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The Besicovitch covering theorem for parabolic balls in Euclidean space

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Abstract. The Besicovitch covering theorem is well known to be the useful tools in many fields of analysis. Federer extended the result of Besicovitch to a directionally limited metric space. In this paper, we prove the Besicovitch covering theorem for parabolic balls in Euclidean space, although the parabolic metric is not directionally limited.

1. Introduction

Covering theorems are well known to be fundamental tools in many fields of analysis. Although there are several types of covering results, all have the same purpose; from an arbitrary cover of a set in a metric space, one extracts a subcover as disjointed as possible. In this paper, we consider the so-called Besicovitch covering theorem. The Besicovitch covering theorem is more powerful than the well-known result of Vitali, because it does not require us to enlarge balls. Besicovitch [2] proved this theorem for disks in the plane, and Morse [9, Theorem 5.9] extended it to balls and more general sets in finite dimensional normed vector spaces. (For a simple proof of Morse's result, see [3, Theorem 5.4].) The best constant in the Besicovitch covering theorem was studied by Loeb [7], Sullivan [10] and Füredi-Loeb [6]. Moreover, Federer [5, Theorem 2.8.14] extended the result of Besicovitch to *directionally limited metric spaces* (see Definition 2.1). In this paper, we prove the Besicovitch covering theorem for parabolic balls. Note that the parabolic metric is not directionally limited. See Proposition 2.2. A more general result about the Besicovitch covering theorem was proved by Le Donne and Rigot [4, Theorem 3.16]. In this paper, we give a different simple proof of the Besicovitch covering theorem for parabolic balls in Euclidean space using homogeneity of the parabolic metric.

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A point in the Euclidean n-space $\mathbb{R}^n$, $n \geq 2$, is denoted by $x = (x_1, \dots, x_n)$ or (x', x_n) where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. Let

$$|x'|_{n-1} = \sqrt{x_1^2 + \cdot + x_{n-1}^2}$$

be the Euclidean norm of $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. For $x, y \in \mathbb{R}^n$, we define the parabolic metric $d(x, y)$ by

$$d(x, y) = \max\{|x' - y'|_{n-1}, \sqrt{|x_n - y_n|}\}. \tag{1.1}$$

Let $B(x, r) = \{y \in \mathbb{R}^n : d(x, y) \leq r\}$ denote the closed parabolic ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$. Let m be the n-dimensional Lebesgue measure. Note that there exists a constant $\alpha_n > 0$ such that $m(B(x, r)) = \alpha_n r^{n+1}$ for any $x \in \mathbb{R}^n$ and $r > 0$.

Theorem 1.1. *There exists a constant $N = N_n > 0$, depending only on n, with the following property: If $\mathscr{F}$ is any collection of closed parabolic balls in $\mathbb{R}^n$ with*

$$R = \sup\{\operatorname{diam} B : B \in \mathscr{F}\} < \infty$$

and if A is the set of centers of balls in $\mathscr{F}$, then there exist $\mathscr{G}_1, \dots, \mathscr{G}_N \subset \mathscr{F}$ such that

(i) *each $\mathscr{G}_j$ is a countable collection of disjoint balls,*

(ii) $A \subset \bigcup_{j=1}^{N} \bigcup_{B \in \mathscr{G}_j} B.$

Remark 1.2. Aimar-Forzani [1] proved the following weak version of Besicovitch covering theorem for other parabolic balls. Let $0 < a_1 \leq a_2 \leq \cdots \leq a_n$ and $p \geq 1$. Observe that for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}$ the equation of r

$$\left(\frac{|x_1|}{r^{a_1}}\right)^p + \cdots + \left(\frac{|x_n|}{r^{a_n}}\right)^p = 1$$

has a unique positive solution, which we call r_x. We define $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\rho(x, y) = \begin{cases} r_{x-y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Although ρ is not a metric in general, ρ is a quasi-metric, that is, there exists a positive constant $C \geq 1$ such that $\rho(x, y) \leq C(\rho(x, z) + \rho(z, y))$ for any $x, y, z \in \mathbb{R}^n$. We define the ρ-ball $B_\rho(x, r)$ centered at $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with radius $r > 0$ by

$$B_\rho(x,r) = \{y \in \mathbb{R}^n : \rho(x,y) \le r\}$$
$$= \left\{ y = (y_1, \dots, y_n) \in \mathbb{R}^n : \left(\frac{|x_1 - y_1|}{r^{a_1}}\right)^p + \cdots + \left(\frac{|x_n - y_n|}{r^{a_n}}\right)^p \le 1 \right\}.$$

They proved that if $a_n/a_1 \le p$, then there exists a constant $C > 0$ with the following property: If $\mathscr{F}$ is any collection of ρ-balls and if the set A of centers of balls in $\mathscr{F}$ is bounded, then there exists $\mathscr{G} \subset \mathscr{F}$ such that the balls in $\mathscr{G}$ cover A, and every point in $\mathbb{R}^n$ belongs to at most C balls in $\mathscr{G}$.

Applying Theorem 1.1, we can prove a weak version of Besicovitch covering theorem for our parabolic balls. Our parabolic balls are used in many fields of analysis more commonly than ρ-balls of Aimar-Forzani. In particular, the parabolic metric (1.1) plays an important role in the study of the mean curvature flow.

2. Proof of Theorem 1.1

Before we prove Theorem 1.1, we show that the parabolic metric is not directionally limited. Federer [5, 2.8.9] introduced the following notion of the directionally limited metric (slightly changed to suit our purposes). We write $\mathrm{Card}(A)$ to denote the cardinality of the set A.

Definition 2.1. Let (X,d) be a metric space, $A \subset X$ and $\xi > 0$, $0 < \eta \le \frac{1}{3}$, $\zeta \in \mathbb{N}$. The metric d is said to be directionally (ξ,η,ζ)-limited at A if the following holds:

(i) If $a,b,c \in A$ with $0 < d(a,c) \le d(a,b)$, then there exists a point $x \in X$ such that

$$d(a,x) = d(a,c) \qquad \text{and} \qquad d(b,x) = d(a,b) - d(a,c). \tag{2.1}$$

(ii) If $a \in A$ and $B \subset A \cap (B(a,\xi) \backslash \{a\})$ such that

$$\frac{d(x,c)}{d(a,c)} \ge \eta$$

whenever $b,c \in B$ with $b \ne c$ and $x \in X$ satisfying (2.1), then $\mathrm{Card}(B) \le \zeta$.

Federer [5, Theorem 2.8.14] proved that the generalized versions of Besicovitch covering theorem for directionally limited metric spaces. However the parabolic metric is not directionally limited. The following proposition was shown by Menne [8].

Proposition 2.2. *Let $A \subset \mathbb{R}^n$. Assume there exist $a,b,c \in A$ such that*

$$0 < d(a,c) < d(a,b) = \sqrt{|a_n - b_n|}. \tag{2.2}$$

Then the parabolic metric is not (ξ, η, ζ)-directionally limited at A for any $\xi > 0$, $0 < \eta \le \frac{1}{3}$, $\zeta \in \mathbb{N}$.

Proof. We prove that there exists no $x \in \mathbb{R}^n$ satisfying (2.1) for $a, b, c \in A$. Assume that there exists $x \in \mathbb{R}^n$ satisfying (2.1), that is,

$$d(a,x) = d(a,c) \qquad \text{and} \qquad d(b,x) = d(a,b) - d(a,c).$$

Then we have

$$\begin{aligned}\sqrt{|a_n - x_n|} + \sqrt{|x_n - b_n|} &\le d(a,x) + d(x,b) \\ &= d(a,b) \\ &= \sqrt{|a_n - b_n|} \le \sqrt{|a_n - x_n|} + \sqrt{|x_n - b_n|}.\end{aligned}$$

Observe that either $x = a$ or $x = b$ holds. This would contradict the assumption (2.2). Hence there exists no $x \in \mathbb{R}^n$ satisfying (2.1) for $a, b, c \in A$ and so Proposition 2.2 holds. □

For $r > 0$ we define the scaling transformation f_r by

$$f_r(x) = f_r(x', x_n) = \left(\frac{x'}{r}, \frac{x_n}{r^2}\right) \qquad \text{for } x \in \mathbb{R}^n.$$

Next we observe the following property of f_r.

Proposition 2.3. *Let $x, y \in \mathbb{R}^n$ and $r, r_1, r_2 > 0$. Then*
(i) $f_r(x + y) = f_r(x) + f_r(y)$,
(ii) $f_{r_1} \circ f_{r_2} = f_{r_2} \circ f_{r_1} = f_{r_1 r_2}$,
(iii) $d(f_r(x), f_r(y)) = \frac{1}{r} d(x, y)$.

Proof. Let $x = (x', x_n)$, $y = (y', y_n) \in \mathbb{R}^n$.
(i)

$$f_r(x+y) = \left(\frac{x'+y'}{r}, \frac{x_n + y_n}{r^2}\right) = \left(\frac{x'}{r}, \frac{x_n}{r^2}\right) + \left(\frac{y'}{r}, \frac{y_n}{r^2}\right) = f_r(x) + f_r(y).$$

(ii)

$$f_{r_1} \circ f_{r_2}(x) = f_{r_1}\left(\frac{x'}{r_2}, \frac{x_n}{r_2^2}\right) = \left(\frac{x'}{r_1 r_2}, \frac{x_n}{(r_1 r_2)^2}\right) = f_{r_1 r_2}(x),$$

$$f_{r_2} \circ f_{r_1}(x) = f_{r_2}\left(\frac{x'}{r_1}, \frac{x_n}{r_1^2}\right) = \left(\frac{x'}{r_1 r_2}, \frac{x_n}{(r_1 r_2)^2}\right) = f_{r_1 r_2}(x).$$

(iii)

$$\begin{aligned} d(f_r(x), f_r(y)) &= \max\left\{\frac{|x'-y'|_{n-1}}{r}, \frac{\sqrt{|x_n-y_n|}}{r}\right\} \\ &= \frac{1}{r}\max\{|x'-y'|_{n-1}, \sqrt{|x_n-y_n|}\} \\ &= \frac{1}{r}d(x,y). \end{aligned}$$

□

We divide the proof of Theorem 1.1 into several lemmas. Our proof is based on the result by Morse [9, Theorem 5.9]. One of the new ingredients in our proof is Lemma 4, which requires the geometric properties specific to the parabolic metric d. Hereafter, let $\mathscr{F}$ be a collection of closed parabolic balls in $\mathbb{R}^n$ with

$$R = \sup\{\operatorname{diam} B : B \in \mathscr{F}\} < \infty$$

and let A be the set of centers of balls in $\mathscr{F}$.

Lemma 1. *If A is bounded, then there exists $\{B(x_j, r_j)\}_{j=1}^J \subset \mathscr{F}$ such that*

(i) *if $i < j$, then $x_j \notin B(x_i, r_i)$ and $r_j \le 2r_i$,*

(ii) *$A \subset \bigcup_{j=1}^J B(x_j, r_j)$.*

Moreover (i) *implies that $\{B(x_j, r_j/3)\}_{j=1}^J$ are disjoint.*

Proof. Choose any ball $B(x_1, r_1) \in \mathscr{F}$ such that $r_1 \ge R/4$. Inductively choose $\{B(x_j, r_j)\}$ as follows. Assume that $B(x_1, r_1), \dots, B(x_{j-1}, r_{j-1})$ are defined. Let $A_j = A \setminus \bigcup_{i=1}^{j-1} B(x_i, r_i)$ and let $R_j = \sup\{r : B(x,r) \in \mathscr{F}, x \in A_j\}$. If $A_j = \varnothing$, then stop and set $J = j-1$. If $A_j \ne \varnothing$, then choose any ball $B(x_j, r_j) \in \mathscr{F}$ such that $x_j \in A_j$ and $r_j \ge R_j/2$. If $A_j \ne \varnothing$ for all j, then set $J = \infty$.

(i) Assume that $i < j$. Then $x_j \in A_j = A \setminus \bigcup_{i=1}^{j-1} B(x_i, r_i)$ and so $x_j \notin B(x_i, r_i)$. Since $x_j \in A_j \subset A_i$,

$$2r_i \ge R_i = \sup\{r : B(x,r) \in \mathscr{F}, x \in A_i\} \ge r_j.$$

Thus the property (i) holds. Moreover, we obtain

$$d(x_i, x_j) > r_i = \frac{r_i}{3} + \frac{2r_i}{3} \ge \frac{r_i}{3} + \frac{r_j}{3}.$$

Therefore $\{B(x_j, r_j/3)\}_{j=1}^J$ are disjoint.

(ii) We prove that $A \subset \bigcup_{j=1}^J B(x_j, r_j)$. If $J < \infty$, this is trivial. Suppose $J = \infty$. Since A is bounded, there is a constant $R_0 > 0$ such that $B(x_j, r_j/3) \subset$

$B(0, R_0)$ for all j. Because $\{B(x_j, r_j/3)\}_{j=1}^{\infty}$ are disjoint, we have

$$\sum_{j=1}^{\infty} \alpha_n (r_j/3)^{n+1} = \sum_{j=1}^{\infty} m(B(x_j, r_j/3))$$
$$= m\left(\bigcup_{j=1}^{\infty} B(x_j, r_j/3)\right) \le m(B(0, R_0)) < \infty.$$

Hence $\lim_{j\to\infty} r_j = 0$.

If $x \in A$, then there is a ball $B(x, r) \in \mathscr{F}$. Since $\lim_{j\to\infty} r_j = 0$, there exists j such that $r_j < r/2$. Assume that $x \notin \bigcup_{i=1}^{j-1} B(x_j, r_j)$. Then $x \in A_j$ and so

$$r_j \ge \frac{R_j}{2} = \frac{1}{2} \sup\{r : B(x, r) \in \mathscr{F}, x \in A_j\} \ge \frac{r}{2},$$

which is a contradiction. Hence we have $x \in \bigcup_{i=1}^{j-1} B(x_j, r_j)$ and so the property (ii) holds. □

Lemma 2. *Given balls $\{B(x_j, r_j)\}_{j=1}^{J}$ and a finite subset $I \subset \{i : i \le J\}$. Then there exists a finite partition $L_1, L_2, \dots, L_K$ of I such that*

(i) *if $j = 1, 2, \dots, K$, $m(j) = \min L_j$ and $i \in L_j$, then $x_{m(j)} \in B(x_i, r_i)$,*

(ii) *if $i < j \le K$, then $m(i) < m(j)$ and $x_{m(i)} \notin B(x_{m(j)}, r_{m(j)})$.*

Proof. Let $m(1) = \min I$ and let $L_1 = \{i \in I : x_{m(1)} \in B(x_i, r_i)\}$. Inductively choose $\{L_j\}$ as follows. Assume that $L_1, \dots, L_{j-1}$ are defined. Let $I_j = I \setminus \bigcup_{i=1}^{j-1} L_i$. If $I_j = \emptyset$, then stop and set $K = j - 1$. If $I_j \ne \emptyset$, then let $m(j) = \min I_j$ and let $L_j = \{i \in I_j : x_{m(j)} \in B(x_i, r_i)\}$. Since I is finite, there is a j such that $I_j = \emptyset$. Obviously, the property (i) holds.

Assume $i < j \le K$. By $I_j \subsetneq I_i$, we have $m(i) = \min I_i < \min I_j = m(j)$. Since $m(j) \in I_i \setminus L_i$, we see $x_{m(i)} \notin B(x_{m(j)}, r_{m(j)})$. Thus the property (ii) holds. □

Lemma 3. *Suppose that balls $\{B(x_j, r_j)\}_{j=1}^{J}$ satisfy the property* (i) *in Lemma* 1, *$k \le J$, $I \subset \{i : i < k\}$ and $B(x_i, r_i) \cap B(x_k, r_k) \ne \emptyset$ for all $i \in I$.*

(i) *If $r_i < 3r_k$ for all $i \in I$, then* $\mathrm{Card}(I) \le 30^{n+1}$.

(ii) *If $I \ne \emptyset$, $m = \min I$ and $x_m \in B(x_i, r_i)$ for all $i \in I$, then* $\mathrm{Card}(I) \le 5^{n+1}$.

(iii) *If $3r_k \le r_i$ for all $i \in I$ and $x_i \notin B(x_j, r_j)$ for all $i, j \in I$ with $i < j$, then* $\mathrm{Card}(I) \le 7^{n+1}$.

We obtain Lemma 3 (iii) by the following crucial lemma.

Lemma 4. *Suppose that* $\{B(x_j, r_j)\}_{j=1,2,3}$ *satisfy*

$$B(x_j, r_j) \cap B(x_3, r_3) \neq \emptyset, \qquad x_3 \notin B(x_j, r_j) \qquad \text{and} \qquad 3r_3 \leq r_j \qquad \text{for } j = 1, 2.$$

Let $d_j = d(x_j, x_3)$ *for* $j = 1, 2$. *If*

$$d_1 \leq d_2 \qquad \text{and} \qquad d(f_{d_1}(x_1 - x_3), f_{d_2}(x_2 - x_3)) \leq 1/3,$$

then $x_1 \in B(x_2, r_2)$.

Proof. Let $0 < l = d_1/d_2 \leq 1$. Fix $y \in B(x_2, r_2) \cap B(x_3, r_3)$ and let

$$z = y + f_l(x_1 - y).$$

We show that $z \in B(x_2, r_2)$. By Proposition 2.3 and the translation invariance of d, we have

$$\begin{aligned} d(x_3 - f_l(x_3), x_2 - f_l(x_1)) &= d(f_l(x_1 - x_3), x_2 - x_3) \\ &= d(f_{1/d_2} \circ f_{d_1}(x_1 - x_3), f_{1/d_2} \circ f_{d_2}(x_2 - x_3)) \\ &= d_2 \cdot d(f_{d_1}(x_1 - x_3), f_{d_2}(x_2 - x_3)) \\ &\leq \frac{d_2}{3}. \end{aligned} \tag{2.3}$$

Since $y \in B(x_3, r_3)$, we obtain that

$$\left| \left(1 - \frac{1}{l}\right) y' - \left(1 - \frac{1}{l}\right)(x_3)' \right|_{n-1} = \left(\frac{1}{l} - 1\right) |y' - (x_3)'|_{n-1} \leq \frac{r_3}{l},$$

$$\sqrt{\left| \left(1 - \frac{1}{l^2}\right) y_n - \left(1 - \frac{1}{l^2}\right)(x_3)_n \right|} = \sqrt{\left(\frac{1}{l^2} - 1\right)} \sqrt{|y_n - (x_3)_n|} \leq \frac{r_3}{l},$$

and so

$$d(y - f_l(y), x_3 - f_l(x_3)) \leq \frac{r_3}{l}. \tag{2.4}$$

By $x_3 \notin B(x_1, r_1)$ and $3r_3 \leq r_1$, we have

$$d_1 = d(x_1, x_3) > r_1 \geq 3r_3. \tag{2.5}$$

Since $y \in B(x_2, r_2) \cap B(x_3, r_3)$ and $3r_3 \leq r_2$, we get

$$d_2 = d(x_2, x_3) \leq d(x_2, y) + d(y, x_3) \leq r_2 + r_3 \leq \frac{4}{3} r_2. \tag{2.6}$$

Combining (2.3), (2.4), (2.5) and (2.6), we obtain

$$\begin{aligned} d(z, x_2) &= d(y - f_l(y), x_2 - f_l(x_1)) \\ &\le d(y - f_l(y), x_3 - f_l(x_3)) + d(x_3 - f_l(x_3), x_2 - f_l(x_1)) \\ &\le \frac{r_3}{l} + \frac{d_2}{3} \le \frac{d_2}{d_1} \cdot \frac{d_1}{3} + \frac{d_2}{3} \le \frac{8r_2}{9} \le r_2. \end{aligned}$$

Hence we see that $z \in B(x_2, r_2)$.

Finally we prove $x_1 \in B(x_2, r_2)$. Observe that

$$x_1 = y + f_{1/l}(z - y) = ((1 - l)y' + lz', (1 - l^2)y_n + l^2 z_n).$$

By $y, z \in B(x_2, r_2)$, we obtain

$$\begin{aligned} |(x_1)' - (x_2)'|_{n-1} &\le (1 - l) \cdot |y' - (x_2)'|_{n-1} + l \cdot |z' - (x_2)'|_{n-1} \\ &\le (1 - l)r_2 + lr_2 = r_2, \end{aligned}$$

$$|(x_1)_n - (x_2)_n| \le (1 - l^2) \cdot |y_n - (x_2)_n| + l^2 \cdot |z_n - (x_2)_n| \le (1 - l^2)r_2^2 + l^2 r_2^2 = r_2^2,$$

and so $x_1 \in B(x_2, r_2)$. □

PROOF OF LEMMA 3. (i) Assume that $r_i < 3r_k$ for all $i \in I$. Fix $i \in I$. By $B(x_i, r_i) \cap B(x_k, r_k) \neq \emptyset$, we have for any $y \in B(x_i, r_i/3)$

$$d(y, x_k) \le d(y, x_i) + d(x_i, x_k) \le \frac{r_i}{3} + r_i + r_k \le 5r_k.$$

Hence we see that $B(x_i, r_i/3) \subset B(x_k, 5r_k)$. Because $\{B(x_j, r_j)\}_{j=1}^{J}$ satisfy the property Lemma 1 (i), $r_k \le 2r_i$ for all $i \in I$ and $\{B(x_i, r_i/3)\}_{i \in I}$ are disjoint. Hence

$$\begin{aligned} \alpha_n (5r_k)^{n+1} = m(B(x_k, 5r_k)) \ge m\left(\bigcup_{i \in I} B(x_i, r_i/3)\right) &= \sum_{i \in I} \alpha_n (r_i/3)^{n+1} \\ &\ge \alpha_n (r_k/6)^{n+1} \cdot \mathrm{Card}(I), \end{aligned}$$

so that $\mathrm{Card}(I) \le 30^{n+1}$.

(ii) Assume that $I \neq \emptyset$, $m = \min I$ and $x_m \in B(x_i, r_i)$ for all $i \in I$. Let $i \in I \setminus \{m\}$. Since $r_i \le 2r_m$ and $x_m \in B(x_i, r_i)$, we obtain for any $y \in B(x_i, r_m/2)$

$$d(y, x_m) \le d(y, x_i) + d(x_i, x_m) \le r_m/2 + r_i \le 5r_m/2.$$

Hence we see that $B(x_i, r_m/2) \subset B(x_m, 5r_m/2)$. If $i, j \in I \setminus \{m\}$ with $i < j$, then $x_j \notin B(x_i, r_i)$, $x_i \notin B(x_m, r_m)$, $x_m \in B(x_i, r_i)$ and so that

$$d(x_i, x_j) > r_i \ge d(x_m, x_i) > r_m.$$

Therefore $\{B(x_i, r_m/2)\}_{i \in I}$ are disjoint. Hence

$$\begin{aligned}\alpha_n(5r_m/2)^{n+1} &= m(B(x_k, 5r_m/2)) \\ &\geq m\left(\bigcup_{i \in I} B(x_i, r_m/2)\right) = \alpha_n(r_m/2)^{n+1} \cdot \mathrm{Card}(I),\end{aligned}$$

so that $\mathrm{Card}(I) \leq 5^{n+1}$.

(iii) Assume $3r_k \leq r_i$ for all $i \in I$ and $x_i \notin B(x_j, r_j)$ for all $i, j \in I$ with $i < j$. Let $i, j \in I$ with $i < j$. Since $x_i \notin B(x_j, r_j)$ and $x_j \notin B(x_i, r_i)$, it follows from Lemma 4 that

$$d(f_{d_i}(x_i - x_k), f_{d_j}(x_j - x_k)) > 1/3,$$

where $d_i = d(x_i, x_k)$ and $d_j = d(x_j, x_k)$. Let $y_i = f_{d_i}(x_i - x_k)$ for $i \in I$. By $d(y_i, y_j) > 1/3$, $\{B(y_i, 1/6)\}_{i \in I}$ are disjoint. Since

$$d(y_i, 0) = d(f_{d_i}(x_i), f_{d_i}(x_k)) = \frac{1}{d_i} d(x_i, x_k) = 1,$$

we have $B(y_i, 1/6) \subset B(0, 7/6)$ for $i \in I$. Hence

$$\alpha_n(7/6)^{n+1} = m(B(0, 7/6)) \geq m\left(\bigcup_{i \in I} B(y_i, 1/6)\right) = \alpha_n(1/6)^{n+1} \cdot \mathrm{Card}(I),$$

so that $\mathrm{Card}(I) \leq 7^{n+1}$. □

Proof of Theorem 1.1. Assume that A is bounded. By Lemma 1, there exists $\{B(x_j, r_j)\}_{j=1}^{J} \subset \mathscr{F}$ such that

(i) if $i < j$, then $x_j \notin B(x_i, r_i)$ and $r_j \leq 2r_i$,

(ii) $A \subset \bigcup_{j=1}^{J} B(x_j, r_j)$.

Fix $k \geq 2$. Let $I_k = \{1 \leq i < k : B(x_i, r_i) \cap B(x_k, r_k) \neq \emptyset\}$ and let $L_0 = \{i \in I_k : r_i < 3r_k\}$. Then there exists a finite partition $L_1, L_2, \ldots, L_K$ of $I_k \backslash L_0$ satisfying the properties (i), (ii) in Lemma 2. It follows from Lemma 3 that $\mathrm{Card}(L_0) \leq 30^{n+1}$, $\mathrm{Card}(L_j) \leq 5^{n+1}$ for $j = 1, \ldots, K$ and $K \leq 7^{n+1}$. Therefore we obtain

$$\mathrm{Card}(I_k) = \mathrm{Card}(L_0) + \sum_{j=1}^{K} L_j \leq 30^{n+1} + 35^{n+1}.$$

The right hand side of this inequality is independent of $k \geq 2$. Set $N = N_n = 30^{n+1} + 35^{n+1} + 1$. Next we determine $\mathscr{G}_1, \ldots, \mathscr{G}_N$. We define $\sigma : \{1, 2, \ldots\} \to \{1, 2, \ldots, N\}$. Let $\sigma(i) = i$ for $i = 1, \ldots, N$. For $k > N$ inductively define $\sigma(k)$ as follows. Assume that $\sigma(1), \ldots, \sigma(k-1)$ are defined. Since

$$\mathrm{Card}(I_k) = \mathrm{Card}(\{1 \leq i < k : B(x_i, r_i) \cap B(x_k, r_k) \neq \emptyset\}) < N,$$

there exists $l \in \{1, 2, \dots, N\}$ such that $B(x_i, r_i) \cap B(x_k, r_k) = \emptyset$ for all j with $\sigma(j) = l$ $(1 \le j < k)$. Set $\sigma(k) = l$. Now, let $\mathscr{G}_j = \{B(x_i, r_i) : \sigma(i) = j\}$ for $j = 1, \dots, N$. By the construction of σ, each $\mathscr{G}_j$ consists of disjoint balls in $\mathscr{F}$. Moreover, we see that

$$A \subset \bigcup_{j=1}^{J} B(x_j, r_j) = \bigcup_{j=1}^{N} \bigcup_{B \in \mathscr{G}_j} B.$$

Thus Theorem 1.1 holds for the case that A is bounded.

Finally we extend the result to general (unbounded) A. For $l \in \mathbb{N}$, set $A_l = \{x \in A : 3R(l-1) \le d(x, 0) < 3Rl\}$ and $\mathscr{F}^l = \{B(x, r) \in \mathscr{F} : x \in A_l\}$. Then there exist countable collections $\mathscr{G}_1^l, \dots, \mathscr{G}_N^l$ of disjoint balls in $\mathscr{F}^l$ such that

$$A_l \subset \bigcup_{j=1}^{N} \bigcup_{B \in \mathscr{G}_j^l} B.$$

For $j = 1, \dots, N$, let

$$\mathscr{G}_j = \bigcup_{l=1}^{\infty} \mathscr{G}_j^{2l-1}, \qquad \text{and} \qquad \mathscr{G}_{j+N} = \bigcup_{l=1}^{\infty} \mathscr{G}_j^{2l}.$$

If $B \in \mathscr{G}_j^l$, then $B \subset \{x \in \mathbb{R}^n : R(3l-1) \le d(x, 0) < R(3l+1)\}$. Therefore each $\mathscr{G}_i$ $(i = 1, 2, \dots, 2N)$ is a countable collection of disjoint balls in $\mathscr{F}$. Moreover we see that

$$A \subset \bigcup_{j=1}^{2N} \bigcup_{B \in \mathscr{G}_j} B.$$

Thus Theorem 1.1 holds. □

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