

Existence of supersingular reduction for families of $K3$ surfaces with large Picard number in positive characteristic

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ABSTRACT. We study non-isotrivial families of $K3$ surfaces in positive characteristic p whose geometric generic fibers satisfy $\rho \geq 21 - 2h$ and $h \geq 3$, where ρ is the Picard number and h is the height of the formal Brauer group. We show that, under a mild assumption on the characteristic of the base field, they have potential supersingular reduction. Our methods rely on Maulik’s results on moduli spaces of $K3$ surfaces and the construction of sections of powers of Hodge bundles due to van der Geer and Katsura. For large p and each $2 \leq h \leq 10$, using deformation theory and Taelman’s methods, we construct non-isotrivial families of $K3$ surfaces satisfying $\rho = 22 - 2h$.

1. Introduction

We study the variation of heights in non-isotrivial families of $K3$ surfaces in characteristic $p > 0$ whose geometric generic fibers have large Picard number. Recall that a $K3$ surface X over a field is a projective smooth surface with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = 0$. Let k be an algebraically closed field of positive characteristic $p > 0$. For a $K3$ surface X over k , let $h(X)$ be the *height* of the formal Brauer group. (When k is not algebraically closed, the height of X is defined to be the height of $X_{\bar{k}} := X \otimes_k \bar{k}$.) We have $1 \leq h(X) \leq 10$ or $h(X) = \infty$. When $h(X) \neq \infty$, the Artin-Mazur-Igusa inequality

$$\rho(X) \leq 22 - 2h(X)$$

is satisfied [1, Theorem 0.1], where $\rho(X)$ is the Picard number of X . A $K3$ surface X over k is called *supersingular* if $h(X) = \infty$. The Tate conjecture for $K3$ surfaces [5], [12], [16], [18], [21], [22] implies that X is supersingular if and only if $\rho(X) = 22$. (See also [4, Corollaire 0.5], [9, Corollary 17.3.7].)

Let C be a proper smooth curve over k with function field $K := k(C)$. Let X be a $K3$ surface over K . Let $v \in C$ be a closed point where X has potential good reduction. We say the *height of X jumps at v* (resp. X has

potential supersingular reduction at v) if there exist a finite extension L/K and a valuation w of L extending the valuation of v , and a smooth family of $K3$ surfaces \mathcal{X} over the valuation ring \mathcal{O}_w such that $\mathcal{X} \otimes_{\mathcal{O}_w} L \simeq X \otimes_K L$, and $h(\mathcal{X}_s) > h(X)$ (resp. $h(\mathcal{X}_s) = \infty$). Here, \mathcal{X}_s is the special fiber of \mathcal{X} . We say X is *non-isotrivial* if there does not exist a $K3$ surface Y over k such that $X \otimes_K \bar{K} \simeq Y \otimes_K \bar{K}$.

The first main result of this paper is as follows.

THEOREM 1.1. *Let k be an algebraically closed field of characteristic $p > 0$, C a projective smooth curve with function field $K := k(C)$. Let X be a non-isotrivial $K3$ surface over K that admits an ample line bundle \mathcal{L} of degree $2d$. Assume that $p > 18d + 4$ and $3 \leq h(X) \leq 10$. Then the height of X jumps at some closed point $v \in C$.*

As a direct consequence of Theorem 1.1 and the Artin-Mazur-Igusa inequality $\rho \leq 22 - 2h$, we have the following corollary.

COROLLARY 1.2. *Under the assumptions of Theorem 1.1, assume moreover that $\rho(X_{\bar{K}}) \geq 21 - 2h(X)$. Then X has potential supersingular reduction at some closed point $v \in C$.*

The following theorem is the second main result of this paper. We shall prove the existence of non-isotrivial $K3$ surfaces over function fields with $\rho = 22 - 2h$ if $h \geq 2$.

THEOREM 1.3. *Let p be a prime number and h a positive integer with $2 \leq h \leq 10$. There exist non-isotrivial $K3$ surfaces X over function fields of characteristic p satisfying $\rho(X) = 22 - 2h(X)$ and $h(X) = h$ if at least one of the following conditions holds:*

- $p = 3$ and $h = 10$, or
- $p \geq 5$.

REMARK 1.4. *When $p \geq 3$, Jang and Liedtke independently proved that there do not exist non-isotrivial $K3$ surfaces X with $\rho(X) = 22 - 2h(X)$ and $h(X) = 1$ [11, Theorem 3.7], [14, Theorem 2.6]. See also Theorem 4.2.*

REMARK 1.5. The main reason why we assume conditions on prime p in Theorem 1.1 and Theorem 1.3 is that we do not currently know the existence of potential semistable reduction of $K3$ surfaces over discretely valued fields. For the precise statement we need, see [15, Assumption (\star)]. If we assume [15, Assumption (\star)], we can show Theorem 1.1 for $p \geq 3$ and Theorem 1.3 for any p . See also Remark 5.4.

The outline of this paper is as follows. We recall the results of van der Geer-Katsura [7] and Maulik [18] on the variation of heights in families of $K3$

surfaces in Section 2. The proof of Theorem 1.1 is given in Section 3. In Section 4, using deformation theory, we show Theorem 1.3 comes down to show the existence of $K3$ surfaces X over $\overline{\mathbb{F}}_p$ satisfying $\rho(X) = 22 - 2h(X)$ and $h(X) = h$. Assuming semistable reduction, Taelman conditionally proved the existence of $K3$ surfaces over finite fields with given L -function, up to finite extensions of the base field; see [28]. When the characteristic of the base field is not too small, the author recently proved that Taelman's results hold unconditionally; see [10]. Using these results, in Section 5, we construct $K3$ surfaces X over $\overline{\mathbb{F}}_p$ satisfying $\rho(X) = 22 - 2h(X)$ and $h(X) = h$ for large p and each $2 \leq h \leq 10$; see Proposition 5.2. Then we achieve Theorem 1.3.

2. The variation of heights in families of $K3$ surfaces

Let X be a $K3$ surface over an algebraically closed field k of characteristic $p > 0$. The following functor from the category Art_k of local artinian k -algebras with residue field k to the category of abelian groups

$$\begin{aligned} \Phi_X^2: \text{Art}_k &\longrightarrow && (\text{Abelian groups}) \\ R &\longrightarrow && \ker(H_{\text{ét}}^2(X \otimes_k R, \mathbf{G}_m) \rightarrow H_{\text{ét}}^2(X, \mathbf{G}_m)) \end{aligned}$$

is pro-representable by a smooth one-dimensional formal group scheme $\widehat{\text{Br}}(X)$ [3]. The *height* of X is defined to be the height of $\widehat{\text{Br}}(X)$

$$h(X) := h(\widehat{\text{Br}}(X)).$$

We have $1 \leq h(X) \leq 10$ or $h(X) = \infty$. When $h(X) = \infty$, we say X is *supersingular*. When $h(X) \neq \infty$, Artin proved the following inequality

$$\rho(X) \leq 22 - 2h(X),$$

where $\rho(X)$ is the Picard number of X [1, Theorem 0.1]. The Tate conjecture for $K3$ surfaces [5], [12], [16], [18], [21], [22] implies that X is supersingular if and only if $\rho(X) = 22$. (See also [4, Corollaire 0.5], [9, Corollary 17.3.7].) When k is not algebraically closed, the height of X is defined to be the height of $X \otimes_k \bar{k}$.

We say $f: \mathcal{X} \rightarrow S$ is a family of $K3$ surfaces if S is a scheme, \mathcal{X} is an algebraic space, and f is a proper smooth morphism whose geometric fibers are $K3$ surfaces. A *polarization* (resp. *quasi-polarization*) of $f: \mathcal{X} \rightarrow S$ is a section $\xi \in \underline{\text{Pic}}(\mathcal{X}/S)(S)$ of the relative Picard functor whose fiber $\xi(s)$ at every geometric point $s \rightarrow S$ is a polarization (resp. quasi-polarization), which means an ample (resp. big and nef) line bundle on the $K3$ surface \mathcal{X}_s . We say a section $\xi \in \underline{\text{Pic}}(\mathcal{X}/S)(S)$ is *primitive* if, for every geometric point $s \rightarrow S$, the cokernel of the inclusion $\langle \xi(s) \rangle \hookrightarrow \text{Pic}(\mathcal{X}_s)$ is torsion free. For an integral

scheme S over k , we say a family of $K3$ surfaces $f : \mathcal{X} \rightarrow S$ is *non-isotrivial* if there do not exist a $K3$ surface Y over k and a finite flat morphism $S' \rightarrow S$ such that $\mathcal{X} \times_S S' \simeq Y \times_{\mathrm{Spec} k} S'$.

For a positive integer $d \geq 1$, let M_{2d} (resp. M_{2d}°) be the moduli functor that sends a \mathbb{Z} -scheme S to the groupoid consists of tuples $(f : \mathcal{X} \rightarrow S, \xi)$, where $f : \mathcal{X} \rightarrow S$ is a family of $K3$ surfaces and $\xi \in \underline{\mathrm{Pic}}(\mathcal{X}/S)(S)$ is a primitive quasi-polarization (resp. primitive polarization) of degree $2d$ (i.e. for every geometric point $s \rightarrow S$, $(\xi(s))^2 = (\xi(s), \xi(s)) = 2d$, where $(,)$ denotes the intersection pairing on \mathcal{X}_s).

The following theorem is well-known.

THEOREM 2.1 ([23, Theorem 4.3.4], [18, Proposition 2.1]). *The moduli functors M_{2d} and M_{2d}° are Deligne-Mumford stacks of finite type over $\mathrm{Spec} \mathbb{Z}$, and smooth over $\mathrm{Spec} \mathbb{Z}[1/2d]$.*

In the following, we fix an algebraically closed field k of characteristic $p > 0$ and a positive integer $d \geq 1$ such that p does not divide $2d$. We work with the moduli stacks $M_{2d,k} := M_{2d} \otimes_{\mathbb{Z}} k$ and $M_{2d,k}^\circ := M_{2d}^\circ \otimes_{\mathbb{Z}} k$. Let $\pi : \mathcal{X} \rightarrow M_{2d,k}$ be the universal family.

Now we recall the results of van der Geer and Katsura on the description of the locus

$$M_{2d,k}^{(h)} = \{s \in M_{2d,k} \mid h(\mathcal{X}_s) \geq h\}$$

for $1 \leq h \leq 11$. The locus $M_{2d,k}^{(h)}$ is a closed substack of $M_{2d,k}$. The locus $M_{2d,k}^{(11)}$ is called the *supersingular locus*. The loci $M_{2d,k}^{(h)}$ ($1 \leq h \leq 11$) form the so-called *height stratification* of the moduli stack $M_{2d,k}$.

Let $\lambda := \pi_*(\Omega_{\mathcal{X}/M_{2d,k}}^2)$ be the *Hodge bundle* on $M_{2d,k}$. For each $1 \leq h \leq 10$, van der Geer and Katsura constructed an $\mathcal{O}_{M_{2d,k}^{(h)}}$ -linear morphism of line bundles

$$\phi_h : (R^2\pi_*\mathcal{O}_{\mathcal{X}})^{(p^h)}|_{M_{2d,k}^{(h)}} \rightarrow R^2\pi_*\mathcal{O}_{\mathcal{X}}|_{M_{2d,k}^{(h)}}$$

on $M_{2d,k}^{(h)}$ such that the locus where ϕ_h vanishes coincides with $M_{2d,k}^{(h+1)}$ [7, Theorem 15.1]. Here, $(R^2\pi_*\mathcal{O}_{\mathcal{X}})^{(p^h)}$ is the pullback of $R^2\pi_*\mathcal{O}_{\mathcal{X}}$ by the h -th power of the absolute Frobenius morphism $F : M_{2d,k} \rightarrow M_{2d,k}$. Therefore, for each $1 \leq h \leq 10$, there is a global section

$$g_h \in \Gamma(M_{2d,k}^{(h)}, \lambda^{\otimes (p^h-1)})$$

on $M_{2d,k}^{(h)}$ whose zero locus coincides with $M_{2d,k}^{(h+1)}$.

When $p \geq 5$, Maulik showed the following important theorem in [18]. Madapusi Pera proved it when $p \geq 3$ in [16].

THEOREM 2.2 ([16, Corollary 5.16], [18, Theorem 5.1]). *Assume $p \geq 3$. Then the Hodge bundle λ is ample on the polarized locus $M_{2d,k}^\circ$. Furthermore, for any morphism $g: C \rightarrow M_{2d,k}$ from a proper smooth curve C over k such that*

- *C is not contracted in the map to the coarse moduli space, and*
- *C meets the polarized locus $M_{2d,k}^\circ$ nontrivially,*

the pullback of the Hodge bundle λ on $M_{2d,k}$ to C has positive degree.

The following corollary is stated in [7, Theorem 15.3], [18, Corollary 5.5] in the polarized case, but we can prove it in the quasi-polarized (and generically polarized) case in a similar way.

COROLLARY 2.3. *Assume $p \geq 3$. Let C be a proper smooth curve over k , and $f: \mathcal{X} \rightarrow C$ a non-isotrivial family of $K3$ surfaces with a primitive quasi-polarization ξ of degree $2d$ such that the generic fiber $(\mathcal{X}_\eta, \xi(\eta))$ is a polarized $K3$ surface. Then, either*

- *the heights of the geometric fibers \mathcal{X}_t are not constant, or*
- *all geometric fibers \mathcal{X}_t are supersingular.*

PROOF. If the heights of the geometric fibers \mathcal{X}_t are constant, and they are equal to h ($1 \leq h \leq 10$), then the image of C is contained in $M_{2d,k}^{(h)} \setminus M_{2d,k}^{(h+1)}$. The pullback of $\lambda^{\otimes(p^h-1)}$ to C is trivial since the pullback of g_h is a global section which is everywhere nonzero on C . However the pullback of λ to C has positive degree by Theorem 2.2, which is absurd. \square

REMARK 2.4. The polarized locus $M_{2d,k}^\circ$ is separated [23, Theorem 4.3.3], but $M_{2d,k}$ is not separated. The non-separatedness of $M_{2d,k}$ is related to the existence of flops; see [18, last paragraph in p. 2362]. So we do not expect $M_{2d,k}^{(h)} \setminus M_{2d,k}^{(h+1)}$ is “quasi-affine” in a reasonable sense. However, it does not cause any problem to obtain Corollary 2.3 thanks to Theorem 2.2. Compare the proof of Corollary 2.3 with the proof of [7, Theorem 15.3].

3. Proof of Theorem 1.1

In this section, for a discretely valued field L , we denote its valuation ring by \mathcal{O}_L , the generic point by $\eta \in \text{Spec } \mathcal{O}_L$, and the closed point by $s \in \text{Spec } \mathcal{O}_L$.

Let k be an algebraically closed field of characteristic $p \geq 5$, and C a proper smooth curve over k with function field $K = k(C)$. Let (X, \mathcal{L}) be a non-isotrivial primitively polarized $K3$ surface over K of degree $2d$ with $p > 18d + 4$ and $3 \leq h(X) \leq 10$. Then there exist a non-empty Zariski open set $U \subset C$ and a family of $K3$ surfaces

$$\mathcal{X}_U \rightarrow U$$

with a polarization ξ_U of degree $2d$ which extends (X, \mathcal{L}) . The set of closed points outside U is denoted by $C \setminus U = \{v_1, \dots, v_m\}$. Take a closed point $v_i \in C \setminus U$. We put $K_i := K$ and consider it as a discretely valued field with respect to the valuation defined by v_i .

First, we shall extend \mathcal{X}_U to a quasi-polarized family of $K3$ surfaces over C , after replacing C by a finite covering of it. This step is now well-understood thanks to Maulik's work [18]. We briefly recall his argument with a minor modification; see Remark 3.3. In [18, Section 4.3], Maulik constructed a projective scheme

$$\mathcal{X}'_i \rightarrow \mathrm{Spec} \mathcal{O}_{K'_i},$$

after taking the base change to a finite extension K'_i/K_i , which satisfies the following conditions:

- (1) \mathcal{X}'_i is Cohen-Macaulay, \mathbb{Q} -factorial, and regular away from finitely many points,
- (2) the generic fiber $\mathcal{X}'_{i,\eta}$ is isomorphic to $X_{K'_i} := X \otimes_{K_i} K'_i$,
- (3) the closed fiber $\mathcal{X}'_{i,s}$ is reduced and irreducible components of $\mathcal{X}'_{i,s}$ are normal,
- (4) the singularities of $\mathcal{X}'_{i,s}$ are rational double points or of normal crossing type, and
- (5) the relative canonical bundle $K_{\mathcal{X}'_i/\mathcal{O}_{K'_i}}$ is trivial and there is a nef \mathbb{Q} -divisor H on \mathcal{X}'_i such that

$$H|_{\mathcal{X}'_{i,\eta}} \sim_{\mathbb{Q}} \gamma \mathcal{L}_{K'_i}$$

for some positive rational number $\gamma \in \mathbb{Q}_{>0}$, where $\mathcal{L}_{K'_i} := \mathcal{L} \otimes_{K_i} K'_i$. Here we use the fact that $\mathcal{L}^{\otimes 3}$ is very ample [25] and the assumption $p > 18d + 4$ to apply Saito's results [26] on the construction of projective semistable models.

CLAIM 3.1. *If $h(X) \geq 3$, then there exist a finite extension of discretely valued fields L_i/K_i and a primitively quasi-polarized family of $K3$ surfaces*

$$\mathcal{X}_i \rightarrow \mathrm{Spec} \mathcal{O}_{L_i},$$

which extends $(X_{L_i}, \mathcal{L}_{L_i}) := (X \otimes_{K_i} L_i, \mathcal{L} \otimes_{K_i} L_i)$.

PROOF. We shall apply Artin's results on the simultaneous resolution of rational double points in the category of algebraic spaces [2, Theorem 2]. Then we find a finite extension of discretely valued fields L_i/K'_i , a proper smooth family of $K3$ surfaces

$$\mathcal{X}_i \rightarrow \mathrm{Spec} \mathcal{O}_{L_i},$$

and a simultaneous resolution over $\text{Spec } \mathcal{O}_{L_i}$

$$\mathcal{X}_i \rightarrow \mathcal{X}'_i \otimes_{\mathcal{O}_{K'_i}} \mathcal{O}_{L_i}.$$

Here \mathcal{X}_i is an algebraic space over $\text{Spec } \mathcal{O}_{L_i}$, and $\mathcal{X}_i \rightarrow \mathcal{X}'_i \otimes_{\mathcal{O}_{K'_i}} \mathcal{O}_{L_i}$ induces an isomorphism on the generic fiber and the minimal resolution of rational double points on the closed fiber.

We apply Nakkajima's results [20, Proposition 3.4], and see that the closed fiber $\mathcal{X}_{i,s}$ is a combinatorial $K3$ surface, which means one of the following types:

- (1) smooth $K3$ surface (type I),
- (2) two rational surfaces joined by a chain of ruled surfaces over an elliptic curve (type II),
- (3) union of rational surfaces, whose dual graph of the configuration is a triangulation of S^2 (type III).

In [24, Proposition 3], Rudakov, Zink, and Shafarevich showed that the height $h(\mathcal{X}_{i,s})$ of the closed fiber satisfies $h(\mathcal{X}_{i,s}) \geq h(\mathcal{X}_{i,\eta})$. Moreover, $1 \leq h(\mathcal{X}_{i,s}) \leq 2$ is satisfied if the closed fiber $\mathcal{X}_{i,s}$ is not smooth. So by our assumption $h(X) \geq 3$, the family of $K3$ surfaces $\mathcal{X}_i \rightarrow \text{Spec } \mathcal{O}_{L_i}$ is necessarily smooth.

We shall show that the line bundle \mathcal{L}_i on \mathcal{X}_i which extends \mathcal{L}_{L_i} is a quasi-polarization. We follow Maulik's argument as in [18, Lemma 4.10]. We denote the natural morphism $\mathcal{X}_i \rightarrow \mathcal{X}'_i$ by g . The pullback g^*H of the nef \mathbb{Q} -divisor H is also nef. Since we have $H|_{\mathcal{X}'_{i,\eta}} \sim_{\mathbb{Q}} \gamma \mathcal{L}_{K'_i}$ on $\mathcal{X}'_{i,\eta}$, we also have

$$g^*H \sim_{\mathbb{Q}} \gamma \mathcal{L}_i$$

and hence \mathcal{L}_i is nef by [18, Lemma 5.12]. The bigness of \mathcal{L}_i follows from the fact that the Euler-Poincaré characteristics are locally constant in a flat family. Since the order of a torsion element in the cokernel of

$$\text{Pic}(\mathcal{X}_{i,\bar{\eta}}) \hookrightarrow \text{Pic}(\mathcal{X}_{i,s})$$

is a power of p [19, Proposition 3.6] and

$$p > 18d + 4 > 2d = (\mathcal{L})^2 = (\mathcal{L}_i)^2,$$

the line bundle \mathcal{L}_i is also primitive on the closed fiber. \square

PROOF OF THEOREM 1.1. We now give a proof of Theorem 1.1. The family $\mathcal{X}_U \rightarrow U$ can be extended to a primitively quasi-polarized family of $K3$ surfaces

$$\mathcal{X} \rightarrow C$$

over a proper smooth curve C , after replacing it by a finite covering of it [18, Lemma 7.2].

By Corollary 2.3, there exists a closed point $v \in C$ such that the height of X jumps at v . So we achieve Theorem 1.1. \square

PROOF OF COROLLARY 1.2. Assume

$$\rho(X_{\bar{K}}) \geq 21 - 2h(X),$$

and take a closed point $v \in C$ such that the height of X jumps at v , namely, $h(\mathcal{X}_v) > h(X)$. Then X necessarily has supersingular reduction at v . Indeed, if $h(\mathcal{X}_v) \neq \infty$, then we have

$$22 - 2h(\mathcal{X}_v) \geq \rho(\mathcal{X}_v) \geq \rho(X_{\bar{K}}) \geq 21 - 2h(X).$$

Hence we have $h(X) \geq h(\mathcal{X}_v) - 1/2$, which is absurd. \square

In Theorem 1.1, we assume $p > 18d + 4$ and $h(X) \geq 3$, but one expects that the following conjecture is true without any assumptions on p , d , and $h(X)$.

CONJECTURE 3.2 ([7, p.288], [18, p.2390]). *Let k be an algebraically closed field of positive characteristic $p > 0$, and C a proper smooth curve over k with function field $K = k(C)$. Let X be a non-isotrivial non-supersingular $K3$ surface over K . Then, after replacing C by a finite covering of it, there exists a semistable family of combinatorial $K3$ surfaces $\mathcal{X} \rightarrow C$ which extends X and satisfies $h(\mathcal{X}_v) > h(X)$ at some closed point $v \in C$.*

A natural strategy to prove Conjecture 3.2 is to generalize Maulik's results (Theorem 2.2) to an appropriate compactification of the moduli stack $M_{2d,k}$, but such results are not currently available.

REMARK 3.3. In the proof of Theorem 1.1, the construction of a quasi-polarized family of $K3$ surfaces $\mathcal{X} \rightarrow C$ is basically the same as in Maulik's proof of [18, Theorem 4.2]. But there is a slight technical difference. In [18, Theorem 4.2], Maulik considered families of *supersingular* $K3$ surfaces, and he constructed families $\mathcal{X} \rightarrow C$ as *schemes*. In fact, he applied Artin's simultaneous resolution over the completion of the local ring at each $v_i \in C \setminus U$ in order to show that there is no rational double point in the closed fiber \mathcal{X}'_{i,v_i} ; see the proof of [18, Lemma 4.10]. (This argument works only for supersingular $K3$ surfaces.) On the other hand, we apply Artin's simultaneous resolution [2] over (the henselization of) the local ring at each $v_i \in C \setminus U$ rather than the completion of it. (This is possible since Artin's results in [2] are valid over henselian (not necessarily complete) discrete valuation rings.) Consequently, in the proof of Theorem 1.1, we construct families $\mathcal{X} \rightarrow C$ as *algebraic spaces*.

Note that the example discussed by Matsumoto in [17, Example 5.2] shows that one does not expect to construct families $\mathcal{X} \rightarrow C$ as schemes in general. For details, compare the proof of Theorem 1.1 with Maulik's proof of [18, Theorem 4.2, Lemma 4.7].

4. Deformations of $K3$ surfaces with constant height and Picard number

In the rest of this paper, we shall give a proof of Theorem 1.3. We fix an algebraically closed field k of characteristic $p > 0$. In this section we show the following.

PROPOSITION 4.1. *Let (X_0, \mathcal{L}_0) be a polarized $K3$ surface over k satisfying $2 \leq h(X_0) \leq 10$. Then there exists a non-isotrivial polarized $K3$ surface (X, \mathcal{L}) over the function field of a proper smooth curve over k such that $\rho(X) = \rho(X_0)$, $h(X) = h(X_0)$, and $(\mathcal{L})^2 = (\mathcal{L}_0)^2$.*

PROOF. We use the deformation theory to construct a non-isotrivial family. Similar arguments may be found in [13, Section 4]. Let $\pi: \mathcal{X} \rightarrow \mathcal{S}$ be the versal formal deformation of X_0 over k . It is known that \mathcal{S} is formally smooth of dimension 20 over k ; so we have

$$\mathcal{S} \simeq \mathrm{Spf} k[[t_1, \dots, t_{20}]].$$

See [6, Corollary 1.2].

Let $\mathcal{L}_1, \dots, \mathcal{L}_{\rho(X_0)}$ be a \mathbf{Z} -basis of $\mathrm{Pic}(X_0)$. Then the deformation space of the pair (X_0, \mathcal{L}_i) is defined by an equation

$$f_i \in k[[t_1, \dots, t_{20}]].$$

See [6, Proposition 1.5]. A linear combination of $\mathcal{L}_1, \dots, \mathcal{L}_{\rho(X_0)}$ is ample since X_0 is projective. Hence by Grothendieck's algebraization theorem, we get a universal family

$$\mathcal{X}' \rightarrow S' = \mathrm{Spec}(k[[t_1, \dots, t_{20}]]/(f_1, \dots, f_{\rho(X_0)}))$$

of deformations of the tuple $(X_0, \mathcal{L}_1, \dots, \mathcal{L}_{\rho(X_0)})$. For each $1 \leq h \leq 10$, let

$$S'^{(h)} := \{s \in S' \mid h(\mathcal{X}'_s) \geq h\}.$$

In [7], van der Geer and Katsura showed that there are elements

$$g_1, \dots, g_{h(X_0)-1} \in k[[t_1, \dots, t_{20}]],$$

such that, for each $2 \leq h \leq h(X_0)$, the closed subscheme $S'^{(h)} \subset S'$ is written as

$$S'^{(h)} = \mathrm{Spec}(k[[t_1, \dots, t_{20}]]/(f_1, \dots, f_{\rho(X_0)}, g_1, \dots, g_{h-1})).$$

Then the $K3$ surface \mathcal{X}'_s corresponding to a geometric point s of $S'^{(h(X_0))}$ satisfies

$$\rho(\mathcal{X}'_s) = \rho(X_0), \quad h(\mathcal{X}'_s) = h(X_0).$$

Moreover, since \mathcal{L}_0 is a \mathbb{Z} -linear combination of $\mathcal{L}_1, \dots, \mathcal{L}_{\rho(X_0)}$, the $K3$ surface \mathcal{X}'_s has a line bundle \mathcal{L}_s which lifts \mathcal{L}_0 . The line bundle \mathcal{L}_s is ample with $(\mathcal{L}_s)^2 = (\mathcal{L}_0)^2$. Since

$$\rho(X_0) + h(X_0) - 1 \leq 21 - h(X_0) < 20,$$

the dimension of $S'^{(h(X_0))}$ is positive. Hence we find a non-isotrivial family over a complete local ring of dimension ≥ 1 . After passing limits and cutting by hyperplanes, we find a non-isotrivial polarized $K3$ surface (X, \mathcal{L}) over a function field such that $\rho(X) = \rho(X_0)$, $h(X) = h(X_0)$, and $(\mathcal{L})^2 = (\mathcal{L}_0)^2$. \square

By Proposition 4.1, in order to construct non-isotrivial $K3$ surfaces X over function fields satisfying $\rho(X) = 22 - 2h(X)$ and $2 \leq h(X) \leq 10$, it suffices to construct such $K3$ surfaces over k .

We note that non-isotrivial $K3$ surfaces X with $\rho(X) = 22 - 2h(X)$ do not exist when p is odd and $h(X) = 1$. This follows from the following result recently proved by Jang and Liedtke, independently.

THEOREM 4.2 ([11, Theorem 3.7], [14, Theorem 2.6]). *Let X be a $K3$ surface with $\rho(X) = 20$ over an algebraically closed field k of characteristic $p \geq 3$. Then X is defined over a finite field.*

PROOF. See [11, Theorem 3.7] and [14, Theorem 2.6]. \square

5. The existence of non-isotrivial $K3$ surfaces with large Picard number

Assuming the existence of potential semistable reduction, Taelman conditionally proved the existence of $K3$ surfaces over finite fields with given L -function, up to finite extensions of the base field [28]. When the characteristic of the base field is not too small, the author recently proved that Taelman's results hold unconditionally [10]. As an application, the author proved the following theorem.

THEOREM 5.1 ([10, Theorem 7.4]). *Let p be a prime number with $p \geq 5$. Let $\rho \in (2\mathbb{Z})_{>0}$ be a positive even integer and $h \in \mathbb{Z}_{>0}$ a positive integer with*

$$\rho \leq 22 - 2h.$$

Then there exists a $K3$ surface X over $\overline{\mathbb{F}}_p$ such that $\rho(X) = \rho$ and $h(X) = h$.

PROOF. See [10, Theorem 7.4]. \square

PROPOSITION 5.2. *Let p be a prime number and h a positive integer with $1 \leq h \leq 10$. There exists a $K3$ surface X over $\overline{\mathbb{F}}_p$ with $\rho(X) = 22 - 2h(X)$ and $h(X) = h$ if at least one of the following conditions holds:*

- $p = 3$ and $h = 10$, or
- $p \geq 5$.

PROOF. When $p \geq 5$ and $1 \leq h \leq 9$, we have $22 - 2h \geq 4$. Then, there exists a $K3$ surface X over $\overline{\mathbb{F}}_p$ with $\rho(X) = 22 - 2h(X)$ and $h(X) = h$ by Theorem 5.1.

When $p \geq 3$ and $h = 10$, it is well-known that there exists a $K3$ surface X over $\overline{\mathbb{F}}_p$ such that $h(X) = h$ by [27, Corollary 2.2] and [7, Theorem 14.2]. Then, we have $\rho(X) \leq 22 - 2h = 2$. By the Tate conjecture for $K3$ surfaces over finite fields, the Picard number $\rho(X)$ is even; see [1, the paragraph following Definition 0.3]. (See also [9, Chapter 17, Corollary 2.9].) Hence we have $\rho(X) = 2$. \square

Combining Proposition 5.2 with Proposition 4.1, we obtain Theorem 1.3.

PROOF OF THEOREM 1.3. By Proposition 5.2, there exists a $K3$ surface X_0 over $\overline{\mathbb{F}}_p$ with $\rho(X_0) = 22 - 2h(X_0)$ and $h(X_0) = h$. By Proposition 4.1, we have a non-isotrivial $K3$ surface X over the function field of a proper smooth curve over $\overline{\mathbb{F}}_p$ with $\rho(X) = \rho(X_0)$ and $h(X) = h(X_0)$. In particular, X satisfies $\rho(X) = 22 - 2h(X)$ and $h(X) = h$. \square

EXAMPLE 5.3. In [29], [8], Yui and Goto calculated the heights of the formal Brauer groups of $K3$ surfaces over finite fields concretely. In their list, we can find concrete examples of $K3$ surfaces X over $\overline{\mathbb{F}}_p$ satisfying $\rho(X) = 22 - 2h(X)$.

REMARK 5.4. Using Taelman's theorem [28, Theorem 3] and Proposition 4.1, it is easy to see that Theorem 1.3 holds for any p and any $2 \leq h \leq 10$ if we assume [15, Assumption (\star)].

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