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LCM-stability and formal power series

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Abstract. In this paper we study the LCM-stability property and other related concepts, and their universality in the case of polynomial and formal power series extensions.

1. Introduction

Let $A \subseteq B$ be an extension of integral domains, X an indeterminate over B, $A[X]$, $B[X]$ polynomial rings and $A[\![X]\!]$, $B[\![X]\!]$ the formal power series rings. As in [16] we say that $A \subseteq B$ is LCM-stable if for any couple $(a,b) \in A^2$, $(aA \cap bA)B = aB \cap bB$ equivalently $(a :_A b)B = (a :_B b)$. This concept was first introduced by R. Gilmer in [16] and well studied in [22, 23, 10, 21], it was recently generalized in [7]. Flatness implies LCM-stability but in general the converse is false ([22], Example 4.8). Except in the case where B is an overring of A, that is $A \subseteq B \subseteq qf(A)$, ([22], Proposition 1.7). This implies that $A \subseteq A[X]$ is always LCM-stable. In this paper, we show that it is not true for power series rings giving an example of extension $A \subseteq A[\![X]\!]$ that is not LCM-stable even for the Krull case. The LCM-stability is shown to be equivalent to another concept in this case and the relation between LCM-stability, D-stability, and t-linkedness in the PvMD case highlighted. It is natural to ask whether the LCM-Stability of an extension $A \subseteq B$ entails the LCM-stability of $A[X] \subseteq B[X]$ or the LCM-stability of $A[\![X]\!] \subseteq B[\![X]\!]$. These questions were studied and proved true in the polynomial case when A is a locally GCD or a Krull domain ([22, 23]), and in the power series case when A is a Dedekind domain [10]. A domain A is said to satisfy the universality of LCM-stability if for any domain B such that $A \subseteq B$ is LCM-stable this implies that the polynomial extensions and power series extensions remain LCM-stable. We prove that the LCM-stableness of $A \subseteq B$ entails the LCM-stability of a particular polynomial extensions for the case where A is a GCD domain.

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Some conditions in particular cases for the non-universality in power series ring are given then the PvMD case investigated generalizing the result of Condo [10]. Finally, the result of Uda about the universality of t-linkedness ([22], Theorem 3.5) is shown not to hold for power series extension and necessary and sufficient conditions are given in the case of Krull extension rings for the t-linkedness in formal power series extensions.

As some of our work involves star operations, it seems useful to give the reader an overview of some known facts. Let A be an integral domain with quotient field K, and let $F(A)$ (resp. $f(A)$) be the set of nonzero fractional ideals (resp. nonzero finitely generated fractional ideals) of A.

A star operation on A is a function $*: F(A) \to F(A)$ that satisfies the following properties for every $J, L \in F(A)$ and $0 \neq u \in K$:
(i) $(u)^* = (u)$ and $(uJ)^* = uJ^*$.
(ii) $J \subseteq J^*$ and $(J \subseteq L \Rightarrow J^* \subseteq L^*)$.
(iii) $(J^*)^* = J^*$.
An $I \in F(A)$ is called $*$-ideal if $I^* = I$ and a $*$-ideal of finite type if $I = J^*$ for some $J \in f(A)$. An $I \in F(A)$ is called integral $*$-ideal if $I^* = I$ and $I \subseteq A$. A prime ideal of A which is a $*$-ideal is called $*$-prime. A maximal proper integral $*$-ideal, under inclusion, is prime. Let $*$-Max(A) denote the set of maximal proper integral $*$-ideals of A. $*$-Max(A) can be empty. For $I \in F(A)$, $I^{-1} = (A :_K I) = \{u \in K \mid uI \subseteq A\}$, $I_v = (I^{-1})^{-1}$, $I_t = \bigcup\{J_v \mid J \subseteq I$ *and* $J \in f(A)\}$. $I \to I_v$ and $I \to I_t$ are examples of star operations. Height one prime ideals are t-ideals. A v-ideal is also called divisorial ideal; for all $I \in F(A)$, I^{-1} is always a v-ideal. A domain satisfying the ascending chain condition on integral divisorial ideals is called Mori domain. Noetherian domains and Krull domains are both examples of Mori domains. If A is a Mori domain then the t-operation and the v-operation on A are the same. In general for $I \in F(A)$ we have $I \subseteq I_t \subseteq I_v$ and the inclusions may be strict. While v-Max(A) can be empty (for example if A is a rank one non-discrete valuation domain), t-Max(A) is never empty (except for the case where A is a field). Every nonunit element in a domain A is included in a t-maximal ideal of A. We say that the domain A is of finite t-character if every nonzero nonunit element of A is contained in only finitely many maximal t-ideals of A. A Mori domain has the finite t-character property. A domain A is called PvMD if A_P is a valuation domain for every t-prime ideal P of A ([20], Corollary 4.3). The t-dimension of a domain A is the supremum of the lengths of the chains of t-prime ideals. For example the t-dimension of a Krull domain is equal to 1. The reader in need of more introduction on star operations is referred to ([17], sections 32 and 34). For Mori domains the reader is referred to [5].

2. Results

Definition 1. Let $A \subseteq B$ be an extension of integral domains, we say that $A \subseteq B$ is LCM-stable if for any couple $(a,b) \in A^2$, $(aA \cap bA)B = aB \cap bB$ equivalently $(a :_A b)B = (a :_B b)$.

We say that $A \subseteq B$ is R_2-stable if for any couple $(a,b) \in A^2$, $(a :_A b) = aA$ implies $(a :_B b) = aB$ equivalently $(a,b)^{-1} = A$ implies $((a,b)B)^{-1} = B$.

If $(a,b)^{-1} = A$ we say that a, b are v-coprime elements of A.

Remark 1. *It is clear that LCM-stableness implies R_2-stableness. The converse is true when A is a GCD domain (*[22]*, remark before Lemma* 3.1*).*

Example 1. *Let K be a field, X an indeterminate, $A = K[X^2, X^3]$ and $B = K[X]$, then the extension $A \subseteq B$ is not LCM-stable, $(X^2A \cap X^3A)B = X^5B$ but $X^2B \cap X^3B = X^3B$ so $(X^2A \cap X^3A)B \neq X^2B \cap X^3B$.*

Definition 2. Let $A \subseteq B$ be an extension of integral domains. As in [21], where the concept was first introduced, we say that the extension is D-stable if for any divisorial ideal I of A we have $I^{-1}B = (IB)^{-1}$.

Example 2. *If A is a factorial domain then for any domain B that contains A, $A \subseteq B$ is D-stable since all divisorial ideals in a factorial domain are principal.*

Definition 3. Let $A \subseteq B$ be an extension of integral domains. We say that the extension is F-stable if for any fractional ideal I of A we have $I^{-1}B = (IB)^{-1}$.

Proposition 1. *Let $A \subseteq B$ be an extension of integral domains. If $A \subseteq B$ is F-stable then $A \subseteq B$ is LCM-stable. If A is a Krull domain then the converse is true.*

Proof. Note that $\left(\left(\frac{1}{a}, \frac{1}{b}\right)A\right)^{-1} = aA \cap bA$ for all $0 \neq a, b \in A$. So if $A \subseteq B$ is F-stable then $(aA \cap bA)B = \left(\left(\frac{1}{a}, \frac{1}{b}\right)A\right)^{-1}B = \left(\left(\frac{1}{a}, \frac{1}{b}\right)B\right)^{-1} = aB \cap bB$. So $(aA \cap bA)B = aB \cap bB$ and $A \subseteq B$ is LCM-stable.

Conversely, suppose that $A \subseteq B$ is LCM-stable and A is a Krull domain. Let I be a fractional ideal, we can suppose that I is an integral ideal. Since A is a Mori domain, there is a finitely generated ideal $J \subseteq I$ such that $I^{-1} = J^{-1}$. Note that $I^{-1}B \subseteq (IB)^{-1}$: Indeed, $I \subseteq A \subseteq B$, then $I^{-1} \subseteq qf(B)$. Let $x \in I^{-1}B$, then there is $n \in \mathbb{N}^*$ and two countable families $(k_i)_{1 \leq i \leq n} \in I^{-1}$ and $(b_i)_{1 \leq i \leq n} \in B$ such that $x = \sum_{i=1}^{n} k_i b_i$. Let $u \in IB$, $u = \sum_{i=0}^{m} q_i b_i'$ with $q_i \in I$ and $b_i' \in B$, so $xu = \sum_{i=0}^{n} \sum_{j=0}^{m} (k_i q_j)(b_i)(b_j')$, $k_i q_j \in I$. And so $ux \in IB$. Then $x \in (IB)^{-1}$. It is sufficient to show the opposite inclusion. By the above, $I^{-1}B = J^{-1}B$. By ([23], Proposition 10), $J^{-1}B = (JB)^{-1}$. Since $J \subseteq I$, this implies $JB \subseteq IB$ and so $(IB)^{-1} \subseteq (JB)^{-1}$. Then $(IB)^{-1} \subseteq I^{-1}B$. Finally $(IB)^{-1} = I^{-1}B$. $\square$

Definition 4. Let $A \subseteq B$ be an extension of integral domains, an ideal I of the domain A is said to be a G.V. ideal if I is finitely generated and $I^{-1} = A$. As in [13], where the term first appeared, we say that the extension is t-linked, if for any G.V. ideal I of A we have $(IB)^{-1} = B$.

Remark 2.

- *In his paper* [22], *Professor Uda introduced the concept of G_2-Stability. Let $A \subseteq B$ be an extension of integral domains, we say that $A \subseteq B$ is G_2-stable if for any finitely generated ideal I of A, $Gr(I) \geq 2$ implies that $Gr(IB) \geq 2$, where Gr stands for the polynomial grade, but as pointed out by Professor Uda in the remark before Lemma* 3.1 *in* [22], *if the ideal I is finitely generated then $Gr(I) \geq 2$ if and only if $I^{-1} = A$. So the G_2-stableness is in fact the t-linkedness.*
- *It is clear that if $A \subseteq B$ is t-linked then $A \subseteq B$ is R_2-stable. The converse is true for the case A is a GCD domain (*[22], *Theorem* 3.6*). But it is not the only case where the converse holds. Recall that an integral domain A is said to be of finite t-character if each nonzero nonunit element of A is contained in only finitely many maximal t-ideals of A, we recall this known result:*

Lemma 1 (cf Corollary 2.8 in [7]). *Let $A \subseteq B$ be an extension of integral domains. If A is of finite t-character, then $A \subseteq B$ R_2-stable implies $A \subseteq B$ t-linked.*

Remark 3. *In Theorem* 4 *in their article* [21], *Sato et al. claim that if $A \subseteq B$ is an extension of Noetherian domains such that A is a Krull domain, then D-stability implies LCM-stability. This result seems to be incorrect. Indeed, note that if A is a factorial domain, then for any domain B that contains A, $A \subseteq B$ is D-stable since divisorial ideals of A are principal. Let A be a Noetherian factorial domain with Krull dimension ≥ 2 (for example take $A = K[X, Y]$, where K is a field). Let P be a prime ideal of A such that $ht(P) \geq 2$, then $P^{-1} = A$. Let V be a DVR overring of A centred in P (that is if M is the maximal ideal of V, $P = M \cap A$) (*[9]*). Then $A \subseteq V$ is D-stable but not LCM-stable. Indeed, note that since A is of finite t-character and $P^{-1} = A$, we can find a nonunit nonzero $x, y \in P$, such that $(x, y)^{-1} = A$ or equivalently $xA \cap yA = xyA$. On the other hand in the valuation domain V, $x|y$ or $y|x$ and both are nonunit, say $x|y$, then $xV \cap yV = yV$ and so $(xA \cap yA)V \neq xV \cap yV$ thus the extension is not LCM-stable.*

Nevertheless the D-stability, along with the t-linkedness, plays an important role in the LCM-stability in the case of PvMD, as shown in the following proposition:

Proposition 2. *Let $A \subseteq B$ be an extension of integral domains, which satisfies the following assertions:*
1- *A is a* PvMD.
2- *$A \subseteq B$ is t-linked.*
3- *$A \subseteq B$ is D-stable.*
Then $A \subseteq B$ is LCM-stable.

Proof. Let I be a divisorial ideal in A, $J = I^{-1}$ is also divisorial. Since $A \subseteq B$ is D-stable then $IB = J^{-1}B = (JB)^{-1}$ is a divisorial ideal in B. By Proposition 8 (2) in [23], since $A \subseteq B$ is t-linked then for each $a, b \in A \backslash \{0\}$ $(a :_B b) = ((a :_A b)B)_v$, by the above, $(a :_A b)B$ is a divisorial ideal of B, so $((a :_A b)B)_v = (a :_A b)B$. This implies that $(a :_B b) = (a :_A b)B$ and the extension $A \subseteq B$ is LCM-stable. □

In the following theorem, we improve the result of Sato et al. about D-stability in polynomial extensions ([21], Theorem 5):

Theorem 1. *Let $A \subseteq B$ be an extension of integral domains. If A is integrally closed, then the following assertions are equivalent:*
(1)- *$A \subseteq B$ is D-stable.*
(2)- *$A[X] \subseteq B[X]$ is D-stable.*

Proof. Let I be a divisorial ideal of $A[X]$. Then there are two possibilities:
(1)- $I \cap A = J \neq 0$, with J a divisorial of A, then $I = J[X]$ and we have:

$$\begin{aligned} I^{-1}B[X] &= (J.A[X])^{-1}.B[X] \\ &= (J^{-1}.A[X]).B[X] \\ &= J^{-1}.B[X] \\ &= J^{-1}.B.B[X] \\ &= (JB)^{-1}B[X] \\ &= ((JB).B[X])^{-1} \\ &= (J.B[X])^{-1} \\ &= (J.A[X].B[X])^{-1} \\ &= (I.B[X])^{-1} \end{aligned}$$

(2)- $I \cap A = 0$, then by ([19], Lemme 2), there is $f \in A[X]$ and a divisorial ideal J of A such that $I = f.J[X]$. The same method is applied as in the first case.

Conversely suppose that $A[X] \subseteq B[X]$ is D-stable, take I a divisorial ideal of A, then $I[X]$ is a divisorial ideal of $A[X]$. $(I[X]B[X])^{-1} = (I.A[X]B[X])^{-1} = ((IB).B[X])^{-1} = ((IB)[X])^{-1} = (IB)^{-1}[X]$.

By the D-stability, $(I[X]B[X])^{-1} = (I[X])^{-1}B[X] = I^{-1}[X]B[X] = (I^{-1}B)[X]$ and so $(IB)^{-1} = I^{-1}B$. □

Let S be a multiplicative set in the domain A and X an indeterminate. Then the subset of $A_S[X]$ defined by $\{f \in A_S[X] \mid f(0) \in A\}$ is a subring of $A_S[X]$ denoted by $A + XA_S[X]$. This construction was studied in [11]. Now the t-linkedness and the LCM-stability of polynomial extensions of the form $A + XA_S[X] \subseteq B + XB_T[X]$ will be investigated:

LEMMA 2. *Let $A \subseteq B$ be an extension of integral domains, S (resp. T) be a multiplicative set of A (resp. B), such that $S \subseteq T$ then if $A + XA_S[X] \subseteq B + XB_T[X]$ is LCM-stable, so is the extension $A \subseteq B$.*

PROOF. The extension $A \subseteq A + XA_S[X]$ is always LCM-stable, in fact $A + XA_S[X]$ is a faithfully-flat A-module. Now, since $A \subseteq A + XA_S[X]$ and $A + XA_S[X] \subseteq B + XB_T[X]$ is LCM-stable then by ([22], Proposition 1.2, (1)), $A \subseteq B + XB_T[X]$ is LCM-stable. Now, since $B + XB_T[X]$ is a faithfully flat B-module then for any ideal I of B, $I(B + XB_T[X]) \cap B = I$ again by ([22], Proposition 1.2, (2)), $A \subseteq B$ is LCM-stable. □

REMARK 4. *While the extension $A \subseteq A + XA_S[X]$ is always LCM-stable, the extension $A[X] \subseteq A + XA_S[X]$ is never LCM-stable (except the trivial case $S \subseteq U(A)$, where $U(A)$ stands for the units in A). More generally, the extension $A[X] \subseteq A + XB[X]$ where B is a domain that contains A, fails to be LCM-stable (in fact it fails to be R_2-stable) once $U(B) \cap A \neq U(A)$. Indeed, take $d \in U(B) \cap A \setminus U(A)$, then $(d, X)^{-1} = A[X]$ by (*[22], *Lemma* 3.1*). But since $d \in U(B)$ then $X = d\frac{1}{d}X$ in $A + XB[X]$ and so $(d, X)(A + XB[X]) = d(A + XB[X])$ then $((d, X)(A + XB[X]))^{-1} \neq A + XB[X]$ so the extension is not R_2-stable and consequently not LCM-stable.*

In the next lemma we give a necessary and sufficient condition to have $(d, X)(A + XA_S[X])$ a G.V. ideal of $A + XA_S[X]$:

LEMMA 3. *Let A be a domain, S a multiplicative set in A, $d \in A$ then (d, X) is a G.V. ideal of $A + XA_S[X]$ if and only if (d, s) are v-coprime for all $s \in S$.*

PROOF. $((d, X)(A + XA_S[X]))^{-1} = \frac{1}{d}A \cap A_S + XA_S[X]$ so $((d, X)(A + XA_S[X]))^{-1} = A + XA_S[X]$ if and only if $\frac{1}{d}A \cap A_S = A$ if and only if $A \cap dA_S = dA$.

If $A \cap dA_S = dA$, take $s \in S$, let $\alpha \in (d,s)^{-1}$, $\alpha = \frac{a}{b} \in K = qf(A)$, $\alpha d \in A$ and $\alpha s \in A$ so there is $e, f \in A$ such that $ad = bf$ and $as = be$ so $ade = bef = asf$ and so $de = sf$ so $d\frac{e}{s} = f$ and $f \in A \cap dA_S = dA$ so $f = f_1 d$ and $a = bf_1$ finally $\alpha = f_1 \in A$.

Conversely if for all $s \in S$, $(d,s)^{-1} = A$ take $f \in A \cap dA_S$ so $f = d\frac{e}{s}$ and $fs = ed$ so $f \in (d :_A s) = dA$. $\square$

Lemma 4. *Let I be a G.V. ideal of $A + XA_S[X]$ then $I \cap A \neq 0$, if $I \cap S \neq \emptyset$ then $I = J(A + XA_S[X])$ with J a G.V. ideal of A.*

Proof. Suppose that $I \cap A = 0$, $I = (f_1, \dots, f_n)$, $f_i \in A + XA_S[X]$, $i = 1, \dots, n$. Let $K = qf(A)$. Then $IK[X] \neq K[X]$, say $IK[X] = f(x)K[X]$, $f(x) \in K[X]$ with $deg(f(x)) \geq 1$. So for all $i = 1, \dots, n$ there exists $h_i(x) \in K[X]$ such that $f_i(x) = f(x)h_i(x)$. So $\frac{I}{f(x)} = (h_1(x), \dots, h_n(x))$ is a fractional ideal of $A + XA_S[X]$. Thus we can find $a \in A \backslash \{0\}$ such that $ah_i(x) \in A[X] \subseteq A + XA_S[X]$. So $\frac{a}{f(x)} I \subseteq A + XA_S[X]$ and $\frac{a}{f(x)} \in I^{-1} \backslash A + XA_S[X]$, since $a \in A$ and $deg(f(x)) \geq 1$, contradiction.

The second part follows from ([4], Lemma 3.7 and its proof). $\square$

Proposition 3. *Let $A \subseteq B$ be an extension of integral domains, K (resp. L) the field of fractions of A (resp. B) then the following statements are equivalent:*

(1) *$A \subseteq B$ is t-linked.*
(2) *$A + XK[X] \subseteq B + XL[X]$ is t-linked.*

Proof. $(1) \Rightarrow (2)$ Let I be a G.V. ideal of $A + XK[X] = A + XA_{A^*}[X]$, then by the previous lemma $I \cap A \neq 0$. Now $I \cap S \neq \emptyset$ so $I = J(A + XK[X])$ with J is a G.V. ideal of A. Now since $A \subseteq B$ is t-linked then JB is a G.V. ideal of B, $B \subseteq B + XL[X]$ is flat implies that $(JB)(B + XL[X])$ is a G.V. ideal of $B + XL[X]$ since $((JB)(B + XL[X])) = I(B + XL[X])$ then $I(B + XL[X])$ is a G.V. ideal of $B + XL[X]$ and the extension $A + XK[X] \subseteq B + XL[X]$ is t-linked.

$(2) \Rightarrow (1)$ Let J be GV ideal of A. Then, by ([4], Lemma 3.7), $J + XK[X] = J(A + XK[X])$ is a GV-ideal of $A + XK[X]$. Since $A + XK[X] \subseteq B + XL[X]$ is t-linked we can conclude that $J(A + XK[X])(B + XL[X])$ is a GV-ideal of $B + XL[X]$. But $J(A + XK[X])(B + XL[X]) = JB + XL[X]$ is a GV-ideal of $B + XL[X]$ and so is JB, by ([4], Lemma 3.7). Thus the extension $A \subseteq B$ is t-linked. $\square$

Corollary 1. *Let $A \subseteq B$ be an extension of integral domains, A a GCD domain, K (resp. L) the field of fractions of A (resp. B) then the following statements are equivalent:*

(1) *$A \subseteq B$ is t-linked.*
(2) *$A \subseteq B$ is R_2-Stable.*
(3) *$A \subseteq B$ is LCM-stable.*
(4) *$A + XK[X] \subseteq B + XL[X]$ is t-linked.*
(5) *$A + XK[X] \subseteq B + XL[X]$ is R_2-Stable.*
(6) *$A + XK[X] \subseteq B + XL[X]$ is LCM-stable.*

Proof. By ([11], Corollary 1.3), if A is a GCD domain then $A + XK[X]$ is a GCD domain. □

Now the formal power series cases are investigated, beginning by studying the LCM-stability of the extension $A \subseteq A[\![X]\!]$:

Definition 5. Let A be an integral domain. A is an almost-finite conductor domain, if for any two elements $a, b \in A$ the ideal $I = aA \cap bA$ verifies the following property: for any countable family $(a_j)_{j \in \mathbb{N}}$ of I, there is an ideal of finite type $F \subseteq I$ that contains this family.

Proposition 4. *For an integral domain A, the extension $A \subseteq A[\![X]\!]$ is LCM-stable if and only if A is an almost-finite conductor domain.*

Proof. It is easy to establish that A has the almost finite conductor property if and only if for all a, b in $A \backslash \{0\}$ we have $(aA \cap bA)[[X]] = (aA \cap bA).A[\![X]\!]$. Suppose that $A \subseteq A[\![X]\!]$ is LCM-stable, let $I = aA \cap bA$, $a, b \in A$ then $I.A[\![X]\!] = (aA \cap bA).A[\![X]\!] = aA[\![X]\!] \cap bA[\![X]\!]$ and so $I.A[\![X]\!]$ is a divisorial ideal of $A[\![X]\!]$. Since $(I.A[\![X]\!])_v = I_v[\![X]\!]$ ([12], Proposition 2.1), $I.A[\![X]\!] = I_v[\![X]\!] = I[\![X]\!]$ since $I_v = I$.

Conversely, let $f \in aA[\![X]\!] \cap bA[\![X]\!]$, $a, b \in A^2$ then there exist $g, h \in A[\![X]\!]$ such that $f(x) = ag(x)$ and $f(x) = bh(x)$. By identifying the coefficients for all $i \geq 0$, $f_i = ag_i = bh_i$ then $f_i \in aA \cap bA$ and so $f \in (aA \cap bA)[\![X]\!]$. Since $(aA \cap bA)[\![X]\!] = (aA \cap bA).A[\![X]\!]$, $aA[\![X]\!] \cap bA[\![X]\!] = (aA \cap bA).A[\![X]\!]$. □

Example 3. *All finite conductor domains such as Prüfer domains, GCD domains, Noetherian domains are almost-finite conductor domains.*

Corollary 2. *Let A be a Krull domain. The extension $A \subseteq A[\![X]\!]$ is LCM-stable if and only if for all divisorial ideal I of A, $I.A[\![X]\!] = I[\![X]\!]$.*

Proof. Let I be a divisorial ideal of A, since A is a Krull domain, by ([17], Corollary 44.6), there are $x, y \in K = qf(A)$ such that $I = xA \cap yA$. There are $a, b, c \in A$ such that $x = \frac{a}{c}$, $y = \frac{b}{c}$ and so $I = \frac{1}{c}(aA \cap bA)$. □

Remark 5. *There is an integral domain A such that $A \subseteq A[\![X]\!]$ is not LCM-stable and consequently the extension is not flat:*

1- *Let K be a field, $A = K[x, xy, yw, y^2w, y^3w, \dots]$, where x, y are indeterminates over K and $w = xy + 1$. Let $Q = qf(A)$, the ideal $I = (xy :_A x) = (xy, yw, y^2w, y^3w, \dots)$ is not a finitely generated ideal of A (cf* [18], *page* 2835*), it is of the form $uA \cap vA$ with $u, v \in Q$. Let $f = xy + \sum_{i=1}^{\infty} y^i wZ^i$. Then $f \in I[\![Z]\!] \backslash I.A[\![Z]\!]$. Suppose by contradiction that we have the equality, then there are $l_1, l_2, \dots, l_n \in I$ and $h_1, \dots h_n \in A[\![Z]\!]$ such that $f(Z) = \sum_{i=1}^{n} l_i h_i(Z)$. By identifying the coefficients, $I \subseteq (l_1, \dots l_n) \subseteq I$ then $I = (l_1, \dots l_n)$. This is a contradiction.*

2- *Let K be a field, $T = K[X_1, X_2, \dots]$ where X_i are indeterminates over K. T is a Krull domain. Let $R = K[X_1^2, X_1X_2, \dots, X_iX_j, \dots]$ a subring of T, then $R = T \cap Q$ with $Q = qf(R)$, thus R is a Krull domain. The extension $R \subseteq T$ is an integral extension since for all $j \in \mathbb{N}$, X_j is a root of the monic polynomial $Q_j(Z) = Z^2 - X_j^2$. Let $P = TX_1 \cap R$, since X_1T is a divisorial prime ideal in the Krull domain T, $ht(X_1T) = 1$. Since R is a Krull domain, R is a completely integrally closed domain, so by the Going-Down theorem, $ht(P) = 1$. So P is a divisorial ideal. $P = (X_1X_i, i \in \mathbb{N}^*)$ and hence is not a finitely generated ideal of R. Then $P.R[\![Z]\!] \neq P[\![Z]\!]$. Indeed, let $g = \sum_{i \geq 1} X_1X_iZ^i \in P[\![Z]\!]$. If $g \in P.R[\![Z]\!]$, then there is $m \in \mathbb{N}$, $p_i \in P$, $f_i \in R[\![Z]\!]$ such that $g = \sum_{i=1}^{m} p_i f_i$. By identifying the coefficients, P is a finitely generated ideal, contradiction.*

Corollary 3. *Let A be a* PvMD*, if $A \subseteq A[\![X]\!]$ is D-stable, then it is LCM-stable. If A is Krull the converse is true.*

Proof. The extension $A \subseteq A[\![X]\!]$ is always t-linked: If I is a finitely generated ideal of A, $I^{-1} = A$ then by ([12], Proposition 2.1), $(I.A[\![X]\!])^{-1} = I[\![X]\!]^{-1} = I^{-1}[\![X]\!] = A[\![X]\!]$. If A is Krull and $A \subseteq A[\![X]\!]$ is LCM-stable, then by Proposition 1, $A \subseteq A[\![X]\!]$ is F-stable, and so D-stable. □

Now we prove the universality of D-stability for the formal power series but in a much smaller setting than the polynomial case:

Theorem 2. *Let $A \subseteq B$ be an extension of integral domains. If A is a regular ring then the following assertions are equivalent:*
(1)- *$A \subseteq B$ is D-stable.*
(2)- *$A[\![X]\!] \subseteq B[\![X]\!]$ is D-stable.*

Proof. Let J be a divisorial ideal of $A[\![X]\!]$.

- if $J \cap A \neq 0$:
  By ([6], Chapitre 13, Proposition 6.19), there is a divisorial ideal I of A such that $J = I[\![X]\!]$.

$$J^{-1}B[\![X]\!] = (I[\![X]\!])^{-1}.B[\![X]\!]$$
$$= (I.A[\![X]\!])^{-1}.B[\![X]\!] \qquad \text{by ([12], Proposition 2.1).}$$

Since $A \subseteq A[\![X]\!]$ is LCM-stable, and A is Krull, the extension is D-stable, then:

$$(I.A[\![X]\!])^{-1}.B[\![X]\!] = (I^{-1}.A[\![X]\!]).B[\![X]\!]$$
$$(I^{-1}.A[\![X]\!]).B[\![X]\!] = I^{-1}.B[\![X]\!]$$
$$= I^{-1}.B.B[\![X]\!]$$
$$= (IB)^{-1}B[\![X]\!]$$

I^{-1} is finitely generated ideal because A is a Noetherian domain. Then $I^{-1}B$ is a finitely generated ideal, and since $I^{-1}B = (IB)^{-1}$ by the D-stability of the extension $A \subseteq B$, $(IB)^{-1}$ is finitely generated.

$$(IB)^{-1}B[\![X]\!] = ((IB)^{-1}[\![X]\!])$$
$$= (I.B[\![X]\!])^{-1}$$
$$= (I.A[\![X]\!].B[\![X]\!])^{-1}$$
$$= (I[\![X]\!].B[\![X]\!])^{-1}$$

- if $J \cap A = 0$:
  By ([15], Corollary 18.23), there exist $f \in A[\![X]\!]\backslash A$ and a divisorial ideal I of A, such that $J = fI[\![X]\!]$. Note that $J^{-1} = f^{-1}(I[\![X]\!])^{-1}$ using the first case in the theorem the wanted result is proved.
  Conversely take I a divisorial of A. Since A is a Noetherian domain, I and I^{-1} are finitely generated. $I[\![X]\!]$ is a divisorial ideal of $A[\![X]\!]$ ([12], Proposition 2.1). $(I[\![X]\!]B[\![X]\!])^{-1} = (I.A[\![X]\!]B[\![X]\!])^{-1} = (I.B[\![X]\!])^{-1} = ((IB).B[\![X]\!])^{-1} = ((IB)[\![X]\!])^{-1} = (IB)^{-1}[\![X]\!]$. On the other hand using the D-stability of $A[\![X]\!] \subseteq B[\![X]\!]$ we have $(I[\![X]\!]B[\![X]\!])^{-1} = (I[\![X]\!])^{-1}B[\![X]\!] = (I.A[\![X]\!])^{-1}B[\![X]\!]$. Since A is regular domain, then it is a Noetherian Krull domain. Thus the LCM-stability of $A \subseteq A[\![X]\!]$ entails its D-stability. And so $(I.A[\![X]\!])^{-1} = I^{-1}.A[\![X]\!]$. So $(I.A[\![X]\!])^{-1}B[\![X]\!] = I^{-1}A[\![X]\!]B[\![X]\!] = I^{-1}B.B[\![X]\!] = (I^{-1}B)[\![X]\!]$. Thus we have $(IB)^{-1}[\![X]\!] = (I^{-1}B)[\![X]\!]$ and $(IB)^{-1} = (I^{-1}B)$. $\square$

THEOREM 3. *Let A be an integral domain, if A contains an infinite sequence of v-coprime nonunit elements $(p_i)_{i\in\mathbb{N}}$ such that $\bigcap p_1 \dots p_i A \neq 0$ then there is an overring B such that $A \subseteq B$ is LCM-stable but $A[\![X]\!] \subseteq B[\![X]\!]$ is not LCM-stable.*

PROOF. Note that since $(p_i)_{i\in\mathbb{N}}$ is an infinite sequence of v-coprime nonunit elements, no t-maximal ideal contains more than one element of this sequence. So for each p_i there is a maximal t-ideal M_i, such that $p_i \in M_i$ and for all $j \in \mathbb{N}$, $j \neq i$ we have $p_j \notin M_i$. Let S be the multiplicative set composed by the finite product of $(p_i)_{i\in\mathbb{N}}$. Clearly $A \subseteq A_S$ is LCM-stable (in fact it is a flat extension). Since $\bigcap p_1 \dots p_i A \neq 0$, a non-zero element $a \in \bigcap p_1 \dots p_i A \neq 0$ can be chosen. Let $f(X) = a + \sum_{i=1}^{\infty} \frac{a}{p_1 \cdots p_{2^i}} X^i$. It is clear that $\frac{a}{p_1 \cdots p_{2^i}} \in A$ so $f \in A[\![X]\!]$. Moreover $f \in A_S[\![X]\!]$, $fa^{-1} = 1 + \sum_{i=1}^{\infty} \frac{1}{p_1 \cdots p_{2^i}} X^i$ $\in U(A_S[\![X]\!])$ so $a = f.(fa^{-1})^{-1} \in f.A_S[\![X]\!]$ whence $a \in a.A_S[\![X]\!] \cap f.A_S[\![X]\!]$. To conclude, we will show that $a \notin (a.A[\![X]\!] \cap f.A[\![X]\!]).A_S[\![X]\!]$. Let $r, s \in A[\![X]\!]$ such that $rf = as$. The equality in the nth coefficient gives:

$$r_0 \frac{a}{p_1 \dots p_{2^n}} + r_1 \frac{a}{p_1 \dots p_{2^{n-1}}} + \dots + r_n a = s_n a$$

$$\frac{r_0}{p_{2^{n-1}+1} \dots p_{2^n}} = (s_n p_1 \dots p_{2^{n-1}}) - r_1 - \dots - r_n(p_1 \dots p_{2^{n-1}}) \in A$$

So for all $n \in \mathbb{N}$, $r_0 \in p_{2^{n-1}+1} \dots p_{2^n} A$. Suppose by contradiction that there is $m \in \mathbb{N}$, $h_i \in a.A[\![X]\!] \cap f.A[\![X]\!]$ and $y_i \in A_S[\![X]\!]$ such that $a = \sum_{i=1}^m h_i y_i$ by the above $h_{i,0} = ar_{i,0}$ with $r_{i,0} \in \bigcap_{n\in\mathbb{N}^*} p_{2^{n-1}+1} \dots p_{2^n} A$. The constant term of the equation gives $a = \sum_{i=1}^m h_{i,0} y_{i,0} = \sum_{i=1}^m a r_{i,0} y_{i,0}$. Whence $1 = \sum_{i=1}^m r_{i,0} y_{i,0}$ every $y_{i,0} \in A_S$, there exist $j, k, l \in \mathbb{N}$ such that for all i $(p_j \dots p_k)^l . y_{i,0} \in A$. Whence $(p_j \dots p_k)^l = \sum_{i=1}^m r_{i,0} (p_j \dots p_k)^l . y_{i,0}$ therefore $(p_j \dots p_k)^l \in \bigcap_{n\in\mathbb{N}^*} p_{2^{n-1}+1} \dots p_{2^n} A$ for any $n > k$ we have that $(p_j \dots p_k)^l \in p_n A$ and so there is $j \leq i \leq k$, $i \neq n$ such that $p_i \in M_n$ contradiction. Whence $a \notin (a.A[\![X]\!] \cap f.A[\![X]\!]) A_S[\![X]\!]$. So $A[\![X]\!] \subseteq A_S[\![X]\!]$ is not LCM-stable whereas $A \subseteq A_S$ is LCM-stable. $\square$

EXAMPLE 4. *Recall that in an integral domain A, a non-zero non-unit element r is said to be rigid if for all x, y if $x|r$ and $y|r$ then $y|x$ or $x|y$. Let A be a GCD domain that is not of Krull type (for example take $A = \mathbb{Z} + Y\mathbb{Q}[Y]$, where $\mathbb{Z}$, $\mathbb{Q}$ are rings of integers and rational numbers respectively. A is a Bezout domain (*[11], *Corollary* 4.13*) and it is not of Krull type by (*[24], *Theorem* D*)). Then by (*[24], *Theorem* A*), there is a non-rigid non-unit element $\alpha \in A$ such that either α is not divisible by any rigid element or α is divisible by an infinity of mutually coprime rigid elements. In both cases, there is an infinity of mutually coprime elements $(p_i)_{i\in\mathbb{N}}$ that divide α. Moreover $\alpha \in \bigcap_{i\in\mathbb{N}} p_i A = \bigcap_{i\in\mathbb{N}} p_0 \dots p_i A$ because they are mutually coprime in a GCD-domain. Take S the multiplicative set defined as in the proof of theorem* 3*, then $A \subseteq A_S$ is LCM-stable but $A[\![X]\!] \subseteq A_S[\![X]\!]$ is not LCM-stable.*

A GCD domain is a special case of PvMD. There is a much stronger result given by the following theorem that summarizes and generalizes the work of Condo by placing it in a broader scope, but first we need this technical lemma:

LEMMA 5. *Let A be an integral domain, P a prime ideal of A, if $A[\![X]\!] \subseteq A_P[\![X]\!]$ is LCM-stable then $A_P[\![X]\!] \cap qf(A[\![X]\!]) \subset A[\![X]\!]_{P[\![X]\!]}$.*

PROOF. $A_P[\![X]\!]$ is a local domain with maximal ideal $M = PA_P + XA_P[\![X]\!]$. Let $Q = M \cap A[\![X]\!]$ then $Q = P + XA[\![X]\!]$, define $B = A[\![X]\!]_{P+XA[\![X]\!]}$, then (B, N) is a local domain and $N = (P + XA[\![X]\!])A[\![X]\!]_{P+XA[\![X]\!]} \subseteq PA_P + XA_P[\![X]\!]$. Now since $A[\![X]\!] \subseteq A_P[\![X]\!]$ is LCM-stable and $A[\![X]\!]_{P+XA[\![X]\!]}$ is an overring of $A[\![X]\!]$ and $A[\![X]\!] \subseteq A[\![X]\!]_{P+XA[\![X]\!]} \subseteq A_P[\![X]\!]$ then by ([22], Proposition 1.4), $A[\![X]\!]_{P+XA[\![X]\!]} \subseteq A_P[\![X]\!]$ is LCM-stable, since $N \subseteq M$ we have $NA_P[\![X]\!] \neq A_P[\![X]\!]$ then by ([22], Proposition 1.11), $A_P[\![X]\!] \cap qf(A[\![X]\!]) = A[\![X]\!]_{P+XA[\![X]\!]} \subseteq A[\![X]\!]_{P[\![X]\!]}$. □

THEOREM 4. *Let A be a* PvMD. *If A verifies the universality of the LCM-stability for power series extensions, then A is a Krull domain.*

PROOF. Since A is a PvMD then for every t-prime ideal P of A, A_P is a valuation domain. Now Theorems 2.6–2.9 in [10] show that A_P is a DVR, and thus t-dimension $A = 1$. Whence by ([8], Lemma 2), it is sufficient to show that P is the radical of a finitely generated ideal for each maximal t-ideal P of A. Suppose that there is a maximal t-ideal P of A that is not the radical of a finitely generated ideal. Then by ([8], Lemma 3), there exists a maximal t-ideal Q of A with ht $Q[\![X]\!] \geq 2$. Thus $A[\![X]\!]_{Q[\![X]\!]}$ is not a DVR, whence by [3], $A_Q[\![X]\!] \cap qf(A[\![X]\!]) \not\subset A[\![X]\!]_{Q[\![X]\!]}$ by the previous lemma $A[\![X]\!] \subseteq A_Q[\![X]\!]$ is not LCM-stable. □

One can ask what happens in the case of Krull domain. Condo has shown that in the case of one dimensional Krull domain, Dedekind domain, it is true. We have shown that one of the components of the LCM-stability, the D-stability is universal in a particular case of Krull domain (regular domain). But what about the t-linkedness?

LEMMA 6. *Let A be an integral domain and $f \in A[\![X]\!]$ such that $f(0) \neq 0$. Then $(f, X^n)^{-1} = A[\![X]\!]$ for all $n \in N^*$.*

PROOF. Since $f(0) \neq 0$ then f is invertible in $K[\![X]\!]$ where $K = qf(A)$. Let $u \in (f, X^n)^{-1}$ then $uf \in A[\![X]\!]$, whence $u \in f^{-1}A[\![X]\!] \subseteq K[\![X]\!]$, so $u \in K[\![X]\!]$ and since $uX^n \in A[\![X]\!]$ we have that $u \in A[\![X]\!]$. □

Proposition 5. *Let A be an integral domain such that the Krull dimension of $A[\![X]\!]$ is equal to 2, let B be an integral domain that contains A. Then the extension $A[\![X]\!] \subseteq B[\![X]\!]$ is t-linked.*

Proof. Let I be a finitely generated ideal of $A[\![X]\!]$ such that $I^{-1} = A[\![X]\!]$. Whence I is not included in any ideal of height 1 of $A[\![X]\!]$. Thus $\sqrt{I} = \bigcap P_i + XA[\![X]\!]$, and so there is $n \in \mathbb{N}$ such that $X^n \in I$. Since $I^{-1} = A[\![X]\!]$ we have that $I \not\subseteq XA[\![X]\!]$ there is $f \in I$ such that $f(0) \neq 0$. $(f, X^n) \subseteq I$ then $(f, X^n)B[\![X]\!] \subseteq IB[\![X]\!]$ then $(IB[\![X]\!])^{-1} \subseteq ((f, X^n)B[\![X]\!])^{-1} = B[\![X]\!]$, indeed, if $f(0) \in U(B)$ then $((f, X^n)B[\![X]\!]) = B[\![X]\!]$, if not, let $L = qf(B)$ then f is invertible in $L[\![X]\!]$, the proof of the previous lemma shows that $((f, X^n)B)$ is a G.V. ideal of $B[\![X]\!]$. Whence $(IB[\![X]\!])^{-1} = B[\![X]\!]$. $\square$

Now we give an example of the non-universality of t-linkedness in the formal power series extensions:

Example 5. *Let (A, M) be a non-discrete rank one valuation domain, let $(t_i)_{i \in \mathbb{N}^*} \in M$, a sequence such that $(v(t_i))_{i \in \mathbb{N}^*}$ is a strictly decreasing sequence and $\lim_{i \to +\infty} v(t_i) = 0$, let $f(X) = \sum_{i=0}^{\infty} t_{i+1} X^i$ then $f \in M[\![X]\!] \backslash M.A[\![X]\!]$, since $t_1 \in M \backslash 0$, by (*[6], *Chapitre* 13, *Lemme* 2.2*), we have $(t_1, f)^{-1} = A[\![X]\!]$. Now define $B = A[Y, (t_j^{-1} Y)_{j \geq 2}, (t_j t_1^{-1} Y)_{j \geq 2}]$. $A \subseteq B$ is t-linked, but $A[\![X]\!] \subseteq B[\![X]\!]$ is not t-linked. $t_1^{-1} Y \in qf(B) \backslash B$ and $t_1^{-1} Y \in ((t_1, f)B[\![X]\!])^{-1}$ so $((t_1, f)B[\![X]\!])^{-1} \neq B[\![X]\!]$. Note that this example shows that the $A[\![X]\!]$-regular sequence (t_1, f) is not a $B[\![X]\!]$-regular sequence, thus the extension $A[\![X]\!] \subseteq B[\![X]\!]$ is not R_2-Stable.*

Remark 6. *Since Dedekind domain is a regular domain, then Propositions* 1-2-5 *with Theorem* 2 *give another demonstration to (*[10], *Theorem* 2.5*).*

Now since a Krull domain is of finite t-character, it is sufficient to study the R_2-stableness in order to study the t-linkedness.

Theorem 5. *Let A be a domain, K the field of fraction of A, if A verifies for all nonzero $f, g \in A[\![X]\!]$, $(A_{fg})_v = (A_f A_g)_v$ then the following assertions are equivalent:*

(1)- $(f(x) :_{A[\![X]\!]} g(x)) = f(x)A[\![X]\!]$.

(2)- $\begin{cases} \text{(i)- } (f(x) :_{A[\![X]\!]_{A^*}} g(x)) = f(x)A[\![X]\!]_{A^*} \\ \text{(ii)- } (A :_K A_f + A_g) = A \end{cases}$

Proof. $(1) \Rightarrow (2)$ Since $A[\![X]\!] \subseteq A[\![X]\!]_{A^*}$ is LCM-stable then $(f(x) :_{A[\![X]\!]_{A^*}} g(x)) = f(x)A[\![X]\!]_{A^*}$, next, take $\alpha \in (A :_K A_f + A_g)$, $\alpha = \frac{a}{b} \in K$ then $\alpha f(x) = h(x) \in A[\![X]\!]$ and $\alpha g(x) = k(x) \in A[\![X]\!]$, this implies $af(x) = bh(x)$ and $ag(x) = bk(x)$. So $af(x)g(x) = bk(x)f(x) = bh(x)g(x)$ so $k(x)f(x) =$

$h(x)g(x)$, $h(x) \in (f(x) :_A [\![X]\!] g(x)) = f(x)A[\![X]\!]$, so there is a $h_1(x) \in A[\![X]\!]$ such that $h(x) = h_1(x)f(x)$, since $af(x)g(x) = bh(x)g(x) = bh_1(x)f(x)g(x)$ we have $a = bh_1(0)$ and $\alpha = h_1(0) \in A$.

(2) $\Rightarrow$ (1) Let $h(x) \in (f(x) :_{A[\![X]\!]} g(x))$. There exists $\Phi(x) \in A[\![X]\!]$ such that $h(x)g(x) = f(x)\Phi(x)$, since $(f(x) :_{A[\![X]\!]_{A^*}} g(x)) = f(x)A[\![X]\!]_{A^*}$ then there exist $a \in A \backslash \{0\}$ and $\Psi(x) \in A[\![X]\!]$ such that $ah(x) = f(x)\Psi(x)$ so $a\Phi(x) = \Psi(x)g(x)$. Let $F(X) = f(X^2) + Xg(X^2)$ then $A_F = A_f + A_g$ and so $(A_F)^{-1} = A$. According to the above $ah(X^2) = f(X^2)\Psi(X^2)$ and $a\Phi(X^2) = g(X^2)\Psi(X^2)$ and so $a(h(X^2) + X\Phi(X^2)) = F(X)\Psi(X^2)$. Then $(h(X^2) + X\Phi(X^2)) \in F(X)A[\![X]\!]_{A^*} \cap A[\![X]\!]$ but since A verifies that for all $f, g \in A[\![X]\!]$, $(A_{fg})_v = (A_f A_g)_v$ then by ([1], Theorem 2.3), $F(X)A[\![X]\!]_{A^*} \cap A[\![X]\!] = F(X)A_F^{-1}[\![X]\!] = F(X)A[\![X]\!]$. So there exists $l(X) \in A[\![X]\!]$ such that $(h(X^2) + X\Phi(X^2)) = F(X)l(X)$ and so $\Psi(X^2) = al(X)$. This implies that all the powers in $l(X)$ are even and the formal power series $l_1(X) = l(X^{1/2})$ is well defined. Since $ah(X) = f(X)\Psi(X)$ then $h(X^2) = f(X^2)l(X)$. And so $h(X) = f(X)l_1(X)$. $\square$

If we replace (A verifies for all nonzero $f, g \in A[\![X]\!]$, $(A_{fg})_v = (A_f A_g)_v$) by (A is a Noetherian domain), the theorem remains true. First, it is necessary to prove the following lemma:

Lemma 7. *Let A be a Noetherian domain, $K = qf(A)$ and $f \in A[\![X]\!]$, then the following assertions are equivalent:*
(1)- $(a :_{A[\![X]\!]} f) = aA[\![X]\!]$ *for all* $a \in A_f$.
(2)- $(a :_{A[\![X]\!]} f) = aA[\![X]\!]$ *for all* $a \in A$.
(3)- $(a :_A A_f) = aA$ *for all* $a \in A$.
(4)- $(A :_K A_f) = A$.

Proof. (1) $\Rightarrow$ (2) Let $b \in A_f$, then for all $a \in A$, $ab \in A_f$, $(ab :_{A[\![X]\!]} f) = abA[\![X]\!]$, but $(ab :_{A[\![X]\!]} f) = b(a :_{A[\![X]\!]} f)$ and so $(a :_{A[\![X]\!]} f) = aA[\![X]\!]$.

(2) $\Rightarrow$ (1) Trivial.

(2) $\Rightarrow$ (3) Let $b \in (a :_A A_f)$ then $b \in (a :_{A[\![X]\!]} f) = aA[\![X]\!]$ and so $b \in aA$.

(3) $\Rightarrow$ (2) Suppose that there is $g \in (a :_{A[\![X]\!]} f) \backslash aA[\![X]\!]$. Take $B = A/_{aA}$, then B is a Noetherian ring with identity. Since B is Noetherian, every ideal of B has a primary decomposition, thus every ideal of B has a minimal primary decomposition. So the zero ideal $(\overline{0})$ has a minimal primary decomposition (also called shortest primary representation). Now, set $\bar{f}$ the image of f in $B[\![X]\!]$. Since $gf \in aA[\![X]\!]$ and $g \notin aA[\![X]\!]$, then $\bar{g} \neq \overline{0}$ and $\bar{f}$ is a zero divisor in $B[\![X]\!]$. By ([14], Theorem 5, (c)) there exists a nonzero element $\bar{b} \in B$ such that $\bar{b}\bar{f} = \overline{0}$. So $b \in (a :_A A_f) = aA$. Thus $\bar{b} = \overline{0}$. Contradiction.

(3) $\Rightarrow$ (4) Let $u = \frac{x}{y} \in (A_f)^{-1}$. Then $xA_f \subseteq yA$. So $x \in (y :_A A_f) = yA$. Thus $x = ye$ for some $e \in A$. Then $u = e \in A$.

(4) $\Rightarrow$ (3) Let $b \in (a :_A A_f)$ for some $a \in A$, then $bA_f \subseteq aA$ and $\frac{b}{a}A_f \subseteq A$. So $\frac{b}{a} \in A$ and $b \in aA$. $\square$

Theorem 6. *Let A be a domain, K the field of fraction of A, if A is a Noetherian domain, then the following assertions are equivalent:*

(1)- $(f(x) :_{A[\![X]\!]} g(x)) = f(x)A[\![X]\!]$.

(2)- $\begin{cases} \text{(i)- } (f(x) :_{A[\![X]\!]_{A^*}} g(x)) = f(x)A[\![X]\!]_{A^*} \\ \text{(ii)- } (A :_K A_f + A_g) = A \end{cases}$

Proof. (1) $\Rightarrow$ (2) Exactly as in the proof of the previous theorem.

(2) $\Rightarrow$ (1) Let $h(x) \in (f(x) :_{A[\![X]\!]} g(x))$. There exists $\Phi(x) \in A[\![X]\!]$ such that $h(x)g(x) = f(x)\Phi(x)$, since $(f(x) :_{A[\![X]\!]_{A^*}} g(x)) = f(x)A[\![X]\!]_{A^*}$ then there exist $a \in A\backslash\{0\}$ and $\Psi(x) \in A[\![X]\!]$ such that $ah(x) = f(x)\Psi(x)$ so $a\Phi(x) = \Psi(x)g(x)$. Let $F(X) = f(X^2) + Xg(X^2)$ then $A_F = A_f + A_g$ and so $(A_F)^{-1} = A$. According to the above $ah(X^2) = f(X^2)\Psi(X^2)$ and $a\Phi(X^2) = g(X^2)\Psi(X^2)$ and so $a(h(X^2) + X\Phi(X^2)) = F(X)\Psi(X^2)$. By the previous lemma and since $\Psi(X^2) \in (a :_{A[\![X]\!]} F(X))$, then $\Psi(X^2) \in aA[\![X]\!]$. So there exists $l(X) \in A[\![X]\!]$ such that $\Psi(X^2) = al(X)$. This implies that all the powers in $l(X)$ are even and the formal power series $l_1(X) = l(X^{1/2})$ is well defined. Since $ah(X) = f(X)\Psi(X)$ then $h(X^2) = f(X^2)l(X)$. And so $h(X) = f(X)l_1(X)$. $\square$

Corollary 4. *Let $A \subseteq B$ be an extension of Krull domains, the following assertions are equivalent:*

(1)- $A[\![X]\!] \subseteq B[\![X]\!]$ *is t-linked.*

(2)- $\begin{cases} \text{(i)- } A[\![X]\!]_{A^*} \subseteq B[\![X]\!]_{B^*} \text{ is t-linked.} \\ \text{(ii)- } A \subseteq B \text{ is t-linked.} \end{cases}$

Proof. (2) $\Rightarrow$ (1) The Theorem 5.

(1) $\Rightarrow$ (2) Let I be a G.V. ideal of A, then $I[\![X]\!]$ is a G.V. ideal of $A[\![X]\!]$. Since $A[\![X]\!] \subseteq B[\![X]\!]$ is t-linked, then $I[\![X]\!]B[\![X]\!] = IB[\![X]\!]$ is a G.V. ideal of $B[\![X]\!]$ and so $(IB[\![X]\!])^{-1} = (IB)^{-1}[\![X]\!] = B[\![X]\!]$ so $(IB)^{-1} = B$. This proves that $A \subseteq B$ is t-linked.

Next we will use the result of ([2], proposition 2.1). Take P a t-prime ideal of the krull domain $B[\![X]\!]_{B^*}$ then there is $P_1 \in spec(B[\![X]\!])$ such that $P = P_1B[\![X]\!]_{B^*}$ and $P_1 \cap B = 0$ since $ht(P) = 1$ we have $ht(P_1) = 1$ and P_1 is a prime t-ideal. Indeed $(B[\![X]\!]_{B^*})_P = (B[\![X]\!]_{B^*})_{P_1B[\![X]\!]_{B^*}} = B[\![X]\!]_{P_1}$ and $(B[\![X]\!]_{B^*})_P$ is a DVR. Suppose that $Q = P \cap A[\![X]\!]_{A^*} \neq 0$, take $u \in P \cap A[\![X]\!]_{A^*} \neq 0$ then $u = \frac{f(x)}{a}$ and $f(x) \in P_1 \cap A[\![X]\!]$, set $Q_1 = P_1 \cap A[\![X]\!]$ then $Q \subseteq Q_1.A[\![X]\!]_{A^*}$, note that $Q_1 \cap A = 0$ since $A[\![X]\!] \subseteq B[\![X]\!]$ is t-linked then $ht(Q_1) = 1$, now $(A[\![X]\!]_{A^*})_{Q_1.A[\![X]\!]_{A^*}} = A[\![X]\!]_{Q_1}$ and $A[\![X]\!]_{Q_1}$ is a DVR, so $ht(Q_1.A[\![X]\!]_{A^*}) = 1$, and $Q = Q_1.A[\![X]\!]_{A^*}$ so Q is a t-prime ideal. We have shown that if P is a t-ideal of $B[\![X]\!]_{B^*}$ such that $Q = P \cap A[\![X]\!]_{A^*} \neq \{0\}$, then Q is a t-ideal of $A[\![X]\!]_{A^*}$. By ([2], Proposition 2.1) $A[\![X]\!]_{A^*} \subseteq B[\![X]\!]_{B^*}$ is t-linked. $\square$

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