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A multiple conjugation biquandle and handlebody-links

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Abstract. We introduce a multiple conjugation biquandle, and show that it is the universal algebra for defining a semi-arc coloring invariant for handlebody-links. A multiple conjugation biquandle is a generalization of a multiple conjugation quandle. We extend the notion of n-parallel biquandle operations for any integer n, and show that any biquandle gives a multiple conjugation biquandle with them.

1. Introduction

A quandle [16, 19], biquandle [7, 8, 18], and multiple conjugation quandle [10] are algebras having certain universal properties related to topological objects in geometric topology. A quandle is a universal algebra for defining an arc coloring invariant for oriented knots, where an arc coloring is a map from the set of arcs of a knot diagram to the algebra. The axioms of a quandle correspond to the Reidemeister moves on oriented knot diagrams. A biquandle is a generalization of a quandle, which is universal with respect to semi-arc colorings, and the axioms of a biquandle correspond to the Reidemeister moves.

A handlebody-knot is a handlebody embedded in the 3-sphere S^3, whose diagram is given by a diagram of a spatial trivalent graph which is a spine of the handlebody. A multiple conjugation quandle (MCQ) is a universal symmetric quandle with a partial multiplication for defining arc coloring invariants for handlebody-knots, where a partial multiplication is an operation used at trivalent vertices (refer to [10] or Section 5). Some axioms of a multiple

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conjugation quandle are not directly derived from the Reidemeister moves. In general we call conditions on an algebra which are directly derived from the Reidemeister moves primitive conditions. In Section 4 of [10], the first author listed primitive conditions for an arc coloring invariant and proved that the axioms of a multiple conjugation quandle are obtained from the primitive conditions.

In this paper, we introduce a multiple conjugation biquandle (MCB) as a universal biquandle with a partial multiplication for defining semi-arc coloring invariants for handlebody-knots. We list primitive conditions for a semi-arc coloring invariant and prove that the axioms of a multiple conjugation biquandle are obtained from the primitive conditions (Theorem 3). From the axioms of an MCB, it is naturally seen that an MCB is a generalization of an MCQ. In [14], Nelson and the first author introduced a partially multiplicative biquandle to construct a semi-arc coloring invariant, whose axioms are almost identical to the primitive conditions. Theorem 3 brings out the algebraic structure of a partially multiplicative biquandle.

In [14], the notions of G-family of biquandles and n-parallel biquandle operations were introduced for $n \in \mathbb{Z}_{\geq 0}$. We refine the axioms of a G-family of biquandles as a corollary of Theorem 3, and extend the notion of n-parallel biquandle operations for any integer n. We also show that, for any biquandle, the n-parallel biquandle operations yield a $\mathbb{Z}$-family of biquandles, which gives us a multiple conjugation biquandle. We introduce a G-family of (generalized) Alexander biquandles, which also gives us many multiple conjugation biquandles.

(Co)homology theory is developed on quandles [5], multiple conjugation quandles [4], and biquandles [2, 6]. The theory provides quandle cocycle invariants, which give us various information about knots, surface-knots, and handlebody-knots (cf. [1, 3, 5, 12, 15, 20]). (Co)homology theory will be also developed for multiple conjugation biquandles in the consecutive paper [13]. This paper is the basis to develop the (co)homology theory for multiple conjugation biquandles.

The rest of the paper is organized as follows. In Section 2, we recall the definition of a biquandle, and introduce n-parallel biquandle operations, whose well-definedness is given in Section 9. In Section 3, we introduce a multiple conjugation biquandle with two equivalent definitions, and in Section 4, we show that the two definitions are equivalent. In Section 5, we recall the definition of a handlebody-link, and introduce colorings for handlebody-knots. In Section 6, we prove that a multiple conjugation biquandle gives a coloring invariant for handlebody-links. In Sections 7 and 8, we discuss the universality of the algebras used for colorings. In Section 9, we show some properties of n-parallel biquandle operations.

2. Biquandles

We recall the definition of a biquandle and introduce a conjugation biquandle.

Definition 1 ([8, 18]). A *biquandle* is a non-empty set X with binary operations $\underline{*}, \overline{*} : X \times X \to X$ satisfying the following axioms.

(B1) For any $x \in X$, $x \,\underline{*}\, x = x \,\overline{*}\, x$.

(B2) For any $a \in X$, the map $\underline{*}a : X \to X$ sending x to $x \,\underline{*}\, a$ is bijective.
For any $a \in X$, the map $\overline{*}a : X \to X$ sending x to $x \,\overline{*}\, a$ is bijective.
The map $S : X \times X \to X \times X$ defined by $S(x, y) = (y \,\overline{*}\, x, x \,\underline{*}\, y)$ is bijective.

(B3) For any $x, y, z \in X$,

$$(x \,\underline{*}\, y) \,\underline{*}\, (z \,\underline{*}\, y) = (x \,\underline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z),$$

$$(x \,\underline{*}\, y) \,\overline{*}\, (z \,\underline{*}\, y) = (x \,\overline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z),$$

$$(x \,\overline{*}\, y) \,\overline{*}\, (z \,\overline{*}\, y) = (x \,\overline{*}\, z) \,\overline{*}\, (y \,\underline{*}\, z).$$

We remark that $(X, *)$ is a quandle if and only if $(X, *, \overline{*})$ is a biquandle with $x \,\overline{*}\, y = x$. We introduce a conjugation biquandle as an example of a biquandle.

Definition 2. Let G be a group with identity element e, $\overline{*} : G \times G \to G$ a binary operation satisfying the following.

- For any $a \in G$, $\overline{*}a : G \to G$ is a group homomorphism.
- For any $a, b, x \in G$, $x \,\overline{*}\, (ab) = (x \,\overline{*}\, a) \,\overline{*}\, (b \,\overline{*}\, a)$ and $x \,\overline{*}\, e = x$.

Define $a \,\underline{*}\, b := (b^{-1}ab) \,\overline{*}\, b$. Then $(G, \underline{*}, \overline{*})$ is a biquandle. We call it a $\overline{*}$*-conjugation biquandle*, or just call it a *conjugation biquandle*.

It is easy to see that a $\overline{*}$-conjugation biquandle satisfies the conditions in Definition 5. Although the axioms of a $\overline{*}$-conjugation biquandle do not include that of a biquandle, we see that a $\overline{*}$-conjugation biquandle is actually a biquandle by Proposition 3.

In this paper, we often omit brackets. When we omit brackets, we apply binary operations from left on expressions, except for multiplications, which we always apply first. For example, $a *_1 b *_2 cd *_3 (e *_4 f *_5 g)$ stands for $((a *_1 b) *_2 (cd)) *_3 ((e *_4 f) *_5 g)$, where $*_i$ is a binary operation.

We define $\underline{*}^n a := (\underline{*}a)^n$ and $\overline{*}^n a := (\overline{*}a)^n$ for $n \in \mathbb{Z}$. Then $\underline{*}^{-1}a$ and $\overline{*}^{-1}a$ are the inverses of $\underline{*}a$ and $\overline{*}a$, respectively. We also introduce *n-parallel biquandle operations* $\underline{*}^{[n]}$, $\overline{*}^{[n]}$ for any integer n, which are extensions of the operations introduced in [14], where they were defined for $n \in \mathbb{Z}_{\geq 0}$.

Definition 3. Let X be a biquandle. We define two families of binary operations $\underline{*}^{[n]}, \overline{*}^{[n]} : X \times X \to X$ $(n \in \mathbb{Z})$ by the equalities

$$a \underline{*}^{[0]} b = a, \qquad a \underline{*}^{[1]} b = a \underline{*} b, \qquad a \underline{*}^{[i+j]} b = (a \underline{*}^{[i]} b) \underline{*}^{[j]} (b \underline{*}^{[i]} b), \quad (1)$$

$$a \overline{*}^{[0]} b = a, \qquad a \overline{*}^{[1]} b = a \overline{*} b, \qquad a \overline{*}^{[i+j]} b = (a \overline{*}^{[i]} b) \overline{*}^{[j]} (b \overline{*}^{[i]} b) \quad (2)$$

for $i, j \in \mathbb{Z}$.

In Section 9, we see that the binary operations $\underline{*}^{[n]}$ and $\overline{*}^{[n]}$ are well-defined. Since $a = a \underline{*}^{[0]} b = (a \underline{*}^{[-1]} b) \underline{*}^{[1]} (b \underline{*}^{[-1]} b) = (a \underline{*}^{[-1]} b) \underline{*} (b \underline{*}^{[-1]} b)$, we have $a \underline{*}^{[-1]} b = a \underline{*}^{-1} (b \underline{*}^{[-1]} b)$. Then we have the following by using (1).

$$a \underline{*}^{[0]} b = a, \qquad a \underline{*}^{[1]} b = a \underline{*} b, \qquad a \underline{*}^{[2]} b = (a \underline{*} b) \underline{*} (b \underline{*} b),$$

$$a \underline{*}^{[3]} b = ((a \underline{*} b) \underline{*} (b \underline{*} b)) \underline{*} ((b \underline{*} b) \underline{*} (b \underline{*} b)),$$

$$a \underline{*}^{[-1]} b = a \underline{*}^{-1} (b \underline{*}^{[-1]} b), \qquad a \underline{*}^{[-2]} b = (a \underline{*}^{[-1]} b) \underline{*}^{[-1]} (b \underline{*}^{[-1]} b),$$

$$a \underline{*}^{[-3]} b = ((a \underline{*}^{[-1]} b) \underline{*}^{[-1]} (b \underline{*}^{[-1]} b)) \underline{*}^{[-1]} ((b \underline{*}^{[-1]} b) \underline{*}^{[-1]} (b \underline{*}^{[-1]} b)),$$

where we note that $b \underline{*}^{[-1]} b$ is the unique element satisfying $(b \underline{*}^{[-1]} b) \underline{*} (b \underline{*}^{[-1]} b) = b$ (see Lemma 1). We define the *type* of a biquandle X by

$$\text{type } X = \min\{n > 0 \mid a \underline{*}^{[n]} b = a = a \overline{*}^{[n]} b \ (\forall a, b \in X)\}.$$

Any finite biquandle is of finite type [14]. For $m, n \in \mathbb{Z}$, if type $X \mid (m - n)$, then $a \underline{*}^{[m]} b = a \underline{*}^{[n]} b$ and $a \overline{*}^{[m]} b = a \overline{*}^{[n]} b$, since we have

$$a \underline{*}^{[i+\text{type } X]} b = (a \underline{*}^{[i]} b) \underline{*}^{[\text{type } X]} (b \underline{*}^{[i]} b) = a \underline{*}^{[i]} b,$$

$$a \overline{*}^{[i+\text{type } X]} b = (a \overline{*}^{[i]} b) \overline{*}^{[\text{type } X]} (b \overline{*}^{[i]} b) = a \overline{*}^{[i]} b.$$

We give examples of biquandles and their n-parallel biquandle operations below.

Example 1. *Let G be a group, and $X := G^2$. Fix $m, n \in \mathbb{Z}$. We define*

$$(a_1, a_2) \underline{*} (b_1, b_2) = (b_1^{-n} a_1 b_1^n, b_1^{-n} a_2 b_1^n),$$

$$(a_1, a_2) \overline{*} (b_1, b_2) = (a_1, b_1^{-n} b_2^{-m} a_2 b_2^m b_1^n).$$

Then X is a biquandle. We have

$$(a_1, a_2) \underline{*}^{[k]} (b_1, b_2) = (b_1^{-kn} a_1 b_1^{kn}, b_1^{-kn} a_2 b_1^{kn}),$$

$$(a_1, a_2) \overline{*}^{[k]} (b_1, b_2) = (a_1, b_1^{-kn} b_2^{-km} a_2 b_2^{km} b_1^{kn}).$$

Example 2. *Let X be an $R[s^{\pm 1}, t^{\pm 1}]$-module, where R is a commutative ring. We define $a \mathbin{\underline{*}} b = ta + (s-t)b$, $a \mathbin{\overline{*}} b = sa$. Then X is a biquandle, which we call an Alexander biquandle. We have $a \mathbin{\underline{*}}^{[n]} b = t^n a + (s^n - t^n)b$ and $a \mathbin{\overline{*}}^{[n]} b = s^n a$.*

Example 3 ([21]). *A group with the binary operations given in each of the following cases is a biquandle.*

(1) $a \mathbin{\underline{*}} b = a^{-1}$, $a \mathbin{\overline{*}} b = a^{-1}$.

(2) $a \mathbin{\underline{*}} b = b^{-1}ab^{-1}$, $a \mathbin{\overline{*}} b = a^{-1}$.

(3) $a \mathbin{\underline{*}} b = b^{-2}a$, $a \mathbin{\overline{*}} b = b^{-1}a^{-1}b$.

We have

$$a \mathbin{\underline{*}}^{[n]} b = \begin{cases} a \mathbin{\underline{*}} b & \text{if } n \text{ is odd}, \\ a & \text{if } n \text{ is even}, \end{cases} \qquad a \mathbin{\overline{*}}^{[n]} b = \begin{cases} a \mathbin{\overline{*}} b & \text{if } n \text{ is odd}, \\ a & \text{if } n \text{ is even} \end{cases}$$

for each case.

Example 4 ([17]). *Let $R := \{a + bi + cj + dk \in \mathbb{H} \mid a, b, c, d \in \mathbb{Z}\}$, where $\mathbb{H}$ is the ring of quaternions with $i^2 = j^2 = k^2 = ijk = -1$. Let X be an R-module. We define $a \mathbin{\underline{*}} b = -ja + (j+k)b$, $a \mathbin{\overline{*}} b = ja + (k-j)b$. Then X is a biquandle. We have*

$$a \mathbin{\underline{*}}^{[n]} b = \begin{cases} a & \text{if } n = 4m, \\ -ja + (j+k)b & \text{if } n = 4m+1, \\ -a & \text{if } n = 4m+2, \\ ja - (j+k)b & \text{if } n = 4m+3, \end{cases}$$

$$a \mathbin{\overline{*}}^{[n]} b = \begin{cases} a & \text{if } n = 4m, \\ ja + (k-j)b & \text{if } n = 4m+1, \\ -a & \text{if } n = 4m+2, \\ -ja - (k-j)b & \text{if } n = 4m+3. \end{cases}$$

We end this section with a lemma.

Lemma 1. *Let X be a biquandle.*

(1) *For $x, y \in X$, if $x \mathbin{\underline{*}} y = y \mathbin{\overline{*}} x$, then $x = y$.*

(2) *For any $a \in X$, there exists a unique element $\alpha \in X$ such that $\alpha \mathbin{\underline{*}} \alpha = \alpha \mathbin{\overline{*}} \alpha = a$.*

Proof. (1) We have $x = y$ from

$$\begin{aligned} x \mathbin{\overline{*}} x &\overset{\text{(B1)}}{=} x \mathbin{\underline{*}} x = (x \mathbin{\underline{*}} x) \mathbin{\underline{*}} (y \mathbin{\overline{*}} x) \mathbin{\underline{*}}^{-1} (y \mathbin{\overline{*}} x) \\ &\overset{\text{(B3)}}{=} (x \mathbin{\underline{*}} y) \mathbin{\underline{*}} (x \mathbin{\underline{*}} y) \mathbin{\underline{*}}^{-1} (y \mathbin{\overline{*}} x) \\ &= (y \mathbin{\overline{*}} x) \mathbin{\underline{*}} (y \mathbin{\overline{*}} x) \mathbin{\underline{*}}^{-1} (y \mathbin{\overline{*}} x) = y \mathbin{\overline{*}} x. \end{aligned}$$

(2) By axiom (B2), there exists a unique pair $(\alpha_1, \alpha_2) \in X$ such that $(\alpha_2 \mathbin{\overline{*}} \alpha_1, \alpha_1 \mathbin{\underline{*}} \alpha_2) = (a, a)$. Since $\alpha_1 \mathbin{\underline{*}} \alpha_2 = \alpha_2 \mathbin{\overline{*}} \alpha_1$ implies $\alpha_1 = \alpha_2$, we put $\alpha := \alpha_1 = \alpha_2$. Then α is a unique element satisfying $\alpha \mathbin{\underline{*}} \alpha = \alpha \mathbin{\overline{*}} \alpha = a$. □

3. A multiple conjugation biquandle (MCB)

In this section, we introduce the notion of a multiple conjugation biquandle (MCB). We give two equivalent definitions for the multiple conjugation biquandle. The first one is useful to study coloring invariants, and the second one is useful to check that a given algebra is a multiple conjugation biquandle. In the next section, we see that these two definitions are equivalent.

Let X be the disjoint union of groups G_λ $(\lambda \in \Lambda)$. We denote by G_a the group G_λ to which $a \in X$ belongs. We denote by e_λ the identity of G_λ. We also denote it by e_a if $a \in G_\lambda$. The identity of G_a is the element e_a.

Definition 4. A *multiple conjugation biquandle* is a biquandle $(X, \underline{*}, \overline{*})$ which is the disjoint union of groups G_λ $(\lambda \in \Lambda)$ satisfying the following axioms.

- For any $a, x \in X$, $\underline{*}x : G_a \to G_{a \underline{*} x}$ and $\overline{*}x : G_a \to G_{a \overline{*} x}$ are group homomorphisms.
- For any $a, b \in G_\lambda$ and $x \in X$,

$$x \mathbin{\underline{*}} ab = (x \mathbin{\underline{*}} a) \mathbin{\underline{*}} (b \mathbin{\overline{*}} a), \tag{3}$$

$$x \mathbin{\overline{*}} ab = (x \mathbin{\overline{*}} a) \mathbin{\overline{*}} (b \mathbin{\overline{*}} a), \tag{4}$$

$$a^{-1}b \mathbin{\overline{*}} a = ba^{-1} \mathbin{\underline{*}} a. \tag{5}$$

Definition 5. A *multiple conjugation biquandle* X is the disjoint union of groups G_λ $(\lambda \in \Lambda)$ with binary operations $\underline{*}, \overline{*} : X \times X \to X$ satisfying the following axioms.

- For any $x, y, z \in X$,

$$(x \mathbin{\underline{*}} y) \mathbin{\underline{*}} (z \mathbin{\underline{*}} y) = (x \mathbin{\underline{*}} z) \mathbin{\underline{*}} (y \mathbin{\overline{*}} z), \tag{6}$$

$$(x \mathbin{\underline{*}} y) \mathbin{\overline{*}} (z \mathbin{\underline{*}} y) = (x \mathbin{\overline{*}} z) \mathbin{\underline{*}} (y \mathbin{\overline{*}} z), \tag{7}$$

$$(x \mathbin{\overline{*}} y) \mathbin{\overline{*}} (z \mathbin{\overline{*}} y) = (x \mathbin{\overline{*}} z) \mathbin{\overline{*}} (y \mathbin{\underline{*}} z). \tag{8}$$

- For any $a, x \in X$, $\underline{*}x : G_a \to G_{a \underline{*} x}$ and $\overline{*}x : G_a \to G_{a \overline{*} x}$ are group homomorphisms.

- For any $\lambda \in \Lambda$, $a, b \in G_\lambda$ and $x \in X$,

$$x \mathbin{\underline{*}} ab = (x \mathbin{\underline{*}} a) \mathbin{\underline{*}} (b \mathbin{\overline{*}} a), \qquad x \mathbin{\underline{*}} e_\lambda = x, \tag{9}$$

$$x \mathbin{\overline{*}} ab = (x \mathbin{\overline{*}} a) \mathbin{\overline{*}} (b \mathbin{\overline{*}} a), \qquad x \mathbin{\overline{*}} e_\lambda = x, \tag{10}$$

$$a^{-1}b \mathbin{\overline{*}} a = ba^{-1} \mathbin{\underline{*}} a. \tag{11}$$

We remark that a multiple conjugation biquandle consisting of one group is a conjugation biquandle. A G-family of biquandles, defined below, yields a multiple conjugation biquandle (See Proposition 1). We note that the bijectivity in its original axioms of a G-family of biquandles in [14] is replaced with $x \mathbin{\underline{*}}^e y = x \mathbin{\overline{*}}^e y = x$. This refinement is induced from the equivalence of the two definitions of a multiple conjugation biquandle. For details on a G-family of biquandles, we refer the reader to [14].

Definition 6. Let G be a group with identity element e. A *G-family of biquandles* is a non-empty set X with two families of binary operations $\underline{*}^g, \overline{*}^g : X \times X \to X$ $(g \in G)$ satisfying the following axioms.

- For any $x, y, z \in X$ and $g, h \in G$,

$$(x \mathbin{\underline{*}}^g y) \mathbin{\underline{*}}^h (z \mathbin{\overline{*}}^g y) = (x \mathbin{\underline{*}}^h z) \mathbin{\underline{*}}^{h^{-1}gh} (y \mathbin{\underline{*}}^h z),$$

$$(x \mathbin{\overline{*}}^g y) \mathbin{\underline{*}}^h (z \mathbin{\overline{*}}^g y) = (x \mathbin{\underline{*}}^h z) \mathbin{\overline{*}}^{h^{-1}gh} (y \mathbin{\underline{*}}^h z),$$

$$(x \mathbin{\overline{*}}^g y) \mathbin{\overline{*}}^h (z \mathbin{\overline{*}}^g y) = (x \mathbin{\overline{*}}^h z) \mathbin{\overline{*}}^{h^{-1}gh} (y \mathbin{\underline{*}}^h z).$$

- For any $x, y \in X$ and $g, h \in G$,

$$x \mathbin{\underline{*}}^{gh} y = (x \mathbin{\underline{*}}^g y) \mathbin{\underline{*}}^h (y \mathbin{\underline{*}}^g y), \qquad x \mathbin{\underline{*}}^e y = x,$$

$$x \mathbin{\overline{*}}^{gh} y = (x \mathbin{\overline{*}}^g y) \mathbin{\overline{*}}^h (y \mathbin{\overline{*}}^g y), \qquad x \mathbin{\overline{*}}^e y = x,$$

$$x \mathbin{\underline{*}}^g x = x \mathbin{\overline{*}}^g x.$$

Proposition 1 ([14]). *Let* $(X, (\underline{*}^g)_{g \in G}, (\overline{*}^g)_{g \in G})$ *be a G-family of biquandles. Then* $X \times G = \bigsqcup_{x \in X} \{x\} \times G$ *is a multiple conjugation biquandle with the binary operations* $\underline{*}, \overline{*} : (X \times G) \times (X \times G) \to X \times G$ *defined by*

$$(x, g) \mathbin{\underline{*}} (y, h) = (x \mathbin{\underline{*}}^h y, h^{-1}gh), \qquad (x, g) \mathbin{\overline{*}} (y, h) = (x \mathbin{\overline{*}}^h y, g).$$

We call this multiple conjugation biquandle the *associated multiple conjugation biquandle*.

A biquandle turns into a G-family of biquandles with parallel biquandle operations $\underline{*}^{[n]}$, $\overline{*}^{[n]}$ (see Proposition 6). Therefore we can construct a multiple conjugation biquandle from any biquandle. We introduce a G-family of (generalized) Alexander biquandles in the following proposition.

Proposition 2. *Let G be a group with identity e, and let $\varphi : G \to Z(G)$ be a homomorphism, where $Z(G)$ is the center of G.*

(1) *Let X be a group with a right action of G. We denote by x^g the result of g acting on x. We define binary operations $\underline{*}^g, \overline{*}^g : X \times X \to X$ by $x \,\underline{*}^g\, y = (xy^{-1})^g y^{\varphi(g)}$, $x \,\overline{*}^g\, y = x^{\varphi(g)}$. Then X is a G-family of biquandles, which we call a G-family of generalized Alexander biquandles.*

(2) *Let R be a ring and X a right $R[G]$-module, where $R[G]$ is the group ring of G over R. We define binary operations $\underline{*}^g, \overline{*}^g : X \times X \to X$ by $x \,\underline{*}^g\, y = xg + y(\varphi(g) - g)$, $x \,\overline{*}^g\, y = x\varphi(g)$. Then X is a G-family of biquandles, which we call a G-family of Alexander biquandles.*

Proof. It is sufficient to show (1), since (2) follows from (1) with an abelian group X. We note that $(xy)^g = x^g y^g$ holds since the map which sends x to x^g is a group homomorphism. For any $x, y, z \in X$ and $g, h \in G$, we have

$$\begin{aligned}(x \,\underline{*}^g\, y) \,\underline{*}^h\, (z \,\overline{*}^g\, y) &= x^{gh} y^{-gh} y^{\varphi(g)h} z^{-\varphi(g)h} z^{\varphi(g)\varphi(h)} \\ &= (x \,\underline{*}^h\, z) \,\underline{*}^{h^{-1}gh}\, (y \,\underline{*}^h\, z),\end{aligned}$$

$$(x \,\overline{*}^g\, y) \,\underline{*}^h\, (z \,\overline{*}^g\, y) = x^{\varphi(g)h} z^{-\varphi(g)h} z^{\varphi(g)\varphi(h)} = (x \,\underline{*}^h\, z) \,\overline{*}^{h^{-1}gh}\, (y \,\underline{*}^h\, z),$$

$$(x \,\overline{*}^g\, y) \,\overline{*}^h\, (z \,\overline{*}^g\, y) = x^{\varphi(g)\varphi(h)} = (x \,\overline{*}^h\, z) \,\overline{*}^{h^{-1}gh}\, (y \,\underline{*}^h\, z)$$

and

$$x \,\underline{*}^{gh}\, y = (xy^{-1})^{gh} y^{\varphi(g)\varphi(h)} = (x \,\underline{*}^g\, y) \,\underline{*}^h\, (y \,\underline{*}^g\, y), \qquad x \,\underline{*}^e\, y = x,$$

$$x \,\overline{*}^{gh}\, y = x^{\varphi(g)\varphi(h)} = (x \,\overline{*}^g\, y) \,\overline{*}^h\, (y \,\overline{*}^g\, y), \qquad x \,\overline{*}^e\, y = x,$$

$$x \,\underline{*}^g\, x = x^{\varphi(g)} = x \,\overline{*}^g\, x,$$

where x^{-g} denotes $(x^g)^{-1}$, which coincides with $(x^{-1})^g$. □

4. The two definitions are equivalent

In this section, we see that the two definitions of a multiple conjugation biquandle introduced in the previous section are equivalent.

Lemma 2. *Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation biquandle in the sense of Definition* 4.

(1) *For any $x \in X$ and $\lambda \in \Lambda$,*

$$x \,\underline{*}\, e_\lambda = x, \qquad x \,\overline{*}\, e_\lambda = x. \tag{12}$$

(2) *For any* $a, x \in X$, $\underline{*}x : G_a \to G_{a \underline{*} x}$ *and* $\overline{*}x : G_a \to G_{a \overline{*} x}$ *are bijections. Furthermore,* $\underline{*}^{-1}x = \underline{*}(x^{-1} \overline{*} x)$, $\overline{*}^{-1}x = \overline{*}(x^{-1} \overline{*} x)$.

Proof. (1) Let $\alpha \in X$ be the unique element satisfying $\alpha \underline{*} \alpha = \alpha \overline{*} \alpha = e_\lambda$. Then

$$x \underline{*} e_\lambda = x \underline{*} e_{\alpha \overline{*} \alpha} = ((x \underline{*}^{-1} \alpha) \underline{*} \alpha) \underline{*} (e_\alpha \overline{*} \alpha) \overset{(3)}{=} (x \underline{*}^{-1} \alpha) \underline{*} \alpha e_\alpha = x,$$

$$x \overline{*} e_\lambda = x \overline{*} e_{\alpha \overline{*} \alpha} = ((x \overline{*}^{-1} \alpha) \overline{*} \alpha) \overline{*} (e_\alpha \overline{*} \alpha) \overset{(4)}{=} (x \overline{*}^{-1} \alpha) \overline{*} \alpha e_\alpha = x.$$

(2) Since the maps $\underline{*}x : X \to X$, $\overline{*}x : X \to X$ are bijective, it is sufficient to show that

$$b \underline{*} x \in G_{a \underline{*} x} \Leftrightarrow b \in G_a \Leftrightarrow b \overline{*} x \in G_{a \overline{*} x}.$$

We have $b \in G_a \Rightarrow b \underline{*} x \in G_{a \underline{*} x}$ and $b \in G_a \Rightarrow b \overline{*} x \in G_{a \overline{*} x}$ by the well-definedness of the maps $\underline{*}x : G_a \to G_{a \underline{*} x}$ and $\overline{*}x : G_a \to G_{a \overline{*} x}$, respectively. We have $b \underline{*} x \in G_{a \underline{*} x} \Rightarrow b \in G_a$ and $b \overline{*} x \in G_{a \overline{*} x} \Rightarrow b \in G_a$ by the equalities

$$(a \underline{*} x) \underline{*} (x^{-1} \overline{*} x) = a = (a \overline{*} x) \overline{*} (x^{-1} \overline{*} x),$$

$$(b \underline{*} x) \underline{*} (x^{-1} \overline{*} x) = b = (b \overline{*} x) \overline{*} (x^{-1} \overline{*} x),$$

which follow from

$$(y \underline{*} x) \underline{*} (x^{-1} \overline{*} x) \overset{(3)}{=} y \underline{*} xx^{-1} = y \underline{*} e_x \overset{(12)}{=} y,$$

$$(y \overline{*} x) \overline{*} (x^{-1} \overline{*} x) \overset{(4)}{=} y \overline{*} xx^{-1} = y \overline{*} e_x \overset{(12)}{=} y$$

for any $y \in X$. □

Proposition 3. *Let X be the disjoint union of groups G_λ $(\lambda \in \Lambda)$ with binary operations $\underline{*}, \overline{*} : X \times X \to X$. Then X is an MCB in the sense of Definition* 4 *if and only if X is an MCB in the sense of Definition* 5.

Proof. By Lemma 2 (1), it is sufficient to show the "if" part. For any $x \in X$, we have

$$x \underline{*} x = x^2 x^{-1} \underline{*} x \overset{(11)}{=} x^{-1} x^2 \overline{*} x = x \overline{*} x. \tag{13}$$

The map $\underline{*}(a^{-1} \overline{*} a) : X \to X$ is the inverse of $\underline{*}a : X \to X$, since we have

$$(x \underline{*} a) \underline{*} (a^{-1} \overline{*} a) \overset{(9)}{=} x \underline{*} aa^{-1} = x \underline{*} e_a \overset{(9)}{=} x, \tag{14}$$

and

$$
\begin{aligned}
x \underline{*} (a^{-1} \overline{*} a) \underline{*} a &\overset{(14)}{=} (x \underline{*} (a^{-1} \overline{*} a)) \underline{*} ((a \underline{*} a) \underline{*} (a^{-1} \overline{*} a)) \\
&\overset{(6)}{=} (x \underline{*} (a \underline{*} a)) \underline{*} ((a^{-1} \overline{*} a) \overline{*} (a \underline{*} a)) \\
&\overset{(13)}{=} (x \underline{*} (a \overline{*} a)) \underline{*} ((a \overline{*} a)^{-1} \overline{*} (a \overline{*} a)) \\
&\overset{(14)}{=} x.
\end{aligned}
$$

Therefore the map $\underline{*}a : X \to X$ is bijective.

The map $\overline{*}(a^{-1} \overline{*} a) : X \to X$ is the inverse of $\overline{*}a : X \to X$, since we have

$$
(x \overline{*} a) \overline{*} (a^{-1} \overline{*} a) \overset{(10)}{=} x \overline{*} aa^{-1} = x \overline{*} e_a \overset{(10)}{=} x, \tag{15}
$$

and

$$
\begin{aligned}
x \overline{*} (a^{-1} \overline{*} a) \overline{*} a &\overset{(14)}{=} (x \overline{*} (a^{-1} \overline{*} a)) \overline{*} ((a \underline{*} a) \underline{*} (a^{-1} \overline{*} a)) \\
&\overset{(8)}{=} (x \overline{*} (a \underline{*} a)) \overline{*} ((a^{-1} \overline{*} a) \overline{*} (a \underline{*} a)) \\
&\overset{(13)}{=} (x \overline{*} (a \overline{*} a)) \overline{*} ((a \overline{*} a)^{-1} \overline{*} (a \overline{*} a)) \\
&\overset{(15)}{=} x.
\end{aligned}
$$

Therefore the map $\overline{*}a : X \to X$ is bijective.

We show that the map $S : X \times X \to X \times X$ defined by $S(x, y) = (y \overline{*} x, x \underline{*} y)$ is the bijection whose inverse $T : X \times X \to X \times X$ is given by

$$
T(x, y) = (y \underline{*} (x \overline{*} x \overline{*}^{-1} y)^{-1}, x \overline{*} (y \underline{*} y \underline{*}^{-1} x)^{-1}),
$$

where we note that

$$
\begin{aligned}
y \underline{*} (x \overline{*} x \overline{*}^{-1} y)^{-1} &= ((y \overline{*} y) \overline{*}^{-1} y) \underline{*} ((x^{-1} \overline{*} x) \overline{*}^{-1} y) \\
&= ((y \overline{*} y) \overline{*} (y^{-1} \overline{*} y)) \underline{*} ((x^{-1} \overline{*} x) \overline{*} (y^{-1} \overline{*} y)) \\
&\overset{(7)}{=} ((y \overline{*} y) \underline{*} (x^{-1} \overline{*} x)) \overline{*} ((y^{-1} \overline{*} y) \underline{*} (x^{-1} \overline{*} x)) \\
&= ((y \overline{*} y) \underline{*}^{-1} x) \overline{*} ((y^{-1} \overline{*} y) \underline{*}^{-1} x) \\
&= (y \overline{*} y \underline{*}^{-1} x) \overline{*} (y \overline{*} y \underline{*}^{-1} x)^{-1} \\
&\overset{(13)}{=} (y \underline{*} y \underline{*}^{-1} x) \overline{*} (y \underline{*} y \underline{*}^{-1} x)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
x \overline{*} (y \underline{*} y \underline{*}^{-1} x)^{-1} &\overset{(13)}{=} x \overline{*} (y \overline{*} y \underline{*}^{-1} x)^{-1} \\
&= ((x \underline{*} x) \underline{*}^{-1} x) \overline{*} ((y^{-1} \overline{*} y) \underline{*}^{-1} x) \\
&= ((x \underline{*} x) \underline{*} (x^{-1} \overline{*} x)) \overline{*} ((y^{-1} \overline{*} y) \underline{*} (x^{-1} \overline{*} x)) \\
&\overset{(7)}{=} ((x \underline{*} x) \overline{*} (y^{-1} \overline{*} y)) \underline{*} ((x^{-1} \overline{*} x) \overline{*} (y^{-1} \overline{*} y)) \\
&= ((x \underline{*} x) \overline{*}^{-1} y) \underline{*} ((x^{-1} \overline{*} x) \overline{*}^{-1} y) \\
&= (x \underline{*} x \overline{*}^{-1} y) \underline{*} (x \overline{*} x \overline{*}^{-1} y)^{-1} \\
&\overset{(13)}{=} (x \overline{*} x \overline{*}^{-1} y) \underline{*} (x \overline{*} x \overline{*}^{-1} y)^{-1}.
\end{aligned}
$$

Then $T \circ S = \mathrm{id}_{X \times X}$ and $S \circ T = \mathrm{id}_{X \times X}$ follow from

$$
\begin{aligned}
&(x \underline{*} y) \underline{*} ((y \overline{*} x) \overline{*} (y \overline{*} x) \overline{*}^{-1} (x \underline{*} y))^{-1} \\
&\quad \overset{(8)}{=} (x \underline{*} y) \underline{*} ((y \overline{*} y) \overline{*} (x \underline{*} y) \overline{*}^{-1} (x \underline{*} y))^{-1} \\
&\quad = (x \underline{*} y) \underline{*} (y \overline{*} y)^{-1} = (x \underline{*} y) \underline{*} (y^{-1} \overline{*} y) \overset{(14)}{=} x, \\
&(y \overline{*} x) \overline{*} ((x \underline{*} y) \underline{*} (x \underline{*} y) \underline{*}^{-1} (y \overline{*} x))^{-1} \\
&\quad \overset{(6)}{=} (y \overline{*} x) \overline{*} ((x \underline{*} x) \underline{*} (y \overline{*} x) \underline{*}^{-1} (y \overline{*} x))^{-1} \\
&\quad \overset{(13)}{=} (y \overline{*} x) \overline{*} (x \overline{*} x)^{-1} = (y \overline{*} x) \overline{*} (x^{-1} \overline{*} x) \overset{(15)}{=} y
\end{aligned}
$$

and

$$
\begin{aligned}
&(x \overline{*} (y \underline{*} y \underline{*}^{-1} x)^{-1}) \overline{*} (y \underline{*} (x \overline{*} x \overline{*}^{-1} y)^{-1}) \\
&\quad = (x \overline{*} (y \underline{*} y \underline{*}^{-1} x)^{-1}) \overline{*} ((y \underline{*} y \underline{*}^{-1} x) \overline{*} (y \underline{*} y \underline{*}^{-1} x)^{-1}) \overset{(15)}{=} x, \\
&(y \underline{*} (x \overline{*} x \overline{*}^{-1} y)^{-1}) \underline{*} (x \overline{*} (y \underline{*} y \underline{*}^{-1} x)^{-1}) \\
&\quad = (y \underline{*} (x \overline{*} x \overline{*}^{-1} y)^{-1}) \underline{*} ((x \overline{*} x \overline{*}^{-1} y) \underline{*} (x \overline{*} x \overline{*}^{-1} y)^{-1}) \\
&\quad \overset{(6)}{=} (y \underline{*} (x \overline{*} x \overline{*}^{-1} y)) \underline{*} ((x \overline{*} x \overline{*}^{-1} y)^{-1} \overline{*} (x \overline{*} x \overline{*}^{-1} y)) \overset{(14)}{=} y,
\end{aligned}
$$

respectively. This completes the proof. □

5. MCB colorings for handlebody-links

In this section we recall a diagrammatic presentation of a handlebody-link and consider its colorings using a multiple conjugation biquandle.

A *handlebody-link* is the disjoint union of handlebodies embedded in the 3-sphere S^3. A *handlebody-knot* is a one component handlebody-link. In this paper, we assume that every component of a handlebody-link is of genus at least 1. An S^1-*orientation* of a handlebody-link is a collection of S^1-orientations of all genus-1 components, that are solid tori, of the handlebody-link. Here an S^1-orientation of a solid torus means an orientation of its core S^1. Two S^1-oriented handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other preserving the S^1-orientation.

A *Y-orientation* of a trivalent graph G, whose vertices are of valency 3, is a direction of all edges of G satisfying that every vertex of G is both the initial vertex of a directed edge and the terminal vertex of a directed edge (See Figure 1). In this paper, a trivalent graph may have a circle component, which has no vertices.

A finite graph embedded in S^3 is called a *spatial graph*. For a Y-oriented spatial trivalent graph K and an S^1-oriented handlebody-link H, we say that K *represents* H if H is a regular neighborhood of K and the S^1-orientation of H agrees with the Y-orientation. Then any S^1-oriented handlebody-link can be represented by some Y-oriented spatial trivalent graph. *R1–R6 moves* are local moves depicted in Figure 3. *Y-oriented R1–R6 moves* are R1–R6 moves between two diagrams with Y-orientations which are identical except in the disk where the move applied. All Y-oriented R6 moves are listed in Figure 2. The following theorem plays a fundamental role in constructing S^1-oriented handlebody-link invariants.

Theorem 1 ([11]). *For a diagram D_i of a Y-oriented spatial trivalent graph K_i $(i = 1, 2)$, K_1 and K_2 represent an equivalent S^1-oriented handlebody-link if and only if D_1 and D_2 are related by a finite sequence of Y-oriented* R1–R6 *moves.*

For a diagram D of a Y-oriented spatial trivalent graph, we denote by $\mathcal{SA}(D)$ the set of semi-arcs of D, where a semi-arc is a piece of a curve each of whose endpoints is a crossing or a vertex.

Fig. 1. Y-orientations

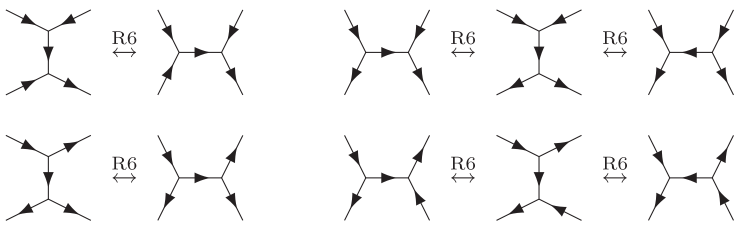

Fig. 2. All Y-oriented R6 moves

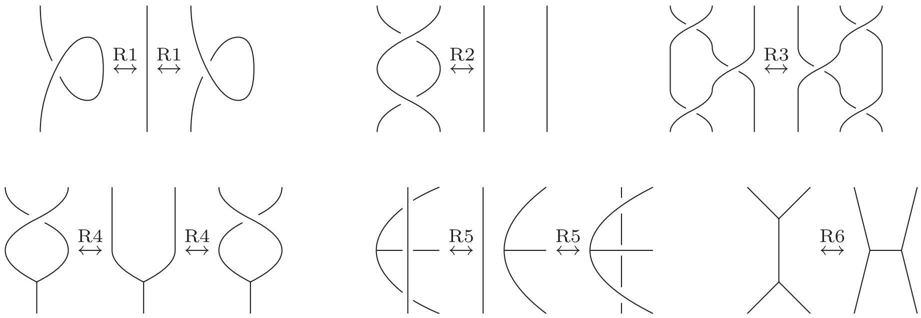

Fig. 3. The Reidemeister moves for handlebody-links

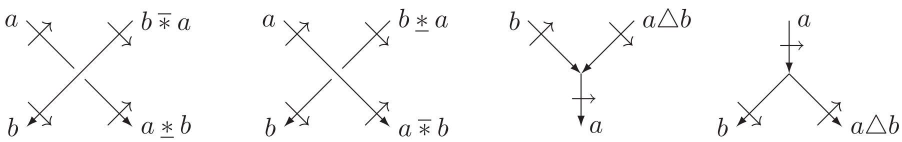

Fig. 4. MCB coloring conditions

Definition 7. Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation biquandle. We define $a \triangle b := b^{-1} a \overline{*} b$ for $a, b \in G_\lambda$. Let D be a diagram of an S^1-oriented handlebody-link H. An X-*coloring* of D is a map $C : \mathscr{SA}(D) \to X$ satisfying the conditions depicted in Figure 4 at each crossing and vertex, where the normal orientation is obtained by rotating the usual orientation counterclockwise by $\pi/2$ on the diagram. We denote by $\mathrm{Col}_X(D)$ the set of X-colorings of D.

Theorem 2. *Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation biquandle. Let D be a diagram of an S^1-oriented handlebody-link H. Let D' be a diagram obtained by applying one of the Y-oriented* R1–R6 *moves to the diagram D once. For an X-coloring C of D, there is a unique X-coloring C' of D' which coincides with C except the place where the move is applied.*

We prove this theorem in the next section. Here we introduce the primitive conditions for the proof and the universality discussed in Section 7.

Let X be a biquandle, P a subset of $X \times X$ and $\triangle : P \to X$ a map. We write $a \sim b$ if $(a, b) \in P$, and denote by $a \triangle b$ the image of (a, b) under the map $\triangle : P \to X$. We define an $(X, P, \triangle)$-coloring to be a map $C : \mathscr{S}\mathscr{A}(D) \to X$ satisfying the conditions in Definition 7 at crossings and vertices. The following conditions (16)–(23), which we call the *primitive conditions*, are the conditions on $(X, P, \triangle)$ which guarantee, for each of the Reidemeister moves R4–R6, that there is a natural one-to-one correspondence between the $(X, P, \triangle)$-colorings of two diagrams related by the move (see Figure 5, where all arcs are directed from top to bottom, except for the Reidemeister moves R4).

(R4) For any $a, b, x \in X$,

$$a \sim b, \qquad x = a \triangle b \Leftrightarrow a \underline{*} b \sim x, \qquad (a \underline{*} b) \triangle x = b \overline{*} a, \tag{16}$$

$$a \sim b, \qquad x = a \triangle b \Leftrightarrow a \overline{*} b \sim x, \qquad (a \overline{*} b) \triangle x = b \underline{*} a. \tag{17}$$

(R5) For any $a, b, x \in X$,

$$a \sim b \Leftrightarrow a \underline{*} x \sim b \underline{*} x$$

$$\Rightarrow (x \overline{*} b) \overline{*} (a \triangle b) = x \overline{*} a, \ (a \triangle b) \underline{*} (x \overline{*} b) = (a \underline{*} x) \triangle (b \underline{*} x), \tag{18}$$

$$a \sim b \Leftrightarrow a \overline{*} x \sim b \overline{*} x$$

$$\Rightarrow (x \underline{*} b) \underline{*} (a \triangle b) = x \underline{*} a, \ (a \triangle b) \overline{*} (x \underline{*} b) = (a \overline{*} x) \triangle (b \overline{*} x). \tag{19}$$

(R6) For any $a, b, c, x \in X$,

$$a \sim b, \ b \sim c, \ x = b \triangle c \Rightarrow a \sim c, \ a \triangle c \sim x, \ (a \triangle c) \triangle x = a \triangle b, \tag{20}$$

$$\exists! b \in X \text{ s.t. } a \sim b, \ b \sim c, \ x = b \triangle c, \ (a \triangle c) \triangle x = a \triangle b \Leftarrow a \sim c, \ a \triangle c \sim x, \tag{21}$$

$$a \sim b, \ a \sim c, \ x = a \triangle c \Rightarrow b \sim c, \ x \sim b \triangle c, \ x \triangle (b \triangle c) = a \triangle b, \tag{22}$$

$$\exists! a \in X \text{ s.t. } a \sim b, \ a \sim c, \ x = a \triangle c, \ x \triangle (b \triangle c) = a \triangle b \Leftarrow b \sim c, \ x \sim b \triangle c. \tag{23}$$

6. Proof of Theorem 2

Lemma 3. *Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be a multiple conjugation biquandle with $a \triangle b := b^{-1} a \overline{*} b$. We have the following.*

- *For any $a \in X$,*

$$\triangle a : G_a \to G_{a \triangle a} \text{ which sends } x \text{ to } x \triangle a \text{ is a bijection.} \tag{24}$$

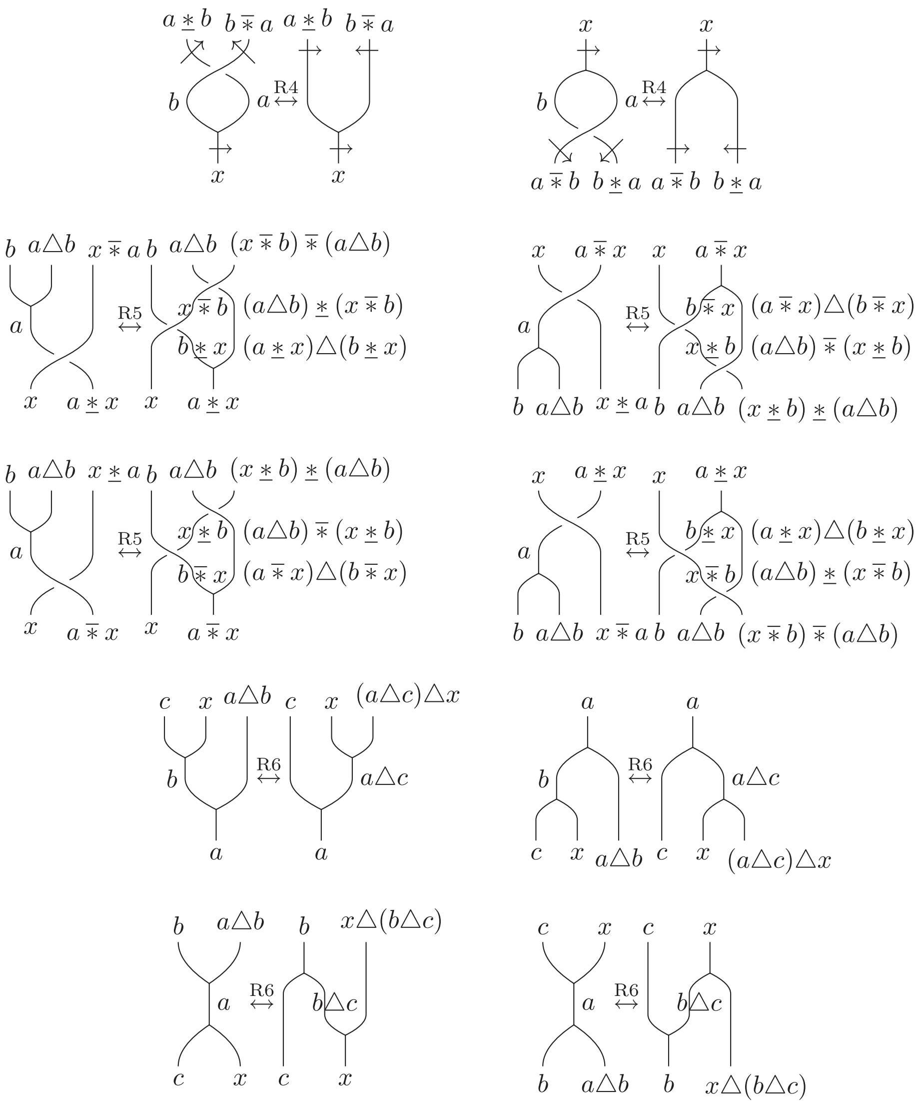

Fig. 5. Colored Reidemeister moves

- *For any* $a, x \in X$,

$$\underline{*}x : G_a \to G_{a \underline{*} x} \ \textit{and} \ \overline{*}x : G_a \to G_{a \overline{*} x} \ \textit{are bijections.} \tag{25}$$

- *For any* $a, b \in G_\lambda$,

$$G_{a \underline{*} b} = G_{a \triangle b}, \qquad (a \underline{*} b) \triangle (a \triangle b) = b \overline{*} a, \tag{26}$$

$$G_{a \overline{*} b} = G_{a \triangle b}, \qquad (a \overline{*} b) \triangle (a \triangle b) = b \underline{*} a. \tag{27}$$

- *For any* $a, b \in G_\lambda$ *and* $x \in X$,

$$(a \triangle b) \mathbin{\underline{*}} (x \mathbin{\overline{*}} b) = (a \mathbin{\underline{*}} x) \triangle (b \mathbin{\underline{*}} x), \tag{28}$$

$$(a \triangle b) \mathbin{\overline{*}} (x \mathbin{\underline{*}} b) = (a \mathbin{\overline{*}} x) \triangle (b \mathbin{\overline{*}} x), \tag{29}$$

$$(x \mathbin{\underline{*}} b) \mathbin{\underline{*}} (a \triangle b) = x \mathbin{\underline{*}} a, \tag{30}$$

$$(x \mathbin{\overline{*}} b) \mathbin{\overline{*}} (a \triangle b) = x \mathbin{\overline{*}} a. \tag{31}$$

- *For any* $a, b, c \in G_\lambda$,

$$(a \triangle c) \triangle (b \triangle c) = a \triangle b. \tag{32}$$

Proof. • The map $\triangle a : G_a \to G_{a \triangle a}$ is a well-defined bijection, since it is the composition of the bijections $a^{-1} \cdot : G_a \to G_a$ defined by $a^{-1} \cdot x = a^{-1}x$ and $\overline{*} a : G_a \to G_{a \overline{*} a} = G_{a \triangle a}$.
- By Lemma 2 (2), $\underline{*} x : G_a \to G_{a \underline{*} x}$ and $\overline{*} x : G_a \to G_{a \overline{*} x}$ are well-defined bijections.
- For $a, b \in G_\lambda$, we have $G_{a \underline{*} b} = G_{a \triangle b} = G_{a \overline{*} b}$, since

$$ab^{-1} \in G_a, \qquad a \triangle b \overset{(5)}{=} ab^{-1} \mathbin{\underline{*}} b \in G_{a \underline{*} b},$$

$$b^{-1}a \in G_a, \qquad a \triangle b = b^{-1}a \mathbin{\overline{*}} b \in G_{a \overline{*} b}.$$

For $a, b \in G_\lambda$, we have

$$\begin{aligned}(a \mathbin{\underline{*}} b) \triangle (a \triangle b) &\overset{(5)}{=} (b^{-1}ab \mathbin{\overline{*}} b) \triangle (b^{-1}a \mathbin{\overline{*}} b) \\ &= (b^{-1}a \mathbin{\overline{*}} b)^{-1}(b^{-1}ab \mathbin{\overline{*}} b) \mathbin{\overline{*}} (b^{-1}a \mathbin{\overline{*}} b) \\ &= (b \mathbin{\overline{*}} b) \mathbin{\overline{*}} (b^{-1}a \mathbin{\overline{*}} b) \overset{(4)}{=} b \mathbin{\overline{*}} a, \\ (a \mathbin{\overline{*}} b) \triangle (a \triangle b) &= (b^{-1}a \mathbin{\overline{*}} b)^{-1}(a \mathbin{\overline{*}} b) \mathbin{\overline{*}} (b^{-1}a \mathbin{\overline{*}} b) \\ &= (a^{-1}ba \mathbin{\overline{*}} b) \mathbin{\overline{*}} (b^{-1}a \mathbin{\overline{*}} b) \overset{(4)}{=} a^{-1}ba \mathbin{\overline{*}} a \overset{(5)}{=} b \mathbin{\underline{*}} a.\end{aligned}$$

- For $a, b \in G_\lambda$ and $x \in X$, we have

$$\begin{aligned}(a \triangle b) \mathbin{\underline{*}} (x \mathbin{\overline{*}} b) &= (b^{-1}a \mathbin{\overline{*}} b) \mathbin{\underline{*}} (x \mathbin{\overline{*}} b) \overset{\text{(B3)}}{=} (b^{-1}a \mathbin{\underline{*}} x) \mathbin{\overline{*}} (b \mathbin{\underline{*}} x) \\ &= (b \mathbin{\underline{*}} x)^{-1}(a \mathbin{\underline{*}} x) \mathbin{\overline{*}} (b \mathbin{\underline{*}} x) = (a \mathbin{\underline{*}} x) \triangle (b \mathbin{\underline{*}} x), \\ (a \triangle b) \mathbin{\overline{*}} (x \mathbin{\underline{*}} b) &= (b^{-1}a \mathbin{\overline{*}} b) \mathbin{\overline{*}} (x \mathbin{\underline{*}} b) \overset{\text{(B3)}}{=} (b^{-1}a \mathbin{\overline{*}} x) \mathbin{\overline{*}} (b \mathbin{\overline{*}} x) \\ &= (b \mathbin{\overline{*}} x)^{-1}(a \mathbin{\overline{*}} x) \mathbin{\overline{*}} (b \mathbin{\overline{*}} x) = (a \mathbin{\overline{*}} x) \triangle (b \mathbin{\overline{*}} x).\end{aligned}$$

- For $a, b \in G_\lambda$ and $x \in X$, we have

$$(x \underline{*} b) \underline{*} (a \triangle b) = (x \underline{*} b) \underline{*} (b^{-1} a \overline{*} b) \overset{(3)}{=} x \underline{*} a,$$

$$(x \overline{*} b) \overline{*} (a \triangle b) = (x \overline{*} b) \overline{*} (b^{-1} a \overline{*} b) \overset{(4)}{=} x \overline{*} a.$$

- For $a, b, c \in G_\lambda$, we have

$$\begin{aligned} (a\triangle c)\triangle(b\triangle c) &= (c^{-1}a \overline{*} c)\triangle(c^{-1}b \overline{*} c) \\ &= (c^{-1}b \overline{*} c)^{-1}(c^{-1}a \overline{*} c) \overline{*} (c^{-1}b \overline{*} c) \\ &= (b^{-1}a \overline{*} c) \overline{*} (c^{-1}b \overline{*} c) \\ &\overset{(4)}{=} b^{-1}a \overline{*} b = a\triangle b. \end{aligned}$$ □

Proof (Proof of Theorem 2). We see that $(X, \bigsqcup_{\lambda \in \Lambda} G_\lambda^2, \triangle)$ satisfies the primitive conditions (16)–(23). By Lemma 3, it is sufficient to show

$$b \in G_a, \qquad x = a\triangle b \Leftarrow x \in G_{a \underline{*} b}, \qquad (a \underline{*} b)\triangle x = b \overline{*} a, \tag{33}$$

$$b \in G_a, \qquad x = a\triangle b \Leftarrow x \in G_{a \overline{*} b}, \qquad (a \overline{*} b)\triangle x = b \underline{*} a \tag{34}$$

for $a, b, c, x \in X$. The other conditions are easily verified, where we note that $b = x\triangle^{-1}c \sim a$ and $a = x\triangle^{-1}c \sim b$ for (21) and (23), respectively.

We show (33). Put $c := x \underline{*}^{-1} b \in G_a$. Then

$$(a\triangle c) \underline{*} (b \overline{*} c) \overset{(28)}{=} (a \underline{*} b)\triangle(c \underline{*} b) = (a \underline{*} b)\triangle x = b \overline{*} a \overset{(31)}{=} (b \overline{*} c) \overline{*} (a\triangle c).$$

By Lemma 1, we have $a\triangle c = b \overline{*} c$. Since $a\triangle c \in G_{a\triangle c} = G_{a \overline{*} c}$, we have $b = (a\triangle c) \overline{*}^{-1} c \in G_a$. The equality $x = a\triangle b$ follows from

$$x\triangle(c\triangle b) = (c \underline{*} b)\triangle(c\triangle b) \overset{(26)}{=} b \overline{*} c = a\triangle c \overset{(32)}{=} (a\triangle b)\triangle(c\triangle b).$$

Then we have (33). We show (34). Put $c := x \overline{*}^{-1} b \in G_a$. Then

$$(a\triangle c) \overline{*} (b \underline{*} c) \overset{(29)}{=} (a \overline{*} b)\triangle(c \overline{*} b) = (a \overline{*} b)\triangle x = b \underline{*} a \overset{(30)}{=} (b \underline{*} c) \underline{*} (a\triangle c).$$

By Lemma 1, we have $a\triangle c = b \underline{*} c$. Since $a\triangle c \in G_{a\triangle c} = G_{a \underline{*} c}$, we have $b = (a\triangle c) \underline{*}^{-1} c \in G_a$. The equality $x = a\triangle b$ follows from

$$x\triangle(c\triangle b) = (c \overline{*} b)\triangle(c\triangle b) \overset{(27)}{=} b \underline{*} c = a\triangle c \overset{(32)}{=} (a\triangle b)\triangle(c\triangle b).$$

Then we have (34). □

7. The universality of an MCB

In this section, we see that a multiple conjugation biquandle is the universal biquandle for defining coloring invariants for S^1-oriented handlebody-links.

Theorem 3. *Let X be a biquandle, P a subset of $X \times X$ and $\triangle : P \to X$ a map. We write $a \sim b$ if $(a, b) \in P$. Suppose $(X, P, \triangle)$ satisfies the primitive conditions* (16)–(23).

(1) *We define $X_1 := \{b \in X \mid \text{there exists } a \in X \text{ such that } a \sim b\}$, $X_2 := X - X_1$. Then X_1, X_2 are subbiquandles of X satisfying*

$$X_1 \,\underline{*}\, a = X_1 \,\overline{*}\, a = X_1, \qquad X_2 \,\underline{*}\, a = X_2 \,\overline{*}\, a = X_2$$

for any $a \in X$, where $X_i \,\underline{*}\, a = \{x \,\underline{*}\, a \mid x \in X_i\}$, $X_i \,\overline{*}\, a = \{x \,\overline{*}\, a \mid x \in X_i\}$.

(2) *The relation $\sim$ is an equivalence relation on X_1.*

(3) *Let $X_1 = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be the partition of X_1 determined by the equivalence relation $\sim$, that is, $a \sim b$ if and only if $a, b \in G_\lambda$ for some $\lambda \in \Lambda$. Then X_1 is a multiple conjugation biquandle, where the group structure of G_λ is given by $ab = a \,\underline{*}\, b \triangle^{-1} b$, $e_\lambda = a \triangle a \,\underline{*}^{-1}\, a$, $a^{-1} = a \triangle a \,\underline{*}^{-1}\, a \triangle a \,\underline{*}^{-1}\, a$.*

By the definition, elements in X_2 cannot be used for colorings at a vertex. For a handlebody-knot of genus greater than one, we see that they also cannot be used for colors of any arcs. In this sense, an MCB is the universal biquandle for S^1-oriented handlebody-links. A multiple conjugation quandle (MCQ) [10] was introduced as the universal symmetric quandle for unoriented handlebody-links in the same sense, where we note that the axioms of an MCQ coincide with that of an MCB under the assumption that $x \,\overline{*}\, y = x$.

In [9], Iijima showed that an MCQ is also the universal quandle for S^1-oriented handlebody-links, although it was introduced as the universal symmetric quandle for unoriented handlebody-links. As a corollary of Theorem 3, we also have this universality. In [14], Nelson and the first author introduced the notion of a partially multiplicative biquandle.

Definition 8 ([14]). A *partially multiplicative biquandle* (PMB) is a biquandle X with a subset $\tilde{P}$ of $X \times X$ and a map $\bullet : \tilde{P} \to X$ satisfying the following axioms, where $a \bullet b$ stands for $\bullet(a, b)$.

(i) $x \mapsto a \bullet x$, $x \mapsto x \bullet b$ are injective.

(ii) $(a, b \,\underline{*}\, a) \in \tilde{P} \Leftrightarrow (b, a \,\overline{*}\, b) \in \tilde{P} \Rightarrow a \bullet (b \,\underline{*}\, a) = b \bullet (a \,\overline{*}\, b)$.

(iii) $(a, b) \in \tilde{P} \Leftrightarrow (a \,\underline{*}\, x, b \,\underline{*}\, (x \,\overline{*}\, a)) \in \tilde{P} \Leftrightarrow (a \,\overline{*}\, x, b \,\overline{*}\, (x \,\underline{*}\, a)) \in \tilde{P} \Rightarrow$

$$x \,\underline{*}\, (a \bullet b) = (x \,\underline{*}\, a) \,\underline{*}\, b, \qquad (a \bullet b) \,\underline{*}\, x = (a \,\underline{*}\, x) \bullet (b \,\underline{*}\, (x \,\overline{*}\, a)),$$

$$x \,\overline{*}\, (a \bullet b) = (x \,\overline{*}\, a) \,\overline{*}\, b, \qquad (a \bullet b) \,\overline{*}\, x = (a \,\overline{*}\, x) \bullet (b \,\overline{*}\, (x \,\underline{*}\, a)).$$

(iv) $(a, b), (a \bullet b, c) \in \tilde{P} \Leftrightarrow (b, c), (a, b \bullet c) \in \tilde{P} \Rightarrow (a \bullet b) \bullet c = a \bullet (b \bullet c)$.

(v) $(a, b), (c, d) \in \tilde{P}, a \bullet b = c \bullet d \Leftrightarrow \exists e \in X$ such that $(a, e), (e, d) \in \tilde{P}$, $a \bullet e = c, e \bullet d = b$.

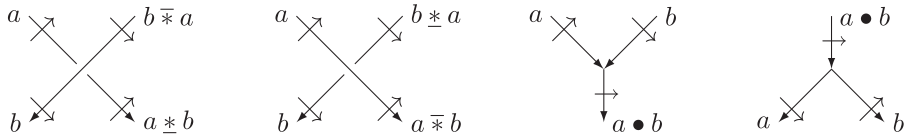

Fig. 6. PMB coloring conditions

The axioms of a partially multiplicative biquandle is obtained from colored Reidemeister moves like the primitive conditions (16)–(23), where the coloring is defined to be a map satisfying the conditions depicted in Figure 6 at each crossing and vertex. Although the axioms of a partially multiplicative biquandle are almost identical to the primitive conditions (16)–(23) under the correspondence

$$a \bullet b = b \triangle^{-1} a = a(b \,\overline{*}^{-1}\, a),$$

$$\tilde{P} = \{(a, b \triangle a) \mid (b, a) \in P\} = \{(a, a^{-1} b \,\overline{*}\, a) \mid (a, b) \in \textstyle\bigsqcup_{\lambda \in \Lambda} G_\lambda^2\},$$

the axiom (i) is an additional axiom to simplified the axioms. Fortunately, we see that the axiom (i) is a necessary condition as follows. By Theorem 3, a partially multiplicative biquandle consists of a multiple conjugation biquandle and a biquandle. Then $a \bullet x_1 = a \bullet x_2$ implies

$$x_1 = a^{-1}(a \bullet x_1) \,\overline{*}\, a = a^{-1}(a \bullet x_2) \,\overline{*}\, a = x_2$$

and $x_1 \bullet b = x_2 \bullet b$ implies $x_1 = x_2$, since

$$
\begin{aligned}
&(x \bullet b)(b^{-1} \,\overline{*}\, b \,\overline{*}^{-1}\, (x \bullet b)) \\
&\quad = x(b \,\overline{*}^{-1}\, x)((b^{-1} \,\overline{*}^{-1}\, x) \,\overline{*}\, x \,\overline{*}\, (b \,\overline{*}^{-1}\, x \,\overline{*}\, x) \,\overline{*}^{-1}\, x(b \,\overline{*}^{-1}\, x)) \\
&\quad \overset{(4)}{=} x(b \,\overline{*}^{-1}\, x)((b^{-1} \,\overline{*}^{-1}\, x) \,\overline{*}\, x(b \,\overline{*}^{-1}\, x) \,\overline{*}^{-1}\, x(b \,\overline{*}^{-1}\, x)) \\
&\quad = x(b \,\overline{*}^{-1}\, x)(b \,\overline{*}^{-1}\, x)^{-1} = x.
\end{aligned}
$$

We prove Theorem 3 (1), (2) below, and (3) in the next section.

Proof. (1) We show that $\underline{*}x : X_1 \to X_1$ is a well-defined bijection for any $x \in X$. For any $b \in X_1$, there exists $a \in X$ such that $a \sim b$. By (18), we have $a \,\underline{*}\, x \sim b \,\underline{*}\, x$ and $a \,\underline{*}^{-1}\, x \sim b \,\underline{*}^{-1}\, x$, which imply $b \,\underline{*}\, x, b \,\underline{*}^{-1}\, x \in X_1$. Therefore $\underline{*}x, \underline{*}^{-1}x : X_1 \to X_1$ are well-defined bijections. In the same way, we see that $\overline{*}x : X_1 \to X_1$ is a well-defined bijection for any $x \in X$. Since

$$\underline{*}x, \overline{*}x : X_1 \to X_1, \qquad \underline{*}x, \overline{*}x : X_1 \sqcup X_2 \to X_1 \sqcup X_2$$

are bijections, $\underline{*}x, \overline{*}x : X_2 \to X_2$ are well-defined bijections. On $X \times X = (X_1 \times X_1) \sqcup (X_1 \times X_2) \sqcup (X_2 \times X_1) \sqcup (X_2 \times X_2)$, the bijection $S : X \times X \to X \times X$ defined by $S(x, y) = (y \overline{*} x, x \underline{*} y)$ is decomposed into the four bijections

$$S : X_1 \times X_1 \to X_1 \times X_1, \qquad S : X_1 \times X_2 \to X_2 \times X_1,$$

$$S : X_2 \times X_1 \to X_1 \times X_2, \qquad S : X_2 \times X_2 \to X_2 \times X_2.$$

Therefore X_1, X_2 are subbiquandles of X.

(2) For any $a \in X_1$, there exists $b \in X$ such that $b \sim a$ by the assumption. From (22), $b \sim a, b \sim a, x = b\triangle a \Rightarrow a \sim a$. By (22), $a \sim b$ with $a \sim a$ implies $b \sim a$. Suppose $a \sim b$, $b \sim c$. By (20), we have $a \sim c$. Thus $\sim$ is an equivalence relation on X_1. □

8. Proof of Theorem 3 (3)

We introduce the notion of a triangle MCB. Although it is defined as the disjoint union of sets, it turns out that a triangle MCB consists of the disjoint union of groups. Furthermore, we show that a triangle MCB is an MCB. At the end of this section, we prove Theorem 3 (3).

Definition 9. A *triangle MCB* $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ is a biquandle $(X, \underline{*}, \overline{*})$ with a map $\triangle : \bigsqcup_{\lambda \in \Lambda} G_\lambda^2 \to X$ satisfying (24)–(32), where G_λ is not necessarily a group.

Lemma 4. *Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be a triangle MCB. For $a, b \in G_\lambda$, we have*

$$a \underline{*} b\triangle^{-1}b = b \overline{*} a\triangle^{-1}a \in G_\lambda, \tag{35}$$

$$a\triangle a \underline{*}^{-1} a = b\triangle b \underline{*}^{-1} b \in G_\lambda. \tag{36}$$

For $a \in G_\lambda$, we have

$$a\triangle a \underline{*}^{-1} a = \alpha\triangle\alpha = a\triangle a \overline{*}^{-1} a, \tag{37}$$

$$a\triangle a \underline{*}^{-1} a\triangle a \underline{*}^{-1} a = a\triangle a \overline{*}^{-1} a\triangle a \overline{*}^{-1} a \in G_\lambda, \tag{38}$$

where $\alpha \in X$ is the unique element satisfying $\alpha \underline{*} \alpha = \alpha \overline{*} \alpha = a$.

Proof. The equality (35) follows from

$$a \underline{*} b\triangle^{-1}b = (a \underline{*} b\triangle^{-1}b)\triangle(a\triangle b\triangle^{-1}b)\triangle^{-1}a$$

$$\overset{(32)}{=} (a \underline{*} b)\triangle(a\triangle b)\triangle^{-1}a \overset{(26)}{=} b \overline{*} a\triangle^{-1}a.$$

The equality (36) follows from

$$\begin{aligned} a\triangle a \,\underline{*}^{-1}\, a &= (a\triangle a \,\underline{*}^{-1}\, a) \,\underline{*}\, b \,\underline{*}^{-1}\, b \\ &\overset{(30)}{=} (a\triangle a) \,\underline{*}\, (b\triangle a) \,\underline{*}^{-1}\, b \\ &= (a\triangle a) \,\underline{*}\, (b\triangle a)\triangle^{-1}(b\triangle a)\triangle(b\triangle a) \,\underline{*}^{-1}\, b \\ &\overset{(35)}{=} (b\triangle a) \,\overline{*}\, (a\triangle a)\triangle^{-1}(a\triangle a)\triangle(b\triangle a) \,\underline{*}^{-1}\, b \\ &\overset{(31)}{=} (b\triangle a \,\overline{*}^{-1}\, a) \,\overline{*}\, a\triangle^{-1}(a\triangle a)\triangle(b\triangle a) \,\underline{*}^{-1}\, b \\ &= (b\triangle a)\triangle^{-1}(a\triangle a)\triangle(b\triangle a) \,\underline{*}^{-1}\, b \\ &\overset{(32)}{=} (b\triangle a)\triangle(a\triangle a)\triangle^{-1}(a\triangle a)\triangle(b\triangle a) \,\underline{*}^{-1}\, b \\ &= (b\triangle a)\triangle(b\triangle a) \,\underline{*}^{-1}\, b \\ &\overset{(32)}{=} b\triangle b \,\underline{*}^{-1}\, b. \end{aligned}$$

Then (37) follows from

$$a\triangle a \,\underline{*}^{-1}\, a = (\alpha \,\underline{*}\, \alpha)\triangle(\alpha \,\underline{*}\, \alpha) \,\underline{*}^{-1}\, a \overset{(28)}{=} (\alpha\triangle\alpha) \,\underline{*}\, (\alpha \,\overline{*}\, \alpha) \,\underline{*}^{-1}\, a = \alpha\triangle\alpha,$$

$$a\triangle a \,\overline{*}^{-1}\, a = (\alpha \,\overline{*}\, \alpha)\triangle(\alpha \,\overline{*}\, \alpha) \,\overline{*}^{-1}\, a \overset{(29)}{=} (\alpha\triangle\alpha) \,\overline{*}\, (\alpha \,\underline{*}\, \alpha) \,\overline{*}^{-1}\, a = \alpha\triangle\alpha.$$

The equality (38) follows from (37) and

$$\begin{aligned} a\triangle a \,\underline{*}^{-1}\, a\triangle a \,\underline{*}^{-1}\, a &\overset{(37)}{=} \alpha\triangle\alpha\triangle a \,\underline{*}^{-1}\, a = (\alpha\triangle\alpha \,\underline{*}^{-1}\, \alpha \,\underline{*}\, \alpha)\triangle(\alpha \,\underline{*}\, \alpha) \,\underline{*}^{-1}\, a \\ &\overset{(28)}{=} (\alpha\triangle\alpha \,\underline{*}^{-1}\, \alpha\triangle\alpha) \,\underline{*}\, (\alpha \,\overline{*}\, \alpha) \,\underline{*}^{-1}\, a = \alpha\triangle\alpha \,\underline{*}^{-1}\, \alpha\triangle\alpha, \\ a\triangle a \,\overline{*}^{-1}\, a\triangle a \,\overline{*}^{-1}\, a &\overset{(37)}{=} \alpha\triangle\alpha\triangle a \,\overline{*}^{-1}\, a = (\alpha\triangle\alpha \,\overline{*}^{-1}\, \alpha \,\overline{*}\, \alpha)\triangle(\alpha \,\overline{*}\, \alpha) \,\overline{*}^{-1}\, a \\ &\overset{(29)}{=} (\alpha\triangle\alpha \,\overline{*}^{-1}\, \alpha\triangle\alpha) \,\overline{*}\, (\alpha \,\underline{*}\, \alpha) \,\overline{*}^{-1}\, a = \alpha\triangle\alpha \,\overline{*}^{-1}\, \alpha\triangle\alpha. \end{aligned}$$

We have $b \,\overline{*}\, a\triangle^{-1}a, a\triangle a \,\underline{*}^{-1}\, a, a\triangle a \,\underline{*}^{-1}\, a\triangle a \,\underline{*}^{-1}\, a \in G_\lambda$, since $\underline{*}a, \overline{*}a, \triangle a$ are bijections from G_λ to $G_{a\underline{*}a} = G_{a\overline{*}a} = G_{a\triangle a}$. □

Proposition 4. *Let* $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ *be a triangle MCB.*

(1) *For any* $\lambda \in \Lambda$, G_λ *is a group with*

$$ab := a \underline{*} b \triangle^{-1} b = b \overline{*} a \triangle^{-1} a \in G_\lambda,$$

$$e_\lambda := a \triangle a \underline{*}^{-1} a = a \triangle a \overline{*}^{-1} a \in G_\lambda,$$

$$a^{-1} := a \triangle a \underline{*}^{-1} a \triangle a \underline{*}^{-1} a = a \triangle a \overline{*}^{-1} a \triangle a \overline{*}^{-1} a \in G_\lambda$$

for $a, b \in G_\lambda$.

(2) *The triangle MCB* $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ *is a multiple conjugation biquandle.*

Proof. (1) By Lemma 4, the multiplication, identity, and inverse are well-defined. The associativity $(ab)c = a(bc)$ follows from

$$\begin{aligned}(ab)c \triangle (ab) &= c \overline{*} (ab) = c \overline{*} (b \overline{*} a \triangle^{-1} a) \overset{(31)}{=} (c \overline{*} a) \overline{*} (b \overline{*} a) \\ &\overset{(B3)}{=} (c \overline{*} b) \overline{*} (a \underline{*} b) = (bc \triangle b) \overline{*} (a \underline{*} b) \overset{(29)}{=} (bc \overline{*} a) \triangle (b \overline{*} a) \\ &\overset{(32)}{=} (bc \overline{*} a \triangle^{-1} a) \triangle (b \overline{*} a \triangle^{-1} a) = a(bc) \triangle (ab).\end{aligned}$$

We have

$$e_\lambda a = (a \triangle a \underline{*}^{-1} a) \underline{*} a \triangle^{-1} a = a,$$

$$a e_\lambda = (a \triangle a \overline{*}^{-1} a) \overline{*} a \triangle^{-1} a = a.$$

We have

$$a^{-1} a = (a \triangle a \underline{*}^{-1} a \triangle a \underline{*}^{-1} a) \underline{*} a \triangle^{-1} a = a \triangle a \underline{*}^{-1} a = e_\lambda,$$

$$a a^{-1} = (a \triangle a \overline{*}^{-1} a \triangle a \overline{*}^{-1} a) \overline{*} a \triangle^{-1} a = a \triangle a \overline{*}^{-1} a = e_\lambda.$$

(2) The maps $\underline{*} x : G_a \to G_{a \underline{*} x}$ and $\overline{*} x : G_a \to G_{a \overline{*} x}$ are group homomorphism, since $b^{-1} a \underline{*} x = (b \underline{*} x)^{-1} (a \underline{*} x)$ and $b^{-1} a \overline{*} x = (b \overline{*} x)^{-1} (a \overline{*} x)$ follow from

$$\begin{aligned}(b \underline{*} x)^{-1} (a \underline{*} x) \overline{*} (b \underline{*} x) &= (a \underline{*} x) \triangle (b \underline{*} x) \\ &\overset{(28)}{=} (a \triangle b) \underline{*} (x \overline{*} b) = (b^{-1} a \overline{*} b) \underline{*} (x \overline{*} b) \\ &\overset{(B3)}{=} (b^{-1} a \underline{*} x) \overline{*} (b \underline{*} x),\end{aligned}$$

$$\begin{aligned}(b \overline{*} x)^{-1} (a \overline{*} x) \overline{*} (b \overline{*} x) &= (a \overline{*} x) \triangle (b \overline{*} x) \\ &\overset{(29)}{=} (a \triangle b) \overline{*} (x \underline{*} b) = (b^{-1} a \overline{*} b) \overline{*} (x \underline{*} b) \\ &\overset{(B3)}{=} (b^{-1} a \overline{*} x) \overline{*} (b \overline{*} x).\end{aligned}$$

For $a, b \in G_\lambda$ and $x \in X$, we have

$$x \underline{*}\, ab \overset{(30)}{=} (x \underline{*}\, a) \underline{*}\, (ab \triangle a) = (x \underline{*}\, a) \underline{*}\, (b \,\overline{*}\, a),$$

$$x \,\overline{*}\, ab \overset{(31)}{=} (x \,\overline{*}\, a) \,\overline{*}\, (ab \triangle a) = (x \,\overline{*}\, a) \,\overline{*}\, (b \,\overline{*}\, a),$$

and

$$\begin{aligned}
a^{-1}b \,\overline{*}\, a &\overset{(31)}{=} (a^{-1}b \,\overline{*}\, ba^{-1}) \,\overline{*}\, (a \triangle ba^{-1}) \\
&= (a^{-1}b \,\overline{*}\, ba^{-1}) \,\overline{*}\, (ab^{-1}a \,\overline{*}\, ba^{-1}) \\
&\overset{(B3)}{=} (a^{-1}b \,\overline{*}\, ab^{-1}a) \,\overline{*}\, (ba^{-1} \underline{*}\, ab^{-1}a) \\
&= (a \triangle ab^{-1}a) \,\overline{*}\, (ba^{-1} \underline{*}\, ab^{-1}a) \\
&\overset{(29)}{=} (a \,\overline{*}\, ba^{-1}) \triangle (ab^{-1}a \,\overline{*}\, ba^{-1}) \\
&= (a \,\overline{*}\, ba^{-1}) \triangle (a \triangle ba^{-1}) \\
&\overset{(27)}{=} ba^{-1} \underline{*}\, a.
\end{aligned}$$

□

Proof (Proof of Theorem 3 (3)). By Proposition 4, it is sufficient to show that X_1 is a triangle MCB. We show (24), (25), and $G_{a \triangle b} = G_{a \underline{*} b} = G_{a \overline{*} b}$ for $a, b \in G_\lambda$. The other equalities (26)–(32) follow directly from the primitive conditions (16)–(23). For $a, b \in G_\lambda$, $a \sim b$ implies $a \underline{*}\, b \sim a \triangle b$ and $a \,\overline{*}\, b \sim a \triangle b$ by (16) and (17), respectively. Then $G_{a \triangle b} = G_{a \underline{*} b} = G_{a \overline{*} b}$.

We verify (24). The map $\triangle a : G_a \to G_{a \triangle a}$ is well-defined, since $x \triangle a \sim a \triangle a$ follows from $x \sim a$ and $a \sim a$ by (20). Let $y \in G_{a \triangle a}$. Then $a \sim a$, $y \sim a \triangle a$. By (23),

$$\exists! x \in X \text{ s.t. } x \sim a,\, y = x \triangle a,\, y \triangle (a \triangle a) = x \triangle a.$$

Since $(x \triangle a) \triangle (a \triangle a) = x \triangle a$ follows from (20), we can remove the condition $y \triangle (a \triangle a) = x \triangle a$, that is,

$$\exists! x \in X \text{ s.t. } x \sim a,\, y = x \triangle a.$$

Then $\triangle a$ is bijective.

We verify (25). By (18), $\underline{*}\, x : G_a \to G_{a \underline{*} x}$ and $\underline{*}^{-1} x : G_{a \underline{*} x} \to G_a$ are well-defined. By (19), $\overline{*}\, x : G_a \to G_{a \overline{*} x}$ and $\overline{*}^{-1} x : G_{a \overline{*} x} \to G_a$ are well-defined. Therefore $\underline{*}\, x : G_a \to G_{a \underline{*} x}$ and $\overline{*}\, x : G_a \to G_{a \overline{*} x}$ are well-defined bijections. □

9. Parallel biquandle operations

In this section, we show that the n-parallel biquandle operations are well-defined and that $(X, (\underline{*}^{[n]})_{n \in \mathbb{Z}}, (\overline{*}^{[n]})_{n \in \mathbb{Z}})$ is a $\mathbb{Z}$-family of biquandles.

Proposition 5. *The binary operations* $\underline{*}^{[n]}, \overline{*}^{[n]} : X \times X \to X$ *are well-defined for any* $n \in \mathbb{Z}$.

Proof. Let $\varphi : X \times X \to X \times X$ be the bijection defined by $\varphi(x, y) = (x \underline{*} y, y \underline{*} y)$, where the bijectivity follows from Lemma 1. For $n \in \mathbb{Z}$, we define $f_n, g_n : X \times X \to X$ by $\varphi^n(x, y) = (f_n(x, y), g_n(x, y))$. Then

$$f_{n+1}(x, y) = f_n(x, y) \underline{*} g_n(x, y), \qquad g_{n+1}(x, y) = g_n(x, y) \underline{*} g_n(x, y).$$

We show that $\underline{*}^{[n]}, f_n : X \times X \to X$ coincide. Since $a \underline{*}^{[n]} b$ can be calculated by using (1), it is sufficient to show the equalities

$$f_0(a, b) = a, \qquad f_1(a, b) = a \underline{*} b, \qquad f_{i+j}(a, b) = f_j(f_i(a, b), f_i(b, b)),$$

which correspond to (1).

We show the equality $g_n(x, y) = f_n(y, y)$ by induction on n. When $n = 0$, the both sides coincide with y. We assume that the equality holds when $n = k$ for some $k \in \mathbb{Z}_{\geq 0}$. Then we have

$$\begin{aligned} g_{k+1}(x, y) &= g_k(x, y) \underline{*} g_k(x, y) = f_k(y, y) \underline{*} f_k(y, y) \\ &= f_k(y, y) \underline{*} g_k(y, y) = f_{k+1}(y, y). \end{aligned}$$

We assume that the equality holds when $n = -k$ for some $k \in \mathbb{Z}_{\geq 0}$. By Lemma 1, we have $g_{-k-1}(y, y) = g_{-k-1}(x, y)$ from

$$\begin{aligned} g_{-k-1}(y, y) \underline{*} g_{-k-1}(y, y) &= g_{-k}(y, y) = f_{-k}(y, y) \\ &= g_{-k}(x, y) = g_{-k-1}(x, y) \underline{*} g_{-k-1}(x, y). \end{aligned}$$

Then the equality $g_{-k-1}(x, y) = f_{-k-1}(y, y)$ follows from

$$\begin{aligned} g_{-k-1}(x, y) \underline{*} g_{-k-1}(x, y) &= g_{-k}(x, y) = f_{-k}(y, y) \\ &= f_{-k-1}(y, y) \underline{*} g_{-k-1}(y, y) = f_{-k-1}(y, y) \underline{*} g_{-k-1}(x, y). \end{aligned}$$

Then we have

$$f_j(f_i(a, b), f_i(b, b)) = f_j(f_i(a, b), g_i(a, b)) = f_{i+j}(a, b),$$

where the last equality follows from

$$\begin{aligned}(f_{i+j}(x,y), g_{i+j}(x,y)) &= \varphi^{i+j}(x,y) = \varphi^j(\varphi^i(x,y)) \\ &= \varphi^j(f_i(x,y), g_i(x,y)) \\ &= (f_j(f_i(x,y), g_i(x,y)), g_j(f_i(x,y), g_i(x,y))).\end{aligned}$$

Therefore $\underline{*}^{[n]}$ coincides with f_n, which is well-defined. In a similar manner, we see that $\overline{*}^{[n]}$ is well-defined. □

LEMMA 5. *For $n \in \mathbb{Z}$, we have the following.*

- *If $a \,\underline{*}^{[n]}\, b = c$, then $a = c \,\underline{*}^{[-n]}\, (b \,\underline{*}^{[n]}\, b)$.*
- *If $a \,\overline{*}^{[n]}\, b = c$, then $a = c \,\overline{*}^{[-n]}\, (b \,\overline{*}^{[n]}\, b)$.*

In particular, for $n \in \mathbb{Z}$, we have the following.

- *If $a \,\underline{*}^{[n]}\, a = c$, then $a = c \,\underline{*}^{[-n]}\, c$.*
- *If $a \,\overline{*}^{[n]}\, a = c$, then $a = c \,\overline{*}^{[-n]}\, c$.*

PROOF. We have $a = a \,\underline{*}^{[0]}\, b = (a \,\underline{*}^{[n]}\, b) \,\underline{*}^{[-n]}\, (b \,\underline{*}^{[n]}\, b) = c \,\underline{*}^{[-n]}\, (b \,\underline{*}^{[n]}\, b)$. If $a = b$, then $a = c \,\underline{*}^{[-n]}\, (b \,\underline{*}^{[n]}\, b) = c \,\underline{*}^{[-n]}\, c$. In the same way, we see the remaining part. □

PROPOSITION 6. *Let $(X, \underline{*}, \overline{*})$ be a biquandle. Then $(X, (\underline{*}^{[n]})_{n\in\mathbb{Z}}, (\overline{*}^{[n]})_{n\in\mathbb{Z}})$ is a $\mathbb{Z}$-family of biquandles.*

PROOF. We show $a \,\underline{*}^{[n]}\, a = a \,\overline{*}^{[n]}\, a$ for $n \in \mathbb{Z}$ and $a \in X$. Let $f, g : X \to X$ be the bijections defined by $f(x) = x \,\underline{*}\, x$, $g(x) = x \,\overline{*}\, x$. Then we have $f^n(x) = x \,\underline{*}^{[n]}\, x$ and $g^n(x) = x \,\overline{*}^{[n]}\, x$. Since $f = g$ follows from (B1), we have

$$a \,\underline{*}^{[n]}\, a = f^n(a) = g^n(a) = a \,\overline{*}^{[n]}\, a.$$

Then, by the definition of $\underline{*}^{[n]}$ and $\overline{*}^{[n]}$, it is sufficient to show that

$$(a \,\underline{*}^{[m]}\, b) \,\underline{*}^{[n]}\, (c \,\overline{*}^{[m]}\, b) = (a \,\underline{*}^{[n]}\, c) \,\underline{*}^{[m]}\, (b \,\underline{*}^{[n]}\, c), \tag{39}$$

$$(a \,\overline{*}^{[m]}\, b) \,\underline{*}^{[n]}\, (c \,\overline{*}^{[m]}\, b) = (a \,\underline{*}^{[n]}\, c) \,\overline{*}^{[m]}\, (b \,\underline{*}^{[n]}\, c), \tag{40}$$

$$(a \,\overline{*}^{[m]}\, b) \,\overline{*}^{[n]}\, (c \,\overline{*}^{[m]}\, b) = (a \,\overline{*}^{[n]}\, c) \,\overline{*}^{[m]}\, (b \,\underline{*}^{[n]}\, c) \tag{41}$$

for $m, n \in \mathbb{Z}$ and $a, b, c \in X$. These equalities were verified for $m, n \geq 0$ in [14]. We note that

$$x = x \,\underline{*}^{[0]}\, y = (x \,\underline{*}^{[n]}\, y) \,\underline{*}^{[-n]}\, (y \,\underline{*}^{[n]}\, y) = (x \,\underline{*}^{[n]}\, y) \,\underline{*}^{[-n]}\, (y \,\overline{*}^{[n]}\, y),$$

$$x = x \,\overline{*}^{[0]}\, y = (x \,\overline{*}^{[n]}\, y) \,\overline{*}^{[-n]}\, (y \,\overline{*}^{[n]}\, y) = (x \,\overline{*}^{[n]}\, y) \,\overline{*}^{[-n]}\, (y \,\underline{*}^{[n]}\, y).$$

We show the equality (40). Let $m \geq 0$, $n = -k \leq 0$. By Lemma 5, the equality

$$(c \,\underline{*}^{[-k]}\, c) \,\overline{*}^{[m]}\, (b \,\underline{*}^{[-k]}\, c) = (c \,\overline{*}^{[m]}\, b) \,\underline{*}^{[-k]}\, (c \,\overline{*}^{[m]}\, b) \tag{42}$$

follows from

$$\begin{aligned} c \mathbin{\overline{*}}^{[m]} b &= ((c \mathbin{\underline{*}}^{[-k]} c) \mathbin{\underline{*}}^{[k]} (c \mathbin{\underline{*}}^{[-k]} c)) \mathbin{\overline{*}}^{[m]} ((b \mathbin{\underline{*}}^{[-k]} c) \mathbin{\underline{*}}^{[k]} (c \mathbin{\underline{*}}^{[-k]} c)) \\ &= ((c \mathbin{\underline{*}}^{[-k]} c) \mathbin{\overline{*}}^{[m]} (b \mathbin{\underline{*}}^{[-k]} c)) \mathbin{\underline{*}}^{[k]} ((c \mathbin{\underline{*}}^{[-k]} c) \mathbin{\overline{*}}^{[m]} (b \mathbin{\underline{*}}^{[-k]} c)). \end{aligned}$$

Then we have

$$\begin{aligned} a \mathbin{\overline{*}}^{[m]} b &= ((a \mathbin{\underline{*}}^{[-k]} c) \mathbin{\underline{*}}^{[k]} (c \mathbin{\underline{*}}^{[-k]} c)) \mathbin{\overline{*}}^{[m]} ((b \mathbin{\underline{*}}^{[-k]} c) \mathbin{\underline{*}}^{[k]} (c \mathbin{\underline{*}}^{[-k]} c)) \\ &= ((a \mathbin{\underline{*}}^{[-k]} c) \mathbin{\overline{*}}^{[m]} (b \mathbin{\underline{*}}^{[-k]} c)) \mathbin{\underline{*}}^{[k]} ((c \mathbin{\underline{*}}^{[-k]} c) \mathbin{\overline{*}}^{[m]} (b \mathbin{\underline{*}}^{[-k]} c)) \\ &\overset{(42)}{=} ((a \mathbin{\underline{*}}^{[-k]} c) \mathbin{\overline{*}}^{[m]} (b \mathbin{\underline{*}}^{[-k]} c)) \mathbin{\underline{*}}^{[k]} ((c \mathbin{\overline{*}}^{[m]} b) \mathbin{\underline{*}}^{[-k]} (c \mathbin{\overline{*}}^{[m]} b)) \end{aligned}$$

By Lemma 5, we have

$$\begin{aligned} &(a \mathbin{\underline{*}}^{[-k]} c) \mathbin{\overline{*}}^{[m]} (b \mathbin{\underline{*}}^{[-k]} c) \\ &\quad = (a \mathbin{\overline{*}}^{[m]} b) \mathbin{\underline{*}}^{[-k]} (((c \mathbin{\overline{*}}^{[m]} b) \mathbin{\underline{*}}^{[-k]} (c \mathbin{\overline{*}}^{[m]} b)) \mathbin{\underline{*}}^{[k]} ((c \mathbin{\overline{*}}^{[m]} b) \mathbin{\underline{*}}^{[-k]} (c \mathbin{\overline{*}}^{[m]} b))) \\ &\quad = (a \mathbin{\overline{*}}^{[m]} b) \mathbin{\underline{*}}^{[-k]} (c \mathbin{\overline{*}}^{[m]} b). \end{aligned}$$

Let $m = -k \leq 0$, $n \geq 0$. By Lemma 5, the equality

$$(b \mathbin{\overline{*}}^{[-k]} b) \mathbin{\underline{*}}^{[n]} (c \mathbin{\overline{*}}^{[-k]} b) = (b \mathbin{\underline{*}}^{[n]} c) \mathbin{\overline{*}}^{[-k]} (b \mathbin{\underline{*}}^{[n]} c) \tag{43}$$

follows from

$$\begin{aligned} b \mathbin{\underline{*}}^{[n]} c &= ((b \mathbin{\overline{*}}^{[-k]} b) \mathbin{\overline{*}}^{[k]} (b \mathbin{\overline{*}}^{[-k]} b)) \mathbin{\underline{*}}^{[n]} ((c \mathbin{\overline{*}}^{[-k]} b) \mathbin{\overline{*}}^{[k]} (b \mathbin{\overline{*}}^{[-k]} b)) \\ &= ((b \mathbin{\overline{*}}^{[-k]} b) \mathbin{\underline{*}}^{[n]} (c \mathbin{\overline{*}}^{[-k]} b)) \mathbin{\overline{*}}^{[k]} ((b \mathbin{\overline{*}}^{[-k]} b) \mathbin{\underline{*}}^{[n]} (c \mathbin{\overline{*}}^{[-k]} b)). \end{aligned}$$

Then we have

$$\begin{aligned} a \mathbin{\underline{*}}^{[n]} c &= ((a \mathbin{\overline{*}}^{[-k]} b) \mathbin{\overline{*}}^{[k]} (b \mathbin{\overline{*}}^{[-k]} b)) \mathbin{\underline{*}}^{[n]} ((c \mathbin{\overline{*}}^{[-k]} b) \mathbin{\overline{*}}^{[k]} (b \mathbin{\overline{*}}^{[-k]} b)) \\ &= ((a \mathbin{\overline{*}}^{[-k]} b) \mathbin{\underline{*}}^{[n]} (c \mathbin{\overline{*}}^{[-k]} b)) \mathbin{\overline{*}}^{[k]} ((b \mathbin{\overline{*}}^{[-k]} b) \mathbin{\underline{*}}^{[n]} (c \mathbin{\overline{*}}^{[-k]} b)) \\ &\overset{(43)}{=} ((a \mathbin{\overline{*}}^{[-k]} b) \mathbin{\underline{*}}^{[n]} (c \mathbin{\overline{*}}^{[-k]} b)) \mathbin{\overline{*}}^{[k]} ((b \mathbin{\underline{*}}^{[n]} c) \mathbin{\overline{*}}^{[-k]} (b \mathbin{\underline{*}}^{[n]} c)). \end{aligned}$$

By Lemma 5, we have

$$\begin{aligned} &(a \mathbin{\overline{*}}^{[-k]} b) \mathbin{\underline{*}}^{[n]} (c \mathbin{\overline{*}}^{[-k]} b) \\ &\quad = (a \mathbin{\underline{*}}^{[n]} c) \mathbin{\overline{*}}^{[-k]} (((b \mathbin{\underline{*}}^{[n]} c) \mathbin{\overline{*}}^{[-k]} (b \mathbin{\underline{*}}^{[n]} c)) \mathbin{\overline{*}}^{[k]} ((b \mathbin{\underline{*}}^{[n]} c) \mathbin{\overline{*}}^{[-k]} (b \mathbin{\underline{*}}^{[n]} c))) \\ &\quad = (a \mathbin{\underline{*}}^{[n]} c) \mathbin{\overline{*}}^{[-k]} (b \mathbin{\underline{*}}^{[n]} c). \end{aligned}$$

Let $m = -k \leq 0$, $n = -l \leq 0$. By Lemma 5, the equality

$$(b \,\overline{*}^{[-k]}\, b) \,\underline{*}^{[-l]}\, (c \,\overline{*}^{[-k]}\, b) = (b \,\underline{*}^{[-l]}\, c) \,\overline{*}^{[-k]}\, (b \,\underline{*}^{[-l]}\, c) \tag{44}$$

follows from

$$\begin{aligned} b \,\underline{*}^{[-l]}\, c &= ((b \,\overline{*}^{[-k]}\, b) \,\overline{*}^{[k]}\, (b \,\overline{*}^{[-k]}\, b)) \,\underline{*}^{[-l]}\, ((c \,\overline{*}^{[-k]}\, b) \,\overline{*}^{[k]}\, (b \,\overline{*}^{[-k]}\, b)) \\ &= ((b \,\overline{*}^{[-k]}\, b) \,\underline{*}^{[-l]}\, (c \,\overline{*}^{[-k]}\, b)) \,\overline{*}^{[k]}\, ((b \,\overline{*}^{[-k]}\, b) \,\underline{*}^{[-l]}\, (c \,\overline{*}^{[-k]}\, b)). \end{aligned}$$

Then we have

$$\begin{aligned} a \,\underline{*}^{[-l]}\, c &= ((a \,\overline{*}^{[-k]}\, b) \,\overline{*}^{[k]}\, (b \,\overline{*}^{[-k]}\, b)) \,\underline{*}^{[-l]}\, ((c \,\overline{*}^{[-k]}\, b) \,\overline{*}^{[k]}\, (b \,\overline{*}^{[-k]}\, b)) \\ &= ((a \,\overline{*}^{[-k]}\, b) \,\underline{*}^{[-l]}\, (c \,\overline{*}^{[-k]}\, b)) \,\overline{*}^{[k]}\, ((b \,\overline{*}^{[-k]}\, b) \,\underline{*}^{[-l]}\, (c \,\overline{*}^{[-k]}\, b)) \\ &\overset{(44)}{=} ((a \,\overline{*}^{[-k]}\, b) \,\underline{*}^{[-l]}\, (c \,\overline{*}^{[-k]}\, b)) \,\overline{*}^{[k]}\, ((b \,\underline{*}^{[-l]}\, c) \,\overline{*}^{[-k]}\, (b \,\underline{*}^{[-l]}\, c)). \end{aligned}$$

By Lemma 5, we have

$$\begin{aligned} &(a \,\overline{*}^{[-k]}\, b) \,\underline{*}^{[-l]}\, (c \,\overline{*}^{[-k]}\, b) \\ &\quad = (a \,\underline{*}^{[-l]}\, c) \,\overline{*}^{[-k]}\, (((b \,\underline{*}^{[-l]}\, c) \,\overline{*}^{[-k]}\, (b \,\underline{*}^{[-l]}\, c)) \\ &\qquad\qquad \overline{*}^{[k]}\, ((b \,\underline{*}^{[-l]}\, c) \,\overline{*}^{[-k]}\, (b \,\underline{*}^{[-l]}\, c))) \\ &\quad = (a \,\underline{*}^{[-l]}\, c) \,\overline{*}^{[-k]}\, (b \,\underline{*}^{[-l]}\, c). \end{aligned}$$

This completes the proof of (40). In a similar manner, we can verify (39) and (41) by using the equalities (42)–(44).

This completes the proof. □

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