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On a Riemannian submanifold whose slice representation has no nonzero fixed points

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Abstract. In this paper, we define a new class of Riemannian submanifolds which we call arid submanifolds. A Riemannian submanifold is called an arid submanifold if no nonzero normal vectors are invariant under the full slice representation. We see that arid submanifolds are a generalization of weakly reflective submanifolds, and arid submanifolds are minimal submanifolds. We also introduce an application of arid submanifolds to the study of left-invariant metrics on Lie groups. We give a sufficient condition for a left-invariant metric on an arbitrary Lie group to be a Ricci soliton.

1. Introduction

Let X be a Riemannian manifold, and Y be a Riemannian submanifold in X. Denote by $\mathrm{Isom}(X)$ the group of isometries of X, and let $N(Y) \subset \mathrm{Isom}(X)$ be the subgroup which normalizes the Riemannian submanifold Y. That is,

$$N(Y) := \{\varphi \in \mathrm{Isom}(X) \mid \varphi(Y) = Y\}.$$

The group $N(Y)$ is sometimes called the *extrinsic isometry group* of Y, and an element of $N(Y)$ is called an *extrinsic isometry* of Y.

Definition 1.1. Take any subgroup H of $N(Y)$, and fix a point $p \in Y$. Denote by $T_p^{\perp} Y$ the normal space of Y at p. The action of the stabilizer $H_p := \{\varphi \in H \mid \varphi(p) = p\}$ on $T_p^{\perp} Y$ by differential

$$g.\xi := (dg)_p \xi \qquad (g \in H_p, \xi \in T_p^{\perp} Y)$$

is called the *H-slice representation* of Y at $p \in Y$. We also call the $N(Y)$-slice representation the *full slice representation*.

Remark 1.2. The above definition of slice representations seems to be slightly different from the usual one; the notion of slice representations is

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usually defined for an isometric action on a Riemannian manifold. Recall that the slice representation of an isometric G-action at a point p is the action of the stabilizer G_p on the normal space of the orbit $G.p$ at p by differential. We remark that the notion of usual slice representations is contained in our H-slice representations. In fact, the slice representation of a G-action at a point p is nothing but the G-slice representation of the Riemannian submanifold $G.p$ at p.

The following Riemannian submanifold is the one which we consider in this paper.

Definition 1.3. Take any subgroup $H \subset N(Y)$. A Riemannian submanifold Y is called an *H-arid submanifold* if the H-slice representation of Y at any point has no nonzero fixed points. We call an $N(Y)$-arid submanifold just an *arid submanifold*.

Remark 1.4. A Riemannian submanifold Y is an H-arid submanifold for $H \subset N(Y)$ if and only if Y satisfies the following condition: for all $p \in Y$ and for all $0 \neq \xi \in T_p^{\perp} Y$, there exists $\varphi \in H_p$ such that $\varphi.\xi \neq \xi$.

An arid submanifold holds an interesting position in the theory of Riemannian submanifolds. One can see that the notion of arid submanifolds is a generalization of weakly reflective submanifolds. On the other hand, any arid submanifold is a minimal submanifold. For more details on the positioning of arid submanifolds in the submanifold theory, see Section 2. For simple examples of arid submanifolds, see Section 3.

In general, a homogeneous arid submanifold can be characterized as follows:

Theorem 1.5. *Let Y be a closed homogeneous submanifold in X. Then the followings are equivalent:*
(1) *Y is an arid submanifold.*
(2) *There exists some closed subgroup $G \subset \mathrm{Isom}(X)$ such that Y is an isolated orbit of the G-action.*

In Section 4, we prove this theorem. In particular, Theorem 1.5 says that any isolated orbit of any isometric proper action is an arid submanifold. Hence, isolated orbits provide many examples of arid submanifolds.

In Section 5, we introduce an application of the notion of arid submanifolds. Namely, we give a sufficient condition for a left-invariant metric on a Lie group to be a Ricci soliton. This sufficient condition comes from a framework to study left-invariant metrics via the action of the group of automorphisms and scalings on the set of all left-invariant metrics. Now we describe the framework. Let G be a simply connected Lie group with

Lie algebra $\mathfrak{g}$. Denote by $\mathfrak{M}(\mathfrak{g})$ the set of all positive definite inner products on $\mathfrak{g}$. Recall that a left-invariant metric on G is canonically identified with an inner product on $\mathfrak{g}$. Hence, we can regard $\mathfrak{M}(\mathfrak{g})$ as the set of all left-invariant metrics on G. Also, $\mathrm{GL}(\mathfrak{g})$ acts on $\mathfrak{M}(\mathfrak{g})$ by base changing:

$$g.\langle\,,\rangle := \langle g^{-1}, g^{-1}\rangle \qquad (g \in \mathrm{GL}(\mathfrak{g})). \tag{1.1}$$

We also note that the $\mathrm{GL}(\mathfrak{g})$-action on $\mathfrak{M}(\mathfrak{g})$ is transitive, and $\mathfrak{M}(\mathfrak{g})$ endows with a $\mathrm{GL}(\mathfrak{g})$-homogeneous Riemannian structure. In fact, by choosing a basis of $\mathfrak{g}$, one can identify $\mathfrak{M}(\mathfrak{g})$ with the Riemannian symmetric space $\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$. Denote by $\mathrm{Aut}(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$. Let $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ be the subgroup of $\mathrm{GL}(\mathfrak{g})$ given by

$$\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) := \{c\varphi \in \mathrm{GL}(\mathfrak{g}) \mid c \in \mathbb{R}\backslash\{0\}, \varphi \in \mathrm{Aut}(\mathfrak{g})\},$$

and consider the isometric action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ on $\mathfrak{M}(\mathfrak{g})$ given in (1.1). We note that, for any left-invariant metric $\langle\,,\rangle$, the orbit $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle\,,\rangle$ is a submanifold in $\mathfrak{M}(\mathfrak{g})$. Now we state a sufficient condition for obtaining left-invariant Ricci solitons as follows:

Theorem 1.6. *Let $\langle\,,\rangle \in \mathfrak{M}(\mathfrak{g})$ be a left-invariant metric on G. If the orbit $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle\,,\rangle$ is an* $\mathrm{Aut}(\mathfrak{g})$*-arid submanifold in $\mathfrak{M}(\mathfrak{g})$, then the left-invariant metric $\langle\,,\rangle$ is a Ricci soliton.*

Left-invariant Ricci solitons on Lie groups have been studied actively by many geometers (*e.g.* [14, 15, 24]). In particular, left-invariant Ricci solitons on solvable Lie groups have been deeply studied. On the other hand, it seems that little result is known for left-invariant Ricci solitons on general Lie groups. We note that one can apply Theorem 1.6 for any Lie group.

Theorem 1.6 gives a kind of extension of works by Hashinaga and Tamaru in [7]. They have been studying left-invariant Ricci solitons via studying the minimality of the orbits of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$-actions. They have proved that

Theorem 1.7 ([7]). *Let G be a three-dimensional simply connected solvable Lie group, and $\mathfrak{g}$ be the Lie algebra of G. Then for any left-invariant metric $\langle\,,\rangle \in \mathfrak{M}(\mathfrak{g})$, the followings are equivalent:*

(1) *the metric $\langle\,,\rangle$ is a solvsoliton. That is, there exists some $\lambda \in \mathbb{R}$ and some $D \in \mathrm{Der}(\mathfrak{g})$ such that*

$$\mathrm{Ric}_{\langle,\rangle}(\,,) = \lambda \cdot \langle\,,\rangle + \langle D,\rangle.$$

(2) *the orbit $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle\,,\rangle$ is a minimal submanifold in $\mathfrak{M}(\mathfrak{g})$.*

Note that a solvsoliton is in fact a left-invariant Ricci soliton ([15]). The above theorem makes us expected that left-invariant Ricci solitons can be characterized by the minimality of the $\mathbb{R}^{\times}\,\mathrm{Aut}(\mathfrak{g})$-orbits. Unfortunately, it has shown that the expectation is wrong: both "minimal $\Rightarrow$ Ricci soliton" and "minimal $\Leftarrow$ Ricci soliton" fail in general cases ([6]). Theorem 1.6 asserts that if one strengthen the assumption from "minimal" to "$\mathrm{Aut}(\mathfrak{g})$-arid", then the implication "$\Rightarrow$" holds for general Lie groups at least.

2. The positioning of arid submanifolds in the theory of submanifolds

The positioning of arid submanifolds in the theory of Riemannian submanifolds is organized as follows.

$$\begin{array}{ccc} \text{reflective submanifold} & \Rightarrow & \text{totally geodesic submanifold} \\ \Downarrow & & \Downarrow \\ \text{weakly reflective submanifold} & \Rightarrow & \text{austere submanifold} \\ \Downarrow & & \Downarrow \\ \text{arid submanifold} & \Rightarrow & \text{minimal submanifold} \end{array} \tag{2.1}$$

A Riemannian submanifold Y in X is called a *reflective submanifold* if there exists some $\sigma \in \mathrm{Isom}(X)$ with $\sigma \circ \sigma = \mathrm{id}$ such that Y is the connected component of the set of fixed points of σ. The isometry σ is called a reflection of Y. The notion of reflective submanifolds has been introduced in [16]. Note that a reflective submanifold is totally geodesic.

A Riemannian submanifold Y is called an *austere submanifold* if Y satisfies the following property; for all $p \in Y$ and for all $v \in T_p^{\perp} Y$, the set of eigenvalues with multiplicities of the shape operator A_v is invariant under the multiplication by -1. The notion of austere submanifolds is motivated by the study of special Lagrangian submanifolds ([5]). Clearly, a totally geodesic submanifold is an austere submanifold.

Next, we recall the notion of weakly reflective submanifolds.

Definition 2.1 ([9]). A Riemannian submanifold Y is called a *weakly reflective submanifold* if Y satisfies the following property; for all $p \in Y$ and $\xi \in T_p^{\perp} Y$, there exists some isometry $\varphi \in \mathrm{Isom}(X)$ such that

$$\varphi(p) = p, \qquad \varphi(Y) = Y, \qquad (d\varphi)_p(\xi) = -\xi.$$

In other words, a weakly reflective submanifold is a submanifold whose full slice representation can invert any normal vector. Also, a reflective submanifold with a reflection σ is a weakly reflective submanifold, since any normal vectors are inverted by σ at the same time. It has been shown in [9] that a weakly reflective submanifold is an austere submanifold.

Recall that a *minimal submanifold* is a Riemannian submanifold whose mean curvature vector vanishes identically. One can easily see that austere submanifolds are minimal submanifolds.

We now prove the parts in (2.1) relating to arid submanifolds:

Proposition 2.2. *One has*

(1) *A weakly reflective submanifold is an arid submanifold.*

(2) *An arid submanifold is a minimal submanifold.*

Proof. By the definition of weakly reflective submanifolds, the first assertion is obvious. The second assertion follows from the fact that the mean curvature vector is invariant under the full slice representation. □

Remark 2.3. It is easy to see that any codimension one arid submanifold is weakly reflective. However, there exist arid submanifolds which are not weak reflective. We see an example in Section 3. Also, there exist minimal submanifolds which are not arid. For examples, one can see that the catenoid surface in $\mathbb{R}^3$ is not arid.

The right three submanifolds appeared in (2.1) are defined by curvature properties. Reflective submanifolds are the special case of totally geodesic submanifolds, which are defined by extrinsic symmetry. Also, weakly reflective submanifolds are "extrinsic symmetry version" of austere submanifolds. In this paper, we defined a class of submanifold which is corresponding to minimal submanifolds.

3. Simple examples of arid submanifolds

In this section, we introduce simple examples of arid submanifolds which are not weakly reflective. Denote by $S^k(r) \subset \mathbb{R}^{k+1}$ the k-dimensional sphere with radius r. Fix two integers $m, n \geq 2$. Let us denote by $X := S^{mn-1}(\sqrt{m})$. Set $Y \subset X$ be the m-times direct product of $S^{n-1}(1)$. That is,

$$Y := \{(x_1, \dots, x_m) \in \mathbb{R}^{mn} \mid \forall i \in \{1, \dots, m\}, x_i \in S^{n-1}(1)\}.$$

Remark that, in this section, we always regard an element of $\mathbb{R}^{mn}$ as an m-tuple of elements of $\mathbb{R}^n$. Note that Y is a submanifold of X with codimension $(m-1)$. We claim that

Proposition 3.1. *One has*

(1) *Y is an arid submanifold in X,*

(2) *Y is not austere if $m \geq 3$. In particular, Y is not weakly reflective if $m \geq 3$.*

Now we introduce some extrinsic isometries of $Y \subset X$ which play key roles to prove Proposition 3.1. Firstly, denote by $H := O(n) \times \cdots \times O(n)$ the m-times direct product of $O(n)$. Then H acts on $\mathbb{R}^{mn}$ by

$$g.x := (g_1x_1, \dots, g_mx_m) \qquad (g := (g_1, \dots, g_m) \in H, x := (x_1, \dots, x_m) \in \mathbb{R}^{mn}).$$

Then one has $H \subset N(Y)$. Secondly, denote by $\mathfrak{S}_m$ the symmetric group on $\{1, \dots, m\}$. Then $\mathfrak{S}_m$ acts on $\mathbb{R}^{mn}$ by

$$\sigma.x := (x_{\sigma(1)}, \dots, x_{\sigma(m)}) \qquad (\sigma \in \mathfrak{S}_m, x := (x_1, \dots, x_m) \in \mathbb{R}^{mn}).$$

Hence, $\mathfrak{S}_m$ is also a subgroup of $N(Y)$. Among the elements of $\mathfrak{S}_m$, we especially use transpositions. For $i, j \in \{1, \dots, m\}$, let us denote by $\sigma_{ij} \in \mathfrak{S}_m$ the transposition with respect to i and j. That is,

$$\sigma_{ij}(i) = j, \qquad \sigma_{ij}(j) = i, \qquad \sigma_{ij}(k) = k \qquad (k \neq i, j).$$

Denote by $e_1 := {}^t(1, 0, \dots, 0) \in \mathbb{R}^n$, and put $p := (e_1, \dots, e_1) \in \mathbb{R}^{mn}$. Note that the stabilizers H_p and $(\mathfrak{S}_m)_p$ act on T_pY and $T_p^\perp Y$ by differential. Our strategy to prove Proposition 3.1 is to analyse these actions. One can see that

$$T_pX = \{v \in \mathbb{R}^{mn} \mid \langle v, p \rangle = 0\} = \left\{ (v_1, \dots, v_m) \in \mathbb{R}^{mn} \,\middle|\, \sum_i \langle v_i, e_1 \rangle = 0 \right\}.$$

Here, $\langle\, , \rangle$ is the canonical inner product on the Euclidean space. The tangent space T_pY is given by

$$T_pY = \{(w_1, \dots, w_m) \in \mathbb{R}^{mn} \mid \forall i \in \{1, \dots, m\}, \langle w_i, e_1 \rangle = 0\}.$$

Also, the normal space $T_p^\perp Y$ is obtained as follows:

$$T_p^\perp Y = \{(\xi_1 e_1, \dots, \xi_m e_1) \in \mathbb{R}^{mn} \mid \xi_1 + \cdots + \xi_m = 0\}.$$

Next, we determine the stabilizers of the actions of H and $\mathfrak{S}_m$, and describe the actions. One can see that

$$H_p = \{(g_1, \dots, g_m) \in H \mid g_i e_1 = e_1\} \cong O(n-1) \times \cdots \times O(n-1),$$

and the actions on T_pY and $T_p^\perp Y$ by differential are given by

$$g.v = (g_1v_1, \dots, g_mv_m) \qquad (v := (v_1, \dots, v_m) \in T_pX), \tag{3.1}$$

where $g := (g_1, \dots, g_m) \in H_p$. On the other hand, the actions of $(\mathfrak{S}_m)_p = \mathfrak{S}_m$ on T_pY and $T_p^\perp Y$ by differential are

$$\sigma.v = (v_{\sigma(1)}, \dots, v_{\sigma(m)}) \qquad (v := (v_1, \dots, v_m) \in T_pX), \tag{3.2}$$

where $\sigma \in \mathfrak{S}_m$. Now, we are in the position to prove the first assertion of Proposition 3.1.

PROOF (of (1) of Proposition 3.1). Since H acts on Y transitively, the full slice representation at each $x \in Y$ is H-equivalent to the full slice representation at $p = (e_1, \dots, e_1)$. Hence, one has only to prove that for all $\xi \in T_p^\perp Y \setminus \{0\}$ there exists $g \in N(Y)_p$ such that $g.\xi \neq \xi$.

Take any $\xi = (\xi_1 e_1, \dots, \xi_m e_1) \in T_p^\perp Y \setminus \{0\}$. Since $\xi \neq 0$ and $\xi_1 + \cdots + \xi_m = 0$, there exist $i, j \in \{1, \dots, m\}$ such that $\xi_i \neq \xi_j$. Let us put $g := \sigma_{ij} \in N(Y)_p$. By Equation (3.2), one has

$$g.\xi = (\xi_1 e_1, \dots, \xi_j e_1, \dots, \xi_i e_1, \dots, \xi_m e_1) \neq \xi,$$

which completes the proof. □

Next, we prove the second assertion of Proposition 3.1. For all $i \in \{1, \dots, m\}$, let us define

$$T_p^i := \{(x_1, \dots, x_m) \in T_p Y \mid \forall j \neq i, x_j = 0\} \subset T_p Y.$$

Then one has an orthogonal decomposition $T_p Y = T_p^1 \oplus \cdots \oplus T_p^m$. Denote by $\pi_i : T_p Y \to T_p^i$ the natural projection

$$\pi_i(x_1, \dots, x_m) = (0, \dots, 0, x_i, 0, \dots, 0) \qquad ((x_1, \dots, x_m) \in T_p Y).$$

Also, for all $i \in \{1, \dots, m\}$, we define $\eta_i \in T_p^\perp Y$ by

$$\eta_i := (e_1, \dots, e_1, -(m-1)e_1, e_1, \dots, e_1).$$

Here, the i-th component of η_i is $-(m-1)e_1$, and the other components are e_1. Firstly, we claim that

LEMMA 3.2. *Let $\alpha : T_p Y \times T_p Y \to T_p^\perp Y$ be a symmetric bilinear map. Assume that α is equivariant under the H_p-action and $\mathfrak{S}_m$-action. Then there exists some $\lambda \in \mathbb{R}$ such that*

$$\alpha(x, y) = \lambda \sum_{i=1}^{m} \langle \pi_i(x), \pi_i(y) \rangle \eta_i \qquad (x, y \in T_p Y).$$

PROOF. Take any symmetric bilinear map $\alpha : T_p Y \times T_p Y \to T_p^\perp Y$ which is equivariant under the actions of H_p and $\mathfrak{S}_m$. This proof consists of four steps. Firstly, we claim that

STEP 1. *For all $i, j \in \{1, \dots, m\}$ with $i \neq j$, $\alpha(T_p^i, T_p^j) = \{0\}$.*

Take any $i, j \in \{1, \dots, m\}$ with $i \neq j$. Also, take any $X_i \in T_p^i$ and $X_j \in T_p^j$. We show that $\alpha(X_i, X_j) = 0$. Denote by $g = \mathrm{diag}(1, -1, \dots, -1) \in O(n)$. Let us define $\hat{g} \in H$ by

$$\hat{g} := (\mathrm{id}, \dots, \mathrm{id}, g, \mathrm{id}, \dots, \mathrm{id}),$$

where the i-th component of $\hat{g}$ is g, and the other components are id. Since $ge_1 = e_1$, one has $\hat{g} \in H_p$. Then Equation (3.1) and H_p-equivariancy of α yield that

$$\alpha(X_i, X_j) = \hat{g}.\alpha(X_i, X_j) = \alpha(\hat{g}.X_i, \hat{g}.X_j) = \alpha(-X_i, X_j).$$

This concludes that $\alpha(X_i, X_j) = 0$.

By the assertion of Step 1, one has $\alpha(,) = \sum_i \alpha(\pi_i, \pi_i)$. Now we study each $\alpha_i := \alpha|_{T_p^i \times T_p^i}$. Next, we prove

Step 2. $\alpha_i(T_p^i, T_p^i) \subset \mathrm{span}_{\mathbb{R}}\{\eta_i\}$ *for all* $i \in \{1, \ldots, m\}$.

Take any $i \in \{1, \ldots, m\}$. By Equation (3.2), one can see that

$$\sigma_{jk}.v = v \qquad (v \in T_p^i, j, k \in \{1, \ldots, m\} \backslash \{i\}).$$

This and $\mathfrak{S}_m$-equivariancy of α yield that

$$\sigma_{jk}.\alpha_i(x, y) = \alpha(\sigma_{jk}.x, \sigma_{jk}.y) = \alpha_i(x, y)$$

for all $x, y \in T_p^i$ and $j, k \in \{1, \ldots, m\} \backslash \{i\}$. Hence one has

$$\alpha_i(T_p^i, T_p^i) \subset \bigcap_{j \neq i, k \neq i} \{\xi \in T_p^{\perp} Y \mid \sigma_{jk}.\xi = \xi\} = \mathrm{span}_{\mathbb{R}}\{\eta_i\}.$$

Then one can obtain a simple expression of α_i as follows:

Step 3. *For all* $i \in \{1, \ldots, m\}$, *there exists* $\lambda_i \in \mathbb{R}$ *such that* $\alpha_i(,) = \lambda_i \langle , \rangle \eta_i$.

Take any $i \in \{1, \ldots, m\}$. Let us put $\theta_i(,) := (1/\langle \eta_i, \eta_i \rangle)\langle \alpha_i(,), \eta_i \rangle$. By the assertion of Step 2, one can see that $\alpha_i(,) = \theta_i(,)\eta_i$. To prove Step 3, we have only to show that there exists some $\lambda_i \in \mathbb{R}$ such that $\theta_i(,) = \lambda_i \langle , \rangle$. Now let us put

$$H^i := \{(g_1, \ldots, g_m) \in H \mid g_i e_1 = e_1, \forall j \neq i, g_j = \mathrm{id}\} \cong O(n-1).$$

Then $H^i \subset H_p$ acts on $T_p^i \cong \mathbb{R}^{n-1}$ irreducibly. On the other hand, $H^i \subset H_p$ acts on $T_p^{\perp} Y$ trivially. Hence one has

$$\theta_i(g.x, g.y) = (1/\langle \eta_i, \eta_i \rangle)\langle \alpha_i(g.x, g.y), \eta_i \rangle = (1/\langle \eta_i, \eta_i \rangle)\langle g.\alpha_i(x, y), \eta_i \rangle = \theta_i(x, y)$$

for all $g \in H^i$ and $x, y \in T_p^i$. This concludes that θ_i is a symmetric bilinear form on T_p^i which is invariant under the irreducible representation of H^i. Hence, by the Schur's lemma, there exists some $\lambda_i \in \mathbb{R}$ such that $\theta_i(,) = \lambda_i \langle , \rangle$.

Finally we study each constant λ_i. We show that

Step 4. $\lambda_1 = \cdots = \lambda_m$.

Take any $i, j \in \{1, \dots, m\}$. We prove that $\lambda_i = \lambda_j$. Take any $x \in T_p^i$ with $\langle x, x \rangle = 1$. By the $\mathfrak{S}_m$-equivariancy of α, one has

$$\sigma_{ij}.\alpha(x, x) = \alpha(\sigma_{ij}.x, \sigma_{ij}.x). \tag{3.3}$$

We firstly study the right hand side of (3.3). From (3.2), one has $\sigma_{ij}.x \in T_p^j$ and $\langle \sigma_{ij}.x, \sigma_{ij}.x \rangle = 1$. This yields that

$$\alpha(\sigma_{ij}.x, \sigma_{ij}.x) = \alpha_j(\sigma_{ij}.x, \sigma_{ij}.x) = \lambda_j \langle \sigma_{ij}.x, \sigma_{ij}.x \rangle \eta_j = \lambda_j \eta_j.$$

Next we study the left hand side of (3.3). Equation (3.2) yields that $\sigma_{ij}.\eta_i = \eta_j$. Then one has

$$\sigma_{ij}.\alpha(x, x) = \sigma_{ij}.\alpha_i(x, x) = \lambda_i \langle x, x \rangle \sigma_{ij}.\eta_i = \lambda_i \langle x, x \rangle \eta_j = \lambda_i \eta_j.$$

Since $\eta_j \neq 0$, one has $\lambda_j = \lambda_i$.

By the assertions of Step 1 to Step 4, one has

$$\alpha(x, y) = \sum_{i=1}^{m} \alpha_i(\pi_i(x), \pi_i(y)) = \sum_{i=1}^{m} \lambda \langle \pi_i(x), \pi_i(y) \rangle \eta_i,$$

which completes the proof. □

Since H_p and $\mathfrak{S}_m$ are the subgroups of $N(Y)_p$, the second fundamental form $S_p : T_p Y \times T_p Y \to T_p^{\perp} Y$ is equivariant under the actions of H_p and $\mathfrak{S}_m$. Hence Lemma 3.2 determines the second fundamental form of Y up to scaling. In particular, we obtain an explicit representation of the shape operator of Y as follows:

PROPOSITION 3.3. *There exists some $\lambda \in \mathbb{R} \setminus \{0\}$ such that for all $\xi \in T_p^{\perp} Y$, the shape operator $A_\xi : T_p Y \to T_p Y$ is given by*

$$A_\xi x = \lambda \sum_{i=1}^{m} \langle \eta_i, \xi \rangle \pi_i(x) \qquad (x \in T_p Y).$$

In particular, the eigenvalues of A_ξ are given by

$$\underbrace{\lambda\langle \eta_1, \xi \rangle, \dots, \lambda\langle \eta_1, \xi \rangle}_{(n-1)\text{-times}}, \dots, \underbrace{\lambda\langle \eta_k, \xi \rangle, \dots, \lambda\langle \eta_k, \xi \rangle}_{(n-1)\text{-times}}, \dots, \underbrace{\lambda\langle \eta_m, \xi \rangle, \dots, \lambda\langle \eta_m, \xi \rangle}_{(n-1)\text{-times}}.$$

PROOF. By the assertion of Lemma 3.2, there exists some $\lambda \in \mathbb{R}$ such that the second fundamental form $S : T_p Y \times T_p Y \to T_p^{\perp} Y$ is given by $S(,) = \lambda \sum_i \langle \pi_i, \pi_i \rangle \eta_i$. Since Y is not totally geodesic, and is H-homogeneous, one has $\lambda \neq 0$. Then by the definition of the shape operator, one has

$$\langle A_\xi x, y\rangle = \left\langle \lambda \sum_i \langle \pi_i(x), \pi_i(y)\rangle \eta_i, \xi \right\rangle = \lambda \sum_i \langle \pi_i(x), y\rangle\langle \eta_i, \xi\rangle$$
$$= \left\langle \lambda \sum_i \langle \eta_i, \xi\rangle \pi_i(x), y \right\rangle$$

for all $x, y \in T_p Y$. Thus we obtain that $A_\xi = \lambda \sum_i \langle \eta_i, \xi\rangle \pi_i$. Our claim for the eigenvalues easily follows from $A_\xi|_{T_p^i} = \lambda\langle \eta_i, \xi\rangle \,\mathrm{id}_{T_p^i}$, and $\dim T_p^i = n-1$ for all $i \in \{1, \dots, m\}$. □

Now we are in the position to prove the remaining assertion of Proposition 3.1.

PROOF (of (2) of Proposition 3.1). Assume that $m \geq 3$. We have only to prove that there exists some $\xi \in T_p^\perp Y$ such that the set of eigenvalues of the shape operator A_ξ is not invariant under the multiplication by -1. Let us put $\xi := \eta_m \in T_p^\perp Y$. By the assertion of Proposition 3.3, there exists some $\lambda \in \mathbb{R}\setminus\{0\}$ such that the eigenvalues of A_ξ are given by

$$\underbrace{\lambda\langle \eta_1, \eta_m\rangle, \dots, \lambda\langle \eta_1, \eta_m\rangle}_{(n-1)\text{-times}}, \dots, \underbrace{\lambda\langle \eta_m, \eta_m\rangle, \dots, \lambda\langle \eta_m, \eta_m\rangle}_{(n-1)\text{-times}}.$$

On the other hand, one has

$$\langle \eta_1, \eta_m\rangle = \cdots = \langle \eta_{m-1}, \eta_m\rangle = -m, \qquad \langle \eta_m, \eta_m\rangle = m(m-1).$$

Thus we obtain that the eigenvalues of A_ξ are

$$\underbrace{-\lambda m, \dots, -\lambda m}_{(n-1)(m-1)\text{-times}}, \underbrace{\lambda m(m-1), \dots, \lambda m(m-1)}_{(n-1)\text{-times}}.$$

Since $\lambda \neq 0$, and $m \geq 3$, one can see that the set of eigenvalues is not invariant under the multiplication by -1. □

REMARK 3.4. If $m = 2$, then it has been shown that $Y := S^{n-1}(1) \times S^{n-1}(1)$ is weakly reflective ([9]). In fact, the first assertion of Proposition 3.1 claims that Y is an arid submanifold of codimension $m-1$, and as mentioned in Remark 2.3, a codimension one arid submanifold is weakly reflective.

4. A characterization of homogeneous arid submanifolds

In this section, we prove Theorem 1.5. We here recall the notion of isolated orbits. Let G be a Lie group, acting on a manifold X. Denote by $G\backslash X$ the orbit space of the G-action. For two G-orbits $G.p, G.q \in G\backslash X$, we denote by $G.p \sim G.q$ if G_p and G_q are G-conjugate. Then "$\sim$" is an equivalence relation on $G \backslash M$. The equivalence class $[G.p]$ is called the *orbit type* of $G.p$. Here, we make the orbit space $G \backslash X$ a topological space by endowing $G \backslash X$ with the natural quotient topology. An orbit $G.p$ is called an *isolated orbit* if there exists some open subset $U \subset G \backslash X$ such that $U \cap [G.p] = \{G.p\}$ (*i.e.* $G.p$ is an isolated point of $[G.p] \subset G \backslash X$).

4.1. Preliminary on proper actions. In order to prove Theorem 1.5, we use some general theory of proper actions. We here give a review of them. Recall that a G-action on M is called a *proper action* if the map

$$G \times M \to M \times M, \qquad (g, p) \mapsto (g.p, p)$$

is proper. That is, the inverse image of any compact subset in $M \times M$ is also compact. It has been proved that an isometric G-action on a connected complete Riemannian manifold X is proper if and only if G is a closed subgroup of $\mathrm{Isom}(X)$ ([3, 19]). In the following arguments, we fix a closed subgroup G of $\mathrm{Isom}(X)$, and consider isometric proper G-action on a connected complete Riemannian manifold X.

An important consequence of a G-action being proper is the "G-equivariant tubular neighborhood theorem", which we described below. For each $p \in X$, let $\mathfrak{N}(G.p)$ be the total space of the normal bundle of $G.p$. That is,

$$\mathfrak{N}(G.p) := \{(q, \xi) \mid q \in G.p, \xi \in T_q^{\perp} G.p\}.$$

Also, for $\lambda > 0$ and $p \in X$, let $\mathfrak{N}^{\lambda}(G.p)$ be the total space of the normal disk bundle of $G.p$ with radius λ, and denote by $\mathfrak{N}_p^{\lambda}(G.p)$ the fiber at p:

$$\mathfrak{N}^{\lambda}(G.p) := \{(q, \eta) \in \mathfrak{N}(G.p) \mid \langle \eta, \eta \rangle_q < \lambda\},$$

$$\mathfrak{N}_p^{\lambda}(G.p) := \{(p, \eta) \mid \eta \in T_p^{\perp} G.p, \langle \eta, \eta \rangle_p < \lambda\}.$$

Note that G acts on $\mathfrak{N}(G.p)$ by $g.(q, \eta) := (g.q, (dg)_q \eta)$, and $\mathfrak{N}^{\lambda}(G.p) = G.\mathfrak{N}_p^{\lambda}(G.p)$. Let us define a map

$$\mathrm{Exp} : \mathfrak{N}(G.p) \to X, (p, \xi) \mapsto \exp_p \xi.$$

One can see that this map is G-equivariant. The assertion of the equivariant tubular neighborhood theorem ([4, Theorem B.24, and Remark B.27]) is given as follows:

PROPOSITION 4.1. *For all $p \in X$, there exists some $\lambda > 0$ such that the map $\mathrm{Exp} : \mathfrak{N}^{\lambda}(G.p) \to X$ is a G-equivariant embedding, and the image $\mathrm{Exp}(\mathfrak{N}^{\lambda}(G.p))$ is an open neighborhood of $G.p$.*

Proposition 4.1 provides nice tools to study the geometry of orbits. For examples, the following lemma implies that a G-orbit passing through $p \in X$ cannot "come back" to near p, unlike the irrational winding of a torus. Namely,

LEMMA 4.2. *Fix $p \in X$. Let us take $\lambda > 0$ as in Proposition 4.1, and $(p, \xi) \in \mathfrak{N}_p^\lambda(G.p)$. Then the orbit $G.p$ coincides with $G.\mathrm{Exp}(p, \xi)$ if and only if $\xi = 0$.*

PROOF. It is obvious that $\xi = 0$ implies $G.p = G.\mathrm{Exp}(p, \xi)$. We prove the "only if" part. Take any $(p, \xi) \in \mathfrak{N}_p^\lambda(G.p)$. Assume that $G.\mathrm{Exp}(p, \xi) = G.p$. We prove that $\xi = 0$. Let us denote by $r := \mathrm{Exp}(p, \xi)$. Then one has

$$\mathrm{Exp}(r, 0) = r = \mathrm{Exp}(p, \xi).$$

Also, one knows that $(p, \xi) \in \mathfrak{N}_p^\lambda(G.p) \subset \mathfrak{N}^\lambda(G.p)$, and $(r, 0) \in \mathfrak{N}^\lambda(G.p)$. By Proposition 4.1, the map $\mathrm{Exp} : \mathfrak{N}^\lambda(G.p) \to X$ is injective. This concludes that $(p, \xi) = (r, 0)$, and hence $\xi = 0$. $\square$

Also, Proposition 4.1 provides the tools to study orbit types of G-actions via the slice representations of G-actions (see Remark 1.2) as follows:

LEMMA 4.3. *Fix $p \in X$, and let us take $\lambda > 0$ as in Proposition 4.1. Take any $(p, \xi) \in \mathfrak{N}_p^\lambda(G.p)$. Then one has $G_{\mathrm{Exp}(p,\xi)} \subset G_p$. Moreover, the followings are equivalent:*

(1) *G_p and $G_{\mathrm{Exp}(p,\xi)}$ are G-conjugate. In other words, $G.\mathrm{Exp}(p, \xi) \in [G.p]$.*
(2) *$G_{\mathrm{Exp}(p,\xi)} = G_p$,*
(3) *ξ is invariant under the slice representation of the G-action at p.*

PROOF. Firstly we prove that $G_{\mathrm{Exp}(p,\xi)} \subset G_p$. Take any $g \in G_{\mathrm{Exp}(p,\xi)}$. Then one has

$$\mathrm{Exp}(p, \xi) = g.\mathrm{Exp}(p, \xi) = \mathrm{Exp}(g.p, g.\xi).$$

Since the map $\mathrm{Exp} : \mathfrak{N}^\lambda(G.p) \to X$ is injective by Proposition 4.1, we have $(p, \xi) = (g.p, g.\xi)$. This concludes that $g \in G_p$.

Next, we prove the equivalence of (1) and (2). The assertion (2) $\Rightarrow$ (1) is obvious. We prove (1) $\Rightarrow$ (2). Let us put $r := \mathrm{Exp}(p, \xi)$. Assume that $G_r \cong G_p$. Let us denote by $\mathfrak{g}_r$ and $\mathfrak{g}_p$ the Lie algebras of G_r and G_p, respectively. Since $G_r \subset G_p$, one has $\mathfrak{g}_r \subset \mathfrak{g}_p$. On the other hand, the assumption $G_r \cong G_p$ yields that $\dim(\mathfrak{g}_r) = \dim(\mathfrak{g}_r)$. Thus we obtain that $\mathfrak{g}_r = \mathfrak{g}_p$, and hence one has

$$(G_r)_0 = (G_p)_0. \tag{4.1}$$

Here, $(G_r)_0$ and $(G_p)_0$ are the connected components of G_r and G_p containing the unit element e, respectively. Let $\mathscr{C}_p$ and $\mathscr{C}_r$ be the set of connected components of G_p and G_r, respectively. We have shown that $G_r = G_{\mathrm{Exp}(p,\xi)} \subset G_p$. This and (4.1) yield that

$$\mathscr{C}_r \subset \mathscr{C}_p. \tag{4.2}$$

On the other hand, it is well known that stabilizers of proper actions are always compact. This implies that both G_r and G_p are compact, and hence $\#\mathscr{C}_p$ and $\#\mathscr{C}_r$ are finite. This and the assumption $G_r \cong G_p$ yield that

$$\#\mathscr{C}_r = \#\mathscr{C}_p < \infty. \tag{4.3}$$

By (4.2) and (4.3), one has $\mathscr{C}_r = \mathscr{C}_p$. This concludes that $G_r = G_p$.

We now prove the equivalence of (2) and (3). We show that (2) implies (3). Assume that $G_p = G_{\mathrm{Exp}(p,\xi)}$. Take any $g \in G_p$. We prove that $g.\xi = \xi$. Since $g \in G_{\mathrm{Exp}(p,\xi)} = G_p$, one has

$$\mathrm{Exp}(p,\xi) = g.\mathrm{Exp}(p,\xi) = \mathrm{Exp}(g.p, g.\xi) = \mathrm{Exp}(p, g.\xi).$$

The map $\mathrm{Exp} : \mathfrak{N}^{\lambda}(G.p) \to X$ is injective by Proposition 4.1. These conclude that $g.\xi = \xi$.

Lastly, we show the assertion (3) $\Rightarrow$ (2). Assume that ξ is a fixed normal vector. We prove that $G_{\mathrm{Exp}(p,\xi)} = G_p$. Recall that $G_{\mathrm{Exp}(p,\xi)} \subset G_p$ always holds, and hence we have only to show that $G_p \subset G_{\mathrm{Exp}(p,\xi)}$. Take any $g \in G_p$. Since g fixes p and ξ, it also fixes $\mathrm{Exp}(p,\xi)$. This completes the proof. $\square$

4.2. Isolated orbits and slice representations. In this subsection, we study isolated orbits of proper isometric actions via the arguments in the previous subsection, and prove Theorem 1.5. Continuing from the previous subsection, we fix a closed subgroup G of $\mathrm{Isom}(X)$, and consider isometric proper G-action on a Riemannian manifold X.

Firstly, we give a simple characterization of isolated orbits by the notion of slice representations.

PROPOSITION 4.4. *For all $p \in X$, the followings are equivalent:*
(1) *the orbit $G.p$ is an isolated orbit of the G-action.*
(2) *the slice representation of the G-action at p has no nonzero fixed normal vector.*
(3) *the orbit $G.p$ is a G-arid submanifold.*

PROOF. As seen in Remark 1.2, the slice representation of the G-action at p coincides with the G-slice representation of $G.p$ at p. Hence, the equivalence of (2) and (3) easily follows from the definition of G-arid submanifolds. Therefore, we prove the equivalence of (1) and (2) only.

We prove (1) $\Rightarrow$ (2). Assume that $G.p$ is isolated. Take any $\xi \in T_p^{\perp} G.p \setminus \{0\}$. We prove that there exists some $g \in G_p$ such that $g.\xi \neq \xi$.

We firstly construct a proper neighborhood V of $G.p$. Take $\lambda > 0$ as in Proposition 4.1. Since $G.p$ is an isolated orbit, there exists some open subset $U \subset G \setminus X$ such that $U \cap [G.p] = \{G.p\}$. Let us denote by $\pi : X \to G \setminus X$ the natural projection. Then $\pi^{-1}(U)$ is an open subset of X. Let us define an open subset $V \subset X$ by

$$V := \pi^{-1}(U) \cap \mathrm{Exp}(\mathfrak{N}^{\lambda}(G.p)).$$

By choosing $t > 0$ small enough, we may assume that $\mathrm{Exp}(p, t\xi) \in V$. Next, we claim that $G.\mathrm{Exp}(p, t\xi) \notin [G.p]$. Assume that $G.\mathrm{Exp}(p, t\xi) \in [G.p]$. One knows that $\mathrm{Exp}(p, t\xi) \in V \subset \pi^{-1}(U)$. Hence one has $G.\mathrm{Exp}(p, t\xi) \in U$. On the other hand, one knows that $U \cap [G.p] = \{G.p\}$. Since $G.\mathrm{Exp}(p, t\xi) \in [G.p]$, one has

$$G.\mathrm{Exp}(p, t\xi) \in U \cap [G.p] = \{G.p\}.$$

This yields that $G.\mathrm{Exp}(p, t\xi) = G.p$. Hence, by Lemma 4.2, one has $t\xi = 0$. One knows that $\xi \neq 0$, and hence we have $t = 0$. This contradicts that $t > 0$. This concludes that $G.\mathrm{Exp}(p, t\xi) \notin [G.p]$.

Now we are in the position to give $g \in G_p$, and show that $g.\xi \neq \xi$. By Lemma 4.3 and the previous claim $G.\mathrm{Exp}(p, t\xi) \notin [G.p]$, one has $G_{\mathrm{Exp}(p, t\xi)} \subsetneq G_p$. Hence there exists some $g \in G_p$ such that $g \notin G_{\mathrm{Exp}(p, t\xi)}$. Then one has

$$\mathrm{Exp}(p, g.t\xi) = \mathrm{Exp}(g.p, g.t\xi) = g.\mathrm{Exp}(p, t\xi) \neq \mathrm{Exp}(p, t\xi).$$

This concludes that $g.t\xi \neq t\xi$, and hence $g.\xi \neq \xi$.

It remains to prove (2) $\Rightarrow$ (1). Assume that the slice representation at p has no nonzero fixed normal vector. We prove that there exists some open subset U of $G \setminus X$ such that $U \cap [G.p] = \{G.p\}$. Let us put

$$U := \pi(\mathrm{Exp}(\mathfrak{N}^{\lambda}(G.p))).$$

Note that U is an open subset of $G \setminus X$ since π is an open map. We show that $U \cap [G.p] = \{G.p\}$. By the definition of U, it is obvious that $\{G.p\} \subset U \cap [G.p]$. We prove $U \cap [G.p] \subset \{G.p\}$. Take any $G.q \in U \cap [G.p]$. By the definition of U, there exists some $(p, \xi) \in \mathfrak{N}_p^{\lambda}(G.p)$ such that $G.q = G.\mathrm{Exp}(p, \xi)$. Since $G.\mathrm{Exp}(p, \xi) \in [G.p]$, Lemma 4.3 yields that ξ is a fixed point of the slice representation. On the other hand, by the assumption, there are no nonzero fixed normal vector under the slice representation. This yields that $\xi = 0$.

Hence one has

$$G.q = G.\mathrm{Exp}(p, 0) = G.p \in \{G.p\}.$$

This completes the proof. $\square$

Now we are in the position to prove Theorem 1.5.

PROOF (of Theorem 1.5). We prove (2) $\Rightarrow$ (1). Assume that Y is an isolated orbit of the action of a closed subgroup $G \subset \mathrm{Isom}(X)$. By Proposition 4.4, one has that Y is a G-arid submanifold, and hence is an arid submanifold.

We prove (1) $\Rightarrow$ (2). Assume that Y is a closed homogeneous arid submanifold. Let us put $G = N(Y)$. Since Y is homogeneous, Y is precisely a G-orbit. Also, one can see that G is a closed subgroup of $\mathrm{Isom}(X)$ since Y is closed. On the other hand, since Y is an arid submanifold, Y is an $N(Y)$-arid submanifold. Hence Proposition 4.4 yields that Y is an isolated orbit of the action of G. This completes the proof. $\square$

5. An application to the study of left-invariant Ricci solitons

Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. In this section, we prove Theorem 1.6, and show an example of a Lie algebra to which one can apply Theorem 1.6.

Firstly, we give some review for Ricci solitons. Let $\langle , \rangle$ be a Riemannian metric on a manifold M. Then $\langle , \rangle$ is called a *Ricci soliton* if there exist some $\lambda \in \mathbb{R}$ and some vector field X such that the Ricci tensor $\mathrm{Ric}_{\langle , \rangle}$ is given by

$$\mathrm{Ric}_{\langle , \rangle} = \lambda \cdot \langle , \rangle + \mathscr{L}_X \langle , \rangle.$$

This condition is equivalent to the condition that the metric evolves along scalings and diffeomorphisms under the Ricci flow. Namely, there exist some one parameter families $c_t \in \mathbb{R}$ and $\Phi_t \in \mathrm{Diff}(M)$ such that the solution $\langle , \rangle_t$ of the Ricci flow

$$\frac{\partial}{\partial t} \langle , \rangle_t = -2\, \mathrm{Ric}_{\langle , \rangle_t}$$

starting at $\langle , \rangle$ is given by

$$\langle , \rangle_t = (1/c_t)^2 \cdot \langle (d\Phi_t)^{-1}, (d\Phi_t)^{-1} \rangle, \qquad \langle , \rangle = \langle , \rangle_0.$$

Hence, a Ricci soliton is a fixed point of the Ricci flow (up to isometry and scaling), and have been considered as a distinguished metric from the view point of the theory of Ricci flow.

Our strategy to prove Theorem 1.6 is to observe the relationship between the $\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g})$-action and the Ricci flow for left-invariant metrics. Recall that the Ricci tensor $\mathrm{Ric}_{\langle,\rangle}$ for a left-invariant metric $\langle\,,\rangle$ is naturally identified with the tangent vector of $\mathfrak{M}(\mathfrak{g})$ at $\langle\,,\rangle$. Hence, the Ricci flow for left-invariant metrics on G is just an ODE on $\mathfrak{M}(\mathfrak{g})$ given by the vector field $\langle\,,\rangle \mapsto \mathrm{Ric}_{\langle,\rangle} \in T_{\langle,\rangle}\mathfrak{M}(\mathfrak{g})$. We note that the vector field Ric is invariant under the action of $\operatorname{Aut}(\mathfrak{g})$ on $\mathfrak{M}(\mathfrak{g})$ by (1.1). We are in the position to prove Theorem 1.6.

PROOF (of Theorem 1.6). Take any left-invariant metric $\langle\,,\rangle \in \mathfrak{M}(\mathfrak{g})$. Assume that the orbit $\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).\langle\,,\rangle$ is an $\operatorname{Aut}(\mathfrak{g})$-arid submanifold in $\mathfrak{M}(\mathfrak{g})$.

We firstly claim that, for all $p \in \mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).\langle\,,\rangle$, the tangent vector $\mathrm{Ric}_p \in T_p\mathfrak{M}(\mathfrak{g})$ is tangent to the orbit $\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).\langle\,,\rangle$, that is, $\mathrm{Ric}_{\langle,\rangle} \in T_{\langle,\rangle}\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).\langle\,,\rangle$. Take any $p \in \mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).\langle\,,\rangle$. Let us denote by $\mathrm{Ric}_p^\perp \in T_p^\perp\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).\langle\,,\rangle$ the normal component of Ric_p. Since the vector field Ric is invariant under the $\operatorname{Aut}(\mathfrak{g})$-action on $\mathfrak{M}(\mathfrak{g})$, so is the normal vector field $\mathrm{Ric}^\perp$. This yields that the normal vector $\mathrm{Ric}_p^\perp$ is invariant under the $\operatorname{Aut}(\mathfrak{g})$-slice representation of the orbit $\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).\langle\,,\rangle$ at p. Since the orbit $\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).\langle\,,\rangle$ is an $\operatorname{Aut}(\mathfrak{g})$-arid submanifold, one has $\mathrm{Ric}_p^\perp = 0$, and hence $\mathrm{Ric}_p \in T_p\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).\langle\,,\rangle$.

Since the vector field Ric is tangent to the orbit $\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).\langle\,,\rangle$, there exists some $c_t\varphi_t \in \mathbb{R}^\times \operatorname{Aut}(\mathfrak{g})$ such that the solution $\langle\,,\rangle_t$ of the Ricci flow starting at $\langle\,,\rangle$ is given by

$$\langle\,,\rangle_t = (c_t\varphi_t).\langle\,,\rangle = (1/c_t)^2 \cdot \langle \varphi_t^{-1}, \varphi_t^{-1}\rangle.$$

On the other hand, since G is simply connected, there exists some $\Phi_t \in \operatorname{Aut}(G)$ such that $(d\Phi_t)_e = \varphi_t$. These imply that the initial metric $\langle\,,\rangle$ evolves along scalings and automorphisms of G under the Ricci flow. This completes the proof. □

REMARK 5.1. A G-invariant metric on a homogeneous manifold G/K that evolves along scalings and (K-normalizing) automorphisms of G under the Ricci flow is called a *G-semi-algebraic Ricci soliton*. Theorem 1.6 asserts that if the orbit $\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}).\langle\,,\rangle$ is an $\operatorname{Aut}(\mathfrak{g})$-arid submanifold then the left-invariant metric $\langle\,,\rangle$ on G is a G-semi-algebraic Ricci soliton. It has been shown that any homogeneous Ricci soliton on X is G-semi-algebraic for some $G \subset \operatorname{Isom}(X)$, and any G-semi-algebraic Ricci soliton is a G-algebraic Ricci soliton. For more details on (semi-)algebraic Ricci solitons, we refer to [10, 11].

We now show an example of Lie group that one can apply Theorem 1.6. Let us denote by $\mathfrak{h}_{2n+1} := \operatorname{span}\{x_1, \dots, x_n, y_1, \dots, y_n, z\}$ the $(2n+1)$-

dimensional Heisenberg Lie algebra. Here, the nonzero bracket relations of $\mathfrak{h}_{2n+1}$ are given as follows:

$$[x_i, y_i] = z \qquad (i \in \{1, \dots, n\}).$$

Then one has

Proposition 5.2. *Let p be an inner product of $\mathfrak{h}_{2n+1}$ such that the basis $\{x_1, \dots, x_n, y_1, \dots, y_n, z\}$ is an orthonormal basis with respect to p. Then the orbit $\mathbb{R}^\times \operatorname{Aut}(\mathfrak{h}_{2n+1}).p$ is an $\operatorname{Aut}(\mathfrak{h}_{2n+1})$-arid submanifold.*

Proof. It has been known that the $\mathbb{R}^\times \operatorname{Aut}(\mathfrak{h}_{2n+1})$-action is transitive for the case $n = 1$ ([12], [13]), and hence Proposition 5.2 trivially follows for this case.

Now we assume that $n \geq 2$. We prove that the $\operatorname{Aut}(\mathfrak{h}_{2n+1})$-slice representation at $p \in \mathfrak{M}(\mathfrak{h}_{2n+1})$ has no nonzero fixed points. Recall that the $\operatorname{Aut}(\mathfrak{h}_{2n+1})$-slice representation is the action of $\operatorname{Aut}(\mathfrak{h}_{2n+1})_p := \{\varphi \in \operatorname{Aut}(\mathfrak{h}_{2n+1}) \mid \varphi.p = p\}$ on the normal space $T_p^\perp := T_p^\perp \mathbb{R}^\times \operatorname{Aut}(\mathfrak{h}_{2n+1}).p$. Let K be the connected component of $\operatorname{Aut}(\mathfrak{h}_{2n+1})_p \cong \operatorname{Aut}(\mathfrak{h}_{2n+1}) \cap O(2n+1)$ with $e \in K$. To prove that the action has no nonzero fixed points, it suffices to show that K acts on $T_p^\perp$ irreducibly.

To study the K-action, we determine the normal space $T_p^\perp$. By a direct calculation, the matrix representation of $\operatorname{Der}(\mathfrak{h}_{2n+1})$ with respect to the basis $\{x_i, y_i, z\}$ is given by

$$\operatorname{Der}(\mathfrak{h}_{2n+1}) = \left\{ \begin{pmatrix} c \cdot I_{2n} + A & 0 \\ * & 2c \end{pmatrix} \in \mathfrak{gl}(2n+1, \mathbb{R}) \mid c \in \mathbb{R}, A \in \mathfrak{sp}(2n, \mathbb{R}) \right\}.$$

Here $\mathfrak{sp}(2n, \mathbb{R}) \subset \mathfrak{gl}(2n, \mathbb{R})$ is given as follows:

$$\mathfrak{sp}(2n, \mathbb{R}) := \left\{ \left(\begin{array}{c|c} X & P \\ \hline Q & -{}^t X \end{array} \right) \in \mathfrak{gl}(2n, \mathbb{R}) \mid X \in \mathfrak{gl}(n, \mathbb{R}), P, Q \in \operatorname{sym}(n, \mathbb{R}) \right\}.$$

Also, let us denote by $\mathbb{R} \oplus \operatorname{Der}(\mathfrak{h}_{2n+1})$ the Lie algebra of $\mathbb{R}^\times \operatorname{Aut}(\mathfrak{h}_{2n+1})$. Then the matrix representation of $\mathbb{R} \oplus \operatorname{Der}(\mathfrak{h}_{2n+1})$ is given by

$$\mathbb{R} \oplus \operatorname{Der}(\mathfrak{h}_{2n+1}) = \left\{ \begin{pmatrix} c \cdot I_{2n} + R & 0 \\ * & * \end{pmatrix} \in \mathfrak{gl}(2n+1, \mathbb{R}) \mid c \in \mathbb{R}, R \in \mathfrak{sp}(2n, \mathbb{R}) \right\}.$$

One can see that the tangent space $T_p := T_p \mathbb{R}^\times \operatorname{Aut}(\mathfrak{h}_{2n+1}).p$ is given by

$$\begin{aligned} T_p &= \{X + {}^t X \in \operatorname{sym}(2n+1, \mathbb{R}) \mid X \in \mathbb{R} \oplus \operatorname{Der}(\mathfrak{h}_{2n+1})\} \\ &= \left\{ \left(\begin{array}{c|c|c} A + cI_n & B & * \\ \hline B & -A + cI_n & * \\ \hline * & * & * \end{array} \right) \in \operatorname{sym}(2n+1, \mathbb{R}) \mid A \in \mathfrak{gl}(n, \mathbb{R}), c \in \mathbb{R} \right\}. \end{aligned}$$

Hence, the normal space $T_p^{\perp}$ is obtained by

$$
\begin{aligned}
T_p^{\perp} &= \{A \in \mathrm{sym}(2n+1, \mathbb{R}) \mid \forall X \in T_p, \mathrm{tr}(AX) = 0\} \\
&= \left\{ \left(\begin{array}{c|c|c} A & -B & 0 \\ \hline B & A & 0 \\ \hline 0 & 0 & 0 \end{array} \right) \in \mathrm{sym}(2n+1, \mathbb{R}) \mid A \in \mathfrak{sl}(n, \mathbb{R}) \right\}.
\end{aligned}
$$

Note that the K-action on $T_p^{\perp}$ is given by the conjugate action of $K \subset \mathrm{Aut}(\mathfrak{h}_{2n+1}) \cap \mathrm{O}(2n+1)$ on $T_p^{\perp} \subset \mathrm{sym}(2n+1, \mathbb{R})$.

Denote by $\mathrm{herm}_0(n) \subset \mathfrak{gl}(n, \mathbb{C})$ the set of all trace free hermitian symmetric matrices of degree n. We claim that our K-action on $T_p^{\perp}$ is equivariant to the conjugacy action of $\mathrm{SU}(n)$ on $\mathrm{herm}_0(n)$, and hence irreducible. The identification between the K-action and the $\mathrm{SU}(n)$-action is given as follows. Let us define ρ the natural embedding of $\mathfrak{gl}(n, \mathbb{C})$ to $\mathfrak{gl}(2n+1, \mathbb{R})$ by

$$
\mathfrak{gl}(n, \mathbb{C}) \ni A + iB \mapsto \left(\begin{array}{c|c|c} A & -B & 0 \\ \hline B & A & 0 \\ \hline 0 & 0 & 0 \end{array} \right) \in \mathfrak{gl}(2n+1, \mathbb{R}).
$$

We note that the Lie algebra $\mathfrak{k}$ of K is given by

$$
\mathfrak{k} = \mathrm{Der}(\mathfrak{h}_{2n+1}) \cap (\mathfrak{o}(2n+1)) = \left\{ \left(\begin{array}{c|c|c} A & -B & 0 \\ \hline B & A & 0 \\ \hline 0 & 0 & 0 \end{array} \right) \in \mathfrak{o}(2n+1) \right\},
$$

and $\mathfrak{k} \subset \mathfrak{gl}(2n+1, \mathbb{R})$ is identified with $\mathfrak{su}(n) \subset \mathfrak{gl}(n, \mathbb{C})$ by ρ. This implies that $K \cong \mathrm{SU}(n)$. On the other hand, $T_p^{\perp} \subset \mathfrak{gl}(2n+1, \mathbb{R})$ is identified with $\mathrm{herm}_0(n) \subset \mathfrak{gl}(n, \mathbb{C})$ by ρ. One can see that $\rho : \mathrm{herm}_0(n) \to T_p^{\perp}$ is an $\mathrm{SU}(n)$-equivariant isomorphism, and hence the K-action is equivariant to the $\mathrm{SU}(n)$-action. $\square$

Remark 5.3. By Theorem 1.6, the left-invariant metric p on the $(2n+1)$-dimensional Heisenberg Lie group H_{2n+1} is a Ricci soliton. We note that it is well known that (H_{2n+1}, p) is a Ricci soliton nilmanifold. For examples, we refer to [14].

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