

A fixed contact angle condition for varifolds

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ABSTRACT. We define a generalized fixed contact angle condition for n -varifold and establish a boundary monotonicity formula. The results are natural generalizations of those for the Neumann boundary condition considered by Grüter-Jost [7].

1. Introduction

Allgren proposed varifold as a generalized manifold in [3] and Allard established a number of fundamental properties of varifold such as rectifiability, compactness and regularity theorems in [1]. One of the key tools to analyze the local properties of varifold is the monotonicity formula [1, Section 5], which gives a good control of measure whenever the first variation is well-behaved.

When a varifold has a “boundary” in a suitable sense, one expects to have a modified monotonicity formula under a suitable set of assumptions. Loosely speaking, with prescribed $C^{1,1}$ boundary (which one may regard as “Dirichlet boundary condition”), Allard obtained a monotonicity formula at the boundary [2] as well as the regularity theorem up to the boundary. The result is improved recently by Bourne to $C^{1,\alpha}$ boundary [4]. For Neumann boundary condition, which corresponds roughly to the prescribed right angle condition, Grüter and Jost [7] derived a monotonicity formula by using a reflection technique and obtained the regularity theorem up to the boundary.

As a further inquiry, it is natural to extend the Neumann boundary condition to more general fixed contact angle condition. The condition arises naturally in various capillarity and free boundary problems, where the boundary of domain under consideration has non-trivial amount of surface energy. For such problem, Taylor [11] established the boundary regularity for area minimizing surfaces. More recently, De Philippis and Maggi [5] proved the boundary regularity for minimizers of anisotropic surface energy. For min-

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imizing problems, one does not necessarily need a monotonicity type formula since one can obtain upper and lower energy density ratio bounds by some energy comparison argument. Here, motivated by the dynamical problem such as the mean curvature flow, we would like to investigate the notion of fixed contact angle condition for general varifolds which do not necessarily correspond to energy minimizing case. As far as we know, this aspect has not been studied in a general setting of varifold so far.

In this paper, we introduce a notion of contact angle condition for general varifolds. Our condition is satisfied for smooth hypersurface having a fixed contact angle with the boundary of domain under consideration. It is stated in terms of the first variation of varifolds and generalized mean curvature vectors, and such condition is satisfied for a limit of diffused interface problem ([8]). With a natural integrability condition on the generalized mean curvature vector, we prove that a modified monotonicity formula holds. The results are natural generalizations of those for the Neumann boundary condition considered by Grüter-Jost [7].

The paper is organized as follows. Section 2 lists notation and recalls some well-known results from geometric measure theory. In Section 3, we state the definition of fixed contact angle condition and discuss the implications such as the monotonicity formula. In Section 4, we prove the monotonicity formula and we give a few final remarks in Section 5.

2. Notation and basic definitions

2.1. Basic notation. In this paper, n will be a positive integer. For $0 < r < \infty$ and $a \in \mathbb{R}^{n+1}$ let

$$B_r(a) := \{x \in \mathbb{R}^{n+1} : |x - a| < r\}.$$

We denote by \mathcal{L}^k the Lebesgue measure on \mathbb{R}^k and by \mathcal{H}^k the k -dimensional Hausdorff measure on \mathbb{R}^{n+1} for each positive integer k . The restriction of \mathcal{H}^k to a set A is denoted by $\mathcal{H}^k \llcorner_A$. We let

$$\omega_k := \mathcal{L}^k(\{x \in \mathbb{R}^k : |x| < 1\}).$$

For any Radon measure μ on \mathbb{R}^{n+1} , $\phi \in C_c(\mathbb{R}^{n+1})$ and μ measurable set A , we often write

$$\mu(\phi) := \int_{\mathbb{R}^{n+1}} \phi \, d\mu, \quad \mu(A) := \int_A d\mu.$$

Let the support of μ be

$$\text{spt}(\mu) := \{x \in \mathbb{R}^{n+1} : \mu(B_r(x)) > 0 \text{ for all } r > 0\}.$$

Let $\Theta^k(\mu, x)$ be the k -dimensional density of μ at x , i.e.,

$$\Theta^k(\mu, x) := \lim_{r \rightarrow 0^+} \frac{\mu(B_r(x))}{\omega_k r^k},$$

if the limit exists.

2.2. Homogeneous maps and varifolds. Let $\mathbf{G}(n+1, n)$ be the space of n -dimensional subspaces of \mathbb{R}^{n+1} . For $S \in \mathbf{G}(n+1, n)$, we identify S with the corresponding orthogonal projection of \mathbb{R}^{n+1} onto S . Let S^\perp be the orthogonal complement of S and we sometimes treat S^\perp as the orthogonal projection \mathbb{R}^{n+1} onto S^\perp . For two elements A and B of $\text{Hom}(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$, we define a scalar product as

$$A \cdot B := \sum_{i,j} A_{ij} B_{ij}.$$

The identity of $\text{Hom}(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ is denoted by I . For these elements A and B , we define the product, the operator norm and the spectrum norm as

$$(A \circ B)_{ij} := \sum_k A_{ik} B_{kj}, \quad \|A\| := \sup_{|x|=1} |Ax|, \quad |A| := \sqrt{A \cdot A},$$

respectively.

We recall some notions related to varifold and refer to [1, 10] for more details. In what follows, let $X \subset \mathbb{R}^{n+1}$ be open and $G_n(X) := X \times \mathbf{G}(n+1, n)$. A general n -varifold in X is a Radon measure on $G_n(X)$ and $\mathbf{V}_n(X)$ denotes the set of all general n -varifolds in X . For $V \in \mathbf{V}_n(X)$, let $\|V\|$ be the weight measure of V , namely,

$$\|V\|(\phi) := \int_{G_n(X)} \phi(x) dV(x, S) \quad \text{for } \phi \in C_c(X).$$

For any \mathcal{H}^n measurable countably n -rectifiable set $M \subset X$ with locally finite \mathcal{H}^n measure, there is a natural n -varifold $|M| \in \mathbf{V}_n(X)$ defined by

$$|M|(\phi) := \int_M \phi(x, \text{Tan}_x M) d\mathcal{H}^n(x) \quad \text{for } \phi \in C_c(G_n(X)),$$

where $\text{Tan}_x M \in \mathbf{G}(n+1, n)$ is the approximate tangent space which exists \mathcal{H}^n a.e. on M . In this case, the weight measure of $|M|$ equals to $\mathcal{H}^n \llcorner_M$. We note that n -dimensional density of this varifold is equal to 1 \mathcal{H}^n a.e. on M .

For $V \in \mathbf{V}_n(X)$, let δV be the first variation of V , namely,

$$\delta V(g) := \int_{G_n(X)} \nabla g(x) \cdot S dV(x, S) \quad \text{for } g \in C_c^1(X; \mathbb{R}^{n+1}).$$

Let $\|\delta V\|$ be the total variation when it exists, and if δV is absolutely continuous with respect to $\|V\|$, we have $\|V\|$ measurable h with

$$\delta V(g) = - \int_X h \cdot g d\|V\| \quad \text{for } g \in C_c^1(X; \mathbb{R}^{n+1}).$$

The vector field h is called the generalized mean curvature vector of V .

3. Main results

We first give a definition and then explain why it may be regarded as a generalized fixed contact angle condition for a varifold.

3.1. Fixed angle condition. We assume that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded open set with C^2 boundary $\partial\Omega$.

DEFINITION 3.1. *Given $V \in \mathbf{V}_n(\Omega)$ with $\|V\|(\Omega) < \infty$, \mathcal{H}^n measurable set $B^+ \subset \partial\Omega$ and $\theta \in [0, \pi]$, we say that “ V has a fixed contact angle θ with $\partial\Omega$ at the boundary of B^+ ” if the following conditions hold.*

(A1) *The generalized mean curvature vector h exists, i.e.,*

$$\begin{aligned} & \int_{G_n(\Omega)} \nabla g(x) \cdot S dV(x, S) \\ &= - \int_{\Omega} g(x) \cdot h(x) d\|V\|(x) \quad \text{for } g \in C_c^1(\Omega; \mathbb{R}^{n+1}). \end{aligned}$$

(A2) *By setting $\sigma := \cos \theta$, we have*

$$\begin{aligned} & \int_{G_n(\Omega)} \nabla g(x) \cdot S dV(x, S) + \sigma \int_{B^+} \operatorname{div}_{\partial\Omega} g(x) d\mathcal{H}^n(x) \\ &= - \int_{\Omega} g(x) \cdot h(x) d\|V\|(x) \end{aligned} \tag{3.1}$$

for all $g \in C^1(\bar{\Omega}; \mathbb{R}^{n+1})$ with $g \cdot \nu_{\partial\Omega} = 0$ on $\partial\Omega$, where $\operatorname{div}_{\partial\Omega}$ is the divergence on $\partial\Omega$ and $\nu_{\partial\Omega}$ is the outward unit normal vector on the boundary $\partial\Omega$.

Let us give a justification for the definition. Due to $\|V\|(\Omega) < \infty$, by setting $V = 0$ outside of Ω , we may extend V to the entire \mathbb{R}^{n+1} as an element in $\mathbf{V}_n(\mathbb{R}^{n+1})$ and we will regard V in this way in the following. The condition (A1) means equivalently that the first variation δV is absolutely continuous with respect to $\|V\|$. Geometrically speaking, if $V = |M|$ with some smooth surface M , this means that there is no boundary of M in Ω since any presence

of boundary in Ω gives a singular δV . To better explain a motivation for the notion, let us assume the following boundedness of the first variation in $\bar{\Omega}$, namely,

$$\sup_{g \in C^1(\bar{\Omega}; \mathbb{R}^{n+1}), |g| \leq 1} \delta V(g) < \infty. \tag{3.2}$$

We may include (3.2) as ‘‘(A3)’’ in Definition 3.1, but as we will see, (3.2) is not needed to prove the subsequent monotonicity formula. With the zero extension of V , (3.2) means that V has a bounded first variation δV on \mathbb{R}^{n+1} . We should emphasize that this is typically different from the first variation as an element of $\mathbf{V}_n(\Omega)$. The condition (A1) implies that $\delta V|_{\Omega} = -h\|V\|_{\Omega}$. On the other hand, $\delta V|_{\partial\Omega}$ is singular with respect to $\|V\|$ whenever it is nonzero, since $\|V\|_{\partial\Omega} = 0$. By the definition of the first variation, we have

$$\int_{G_n(\Omega)} \nabla g(x) \cdot S \, dV(x, S) = \int_{G_n(\mathbb{R}^{n+1})} \nabla g(x) \cdot S \, dV(x, S) = \int_{\bar{\Omega}} g(x) \cdot d(\delta V)(x).$$

Since $\delta V|_{\Omega} = -h\|V\|_{\Omega}$, the condition (A2) implies that

$$\int_{\partial\Omega} g \cdot d(\delta V) + \sigma \int_{B^+} \operatorname{div}_{\partial\Omega} g \, d\mathcal{H}^n = 0 \tag{3.3}$$

for all $g \in C^1(\bar{\Omega}; \mathbb{R}^{n+1})$ with $g \cdot \nu_{\partial\Omega} = 0$. If $\sigma = \cos \theta \neq 0$ (or $\theta \neq \pi/2$), (3.3) implies that B^+ has a finite perimeter in $\partial\Omega$. By De Giorgi’s theorem (see [6, Theorem 5.16]), the second term may be expressed as

$$\sigma \int_{\partial^* B^+} g \cdot \mathbf{n}_{B^+} \, d\mathcal{H}^{n-1} \tag{3.4}$$

where $\partial^* B^+$ is the reduced boundary of B^+ which is countably $(n-1)$ -rectifiable and \mathbf{n}_{B^+} is the outer pointing unit normal to $\partial^* B^+$ which exists \mathcal{H}^{n-1} a.e. on $\partial^* B^+$. Now, define $\mathbf{n}_V = \frac{\delta V}{\|\delta V\|}$ on $\partial\Omega$ so that $\mathbf{n}_V \|\delta V\| = \delta V$. Then (3.3) and (3.4) mean that we have

$$\int_{\partial\Omega} g \cdot \mathbf{n}_V \|\delta V\| + \sigma \int_{\partial^* B^+} g \cdot \mathbf{n}_{B^+} \, d\mathcal{H}^{n-1} = 0. \tag{3.5}$$

Since \mathbf{n}_{B^+} is tangent to $\partial\Omega$, (3.5) shows that

$$(\mathbf{n}_V - (\mathbf{n}_V \cdot \nu_{\partial\Omega})\nu_{\partial\Omega})\|\delta V\| = -\sigma \mathbf{n}_{B^+} \mathcal{H}^{n-1}|_{\partial^* B^+} \tag{3.6}$$

on $\partial\Omega$.

Let us see what the above means in the case of smooth surfaces. Suppose that $V = |M|$ with a C^2 n -dimensional surface $M \subset \Omega$ without boundary inside

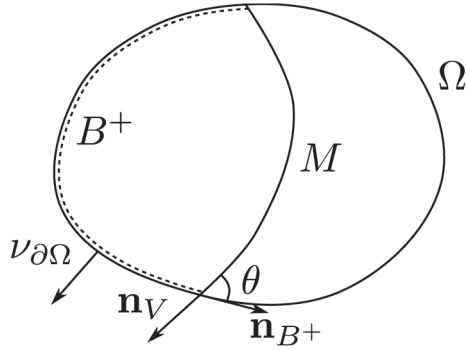


Fig. 1. The figure for the smooth surface M

Ω but with a nontrivial boundary ∂M in $\partial\Omega$. Suppose also that B^+ has a smooth boundary ∂B^+ . Then \mathbf{n}_V corresponds to the unit co-normal of ∂M pointing outwards. We also have $\|\delta V\|_{L^\infty(\partial\Omega)} = \mathcal{H}^{n-1} \llcorner_{\partial M}$ in this case. The reduced boundary $\partial^* B^+$ is the usual boundary ∂B^+ . Then (3.6) means that we need to have $\partial B^+ \subset \partial M$, and

$$\begin{aligned} \mathbf{n}_V &= \nu_{\partial\Omega} && \text{on } \partial M \setminus \partial B^+, \\ \mathbf{n}_V - (\mathbf{n}_V \cdot \nu_{\partial\Omega})\nu_{\partial\Omega} &= -\sigma \mathbf{n}_{B^+} && \text{on } \partial B^+. \end{aligned} \tag{3.7}$$

The first case means that ∂M intersects $\partial\Omega$ with 90 degree if it is not part of ∂B^+ . The second case is precisely the fixed angle condition in the sense that the angle formed by \mathbf{n}_V and $-\mathbf{n}_{B^+}$ is θ (see figure 1). This is the reason for the definition of the fixed angle condition. Above discussion does not make sense unless (3.2) is satisfied, so we understand the definition in a weak sense in this case. Interestingly, for varifolds arising from singular perturbation limit problems, finiteness of the first variation (3.2) is automatically satisfied, see [8, 9]. On the other hand, the subsequent monotonicity formula holds even without (3.2), and we think it better if (3.2) is not included as a part of the definition for a broader applicability of the notion.

If we assume that there exists an open set $U \subset \mathbb{R}^{n+1}$ such that $M = \Omega \cap \partial U$ and $B^+ = \partial\Omega \cap U$, then, $\partial M = \partial B^+$ and the first case of (3.7) does not happen. In the application to the diffused interface limit problem with a fixed angle condition [8], the varifold V arises as a ‘‘phase boundary’’ in a sense and the presence of B^+ follows naturally.

3.2. The monotonicity formula. The following Theorem 3.2 is a natural generalization of monotonicity formula of Grüter-Jost [7] to the fixed angle condition defined in the previous subsection. To present the statement, we

need some more notation. Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a bounded open set with C^2 boundary $\partial\Omega$. Define \varkappa as

$$\varkappa := \|\text{principal curvature of } \partial\Omega\|_{L^\infty(\partial\Omega)}.$$

For $s > 0$, define a subset N_s of \mathbb{R}^{n+1} by

$$N_s := \{x \in \mathbb{R}^{n+1} : \text{dist}(x, \partial\Omega) < s\}.$$

For any boundary point $b \in \partial\Omega$ let

$$\tau(b) := \text{Tan}_b \partial\Omega \quad \text{and} \quad \nu(b) := \tau(b)^\perp.$$

There exists a sufficiently small

$$s_0 \in (0, \varkappa^{-1}]$$

depending only on $\partial\Omega$ such that all points $x \in N_{s_0}$ have a unique point $\zeta(x) \in \partial\Omega$ such that $\text{dist}(x, \partial\Omega) = |x - \zeta(x)|$. By using this $\zeta(x)$, we define the reflection point \tilde{x} of x with respect to $\partial\Omega$ as $\tilde{x} := 2\zeta(x) - x$ and the reflection ball $\tilde{B}_r(a)$ of $B_r(a)$ with respect to $\partial\Omega$ as

$$\tilde{B}_r(a) := \{x \in \mathbb{R}^{n+1} : |\tilde{x} - a| < r\}.$$

In the case for $x \in \partial\Omega$, we note that $\zeta(x) = \tilde{x} = x$. If $y \in \mathbb{R}^{n+1}$, we set

$$i_x(y) := \tau(\zeta(x))y - \nu(\zeta(x))y.$$

THEOREM 3.2. *Given $V \in \mathbf{V}_n(\Omega)$ with $\|V\|(\Omega) < \infty$, \mathcal{H}^n measurable set $B^+ \subset \partial\Omega$ and $\theta \in [0, \pi/2]$, suppose that V has a fixed contact angle θ with $\partial\Omega$ at the boundary of B^+ , as in Definition 3.1. Assume that for some $p > n$ and $\Gamma \geq 0$, we have*

$$\left(\frac{1}{\omega_n} \int_{N_{s_0} \cap \Omega} 2|h(x)|^p d\|V\|(x) \right)^{1/p} \leq \Gamma. \tag{3.8}$$

Then there exists a constant $C \geq 0$ depending only on n such that for any $x \in N_{s_0/6} \cap \bar{\Omega}$,

$$\left\{ \frac{\|V\|(B_\rho(x)) + \|V\|(\tilde{B}_\rho(x)) + 2\sigma \mathcal{H}^n \lfloor_{B^+(B_\rho(x))}}{\omega_n \rho^n} \right\}^{1/p} \left(1 + C\kappa\rho \left(1 + \frac{1}{p-n} \right) \right) + \frac{\Gamma\rho^{1-n/p}}{p-n} \tag{3.9}$$

is a non-decreasing function of ρ in $(0, s_0/6)$. Here $\sigma = \cos \theta$. In the case that $\theta \in (\pi/2, \pi]$, the same claim holds with σ and B^+ replaced by $-\sigma$ and $\partial\Omega \setminus B^+$ in (3.9), respectively.

See Section 5 for more discussion.

4. Proof of Theorem 3.2

The proof of the monotonicity formula (3.9) is similar to that of Grüter-Jost [7] except that we use (3.1) with $\sigma \neq 0$. For completeness, we present the proof in this section.

First, we need to estimate the derivatives of $\zeta(x)$ and i_x , and we cite the following lemma ([2, Lemma 2.2]).

LEMMA 4.1. *The following statements hold.*

- (i) ζ is continuously differentiable in N_{s_0} .
- (ii) For $x \in N_{s_0}$, $Q(x) := \nabla \zeta(x) - \tau(\zeta(x))$ is symmetric,

$$Q(x) \circ v(\zeta(x)) = 0 \quad (4.1)$$

and

$$\|Q(x)\| \leq \varkappa \operatorname{dist}(x, \partial\Omega)(1 - \varkappa \operatorname{dist}(x, \partial\Omega))^{-1}. \quad (4.2)$$

- (iii) For $b \in \partial\Omega$,

$$\|\nabla_{\partial\Omega} v(b)\| = \|\nabla_{\partial\Omega} \tau(b)\| \leq \varkappa \quad (4.3)$$

holds.

In addition, we need the following.

LEMMA 4.2. *Assume $a \in N_{s_0}$ and $\rho > 0$ satisfy $\operatorname{dist}(a, \partial\Omega) \leq \rho$ and $B_\rho(a) \subset N_{s_0}$. Then for any point $x \in \tilde{B}_\rho(a)$*

$$\operatorname{dist}(x, \partial\Omega) \leq 2\rho \quad (4.4)$$

and

$$\tilde{B}_\rho(a) \subset B_{5\rho}(a). \quad (4.5)$$

PROOF. For any $x \in \tilde{B}_\rho(a)$, we have by the assumption $\operatorname{dist}(a, \partial\Omega) \leq \rho$

$$\operatorname{dist}(x, \partial\Omega) = \operatorname{dist}(\tilde{x}, \partial\Omega) \leq |\tilde{x} - a| + \operatorname{dist}(a, \partial\Omega) < 2\rho,$$

which shows (4.4). Using (4.4) we have

$$|x - a| \leq |2\zeta(x) - x - a| + 2|\zeta(x) - x| = |\tilde{x} - a| + 2 \operatorname{dist}(x, \partial\Omega) < 5\rho$$

and hence (4.5) holds.

PROOF (Proof of Theorem 3.2). First, for $\theta \in (\pi/2, \pi]$, using

$$\int_{\partial\Omega} \operatorname{div}_{\partial\Omega} g(x) d\mathcal{H}^n(x) = 0$$

for g tangent to $\partial\Omega$, we may replace σ by $-\sigma$ and B^+ by $\partial\Omega \setminus B^+$ in (3.1). Thus in the following, we may assume without loss of generality that $\sigma \geq 0$.

For any point $a \in N_{s_0/6} \cap \bar{\Omega}$, we choose for (3.1) the test function to prove the monotonicity formula around a . Let $\gamma \in C^1(\mathbb{R})$ satisfy

$$\gamma(t) = \begin{cases} 1, & t \leq \frac{\rho}{2}, \\ 0, & t \geq \rho \end{cases}$$

and $\gamma'(t) \leq 0$ for any t , where a constant $\rho > 0$ satisfies $0 < \rho \leq s_0/6$. Thus the boundary point $\zeta(x)$ is defined for $x \in B_\rho(a)$ from $\text{dist}(x, \partial\Omega) \leq s_0/3$. Let the vector field g be

$$g(x) = \gamma(r)(x - a) + \gamma(\tilde{r})(i_x(\tilde{x} - a)),$$

where $r = |x - a|$ and $\tilde{r} = |\tilde{x} - a|$. This vector field g may satisfy the property $g \cdot \nu_{\partial\Omega} = 0$ on $\partial\Omega$ by the following argument. For $x \in \partial\Omega$, $x = \tilde{x} = \zeta(x)$ yields $r = \tilde{r}$ and

$$i_x(\tilde{x} - a) = \tau(x)(x - a) - \nu(x)(x - a).$$

Thus, we have

$$\begin{aligned} g(x) &= \gamma(r)\{\tau(x)(x - a) + \nu(x)(x - a)\} + \gamma(r)\{\tau(x)(x - a) - \nu(x)(x - a)\} \\ &= 2\gamma(r)\tau(x)(x - a) \end{aligned}$$

and hence $g \cdot \nu_{\partial\Omega} = 0$ holds on $\partial\Omega$. To substitute g in (3.1), we calculate the gradient of g

$$\nabla g(x) = \nabla(\gamma(r)(x - a)) + \nabla(\gamma(\tilde{r})(i_x(\tilde{x} - a))). \quad (4.6)$$

For the first term of right hand side of (4.6), we have by a simple calculation

$$\nabla(\gamma(r)(x - a)) = r\gamma'(r) \left(\frac{x - a}{r} \otimes \frac{x - a}{r} \right) + \gamma(r)I.$$

For the second term of right hand side of (4.6), we calculate the following matrices M and N :

$$\begin{aligned} \nabla(\gamma(\tilde{r})(i_x(\tilde{x} - a))) &= \left(\frac{\partial}{\partial x_i} (\gamma(\tilde{r})(i_x(\tilde{x} - a)))_j \right) + \left(\gamma(\tilde{r}) \frac{\partial (i_x(\tilde{x} - a))_j}{\partial x_i} \right) \\ &=: (M_{ij}) + (N_{ij}). \end{aligned} \quad (4.7)$$

Calculation of M : By the definitions $\tilde{x} = 2\zeta(x) - x$ and $\tilde{r} = |\tilde{x} - a|$, we have

$$\begin{aligned} \frac{\partial}{\partial x_i}(\gamma(\tilde{\mathbf{r}})) &= \gamma'(\tilde{\mathbf{r}}) \frac{\partial}{\partial x_i} \sqrt{(2\xi_1(x) - x_1 - a_1)^2 + \cdots + (2\xi_{n+1}(x) - x_{n+1} - a_{n+1})^2} \\ &= \frac{\gamma'(\tilde{\mathbf{r}})}{\tilde{\mathbf{r}}} \left\{ \left(\sum_k 2 \frac{\partial \xi_k(x)}{\partial x_i} (\tilde{\mathbf{x}} - a)_k \right) - (\tilde{\mathbf{x}} - a)_i \right\}. \end{aligned}$$

Thus, the matrix element of M is represented by

$$M_{ij} = \frac{\gamma'(\tilde{\mathbf{r}})}{\tilde{\mathbf{r}}} \left\{ \left(\sum_k 2 \frac{\partial \xi_k(x)}{\partial x_i} (\tilde{\mathbf{x}} - a)_k (i_x(\tilde{\mathbf{x}} - a))_j \right) - (\tilde{\mathbf{x}} - a)_i (i_x(\tilde{\mathbf{x}} - a))_j \right\}$$

and hence, by $Q(x) = \nabla \xi(x) - \tau(\xi(x))$ and $i_x = \tau(\xi(x)) - \nu(\xi(x))$, we have

$$\begin{aligned} M &= \frac{\gamma'(\tilde{\mathbf{r}})}{\tilde{\mathbf{r}}} \{ 2\nabla \xi(x) \circ ((\tilde{\mathbf{x}} - a) \otimes i_x(\tilde{\mathbf{x}} - a)) - (\tilde{\mathbf{x}} - a) \otimes i_x(\tilde{\mathbf{x}} - a) \} \\ &= \frac{\gamma'(\tilde{\mathbf{r}})}{\tilde{\mathbf{r}}} \{ 2Q(x) \circ ((\tilde{\mathbf{x}} - a) \otimes i_x(\tilde{\mathbf{x}} - a)) + i_x \circ ((\tilde{\mathbf{x}} - a) \otimes i_x(\tilde{\mathbf{x}} - a)) \} \\ &= \tilde{\mathbf{r}}\gamma'(\tilde{\mathbf{r}}) \left\{ 2Q(x) \circ \left(\frac{\tilde{\mathbf{x}} - a}{\tilde{\mathbf{r}}} \otimes i_x \left(\frac{\tilde{\mathbf{x}} - a}{\tilde{\mathbf{r}}} \right) \right) + i_x \left(\frac{\tilde{\mathbf{x}} - a}{\tilde{\mathbf{r}}} \right) \otimes i_x \left(\frac{\tilde{\mathbf{x}} - a}{\tilde{\mathbf{r}}} \right) \right\}. \quad (4.8) \end{aligned}$$

Calculation of N : We define the matrix $L(\xi(x))$ by $L(\xi(x)) = \tau(\xi(x)) - \nu(\xi(x)) = i_x$ and calculate as

$$\begin{aligned} \frac{\partial}{\partial x_i} (i_x(\tilde{\mathbf{x}} - a))_j &= \frac{\partial}{\partial x_i} \left(\sum_k L_{jk}(\xi(x)) (\tilde{\mathbf{x}} - a)_k \right) \\ &= \sum_k \left(\sum_l \frac{\partial L_{jk}}{\partial \xi_l}(\xi(x)) \frac{\partial \xi_l}{\partial x_i}(x) \right) (\tilde{\mathbf{x}} - a)_k \\ &\quad + \sum_k L_{jk}(\xi(x)) \left(2 \frac{\partial \xi_k}{\partial x_i}(x) - \delta_{ik} \right), \end{aligned}$$

where δ_{ik} is the Kronecker delta. By using this calculation, $Q(x) = \nabla \xi(x) - \tau(\xi(x))$ and (4.1), we obtain

$$\begin{aligned} N &= \gamma(\tilde{\mathbf{r}}) \{ \nabla \xi(x) \circ \nabla_{\partial\Omega} L(\xi(x)) \circ (\tilde{\mathbf{x}} - a) + 2\nabla \xi(x) \circ T(\xi(x)) - T(\xi(x)) \} \\ &= \gamma(\tilde{\mathbf{r}}) \{ (Q(x) + \tau(\xi(x))) \circ \nabla_{\partial\Omega} (\tau - \nu)(\xi(x)) \circ (\tilde{\mathbf{x}} - a) \\ &\quad + 2(Q(x) + \tau(\xi(x))) \circ (\tau(\xi(x)) - \nu(\xi(x))) - \tau(\xi(x)) + \nu(\xi(x)) \} \\ &= \gamma(\tilde{\mathbf{r}}) \{ (Q(x) + \tau(\xi(x))) \circ \nabla_{\partial\Omega} (\tau - \nu)(\xi(x)) \circ (\tilde{\mathbf{x}} - a) + 2Q(x) + I \}, \quad (4.9) \end{aligned}$$

where $\cdot \circ \cdot \circ \cdot$ has to be interpreted appropriately.

Substituting (4.8) and (4.9) in (4.7), we have

$$\begin{aligned} \nabla g(x) &= (\gamma(r) + \gamma(\tilde{r}))I + r\gamma'(r)\left(\frac{x-a}{r} \otimes \frac{x-a}{r}\right) \\ &\quad + \tilde{r}\gamma'(\tilde{r})\left\{i_x\left(\frac{\tilde{x}-a}{\tilde{r}}\right) \otimes i_x\left(\frac{\tilde{x}-a}{\tilde{r}}\right)\right\} + \gamma(\tilde{r})\tilde{\varepsilon}_1(x) - \tilde{r}\gamma'(\tilde{r})\tilde{\varepsilon}_2(x), \end{aligned}$$

where

$$\begin{aligned} \tilde{\varepsilon}_1(x) &= 2Q(x) + \{\tau(\xi(x)) + Q(x)\} \circ \nabla_{\partial\Omega}(\tau - \nu)(\xi(x)) \circ (\tilde{x} - a), \\ \tilde{\varepsilon}_2(x) &= -2Q(x) \circ \left(\frac{\tilde{x}-a}{\tilde{r}} \otimes i_x\left(\frac{\tilde{x}-a}{\tilde{r}}\right)\right). \end{aligned}$$

Thus, for any $S \in \mathbf{G}(n+1, n)$

$$\begin{aligned} \nabla g(x) \cdot S &= n(\gamma(r) + \gamma(\tilde{r})) + r\gamma'(r)(1 - |S^\perp(\nabla r)|^2) \\ &\quad + \tilde{r}\gamma'(\tilde{r})\left(1 - \left|S^\perp\left(i_x\left(\frac{\tilde{x}-a}{\tilde{r}}\right)\right)\right|^2\right) \\ &\quad + \gamma(\tilde{r})\varepsilon_1(x, S) - \tilde{r}\gamma'(\tilde{r})\varepsilon_2(x, S), \end{aligned}$$

where

$$\begin{aligned} \varepsilon_1(x, S) &= 2\langle S \cdot Q(x) \rangle + \langle S \circ \{\tau(\xi(x)) + Q(x)\}, \nabla_{\partial\Omega}(\tau - \nu)(\xi(x)), \tilde{x} - a \rangle, \\ \varepsilon_2(x, S) &= -2\left\{Q(x)\left(\frac{\tilde{x}-a}{\tilde{r}}\right)\right\} \cdot \left\{S \circ i_x\left(\frac{\tilde{x}-a}{\tilde{r}}\right)\right\} \end{aligned}$$

and $\langle \cdot, \cdot, \cdot \rangle$ has to be interpreted appropriately. For $x \in \partial\Omega$, the properties $r = \tilde{r}$, $x = \tilde{x}$, $\text{dist}(x, \partial\Omega) = 0$ and (4.2) yield

$$\begin{aligned} \text{div}_{\partial\Omega} g(x) &= 2n\gamma(r) + r\gamma'(r)\{(1 - |v(x)(\nabla r)|^2) + (1 - |v(x)(i_x(\nabla r))|^2)\} \\ &\quad + \gamma(r)\varepsilon_3(x), \end{aligned}$$

where

$$\varepsilon_3(x) = \langle \tau(x), \nabla_{\partial\Omega}(\tau - \nu)(x), x - a \rangle.$$

Thus (3.1), $\sigma \geq 0$ and $\gamma' \leq 0$ imply

$$\begin{aligned} n\left(\int_{\Omega} \gamma(r) + \gamma(\tilde{r})d\|V\| + 2\sigma \int_{B^+} \gamma(r)d\mathcal{H}^n\right) \\ + \int_{\Omega} r\gamma'(r) + \tilde{r}\gamma'(\tilde{r})d\|V\| + 2\sigma \int_{B^+} r\gamma'(r)d\mathcal{H}^n \end{aligned}$$

$$\begin{aligned}
&\leq - \int_{\Omega} \{ \gamma(r)(x-a) + \gamma(\tilde{r})i_x(\tilde{x}-a) \} \cdot hd \|V\| \\
&\quad - \left(\int_{G_n(\Omega)} \gamma(\tilde{r})\varepsilon_1(x, S) - \tilde{r}\gamma'(\tilde{r})\varepsilon_2(x, S) dV(x, S) \right) \\
&\quad - \sigma \int_{B^+} \gamma(r)\varepsilon_3(x) d\mathcal{H}^n. \tag{4.10}
\end{aligned}$$

Now take $\phi \in C^1(\mathbb{R})$ such that

$$\phi(t) = 1 \quad \text{for } t \leq 1/2, \quad \phi(t) = 0 \quad \text{for } t \geq 1$$

and $\phi'(t) \leq 0$ for all t and use (4.10) with $\gamma(r) = \phi(r/\rho)$. For this γ ,

$$r\gamma'(r) = -\rho \frac{\partial}{\partial \rho} \left\{ \phi\left(\frac{r}{\rho}\right) \right\}$$

holds and implies

$$nI(\rho) - \rho I'(\rho) \leq -\{H(\rho) + E_1(\rho) + \rho E_2'(\rho) + E_3(\rho)\},$$

where

$$\begin{aligned}
I(\rho) &= \int_{\Omega} \phi\left(\frac{r}{\rho}\right) + \phi\left(\frac{\tilde{r}}{\rho}\right) d\|V\| + 2\sigma \int_{B^+} \phi\left(\frac{r}{\rho}\right) d\mathcal{H}^n, \\
H(\rho) &= \int_{\Omega} \left\{ \phi\left(\frac{r}{\rho}\right)(x-a) + \phi\left(\frac{\tilde{r}}{\rho}\right)i_x(\tilde{x}-a) \right\} \cdot hd \|V\|, \\
E_i(\rho) &= \int_{G_n(\Omega)} \phi\left(\frac{\tilde{r}}{\rho}\right) \varepsilon_i(x, S) dV(x, S), \quad i = 1, 2, \\
E_3(\rho) &= \sigma \int_{B^+} \phi\left(\frac{r}{\rho}\right) \varepsilon_3(x) d\mathcal{H}^n.
\end{aligned}$$

Multiplying by ρ^{-n-1} we have

$$\frac{d}{d\rho} \{ \rho^{-n} I(\rho) \} \geq \rho^{-n} E_2'(\rho) + \rho^{-n-1} \{ H(\rho) + E_1(\rho) + E_3(\rho) \}. \tag{4.11}$$

To estimate the right hand side of this inequality, we apply (4.2) and (4.4). From $a \in N_{s_0/6} \cap \bar{\Omega}$ and $\rho \leq s_0/6$, the distance $\text{dist}(x, \partial\Omega) \leq s_0/3 \leq 1/(3\kappa)$ for $x \in \tilde{B}_\rho(a)$ with $\rho \geq \text{dist}(a, \partial\Omega)$. Hence $\kappa \text{dist}(x, \partial\Omega) \leq 1/3$ and

$$\begin{aligned}
|\varepsilon_i(x)| &\leq C\kappa\rho, & \text{for } x \in \tilde{B}_\rho(a), \quad i = 1, 2, \\
|\varepsilon_3(x)| &\leq C\kappa\rho & \text{for } x \in B^+ \cap B_\rho(a)
\end{aligned}$$

with some constant C depending only on n whenever $\rho \geq \text{dist}(a, \partial\Omega)$. These inequalities, $\sigma \geq 0$ and $\partial/(\partial\rho)\{\phi(\tilde{r}/\rho)\} \geq 0$ imply

$$\begin{aligned} |E_1(\rho)| &\leq C\kappa\rho \int_{\Omega} \phi\left(\frac{\tilde{r}}{\rho}\right) d\|V\| \leq C\kappa\rho I(\rho), \\ |E_2'(\rho)| &\leq C\kappa\rho \frac{\partial}{\partial\rho} \int_{\Omega} \phi\left(\frac{\tilde{r}}{\rho}\right) d\|V\| \leq C\kappa\rho I'(\rho), \\ |E_3(\rho)| &\leq C\kappa\rho\sigma \int_{B^+} \phi\left(\frac{r}{\rho}\right) d\mathcal{H}^n \leq C\kappa\rho I(\rho) \end{aligned}$$

for any $\rho > 0$ because of $\tilde{B}_\rho(a) \cap \text{spt}\|V\| = \emptyset$ and $B_\rho(a) \cap B^+ = \emptyset$ whenever $\rho < \text{dist}(a, \partial\Omega)$. By applying the Hölder inequality and (4.5), we have

$$\begin{aligned} H(\rho) &\leq \rho \left(\int_{\Omega} \phi\left(\frac{r}{\rho}\right) + \phi\left(\frac{\tilde{r}}{\rho}\right) d\|V\| \right)^{1-1/p} \left(\int_{\Omega} \left(\phi\left(\frac{r}{\rho}\right) + \phi\left(\frac{\tilde{r}}{\rho}\right) \right) |h|^p d\|V\| \right)^{1/p} \\ &\leq \rho I^{1-1/p}(\rho) \left(\int_{N_{s_0} \cap \Omega} 2|h|^p d\|V\| \right)^{1/p} \leq \rho \omega_n^{1/p} \Gamma I^{1-1/p}(\rho). \end{aligned}$$

Combining these inequalities and (4.11), we have

$$\frac{d}{d\rho} \{\rho^{-n} I(\rho)\} \geq -C\kappa\rho^{-n} I(\rho) - C\kappa\rho^{-n+1} I'(\rho) - \omega_n^{1/p} \Gamma \rho^{-n} I^{1-1/p}(\rho)$$

and $p > n$ implies

$$\begin{aligned} &\frac{d}{d\rho} \{\rho^{-n} I(\rho)\}^{1/p} \\ &\geq -\frac{1}{p} \{C\kappa\rho^{-n/p} I^{1/p}(\rho) + C\kappa\rho^{1-n/p} I^{1/p-1}(\rho) I'(\rho) + \omega_n^{1/p} \Gamma \rho^{-n/p}\} \\ &\geq -\frac{C\kappa}{p} \left\{ (1+p-n)\rho^{-n/p} I^{1/p}(\rho) + \left(1 + \frac{1}{p-n}\right) \rho^{1-n/p} I^{1/p-1}(\rho) I'(\rho) \right\} \\ &\quad - \frac{\omega_n^{1/p} \Gamma}{p} \rho^{-n/p}. \end{aligned}$$

Integrating from σ to s and integrating by parts, we may estimate

$$\begin{aligned} \{\rho^{-n} I(\rho)\}^{1/p} - \{s^{-n} I(s)\}^{1/p} &\geq -C\kappa \left(1 + \frac{1}{p-n}\right) (\rho^{1-n/p} I^{1/p}(\rho) - s^{1-n/p} I^{1/p}(s)) \\ &\quad - \omega_n^{1/p} \frac{\Gamma}{p-n} (\rho^{1-n/p} - s^{1-n/p}). \end{aligned}$$

Rearranging terms and letting ϕ increase to $\chi_{(-\infty, 1)}$ we have the monotonicity formula.

5. Additional remarks

As a consequence of the monotonicity formula, we may conclude the following.

COROLLARY 5.1. *Under the same assumption of Theorem 3.2, for any $x \in N_{s_0/6} \cap \bar{\Omega}$,*

$$\lim_{\rho \rightarrow 0^+} \frac{\|V\|(B_\rho(x)) + \|V\|(\tilde{B}_\rho(x)) + 2\sigma \mathcal{H}^n \llcorner_{B^+}(B_\rho(x))}{\omega_n \rho^n} \quad (5.1)$$

exists and it is upper-semicontinuous function of x .

For $x \in \partial\Omega$, $B_\rho(x)$ and $\tilde{B}_\rho(x)$ approach to each other as $\rho \rightarrow 0^+$, thus if $\sigma = 0$, this implies the existence of $\Theta^n(\|V\|, x)$ for $x \in \partial\Omega$. For $\sigma \neq 0$, since the existence of the third term of (5.1) is not guaranteed (only up to \mathcal{H}^n a.e.), we cannot conclude such existence in general on $\partial\Omega$. If we assume, in addition to (A1) and (A2), that (3.2) is satisfied, then B^+ has a finite perimeter as discussed in the definition. Then by De Giorgi's theorem (cf. [6, Theorem 5.19] used for characteristic function) again, for \mathcal{H}^{n-1} a.e. $x \in \partial\Omega$, the limit of the third term exists and is equal to either 2σ , σ or 0 . In particular, on $\partial^* B^+$, it is σ for \mathcal{H}^{n-1} a.e. Thus, in this case, we have $\Theta^n(\|V\|, x)$ for \mathcal{H}^{n-1} a.e. on $\partial\Omega$ instead.

It is also interesting to pursue a boundary regularity theorem under a natural ‘‘closeness to a single sheet’’ assumption. Extrapolating from [2, 7], one may assume that $\|V\|(B_\rho(x))$ is close to $\omega_n \rho^n / 2$. On the other hand, one needs to differentiate two cases, the case away from ‘‘ ∂B^+ ’’ where the right angle condition should be satisfied, and the other case of near ‘‘ ∂B^+ ’’ where the fixed angle condition of θ should be satisfied. Compared to [7], it is less clear what should be the right assumption for the further regularity theorem.

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