Hiroshima Math. J.
47 (2017), 19–41

Link invariant and G_2 web space

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(Received September 1, 2015)
(Revised October 24, 2016)

Abstract. In this paper, we reconstruct Kuperberg's G_2 web space [5, 6]. We introduce a new web diagram (a trivalent graph with only double edges) and new relations between Kuperberg's web diagrams and the new web diagram. Using the web diagrams, we give crossing formulas for the R-matrices associated to some irreducible representations of $U_q(G_2)$ and calculate G_2 quantum link invariants for generalized twist links.

1. Introduction

Suppose that $U_q(G_2)$ is the quantum group of type G_2, where $q \in \mathbb{C}$ is neither zero nor a root of unity [1, 3].

Invariant theory of the $U_q(G_2)$ fundamental representations was studied in a skein theoretic approach by Kuperberg [6] and in a representation theoretic approach by Lehrer–Zhang [7]. (Invariant theory of exceptional Lie group G_2 was studied by Schwarz, Huang–Zhu [2, 11].) As an application of these studies, Kuperberg explicitly gave Reshetikhin–Turaev's quantum link invariant (R-matrix) associated to the $U_q(G_2)$ fundamental representations [9]. (The G_2 quantum link invariant was also obtained in a planar algebra approach by Morrison–Peters–Snyder [8].)

In Kuperberg's approach, diagrams in Figure 1 are introduced, which are called elementary G_2 web diagrams. They are diagrammatizations of intertwiners between tensor representations of the $U_q(G_2)$ fundamental representations [5, 6].

These diagrams correspond to intertwiners in $\mathrm{Hom}_{U_q(G_2)}(V_{\varpi_1}, V_{\varpi_1} \otimes V_{\varpi_1})$ and $\mathrm{Hom}_{U_q(G_2)}(V_{\varpi_2}, V_{\varpi_1} \otimes V_{\varpi_1})$, where V_{ϖ_1} is the first fundamental representation and V_{ϖ_2} is the second fundamental representation.

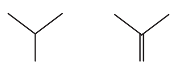


Fig. 1. Kuperberg's elementary G_2 web diagram

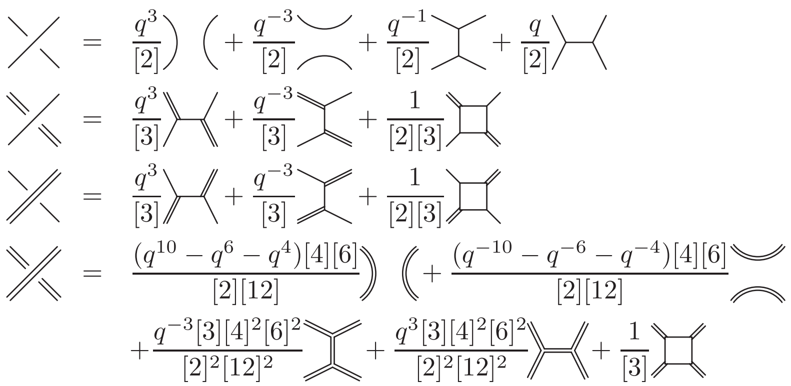
2010 *Mathematics Subject Classification*. Primary 81R50, Secondary 57M25.
Key words and phrases. Quantum link invariant, Quantum group.

Fig. 2. New web diagram

The purpose of this paper is to give a reformulation of Kuperberg's G_2 web space, by introducing a new elementary G_2 web diagram in Figure 2 which corresponds to an intertwiner in $\mathrm{Hom}_{U_q(G_2)}(V_{\varpi_2}, V_{\varpi_2} \otimes V_{\varpi_2})$, and to describe crossings corresponding to the R-matrices associated to some $U_q(G_2)$ irreducible representations in the new G_2 web space.

In Section 2, we introduce the new elementary G_2 web and give relations between Kuperberg's webs and the new web. In Section 3, we define a G_2 web space W_{G_2} which is a $\mathbb{C}$-vector space composed of G_2 web diagrams (G_2 webs embedded in a unit disk) and show that the G_2 web space is isomorphic to an invariant space of tensor representations of the $U_q(G_2)$ fundamental representations.

In Sections 4, we give the following crossing formulas which express the crossing diagrams (R-matrices associated to $U_q(G_2)$) by linear sums of G_2 web diagrams. (The first three crossing formulas are the same as Kuperberg's formulas [6], but the last formula is different from his. His last crossing formula of double edges contains an error.)



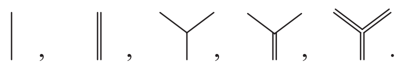
In Section 5, we show that the above crossing formulas induce a braid group action on G_2 web space W_{G_2}, and in Section 6, we give identities which express idempotents in hom space between tensor representations by G_2 web diagrams. Using the expressions of idempotents, we can obtain crossings formulas for R-matrices associated to the $U_q(G_2)$ representation with the highest weight $2\varpi_1$. In Section 7, we calculate G_2 quantum invariant of generalized twist links $TW(m,n)$.

2. G_2 web

First, we introduce G_2 webs in order to define a G_2 web space.

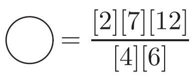
Definition 1 (G_2 web). Let $q \in \mathbb{C}$ be neither zero nor a root of unity. Denote by $[n]$ for $n \in \mathbb{Z}_{\geq 0}$ the q-integer $\frac{q^n - q^{-n}}{q - q^{-1}}$ and put $[n]! := [n][n-1]\dots[1]$ and $\begin{bmatrix} m \\ n \end{bmatrix} := \dfrac{[m]!}{[n]![m-n]!}$ for $0 \leq n \leq m$.

By an *elementary G_2 web*, we mean one of the following two planar univalent graphs or one of three planar uni-trivalent graphs



A *G_2 web* is a planar uni-trivalent graph whose vertex is either one of the elementary G_2 webs with the following local relations:

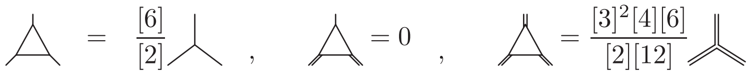
(Loop relation)



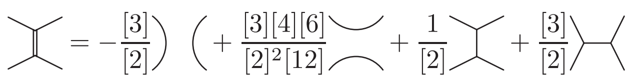
(Monogon relations)

(Digon relations)

(Triangle relations)

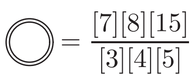


(Double edge elimination)



Using the above relations, we obtain the following additional relations.

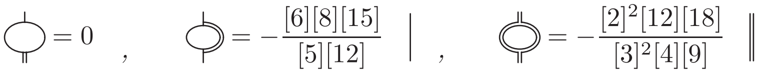
Proposition 1. *(Loop relation)*



(Monogon relation)

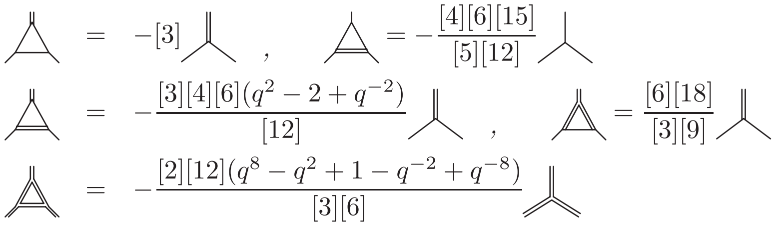
$$= 0$$

(Digon relations)



$$= 0 \quad , \quad = -\frac{[6][8][15]}{[5][12]} \quad , \quad = -\frac{[2]^2[12][18]}{[3]^2[4][9]}$$

(Triangle relations)

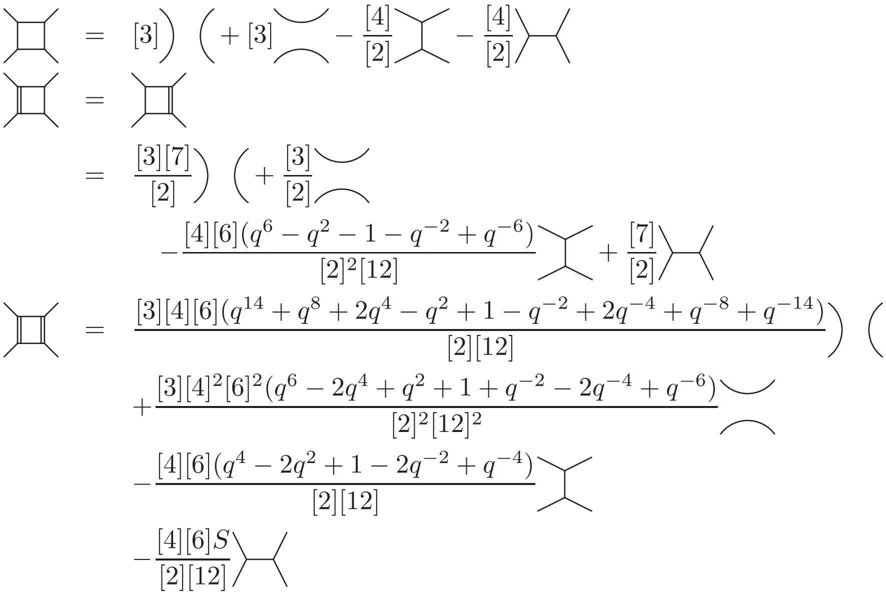


$$= -[3] \quad , \quad = -\frac{[4][6][15]}{[5][12]}$$

$$= -\frac{[3][4][6](q^2 - 2 + q^{-2})}{[12]} \quad , \quad = \frac{[6][18]}{[3][9]}$$

$$= -\frac{[2][12](q^8 - q^2 + 1 - q^{-2} + q^{-8})}{[3][6]}$$

(Square relations)

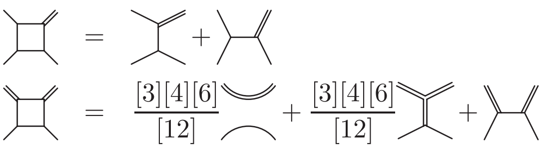


$$= [3] + [3] - \frac{[4]}{[2]} - \frac{[4]}{[2]}$$

$$= \frac{[3][7]}{[2]} + \frac{[3]}{[2]} - \frac{[4][6](q^6 - q^2 - 1 - q^{-2} + q^{-6})}{[2]^2[12]} + \frac{[7]}{[2]}$$

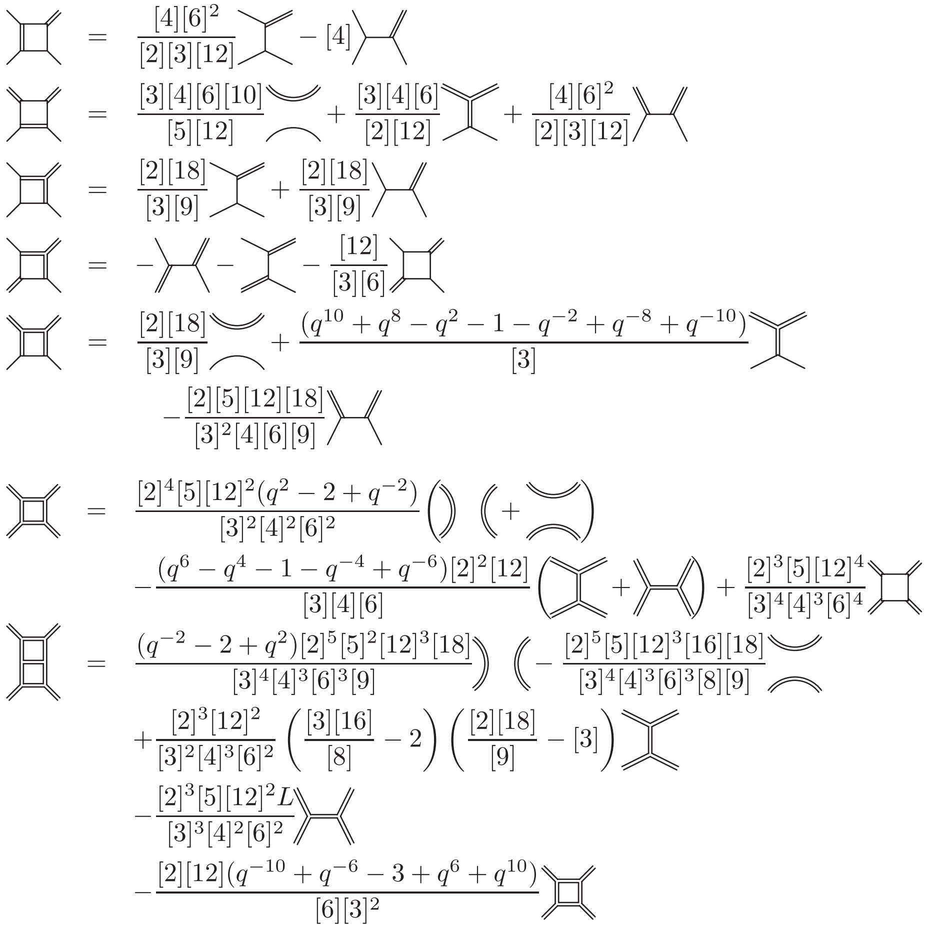
$$= \frac{[3][4][6](q^{14} + q^8 + 2q^4 - q^2 + 1 - q^{-2} + 2q^{-4} + q^{-8} + q^{-14})}{[2][12]} + \frac{[3][4]^2[6]^2(q^6 - 2q^4 + q^2 + 1 + q^{-2} - 2q^{-4} + q^{-6})}{[2]^2[12]^2} - \frac{[4][6](q^4 - 2q^2 + 1 - 2q^{-2} + q^{-4})}{[2][12]} - \frac{[4][6]S}{[2][12]}$$

where $S = q^{12} + q^{10} + q^6 - q^4 + q^2 - 1 + q^{-2} - q^{-4} + q^{-6} + q^{-10} + q^{-12}$



$$= \; + $$

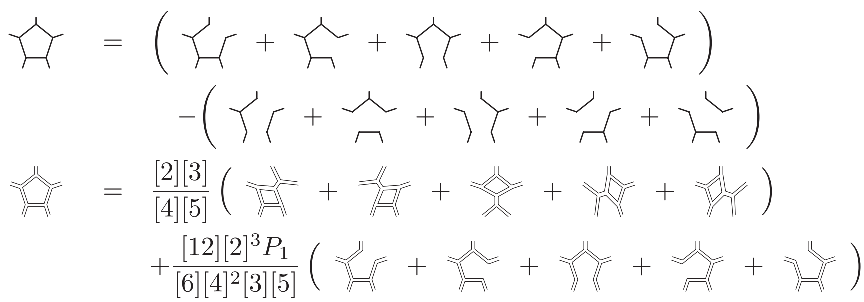
$$= \frac{[3][4][6]}{[12]} + \frac{[3][4][6]}{[12]} + $$

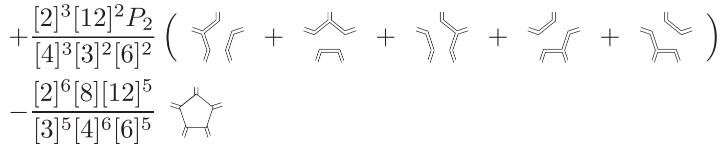


where

$$L = q^{-15} - q^{-13} - q^{-9} - 2q^{-5} + q^{-3} + q^{-1} + q + q^3 - 2q^5 - q^9 - q^{13} + q^{15}.$$

(Pentagon relation)



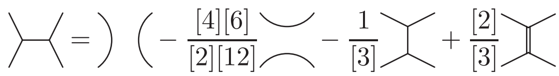


where

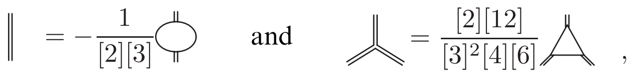
$$P_1 = \frac{2}{q^8} + \frac{1}{q^6} + \frac{1}{q^4} - \frac{2}{q^2} - 2q^2 + q^4 + q^6 + 2q^8$$

$$P_2 = \frac{1}{q^8} - \frac{1}{q^6} - \frac{2}{q^4} + 2 - 2q^4 - q^6 + q^8.$$

PROOF (Sketch of proof). Applying the relation (Double edge elimination) or its rearrangement

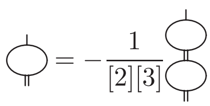


to the left-hand side of an identity in this proposition and using the relations in Definition 1, we obtain the identity. If we can not apply the elimination or the rearrangement to the left-hand side, we first create single edges on the web by using the relations

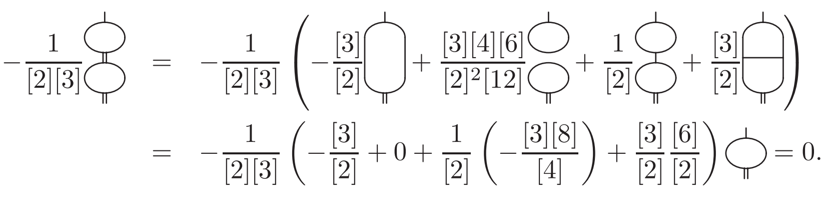


and apply the elimination or its rearrangement.

For example, the first digon relation in this proposition is obtained as follows. First, create single edges on the G_2 web:



By applying (Double edge elimination) to the right-hand side of the above identity and using monogon, digon and triangle relations in Definition 1, we obtain the following identity:



□

3. Web space W_{G_2} and invariant space of representation

In this section, we define a G_2 web space W_{G_2}, which is a $\mathbb{C}$-vector space spanned by G_2 web diagrams (G_2 webs embedded in a unit disk), where G_2 web diagrams are defined as follows.

Let D be a closed unit disk in $\mathbb{R}^2$ with a fixed base point $*$ on the boundary ∂D. A G_2 *web diagram* is the image of an embedding of a G_2 web P in D such that every univalent of P lies in $\partial D \backslash \{*\}$. We do not consider a G_2 web which can not be embedded in the disk D.

For a given G_2 web diagram W, put the number 1 at each intersection of single edges of W with ∂D and put the number 2 at each intersection on double edges of W with ∂D. A *coloring* of W is defined to be the sequence obtained by reading numbers 1 and 2 on ∂D clockwisely from the base point $*$. If W has no univalent, a coloring of W is defined to be the empty sequence $\emptyset$. Denote by $s(W)$ the coloring of W.

For example, the colorings of G_2 web diagrams in Figure 3 are given by $s(W_1) = (1,1,1,1)$, $s(W_2) = (2,1,1)$, $s(W_3) = (1,1,2,1)$, $s(W_4) = (1,2,2,1,1)$.

Two G_2 web diagrams W_1 and W_2 are *isotopic* if there exists a base point-preserving isotopy of D which moves W_1 to W_2.

Hereafter we fix a base point as G_2 web diagrams in Figure 3 and omit the boundary of the unit disk.

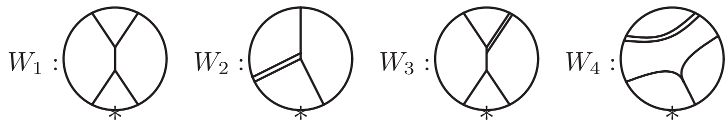


Fig. 3. G_2 web diagrams

Write

$$S := \{s = (s_1, s_2, \dots, s_n) \mid n \geq 1, s_i \in \{1,2\} \ (i = 1, 2, \dots, n)\} \cup \{\emptyset\}.$$

We define a G_2 web space $W_{G_2}(s)$ for $s \in S$ by a $\mathbb{C}$-vector space spanned by the isotopy classes of G_2 web diagrams with the coloring s, modulo the relations in Definition 1.

Remark 1. *The collection of the web spaces* $\{W_{G_2}(s)\}_{s \in S}$ *has the spider structure in the sense of Kuperberg* [6, Section 3]*:*
(Join)

$$\mu_{s,t} : W_{G_2}(s) \times W_{G_2}(t) \to W_{G_2}(st)$$

(Rotation)

$$\rho_{s,t} : W_{G_2}(st) \to W_{G_2}(ts)$$

(Stitch)

$$\sigma_{sst} : \mathrm{W}_{G_2}(sst) \to \mathrm{W}_{G_2}(t).$$

For $s = (s_1, s_2, \ldots, s_n) \in S$, let V_s be the tensor representation of G_2 quantum group $V_{\varpi_{s_1}} \otimes V_{\varpi_{s_2}} \otimes \cdots V_{\varpi_{s_n}}$, where $V_{\varpi_{s_i}}$ is the s_i-th fundamental representation $(i = 1, \ldots, n)$.

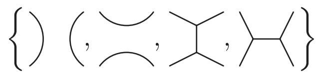
The following theorem is due to [6, Theorem 6.10].

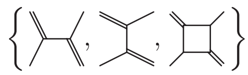
Theorem 1 ([6]). *The vector spaces* $\mathrm{W}_{G_2}(s)$ *and the invariant space* $\mathrm{Inv}(V_s)$ *have the same dimension.*

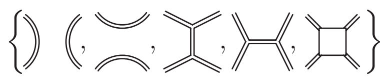
Proof. Replacing numbers 2 in the coloring s into $[1,1]$, we obtain a clasp sequence C (see [6]). Since the web space $\mathrm{W}_{G_2}(s)$ and the clasp web space $\mathrm{W}_{G_2}(C)$ have the same dimension, we obtain the theorem. □

We denote by $\mathrm{B}(s)$ a basis of the vector space $\mathrm{W}_{G_2}(s)$, called a *G_2 web basis*.

Example 1. *For* $s = (1,1,1,1)$, $(1,2,1,2)$ *and* $(2,2,2,2)$, *we have a* G_2 *web basis* $\mathrm{B}(s)$.

$\mathrm{B}(1,1,1,1) =$ 

$\mathrm{B}(1,2,1,2) =$ 

$\mathrm{B}(2,2,2,2) =$ 

4. Crossing formula in the G_2 web space

Let $\#(s)$ be the length of a sequence $s \in S$, and define

$$S[n] := \{s \in S \mid \#(s) = n\}.$$

We define an action of the braid group

$$B_n = \left\langle b_i \ (1 \le i \le n-1) \;\middle|\; \begin{array}{ll} b_i b_j = b_j b_i & (|i-j| > 1), \\ b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} & (1 \le i \le n-2) \end{array} \right\rangle$$

on the collection of the web spaces

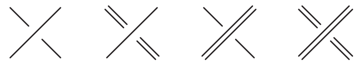
$$\{\mathrm{W}_{G_2}(s)\}_{s \in S[n]}.$$

For each $s = (s_1, s_2 \ldots, s_n) \in S[n]$, we define an action of the braid group B_n on the representation V_s, where V_s is the tensor representation $V_{\varpi_{s_1}} \otimes V_{\varpi_{s_2}} \otimes \cdots \otimes$

$V_{\varpi_{s_n}}$, by setting $\rho_s(b_i)$ to be the invertible intertwiner composed of R-matrix

$$\mathrm{Id}_{V_{\varpi_{s_1}}\otimes\cdots\otimes V_{\varpi_{s_{i-1}}}} \otimes R_{s_i s_{i+1}} \otimes \mathrm{Id}_{V_{\varpi_{s_{i+2}}}\otimes\cdots\otimes V_{\varpi_{s_n}}} \in \mathrm{Hom}_{U_q(G_2)}(V_s, V_{\sigma_i(s)}),$$

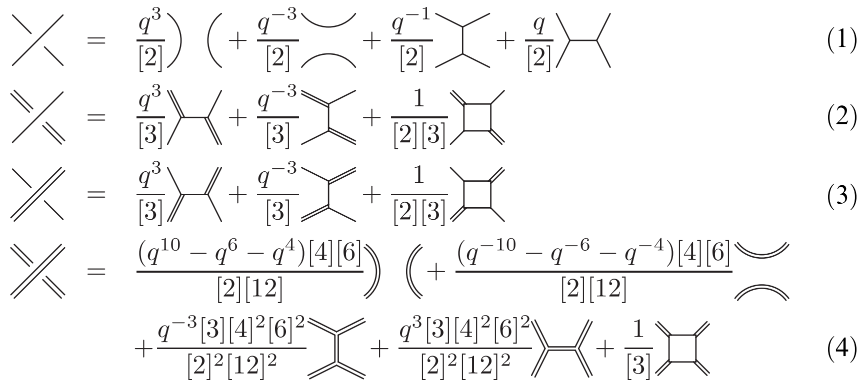
where σ_i is the transposition between i-th and $(i+1)$-th entries. We represent the R-matrices R_{11}, R_{12}, R_{21} and R_{22} by the following crossing diagrams



and represent the inverse R_{11}^{-1}, R_{12}^{-1}, R_{21}^{-1} and R_{22}^{-1} by the diagram obtained by operating $\frac{\pi}{2}$-rotation on the above crossing diagrams.

The vector space $\mathrm{Hom}_{U_q(G_2)}(V_s, V_{s'})$ $(s, s' \in S)$ is isomorphic to the invariant space $\mathrm{Inv}(V_{w(s)} \otimes V_{s'})$, where $w(s)$ is the sequence obtained by reversing the order of the elements in the sequence s. By Theorem 1, each of the above crossing diagrams has a description as a linear sum of the G_2 web diagrams.

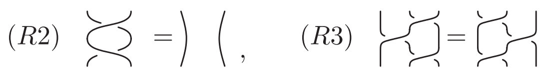
Theorem 2. *The four types of crossings corresponding to the R-matrices have the following descriptions in the G_2 web diagrams:*



Proof. The crossing diagram on the left-hand side of Identity (1) corresponds to the R-matrix R_{11} in $\mathrm{End}_{U_q(G_2)}(V_{(1,1)})$. The vector space $\mathrm{End}_{U_q(G_2)}(V_{(1,1)})$ is isomorphic to the invariant space $\mathrm{Inv}(V_{(1,1,1,1)})$ and, by Theorem 1, is isomorphic to the web space $W_{G_2}(1,1,1,1)$. Therefore, the crossing diagram corresponding to R_{11} is expressed by a linear sum of G_2 web basis $B(1,1,1,1)$. That is, Identity (1) is the expression of the crossing diagram by G_2 web diagrams. The crossing formula about the R-matrix R_{11}^{-1} is the identity obtained by operating the $\frac{\pi}{2}$-rotation on each diagram in Identity (1). Other Identities (2), (3) and (4) are also the expressions of the crossing diagrams corresponding to the R-matrices R_{12}, R_{21} and R_{22} by G_2 web diagrams.

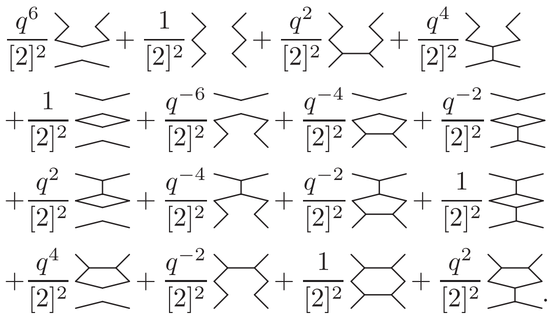
The 2nd and 3rd Reidemeister moves are corresponding to R-matrix invertibility and the Yang-Baxter equation. We have to check that Identities (1), (2), (3) and (4) are well-defined. That is, we need to check that crossing diagrams related by the 2nd and 3rd Reidemeister moves are identical as elements of the G_2 web space, using Identities (1), (2), (3) and (4).

For crossing diagrams with only single edges, it is enough to prove the following identities by the 2nd and 3rd Reidemeister moves with only single edges:



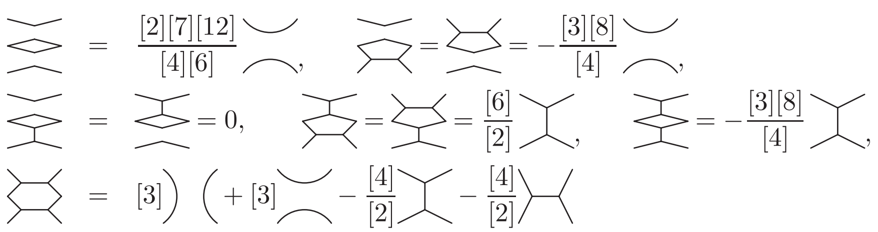
Other identities by the 2nd and 3rd Reidemeister moves including double edge can be obtained by Identities (R2), (R3) and relations in Definition 1.

<u>Proof of Identity (R2):</u> By Identity (1), the left-hand side of Identity (R2) is equal to



(5)

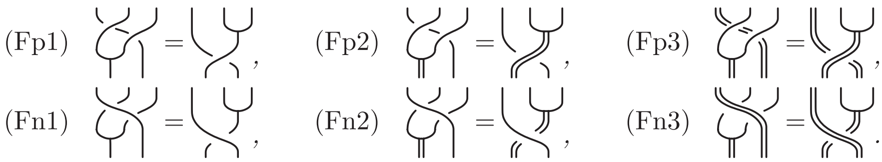
Using relations in Definition 1 and Proposition 1, we have the following identities:



By these identities, we find that the linear sum (5) is equal to the right-hand side of Identity (R2).

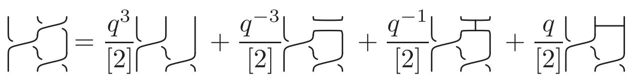
The following lemma is helpful to prove other identities by Reidemeister moves.

Lemma 1. *We have the following identities:*

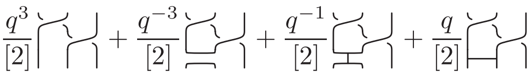


This lemma is proved in Appendix A.

<u>Proof of Identity (R3):</u> By Identity (1), we have the following identity.

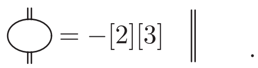


By Identity (R2) and Lemma (1) (Fp1), the right-hand side is equal to the following:

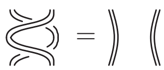


By Identity (1), this linear sum is equal to the right-hand side of Identity (R3).

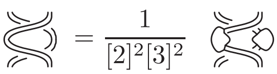
The invariance of crossing diagrams including double edges, as elements of the G_2 web space, by the Reidemeister moves can be proved by using Identities (R2) and (R3), Lemma 1 and the following digon relation



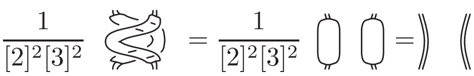
Here, we prove the following identity in the G_2 web space, corresponding to the invertibility of R_{22} in $\mathrm{End}_{U_q(G_2)}(V_{(2,2)})$:

 (6)

By the digon relation, the left-hand side is equal to



and, by Lemma 1 (Fp2), Identity (R2) and the digon relation, this is equal to



Proofs for the identities corresponding to the remaining Reidemeister moves containing double edges can be done in a similar way. □

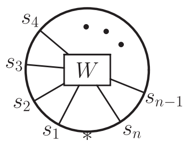
5. Braid action on the G_2 web space W_{G_2}

Using the crossing formulas in Theorem 2, we can define an action of the braid group B_n on $\mathrm{W}_{G_2}[n] = \bigoplus_{s \in S[n]} \mathrm{W}_{G_2}(s)$.

The action of the i-th generators b_i and b_i^{-1} of B_n $(i = 1, \ldots, n-1)$ is defined as follows. For a G_2 web diagram $W \in \mathrm{W}_{G_2}(s) \subset \mathrm{W}_{G_2}[n]$, $b_i(W)$ is the element of $\mathrm{W}_{G_2}(\sigma_i(s))$, where σ_i is the transposition of i-th and $(i+1)$-th entries, obtained from W by gluing the (s_i, s_{i+1})-boundary of W and the positive crossing (as the s_i-univalent of W connects to the over arc of the crossing). Similarly, $b_i^{-1}(W)$ is the element of $\mathrm{W}_{G_2}(\sigma_i(s))$ obtained from W by gluing the (s_i, s_{i+1})-boundary of W and the negative crossing (as the s_i-univalent of W connects to the under arc of the crossing). Then, we replace the obtained knotted diagram into the linear sum of G_2 web diagrams by the formulas in Theorem 2.

In other words, we regard the action of generators b_i and b_i^{-1} as positive and negative crossings and univalents on the unit disc with a hole in Figure 4.

For a G_2 web diagram W diagrammatically denoted by



,

the action of the generators b_i (resp. b_i^{-1}) amounts to putting the diagram W into the hole of the diagram of b_i (resp. b_i^{-1}) in Figure 4 and gluing these diagrams.

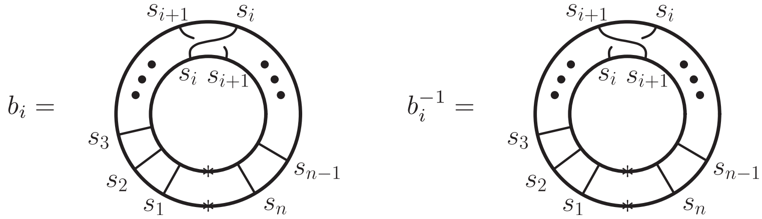
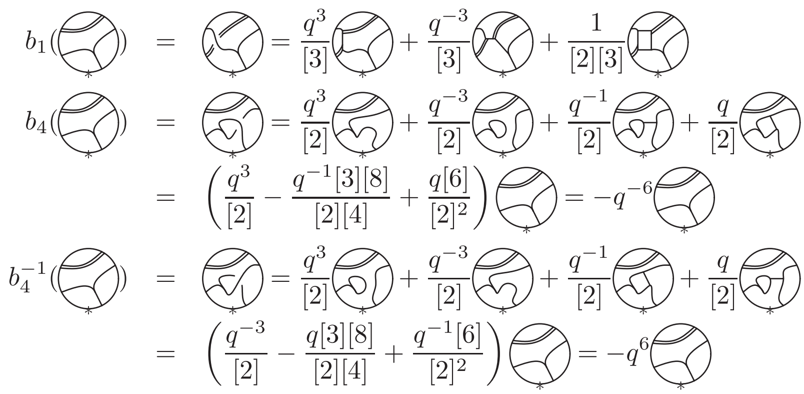


Fig. 4. Diagrammatic description of generator action

For example, the B_5 action on the G_2 web space $\mathrm{W}_{G_2}(1, 2, 2, 1, 1)$ is described as follows. To the G_2 web diagram W_4 in Figure 3, the actions of $b_1, b_4, b_4^{-1} \in B_5$ are given by:



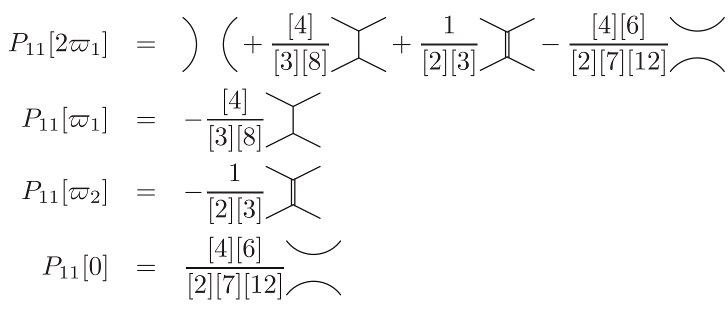
6. Relation to idempotents and R-matrix of other irreducible representations

In this section, we describe a relation between G_2 web diagrams and idempotents in the hom set $\mathrm{Hom}_{U_q(G_2)}(V_{\varpi_i} \otimes V_{\varpi_j}, V_{\varpi_j} \otimes V_{\varpi_i})$, where ϖ_i and ϖ_j are fundamental weights. Using the idempotents, we construct the crossing formulas for the R-matrices associated to other irreducible representations.

Let $P_{11}[\varpi]$ be the idempotent in $\mathrm{End}_{U_q(G_2)}(V_{\varpi_1}^{\otimes 2})$ which factors through the irreducible representation with the highest weight ϖ. Note that the idempotents satisfy

$$P_{11}[\varpi]P_{11}[\varpi'] = \delta_{\varpi,\varpi'}P_{11}[\varpi].$$

By Theorem 1, $\mathrm{End}_{U_q(G_2)}(V_{\varpi_1}^{\otimes 2})$ is isomorphic to the web space $W_{G_2}(1,1,1,1)$. Therefore, we have the following identities which express idempotents $P_{11}[\varpi]$ by linear sums of G_2 web diagrams:



By these identities and Identity (1), the R-matrix R_{11} is expressed by a linear sum of idempotents $P_{11}[\varpi]$ as follows:

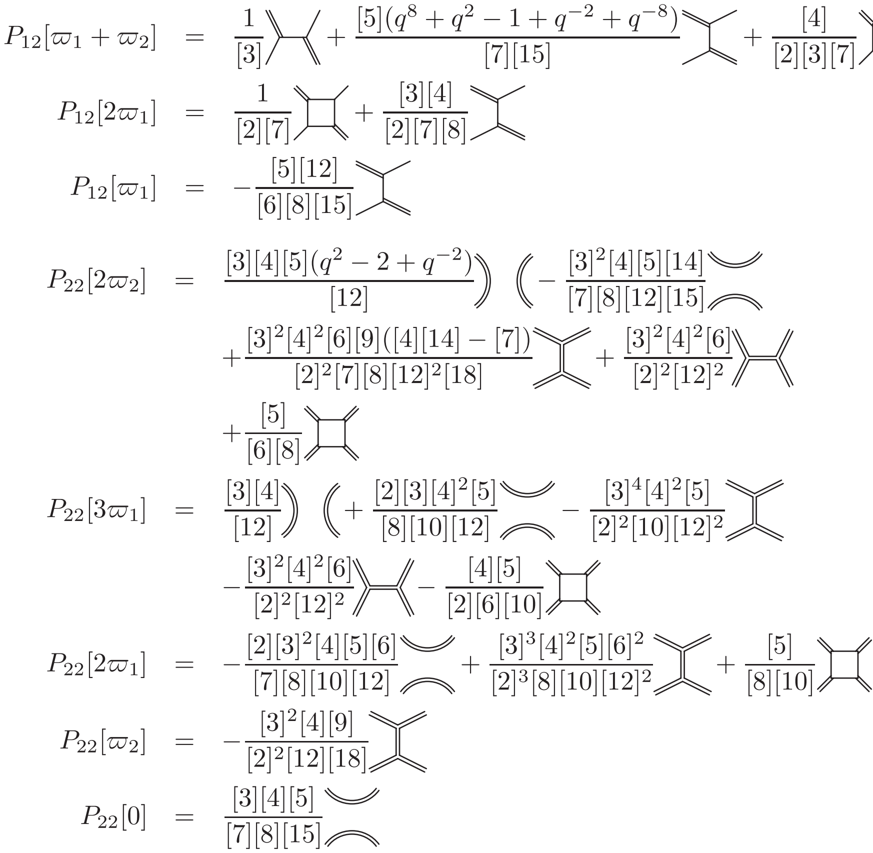
$$R_{11} = q^2 P_{11}[2\varpi_1] - q^{-6}P_{11}[\varpi_1] - P_{11}[\varpi_2] + q^{-12}P_{11}[0].$$

(This identity can be found in [7, Sec. 8.1.1].)

Let $P_{ij}[\varpi]$, $i, j \in \{1, 2\}$, be the idempotent in $\mathrm{Hom}_{U_q(G_2)}(V_{\varpi_i} \otimes V_{\varpi_j}, V_{\varpi_j} \otimes V_{\varpi_i})$ which factors through the representation V_{ϖ}. Note that the idempotents satisfy

$$P_{12}[\varpi]P_{21}[\varpi']P_{12}[\varpi] = \delta_{\varpi,\varpi'}P_{12}[\varpi], \qquad P_{22}[\varpi]P_{22}[\varpi'] = \delta_{\varpi,\varpi'}P_{22}[\varpi].$$

We have the following identities which express the idempotents $P_{ij}[\varpi]$ by linear sums of G_2 web diagrams:



$$P_{12}[\varpi_1+\varpi_2] = \frac{1}{[3]}\cdots + \frac{[5](q^8+q^2-1+q^{-2}+q^{-8})}{[7][15]}\cdots + \frac{[4]}{[2][3][7]}\cdots$$
$$P_{12}[2\varpi_1] = \frac{1}{[2][7]}\cdots + \frac{[3][4]}{[2][7][8]}\cdots$$
$$P_{12}[\varpi_1] = -\frac{[5][12]}{[6][8][15]}\cdots$$
$$P_{22}[2\varpi_2] = \frac{[3][4][5](q^2-2+q^{-2})}{[12]}\cdots\Big(-\frac{[3]^2[4][5][14]}{[7][8][12][15]}\cdots + \frac{[3]^2[4]^2[6][9]([4][14]-[7])}{[2]^2[7][8][12]^2[18]}\cdots + \frac{[3]^2[4]^2[6]}{[2]^2[12]^2}\cdots + \frac{[5]}{[6][8]}\cdots$$
$$P_{22}[3\varpi_1] = \frac{[3][4]}{[12]}\cdots\Big(+\frac{[2][3][4]^2[5]}{[8][10][12]}\cdots - \frac{[3]^4[4]^2[5]}{[2]^2[10][12]^2}\cdots - \frac{[3]^2[4]^2[6]}{[2]^2[12]^2}\cdots - \frac{[4][5]}{[2][6][10]}\cdots$$
$$P_{22}[2\varpi_1] = -\frac{[2][3]^2[4][5][6]}{[7][8][10][12]}\cdots + \frac{[3]^3[4]^2[5][6]^2}{[2]^3[8][10][12]^2}\cdots + \frac{[5]}{[8][10]}\cdots$$
$$P_{22}[\varpi_2] = -\frac{[3]^2[4][9]}{[2]^2[12][18]}\cdots$$
$$P_{22}[0] = \frac{[3][4][5]}{[7][8][15]}\cdots$$

The idempotent $P_{21}[\varpi]$ is equal to the linear sum obtained by operating (left-right) symmetry on each diagram of the right-hand side of $P_{12}[\varpi]$. In other words,

$$P_{21}[\varpi] = R_{21}P_{12}[\varpi]R_{12}^{-1}.$$

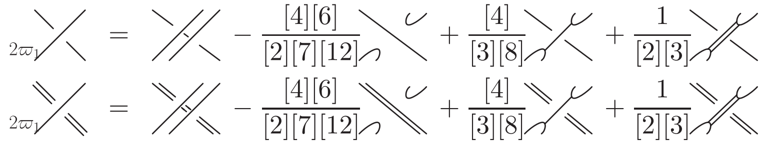
The R-matrices $R_{12} \in \mathrm{Hom}_{U_q(G_2)}(V_{\varpi_1} \otimes V_{\varpi_2}, V_{\varpi_2} \otimes V_{\varpi_1})$, $R_{21} \in \mathrm{Hom}_{U_q(G_2)}(V_{\varpi_2} \otimes V_{\varpi_1}, V_{\varpi_1} \otimes V_{\varpi_2})$ and $R_{22} \in \mathrm{End}_{U_q(G_2)}(V_{\varpi_2}^{\otimes 2})$ are expressed by the following linear sums of idempotents.

$$R_{12} = q^3 P_{12}[\varpi_1 + \varpi_2] + q^{-4} P_{12}[2\varpi_1] - q^{-12} P_{12}[\varpi_1]$$

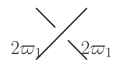
$$R_{21} = q^3 P_{21}[\varpi_1 + \varpi_2] + q^{-4} P_{21}[2\varpi_1] - q^{-12} P_{21}[\varpi_1]$$

$$R_{22} = q^6 P_{22}[2\varpi_2] - P_{22}[3\varpi_1] + q^{-10} P_{22}[2\varpi_1] - q^{-12} P_{22}[\varpi_2] + q^{-24} P_{22}[0],$$

Moreover, using the identities for the idempotents, we obtain crossing formulas for the R-matrices associated to other irreducible representations. For example, using the identity for $P_{11}[2\varpi_1]$, we obtain the following formulas for the R-matrices in $\mathrm{Hom}_{U_q(G_2)}(V_{2\varpi_1} \otimes V_{\varpi_1}, V_{\varpi_1} \otimes V_{2\varpi_1})$ and $\mathrm{Hom}_{U_q(G_2)}(V_{2\varpi_1} \otimes V_{\varpi_2}, V_{\varpi_1} \otimes V_{2\varpi_2})$



We also have a crossing formula which expresses the following crossing corresponding to the R-matrix in $\mathrm{End}_{U_q(G_2)}(V_{2\varpi_1}^{\otimes 2})$

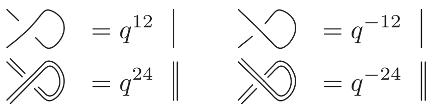


by a linear sum of 16 diagrams. Similarly, we have crossing formulas with colorings $\varpi_1 + \varpi_2$, $2\varpi_2$ and $3\varpi_1$ by using the idempotents $P_{12}[\varpi_1 + \varpi_2]$, $P_{22}[2\varpi_2]$ and $P_{22}[3\varpi_1]$.

An open problem is to construct the idempotents which factor through other irreducible representations as linear sums of G_2 web diagrams. If this problem is solved, we can explicitly construct crossing formulas for the R-matrix associated to other irreducible representations of $U_q(G_2)$ as above.

7. G_2 quantum invariant of generalized twist link

We can obtain the following evaluations of positive and negative crossings curls (diagrams in Reidemeister move 1) by using the crossing formulas (1) and (4) in Theorem 2.



Therefore, to obtain G_2 quantum invariant of an oriented link, we need to normalize the crossing formulas in Theorem 2.

Let L be an oriented link with k components $(L_1, L_2, \dots, L_k)$, and let $D = (D_1, D_2, \dots, D_k)$ be an unoriented link diagram of L. Using the crossing formulas, we define the polynomial evaluation for a link diagram D, denoted by $\langle D \rangle_{(\varpi_{i_1}, \varpi_{i_2}, \dots, \varpi_{i_k})}$, $i_j \in \{1, 2\}$ and $j = 1, \dots, k$, as follows: First, replace each component D_j with the double line of G_2 web diagram if $\varpi_{i_j} = \varpi_2$. (We regard D_j as the single line of G_2 web diagram if $\varpi_{i_j} = \varpi_1$.) Next, apply the crossing formulas in Theorem 2 to all crossings of the replaced diagram of D. The polynomial $\langle D \rangle_{(\varpi_{i_1}, \varpi_{i_2}, \dots, \varpi_{i_k})}$ is defined to be the polynomial which is the evaluation of the above linear sum of G_2 web diagrams by using the relations in Definition 1 and Proposition 1.

Theorem 3. *For an oriented link L,*

$$(q^{-12})^{\omega_{11}(D)}(q^{-24})^{\omega_{22}(D)}\langle D \rangle_{(\varpi_{i_1}, \varpi_{i_2}, \dots, \varpi_{i_k})}$$

is a link invariant of L, where D is a link diagram of L and $\omega_{11}(D)$ (resp. $\omega_{22}(D)$) is the number of positive crossings of single edge on D minus the number of negative crossings of single edge (resp. the number of positive crossings of double edge minus the number of negative crossings of double edge).

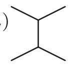
The link invariant is Reshetikhin-Turaev's quantum link invariant associated to the $U_q(G_2)$ fundamental representations, called G_2 *quantum invariant* for short. Denote by $P_{(\varpi_{i_1}, \varpi_{i_2}, \dots, \varpi_{i_k})}(L)$ the G_2 quantum invariant of an oriented link L.

In the following, we determine the G_2 quantum invariant of the generalized twist link $TW(m,n)$ in Figure 5. The box of $TW(m,n)$ is the tangle diagram with n-crossing in Figure 6. Denote by $Cr(n)$ the box illustrated in Figure 6.

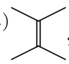
The evaluation $\langle Cr(n) \rangle_{(\varpi_1, \varpi_1)}$ is given by the following formula:

$$\begin{aligned}\langle Cr(n) \rangle_{(\varpi_1, \varpi_1)} &= (q^2 P_{11}[2\varpi_1] - q^{-6} P_{11}[\varpi_1] - P_{11}[\varpi_2] + q^{-12} P_{11}[0])^n \\ &= q^{2n}\Big) \Big(+ A_{11}^{(n)}\end{aligned}$$

$$+ B_{11}^{(n)}$$



$$+ C_{11}^{(n)}$$



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where

$$A_{11}^{(n)} = \frac{[4][6]}{[2][7][12]}(-q^{2n} + q^{-12n}), \qquad B_{11}^{(n)} = \frac{[4]}{[3][8]}(q^{2n} - (-q^{-6})^n),$$

$$C_{11}^{(n)} = \frac{1}{[2][3]}(q^{2n} - (-1)^n).$$

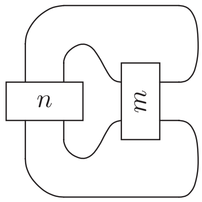


Fig. 5. Generalized twist link $TW(m,n)$

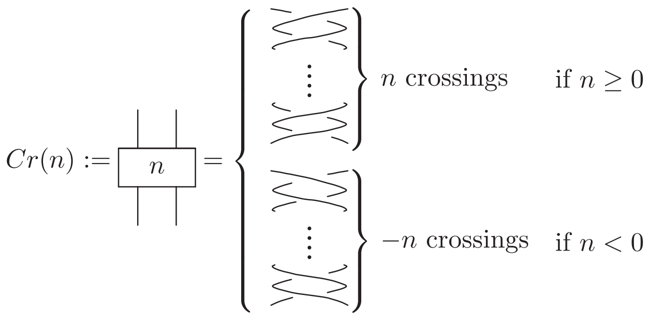


Fig. 6. n-crossing

Using the above evaluation of $\langle Cr(n)\rangle_{(\varpi_1,\varpi_1)}$, we obtain the following:

$$\begin{aligned}
&P_{(\varpi_1,\varpi_1)}(TW(m,n)) = q^{-12\omega(TW(m,n))}\langle TW(m,n)\rangle_{(\varpi_1,\varpi_1)} \\
= \ & q^{-12\omega(TW(m,n))}\left\{q^{2n-12m}\frac{[2][7][12]}{[4][6]}\right. \\
&+A_{11}^{(n)}\frac{[2][7]}{[4]}\left(q^{2m}\frac{[2][7][12]^2}{[4][6]^2}+A_{11}^{(m)}\frac{[12]}{[6]}-B_{11}^{(m)}\frac{[3][8][12]}{[4][6]}-C_{11}^{(m)}\frac{[8][15]}{[5]}\right) \\
&+B_{11}^{(n)}\frac{[3][7][8]}{[4]}\left(-A_{11}^{(m)}\frac{[2][12]}{[4][6]}-B_{11}^{(m)}\frac{[12]}{[4]}+C_{11}^{(m)}\frac{[2][15]}{[5]}\right) \\
&\left.+C_{11}^{(n)}\frac{[2][7][8][15]}{[5]}\left(-A_{11}^{(m)}\frac{1}{[4]}+B_{11}^{(m)}\frac{[3]}{[4]}+C_{11}^{(m)}\frac{[3][6]}{[12]}(q^2-2+q^{-2})\right)\right\}.
\end{aligned}$$

Similarly, when m and n are odd integers, we obtain the following:

$$\begin{aligned}
&P_{(\varpi_1,\varpi_2)}(TW(m,n)) = \langle TW(m,n)\rangle_{(\omega_1,\omega_2)} \\
= \ & A_{12}^{(n)}\frac{[2][6][7][8][15]}{[5][12]}\left(A_{12}^{(m)}[3](q^2-2+q^{-2})-B_{12}^{(m)}\frac{[3][15]}{[5]}+C_{12}^{(m)}\frac{[8][15]}{[4][5]}\right) \\
&-B_{12}^{(n)}\frac{[2][3][6][7][8][15]}{[5][12]}\left(A_{12}^{(m)}\frac{[15]}{[5]}+B_{12}^{(m)}[3]T-C_{12}^{(m)}\frac{[15]}{[5]}\right) \\
&+C_{12}^{(n)}\frac{[2][6][7][8][15]}{[5][12]}\left(A_{12}^{(m)}\frac{[8][15]}{[4][5]}-B_{12}^{(m)}\frac{[3][15]}{[5]}+C_{12}^{(m)}[3](q^2-2+q^{-2})\right),
\end{aligned}$$

where

$$T = q^{-12} - q^{-8} + 2q^{-6} + q^{-4} - q^{-2} + 1 - q^2 + q^4 + 2q^6 - q^8 + q^{12},$$

$$A_{12}^{(m)} = -(-q^{-12})^m \frac{[5][12]}{[6][8][15]} + q^{-4m}\frac{[3][4]}{[2][7][8]} + q^{3m}\frac{[3][5]([4][14]+[7])}{[7]^2[15]},$$

$$B_{12}^{(m)} = q^{-4m}\frac{1}{[2][7]} + q^{3m}\frac{[4]}{[2][3][7]}, \quad C_{12}^{(m)} = q^{3m}\frac{1}{[3]}.$$

When $m, n \in \mathbb{Z}$, we obtain the following:

$$\begin{aligned}
&P_{(\varpi_2,\varpi_2)}(TW(m,n)) \\
= \;& q^{-24\omega(TW(m,n))}\langle TW(m,n)\rangle_{(\omega_2,\omega_2)} := q^{-24\omega(TW(m,n))}\Bigg\{A_{22}^{(n)} q^{-24m}\frac{[7][8][15]}{[3][4][5]} \\
&+B_{22}^{(n)}\frac{[7][8][15]}{[4][5]}\left(A_{22}^{(m)}\frac{[7][8][15]}{[3]^2[4][5]} + B_{22}^{(m)}\frac{1}{[3]} - C_{22}^{(m)}\frac{[2]^2[12][18]}{[3]^3[4][9]} + E_{22}^{(m)}\frac{[2][6][8][15]}{[5][12]}\right) \\
&-C_{22}^{(n)}\frac{[2][7][8][15][18]}{[3]^2[4][5][9]}\left(B_{22}^{(m)}\frac{[2][12]}{[3][4]} + C_{22}^{(m)}\frac{[2]^2[12]^2([3][18]-[2][9])}{[3]^2[4]^2[6][9]}\right. \\
&\qquad\qquad\left. + D_{22}^{(m)}\frac{[2]^3[12]^2[18]}{[3]^3[4]^2[9]} - E_{22}^{(m)}\frac{[6]^2[18]}{[9]}\right) \\
&-D_{22}^{(n)}\frac{[2][7][8][15][18]}{[3]^2[4][5][9]}\left(A_{22}^{(m)}\frac{[2][12]}{[3][4]} + C_{22}^{(m)}\frac{[2]^3[12]^2[18]}{[3]^3[4]^2[9]}\right. \\
&\qquad\qquad\left. + D_{22}^{(m)}\frac{[2]^2[12]^2([3][18]-[2][9])}{[3]^2[4]^2[6][9]} - E_{22}^{(m)}\frac{[6]^2[18]}{[9]}\right) \\
&+E_{22}^{(n)}\frac{[2][6][7][8][15]}{[5]}\left(A_{22}^{(m)}\frac{[8][15]}{[4][5][12]} + B_{22}^{(m)}\frac{[8][15]}{[4][5][12]} - C_{22}^{(m)}\frac{[6][18]^2}{[3]^2[4][9]^2}\right. \\
&\qquad\qquad\left. - D_{22}^{(m)}\frac{[6][18]^2}{[3]^2[4][9]^2} + E_{22}^{(m)}\frac{[3][4][6]^2[7][8][15]}{[5][12]^2}U\right)\Bigg\},
\end{aligned}$$

where

$$\begin{aligned}
U &= \frac{[35]}{[7]} + 2\frac{[3][25]}{[5]} + \frac{[8][21]}{[4][7]} + 2\frac{[2][8][10]}{[4][5]} + 3, \\
A_{22}^{(m)} &= \frac{[3][4]}{[12]}\left(q^{6m}[5](q^2 - 2 + q^-2) + (-1)^m\right), \\
B_{22}^{(m)} &= \frac{[3][4][5]}{[8]}\left(-q^{6m}\frac{[3][14]}{[7][12][15]} + (-1)^m\frac{[2][4]}{[10][12]}\right. \\
&\qquad\left. - q^{-10m}\frac{[2][3][6]}{[7][10][12]} + q^{-24m}\frac{1}{[7][15]}\right), \\
C_{22}^{(m)} &= \frac{[3]^2[4]}{[2]^2[12]}\left(q^{6m}\frac{[4][6][9]([4][14]-[7])}{[7][8][12][18]} - (-1)^m\frac{[3]^2[4][5]}{[10][12]}\right. \\
&\qquad\left. + q^{-10m}\frac{[3][4][5][6]^2}{[2][8][10][12]} - (-q^{-12})^m\frac{[9]}{[18]}\right), \\
D_{22}^{(m)} &= (q^{6m} - (-1)^m)\frac{[3]^2[4]^2[6]}{[2]^2[12]^2}, \\
E_{22}^{(m)} &= q^{6m}\frac{[5]}{[6][8]} - (-1)^m\frac{[4][5]}{[2][6][10]} + q^{-10m}\frac{[5]}{[8][10]}.
\end{aligned}$$

REMARK 2. (1) If either (i) m and n are even or (ii) m is even and n is odd, then $\omega(TW(m,n)) = m - n$. If m is odd and n is even, then $\omega(TW(m,n)) = n - m$. If n and m are odd, $TW(m,n)$ is a link. Therefore the number $\omega(TW(m,n))$ is $n - m$ or $m - n$.

(2) Since $TW(0,0)$ is the 2-component trivial link, $P_{(\omega_1,\omega_1)}(TW(0,0)) = \frac{[2]^2[7]^2[12]^2}{[4]^2[6]^2}$ and $P_{(\omega_2,\omega_2)}(TW(0,0)) = \frac{[7]^2[8]^2[15]^2}{[3]^2[4]^2[5]^2}$.

(3) Since $TW(m,0)$ and $TW(0,m)$ $(m \leq -1,\ 1 \leq m)$ are the trivial knot, we have $P_{(\omega_1,\omega_1)}(TW(m,0)) = P_{(\omega_1,\omega_1)}(TW(0,m)) = \frac{[2][7][12]}{[4][6]}$ and $P_{(\omega_2,\omega_2)}(TW(m,0)) = P_{(\omega_2,\omega_2)}(TW(0,m)) = \frac{[7][8][15]}{[3][4][5]}$.

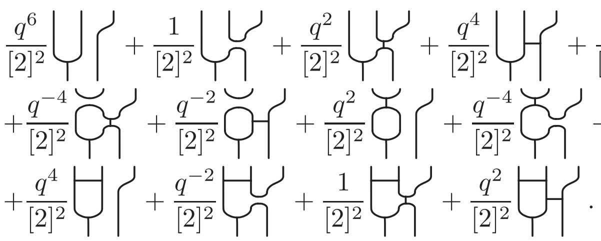
(4) Since $TW(-1,n-1)$, $TW(n-1,-1)$, $TW(1,n+1)$ and $TW(n+1,1)$ are the $(2,n)$-torus link, we find these G_2 link invariant associated to the fundamental representations are the same evaluation.

(5) By the up-down symmetry of the generalized twist link $TW(m,n)$, we have $P_{(\omega_2,\omega_1)}(TW(m,n)) = P_{(\omega_1,\omega_2)}(TW(m,n))$.

Appendix A. Proof of Lemma 1

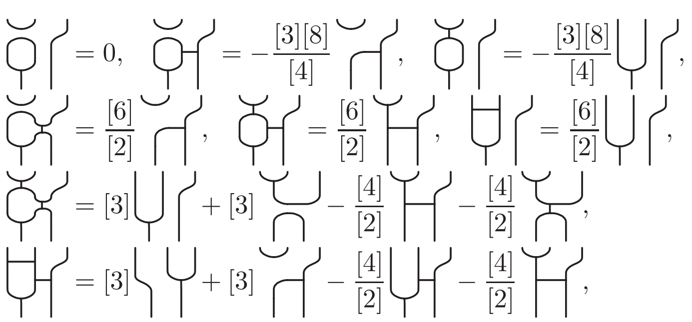
Here, we give proofs of Lemma 1 (Fp1), (Fp2) and (Fp3). (The proofs of Lemma 1 (Fn1), (Fn2) and (Fn3) are similar.)

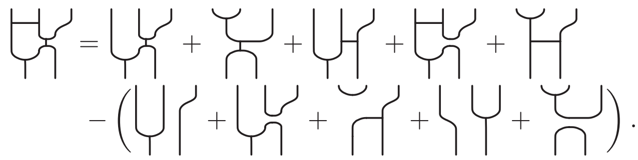
Proof of Lemma 1 (Fp1): By Identity (1), the left-hand side of (Fp1) is equal to



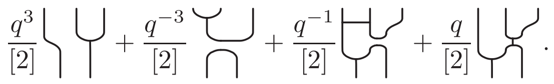
(7)

Using relations in Section 2, we have the following identities:



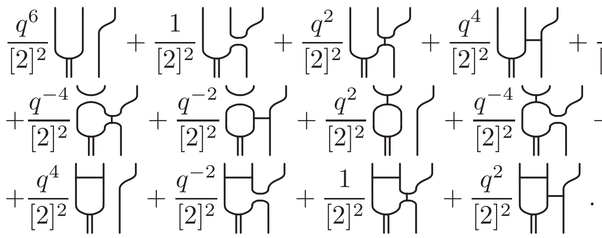


By these identities, the linear sum (7) is equal to



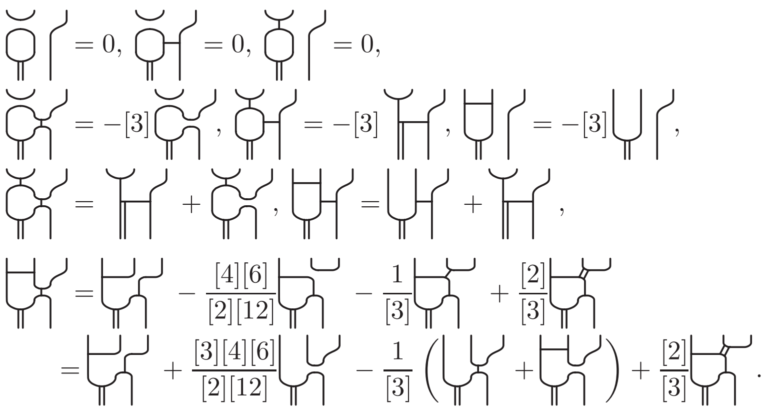
We see, by Identity (1), that this is equal to the right-hand side of Identity (Fp1).

<u>Proof of Lemma 1 (Fp2):</u> By Identity (1), the left-hand side of (Fp2) is equal to

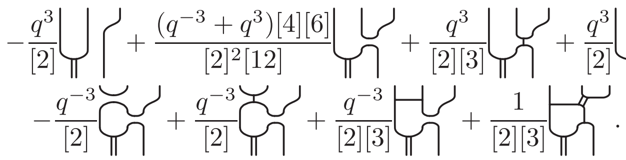


(8)

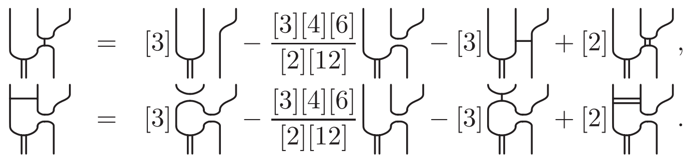
Using relations in Section 2, we have the following identities:



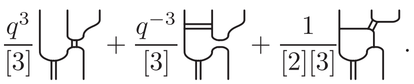
By these identities, the linear sum (8) is equal to



By Relation (Double edge elimination) in Definition 1, we have the following identities:

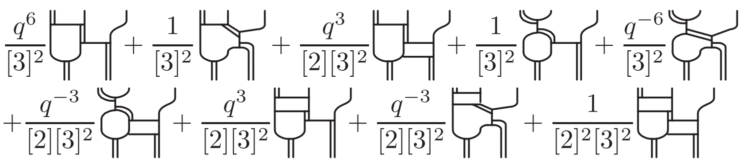


Therefore, the linear sum (8) is equal to



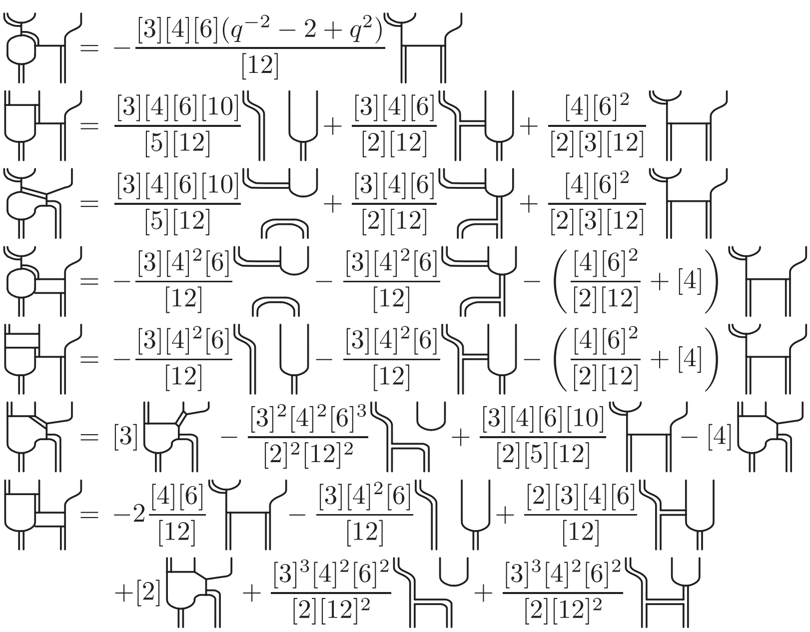
We see, by Identity (3), that this is equal to the right-hand side of Identity (Fp2).

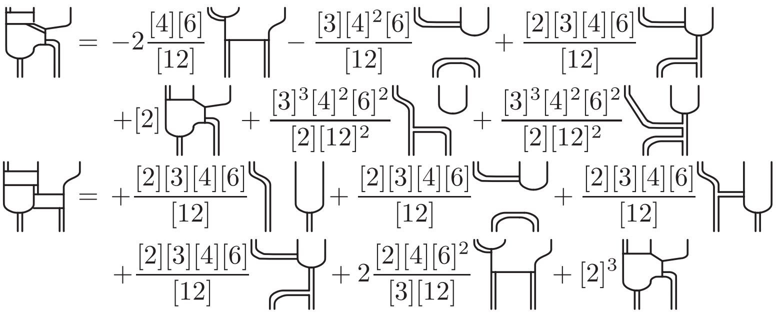
<u>Proof of Lemma 1 (Fp3):</u> By Identity (2), the left-hand side of (Fp3) is equal to



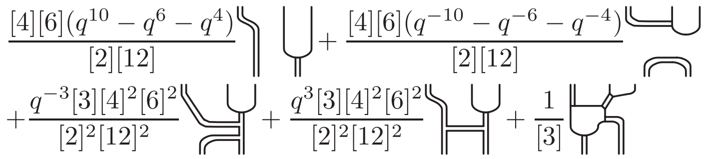
(9)

Using relations in Section 2, we have the following identities:





Therefore, the linear sum (9) is equal to



We see, by Identity (4), that this is equal to the right-hand side of Identity (Fp3).

Acknowledgements

We thank Greg Kuperberg for comments on the earlier version of our paper. We also thank Takahiro Hayashi, Tomoki Nakanishi, Soichi Okada and Kenichi Shimizu for comments on our work and Makoto Sakuma for carefully reading the manuscript. The second author thanks Yoshiyuki Kimura and Jun Murakami for helpful discussions.

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