper, we prove some uniqueness results which improve and generalize the works of Yang and Yi [9] , Wang and Gao [5], and Huang and Huang [3]. Also, a result related to a conjecture of Fang and Wang [1] concerning value distribution of $f^{(k)} - a$, where $k \in \mathbf{N}$ and a $(\not\equiv 0, \infty)$ is a small function of f, is obtained.

2. Main results

Yang and Yi [9, Theorem 3.29, p. 197] proved the following result for class $\mathscr{A}$:

Theorem A. *Let $f, g \in \mathscr{A}$, and a be a non-zero complex number. Furthermore, let k be a positive integer.*

(i) *If $\bar{E}_{1)}(a, f) = \bar{E}_{1)}(a, g)$, then $f \equiv g$ or $fg \equiv a^2$.*

(ii) *If $\bar{E}_{1)}(a, f^{(k)}) = \bar{E}_{1)}(a, g^{(k)})$, then $f \equiv g$ or $f^{(k)}g^{(k)} \equiv a^2$.*

A function f is said to *share a value a partially with g IM* if $\bar{E}(a, f) \subseteq \bar{E}(a, g)$. We use the notation $N_{1)}(r, 1/(f-a)|g \neq a)$, to denote the simple zeros of $f - a$, that are not the zeros of $g - a$. Using this notation and the notion of partial sharing, we improve Theorem A as

Theorem 1. *Let $f, g \in \mathscr{A}$, a be a non-zero complex number and k be a positive integer.*

(i) *If $\bar{E}_{1)}(a, f) \subseteq \bar{E}_{1)}(a, g)$ and $N_{1)}(r, 1/(g-a)|f \neq a) = S(r, g)$, then $f \equiv g$ or $fg \equiv a^2$.*

(ii) *If $\bar{E}_{1)}(a, f^{(k)}) \subseteq \bar{E}_{1)}(a, g^{(k)})$ and $N_{1)}(r, 1/(g^{(k)}-a)|f^{(k)} \neq a) = S(r, g)$, then $f \equiv g$ or $f^{(k)}g^{(k)} \equiv a^2$.*

Example. Consider $f(z) = e^z$ and $g(z) = e^{2z}$. Then $f, g \in \mathscr{A}$, $\bar{E}_{1)}(1, f) \subseteq \bar{E}_{1)}(1, g)$ and $N_{1)}(r, 1/(g-1)|f \neq 1) \neq S(r, g)$, and the conclusion of Theorem 1 does not hold. Thus, the condition "$N_{1)}(r, 1/(g-a)|f \neq a) = S(r, g)$" in Theorem 1, is essential.

In 2011, Huang and Huang [3, Theorem 3, p. 231] improved a result of Yang and Hua [7, Theorem 1, p. 396] as

Theorem B. *Let f and g be two meromorphic functions and $n \geq 19$ be an integer. If $E_{1)}(1, f^n f') = E_{1)}(1, g^n g')$, then either $f = dg$ for some $(n+1)$-th root of unity d or $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

In this paper, we improve Theorem B for functions of class $\mathscr{A}$ as

Theorem 2. *Let $f, g \in \mathscr{A}$, $n \geq 2$ be an integer and $a(\neq 0) \in \mathbf{C}$. If $\bar{E}_{1)}(a, f^n f') = \bar{E}_{1)}(a, g^n g')$, then either $f = dg$ for some $(n+1)$-th root of unity d or $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.*

Concerning sharing of small functions, Wang and Gao [5, Theorem 1.3, p. 2] proved:

Theorem C. *Let f and g be two transcendental meromorphic functions, $a(\not\equiv 0) \in S(f) \cap S(g)$, and let $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share a CM, then either $f^n f' g^n g' \equiv a^2$ or $f = dg$ for some $(n+1)$-th root of unity d.*

Definition. Let f and g be two non-constant meromorphic functions, and a is a small function related to both f and g. We say that f and g share the small function a CM if $f-a$ and $g-a$ assume the same zeros with the same multiplicities.

Here in this paper, we partially extend Theorem C to a more general class of differential polynomials as

Theorem 3. *Let f and g be two transcendental meromorphic functions, $a(\not\equiv 0) \in S(f) \cap S(g)$, and let n, m, k be positive integers satisfying $n > km + 3m + 2k + 8$, and $m > k-1$. If $f^n(f^m)^{(k)}$ and $g^n(g^m)^{(k)}$ share a CM, then either*

$$f^n(f^m)^{(k)} g^n(g^m)^{(k)} \equiv a^2 \qquad \text{or} \qquad f^n(f^m)^{(k)} \equiv g^n(g^m)^{(k)}.$$

For $m > k-1$, we have $n > k^2 + 4k + 5$ so that by substituting $k = 1$, we get $n > 10$. Thus Theorem 3 reduces to Theorem C.

Concerning the value distribution of k-th derivative of a meromorphic function, Fang and Wang [1, Proposition 3, p. 542] proved the following result:

Theorem D. *Let f be a transcendental meromorphic function having at most finitely many simple zeros. Then $f^{(k)}$ takes on every non-zero polynomial infinitely often for $k = 1, 2, 3, \dots$.*

Definition. *A meromorphic function f is said to take a function h infinitely often if $f - h$ has infinitely many zeros.*

Further, Fang and Wang [1, Question 2, p. 543] asked the following question:

Question. *Let f be a transcendental meromorphic function having at most finitely many simple zeros. Must $f^{(k)}$ take on every non-zero rational function infinitely often for $k = 1, 2, 3, \dots$?*

Here, we obtained a result related to the above question involving small function as

Theorem 4. *Let f be a transcendental meromorphic function having at most finitely many simple zeros and $N(r, 1/f'') = S(r, f)$. Let $a(\not\equiv 0, \infty) \in S(f)$, then $f^{(k)} - a$ has infinitely many zeros for $k = 1, 2, 3, \dots$.*

3. Some lemmas

We recall the following results which we shall use in the proof of main results of this paper:

Lemma 1 [7, Theorem 3, p. 396]. *Let f and g be two non-constant entire functions, $n \geq 1$ and $a(\neq 0) \in \mathbb{C}$. If $f^n f' g^n g' = a^2$, then $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.*

Lemma 2 [9, Lemma 1.10, p. 82]. *Let f_1 and f_2 be non-constant meromorphic functions and let c_1, c_2 and c_3 be non-zero constants. If $c_1 f_1 + c_2 f_2 \equiv c_3$, then*

$$T(r, f_1) < \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}(r, f_1) + S(r, f_1).$$

Lemma 3 [9, Lemma 3.8, p. 193]. *If $f \in \mathscr{A}$ and k is a positive integer, then $f^{(k)} \in \mathscr{A}$.*

Lemma 4 [9, Lemma 3.9, p. 194]. *If $f, g \in \mathscr{A}$ and $f^{(k)} = g^{(k)}$, where k is a positive integer, then $f \equiv g$.*

Lemma 5 [9, Lemma 3.10, p. 194]. *If $f \in \mathscr{A}$ and a is a finite non-zero number, then*

$$N_{1)}\left(r, \frac{1}{f - a}\right) = T(r, f) + S(r, f),$$

where $N_{1)}(r, 1/(f - a))$ denotes the simple zeros of $f - a$.

Lemma 6 [9, Theorem 1.24, p. 39]. *Suppose f is a non-constant meromorphic function and k is a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Lemma 7 [5, Lemma 2.3, p. 3]. *Let f and g be two meromorphic functions. If f and g share 1 CM, then one of the following must occur:* i) $T(r, f) + T(r, g) \leq 2\{N_2(r, 1/f) + N_2(r, 1/g) + N_2(r, f) + N_2(r, g)\} + S(r, f) + S(r, g)$, ii) *either $f \equiv g$ or $fg \equiv 1$.*

Lemma 8 [1, Lemma 1, p. 537]. *Let f be a transcendental meromorphic function, let $k \geq 2$ be an integer, and $\varepsilon > 0$. Then*

$$(k - 1)\overline{N}(r, f) + N_1\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{f^{(k)}}\right) + \varepsilon T(r, f).$$

4. Proof of main results

We divide this section into four subsections as follows:

4.1. Proof of Theorem 1. Since $\bar{E}_{1)}(a,f) \subseteq \bar{E}_{1)}(a,g)$,

$$N_{1)}\left(r,\frac{1}{f-a}\right) \le N_{1)}\left(r,\frac{1}{g-a}\right).$$

Since (by Lemma 5)

$$N_{1)}\left(r,\frac{1}{f-a}\right) = T(r,f) + S(r,f)$$

and

$$N_{1)}\left(r,\frac{1}{g-a}\right) = T(r,g) + S(r,g),$$

therefore,

$$N_{(2}\left(r,\frac{1}{f-a}\right) = S(r,f),$$

$$N_{(2}\left(r,\frac{1}{g-a}\right) = S(r,g)$$

and

$$T(r,g) \ge T(r,f) + S(r,f). \tag{1}$$

Define a function $h : \mathbb{C} \to \overline{\mathbb{C}}$ by

$$h(z) = \frac{f(z)-a}{g(z)-a}. \tag{2}$$

Since $\bar{E}_{1)}(a,f) \subseteq \bar{E}_{1)}(a,g)$, we have

$$\overline{N}(r,h) \le \overline{N}(r,f) + \overline{N}_{(2}\left(r,\frac{1}{g-a}\right) + N_{1)}\left(r,\frac{1}{g-a}\,\middle|\, f \ne a\right) = S(r,g) \tag{3}$$

$$\overline{N}\left(r,\frac{1}{h}\right) \le \overline{N}(r,g) + \overline{N}_{(2}\left(r,\frac{1}{f-a}\right) = S(r,g) \tag{4}$$

and

$$T(r,h) \le T(r,f) + T(r,g) + O(1) \le 2T(r,g) + S(r,g).$$

Let $f_1 = (1/a)f$, $f_2 = h$, $f_3 = (-1/a)hg$. Then,

$$\sum_{j=1}^{3} f_j \equiv 1. \tag{5}$$

Combining (2), (3) and (4), we get

$$\sum_{j=1}^{3}\left(\overline{N}(r,f_j)+\overline{N}\left(r,\frac{1}{f_j}\right)\right)=S(r,g).$$

Clearly, f_1, f_2 and f_3 are linearly dependent and so there exist three constants c_1, c_2 and c_3 (at least one of them is not zero) such that

$$\sum_{j=1}^{3} c_j f_j = 0. \tag{6}$$

If $c_1 = 0$, then from (6) we see that $c_2 \neq 0$, $c_3 \neq 0$, and

$$f_3 = -\frac{c_2}{c_3} f_2. \tag{7}$$

Substituting (7) into (5) gives

$$f_1 + \left(1 - \frac{c_2}{c_3}\right) f_2 = 1. \tag{8}$$

From (7) and (8), we get

$$T(r,f_3) = T(r,f_1) + O(1)$$

and thus

$$T(r) = T(r,f_1) + O(1), \tag{9}$$

where $T(r) = \max_{1\le j\le 3}\{T(r,f_j)\}$.

Since f_1 is not a constant, it follows from (8) that $1 - c_2/c_3 \neq 0$. From (8), (9) and Lemma 2, we deduce that

$$T(r) < \overline{N}\left(r,\frac{1}{f_1}\right) + \overline{N}\left(r,\frac{1}{f_2}\right) + \overline{N}(r,f_1) + S(r) = S(r),$$

where $S(r) = o(T(r))$, which is a contradiction and so $c_1 \neq 0$, and then (6) gives

$$f_1 = -\frac{c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3. \tag{10}$$

Now, from (5) and (10), we get

$$\left(1 - \frac{c_2}{c_1}\right) f_2 + \left(1 - \frac{c_3}{c_1}\right) f_3 = 1. \tag{11}$$

We consider the following three cases:

Case 1: $1 - c_2/c_1 \neq 0$ and $1 - c_3/c_1 \neq 0$. In this case, (10) and (11) give

$$f_1 = \frac{c_2 - c_3}{c_1 - c_2} f_3 - \frac{c_2}{c_1 - c_2}. \tag{12}$$

From (11) and (12), we have

$$T(r, f_2) = T(r, f_1) + O(1)$$

and hence

$$T(r) = T(r, f_1) + O(1). \tag{13}$$

Applying Lemma 2 to (11) and using (13), we obtain

$$T(r) < \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}\left(r, \frac{1}{f_3}\right) + \overline{N}(r, f_2) + S(r) = S(r),$$

which is a contradiction.

Case 2: $1 - c_2/c_1 = 0$. From (11), we have $1 - c_3/c_1 \neq 0$, and

$$f_3 = \frac{c_1}{c_1 - c_3}. \tag{14}$$

Since $1 - c_2/c_1 = 0$, we obtain $c_1 = c_2$. Thus from (10) and (14), we obtain

$$f_1 + f_2 = -\frac{c_3}{c_1 - c_3}. \tag{15}$$

If $c_3 \neq 0$, then by applying Lemma 2 to (15), we obtain

$$T(r) < \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}(r, f_1) + S(r) = S(r),$$

which is a contradiction. Hence $c_3 = 0$ and so from (14), it follows that $f_3 \equiv 1$.

Case 3: $1 - c_3/c_1 = 0$. From (11), we have $1 - c_2/c_1 \neq 0$, and

$$f_2 = \frac{c_1}{c_1 - c_2}. \tag{16}$$

Since $1 - c_3/c_1 = 0$, we obtain $c_1 = c_3$. Thus from (10) and (16), we obtain

$$f_1 + f_3 = -\frac{c_2}{c_1 - c_2}. \tag{17}$$

If $c_2 \neq 0$, then by applying Lemma 2 to (17), we obtain

$$T(r) < \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_3}\right) + \overline{N}(r, f_1) + S(r) = S(r),$$

which is a contradiction. Hence $c_2 = 0$ and so from (16), it follows that $f_2 \equiv 1$.

Thus if $f_2 \equiv 1$, then by (2), we get $f \equiv g$. If $f_3 \equiv 1$, then (2) gives $fg \equiv a^2$. This proves (i).

From Lemma 3, we see that $f^{(k)}, g^{(k)} \in \mathscr{A}$. Using the conclusion of (i), we get either

$$f^{(k)} \equiv g^{(k)}$$

or

$$f^{(k)} g^{(k)} \equiv a^2.$$

If $f^{(k)} \equiv g^{(k)}$, then from Lemma 4, we have $f \equiv g$. This completes the proof of (ii). □

4.2. Proof of Theorem 2. Let the functions F and G be given by

$$F = \frac{f^{n+1}}{n+1} \qquad \text{and} \qquad G = \frac{g^{n+1}}{n+1}.$$

By hypothesis, $\overline{E}_{1)}(a, f^n f') = \overline{E}_{1)}(a, g^n g')$, therefore

$$\overline{E}_{1)}(a, F') = \overline{E}_{1)}(a, G').$$

Now

$$\begin{aligned} \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) &= \overline{N}\left(r, \frac{f^{n+1}}{n+1}\right) + \overline{N}\left(r, \frac{n+1}{f^{n+1}}\right) \\ &= \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) \\ &= S(r, f) \\ &= S(r, F). \end{aligned}$$

Similarly by replacing F by G in above equation, we have

$$\overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) = S(r, G).$$

Thus $F, G \in \mathscr{A}$ and so by the Theorem 2.1, it follows that either

$$F'G' \equiv a^2 \qquad \text{or} \qquad F \equiv G.$$

Consider the case $F'G' \equiv a^2$, that is,

$$f^n f' g^n g' \equiv a^2. \tag{18}$$

Suppose that z_1 is a pole of f of order p. Then z_1 is a zero of g of order say q and so from (18), we find that

$$nq + q - 1 = np + p + 1.$$

That is, $(q-p)(n+1) = 2$, which is not possible as $n \geq 2$ and p, q are positive integers. Thus f and g are entire functions and so from Lemma 1, we get $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.

Next consider the case when $F \equiv G$. This gives

$$\frac{f^{n+1}}{n+1} = \frac{g^{n+1}}{n+1}$$

or

$$f^{n+1} = g^{n+1}.$$

Hence $f = dg$ for some $(n+1)$-th root of unity d. □

4.3. Proof of Theorem 3. Let the functions F and G be given by

$$F = \frac{f^n (f^m)^{(k)}}{a} \qquad \text{and} \qquad G = \frac{g^n (g^m)^{(k)}}{a}.$$

Since $f^n (f^m)^{(k)}$ and $g^n (g^m)^{(k)}$ share a CM, F and G share 1 CM. Since (by Lemma 6 and $T(r,a) = S(r,f)$),

$$\begin{aligned}
N_2\left(r, \frac{1}{F}\right) + N_2(r,F) &\leq N_2\left(r, \frac{1}{f^n (f^m)^{(k)}}\right) + N_2(r, f^n (f^m)^{(k)}) + S(r,f) \\
&\leq N_2\left(r, \frac{1}{f^n}\right) + N_2\left(r, \frac{1}{(f^m)^{(k)}}\right) + 2\overline{N}(r, f^n (f^m)^{(k)}) + S(r,f) \\
&\leq 2\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{(f^m)^{(k)}}\right) + 2\overline{N}(r,f) + S(r,f) \\
&\leq 2\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^m}\right) + k\overline{N}(r, f^m) + 2\overline{N}(r,f) + S(r,f) \\
&= 2\overline{N}\left(r, \frac{1}{f}\right) + mN\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + 2\overline{N}(r,f) + S(r,f)
\end{aligned}$$

$$= 2\overline{N}\left(r, \frac{1}{f}\right) + mN\left(r, \frac{1}{f}\right) + (k+2)\overline{N}(r, f) + S(r, f)$$
$$\leq 2T(r, f) + mT(r, f) + (k+2)T(r, f) + S(r, f)$$
$$= (k+m+4)T(r, f) + S(r, f),$$

therefore,

$$N_2\left(r, \frac{1}{F}\right) + N_2(r, F) \leq (k+m+4)T(r, f) + S(r, f). \tag{19}$$

On the similar lines we can write (19) for the function G as

$$N_2\left(r, \frac{1}{G}\right) + N_2(r, G) \leq (k+m+4)T(r, g) + S(r, g). \tag{20}$$

Since

$$nT(r, f) = T(r, f^n) = T\left(r, \frac{f^n (f^m)^{(k)}}{a} \cdot \frac{a}{(f^m)^{(k)}}\right)$$
$$\leq T(r, F) + T\left(r, \frac{1}{(f^m)^{(k)}}\right) + T(r, a) + S(r, f)$$
$$\leq T(r, F) + T\left(r, \frac{1}{(f^m)^{(k)}}\right) + S(r, f)$$
$$\leq T(r, F) + (k+1)T\left(r, \frac{1}{f^m}\right) + S(r, f)$$
$$= T(r, F) + (km+m)T\left(r, \frac{1}{f}\right) + S(r, f),$$

therefore

$$(n - km - m)T(r, f) \leq T(r, F) + S(r, f). \tag{21}$$

Similarly,

$$(n - km - m)T(r, g) \leq T(r, G) + S(r, g). \tag{22}$$

Adding (21) and (22), we get

$$(n - km - m)\{T(r, f) + T(r, g)\} \leq \{T(r, F) + T(r, G)\} + S(r, f) + S(r, g). \tag{23}$$

Suppose that

$$T(r,F)+T(r,G)\le 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)\right\}$$
$$+S(r,F)+S(r,G) \tag{24}$$

holds. Then from (19), (20), (23) and (24), we have

$$\begin{aligned}(n-km-m)&\{T(r,f)+T(r,g)\}\\ &\le 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)\right\}\\ &\quad +S(r,f)+S(r,g)\\ &\le 2(k+m+4)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &=(2k+2m+8)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),\end{aligned}$$

which implies that

$$(n-km-3m-2k-8)\{T(r,f)+T(r,g)\}\le S(r,f)+S(r,g),$$

a contradiction since $n>km+3m+2k+8$, where $m>k-1$.

Thus, by Lemma 7, it follows that either

$$FG\equiv 1$$

or

$$F\equiv G.$$

That is, either

$$f^n(f^m)^{(k)}g^n(g^m)^{(k)}\equiv a^2$$

or

$$f^n(f^m)^{(k)}=g^n(g^m)^{(k)}.$$

□

4.4. Proof of Theorem 2. Since

$$\begin{aligned}m\left(r,\frac{1}{f}\right)&=m\left(r,\frac{f^{(k)}}{f}\cdot\frac{1}{f^{(k)}}\right)\\ &\le m\left(r,\frac{1}{f^{(k)}}\right)+m\left(r,\frac{f^{(k)}}{f}\right)\\ &=m\left(r,\frac{1}{f^{(k)}}\right)+S(r,f),\end{aligned}$$

therefore,

$$T(r,f) - N\left(r,\frac{1}{f}\right) \le T(r,f^{(k)}) - N\left(r,\frac{1}{f^{(k)}}\right) + S(r,f),$$

and so

$$N\left(r,\frac{1}{f^{(k)}}\right) \le T(r,f^{(k)}) - T(r,f) + N\left(r,\frac{1}{f}\right) + S(r,f). \tag{25}$$

Applying the second fundamental theorem of Nevanlinna [2, Theorem 2.5, p. 47] to the function $f^{(k)}$, we get

$$T(r,f^{(k)}) \le \overline{N}(r,f^{(k)}) + \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-a}\right) + S(r,f^{(k)}).$$

That is,

$$T(r,f^{(k)}) \le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-a}\right) + S(r,f). \tag{26}$$

Since $N(r,1/f'') = S(r,f)$, it follows from Lemma 8 with $k=2$ that

$$\begin{aligned}\overline{N}(r,f) + N_1\left(r,\frac{1}{f}\right) &\le N\left(r,\frac{1}{f''}\right) + \varepsilon T(r,f)\\ &= \varepsilon T(r,f) + S(r,f).\end{aligned}$$

Thus, from (25), (26) and the fact that f has finitely many simple zeros, we get

$$\begin{aligned}T(r,f) &\le \overline{N}\left(r,\frac{1}{f^{(k)}-a}\right) + \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + S(r,f)\\ &\le N\left(r,\frac{1}{f^{(k)}-a}\right) + \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + S(r,f)\\ &= N\left(r,\frac{1}{f^{(k)}-a}\right) + \overline{N}(r,f) + N_1\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f}\right) + S(r,f)\\ &\le N\left(r,\frac{1}{f^{(k)}-a}\right) + \varepsilon T(r,f) + \frac{1}{2}N\left(r,\frac{1}{f}\right) + S(r,f)\end{aligned}$$

$$\leq N\left(r, \frac{1}{f^{(k)} - a}\right) + \varepsilon T(r, f) + \frac{1}{2} T(r, f) + S(r, f)$$
$$= N\left(r, \frac{1}{f^{(k)} - a}\right) + \left(\frac{1}{2} + \varepsilon\right) T(r, f) + S(r, f),$$

which implies that

$$\left(\frac{1}{2} - \varepsilon\right) T(r, f) \leq N\left(r, \frac{1}{f^{(k)} - a}\right) + S(r, f). \tag{27}$$

Taking $\varepsilon = 1/4$ in (27), we get

$$T(r, f) \leq 4N\left(r, \frac{1}{f^{(k)} - a}\right) + S(r, f).$$

Hence $f^{(k)} - a$ has infinitely many zeros for $k = 1, 2, 3, \ldots$. □

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