

## Link invariant and $G_2$ web space

Takuro SAKAMOTO and Yasuyoshi YONEZAWA

(Received September 1, 2015)

(Revised October 24, 2016)

**ABSTRACT.** In this paper, we reconstruct Kuperberg’s  $G_2$  web space [5, 6]. We introduce a new web diagram (a trivalent graph with only double edges) and new relations between Kuperberg’s web diagrams and the new web diagram. Using the web diagrams, we give crossing formulas for the  $R$ -matrices associated to some irreducible representations of  $U_q(G_2)$  and calculate  $G_2$  quantum link invariants for generalized twist links.

### 1. Introduction

Suppose that  $U_q(G_2)$  is the quantum group of type  $G_2$ , where  $q \in \mathbb{C}$  is neither zero nor a root of unity [1, 3].

Invariant theory of the  $U_q(G_2)$  fundamental representations was studied in a skein theoretic approach by Kuperberg [6] and in a representation theoretic approach by Lehrer–Zhang [7]. (Invariant theory of exceptional Lie group  $G_2$  was studied by Schwarz, Huang–Zhu [2, 11].) As an application of these studies, Kuperberg explicitly gave Reshetikhin–Turaev’s quantum link invariant ( $R$ -matrix) associated to the  $U_q(G_2)$  fundamental representations [9]. (The  $G_2$  quantum link invariant was also obtained in a planar algebra approach by Morrison–Peters–Snyder [8].)

In Kuperberg’s approach, diagrams in Figure 1 are introduced, which are called elementary  $G_2$  web diagrams. They are diagrammatizations of intertwiners between tensor representations of the  $U_q(G_2)$  fundamental representations [5, 6].

These diagrams correspond to intertwiners in  $\text{Hom}_{U_q(G_2)}(V_{\varpi_1}, V_{\varpi_1} \otimes V_{\varpi_1})$  and  $\text{Hom}_{U_q(G_2)}(V_{\varpi_2}, V_{\varpi_1} \otimes V_{\varpi_1})$ , where  $V_{\varpi_1}$  is the first fundamental representation and  $V_{\varpi_2}$  is the second fundamental representation.



**Fig. 1.** Kuperberg’s elementary  $G_2$  web diagram



Fig. 2. New web diagram

The purpose of this paper is to give a reformulation of Kuperberg's  $G_2$  web space, by introducing a new elementary  $G_2$  web diagram in Figure 2 which corresponds to an intertwiner in  $\text{Hom}_{U_q(G_2)}(V_{\varpi_2}, V_{\varpi_2} \otimes V_{\varpi_2})$ , and to describe crossings corresponding to the  $R$ -matrices associated to some  $U_q(G_2)$  irreducible representations in the new  $G_2$  web space.

In Section 2, we introduce the new elementary  $G_2$  web and give relations between Kuperberg's webs and the new web. In Section 3, we define a  $G_2$  web space  $W_{G_2}$  which is a  $\mathbb{C}$ -vector space composed of  $G_2$  web diagrams ( $G_2$  webs embedded in a unit disk) and show that the  $G_2$  web space is isomorphic to an invariant space of tensor representations of the  $U_q(G_2)$  fundamental representations.

In Sections 4, we give the following crossing formulas which express the crossing diagrams ( $R$ -matrices associated to  $U_q(G_2)$ ) by linear sums of  $G_2$  web diagrams. (The first three crossing formulas are the same as Kuperberg's formulas [6], but the last formula is different from his. His last crossing formula of double edges contains an error.)

$$\begin{aligned}
\begin{array}{c} \diagup \\ \diagdown \end{array} &= \frac{q^3}{[2]} \left( + \frac{q^{-3}}{[2]} \begin{array}{c} \frown \\ \smile \end{array} + \frac{q^{-1}}{[2]} \begin{array}{c} \diagup \\ \diagdown \end{array} + \frac{q}{[2]} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) \\
\begin{array}{c} \diagup \\ \parallel \\ \diagdown \end{array} &= \frac{q^3}{[3]} \begin{array}{c} \diagup \\ \diagdown \end{array} + \frac{q^{-3}}{[3]} \begin{array}{c} \diagdown \\ \diagup \end{array} + \frac{1}{[2][3]} \begin{array}{c} \diagup \\ \diagdown \end{array} \\
\begin{array}{c} \parallel \\ \diagup \\ \diagdown \end{array} &= \frac{q^3}{[3]} \begin{array}{c} \diagdown \\ \diagup \end{array} + \frac{q^{-3}}{[3]} \begin{array}{c} \diagup \\ \diagdown \end{array} + \frac{1}{[2][3]} \begin{array}{c} \diagup \\ \diagdown \end{array} \\
\begin{array}{c} \parallel \\ \parallel \\ \diagup \\ \diagdown \end{array} &= \frac{(q^{10} - q^6 - q^4)[4][6]}{[2][12]} \left( + \frac{(q^{-10} - q^{-6} - q^{-4})[4][6]}{[2][12]} \begin{array}{c} \frown \\ \smile \end{array} \right) \\
&\quad + \frac{q^{-3}[3][4]^2[6]^2}{[2]^2[12]^2} \begin{array}{c} \diagup \\ \diagdown \end{array} + \frac{q^3[3][4]^2[6]^2}{[2]^2[12]^2} \begin{array}{c} \diagdown \\ \diagup \end{array} + \frac{1}{[3]} \begin{array}{c} \diagup \\ \diagdown \end{array}
\end{aligned}$$

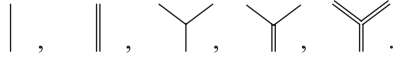
In Section 5, we show that the above crossing formulas induce a braid group action on  $G_2$  web space  $W_{G_2}$ , and in Section 6, we give identities which express idempotents in hom space between tensor representations by  $G_2$  web diagrams. Using the expressions of idempotents, we can obtain crossings formulas for  $R$ -matrices associated to the  $U_q(G_2)$  representation with the highest weight  $2\varpi_1$ . In Section 7, we calculate  $G_2$  quantum invariant of generalized twist links  $TW(m, n)$ .

**2.  $G_2$  web**

First, we introduce  $G_2$  webs in order to define a  $G_2$  web space.

DEFINITION 1 ( $G_2$  web). Let  $q \in \mathbb{C}$  be neither zero nor a root of unity. Denote by  $[n]$  for  $n \in \mathbb{Z}_{\geq 0}$  the  $q$ -integer  $\frac{q^n - q^{-n}}{q - q^{-1}}$  and put  $[n]! := [n][n-1] \dots [1]$  and  $\begin{bmatrix} m \\ n \end{bmatrix} := \frac{[m]!}{[n]![m-n]!}$  for  $0 \leq n \leq m$ .

By an *elementary  $G_2$  web*, we mean one of the following two planar univalent graphs or one of three planar uni-trivalent graphs



A  $G_2$  web is a planar uni-trivalent graph whose vertex is either one of the elementary  $G_2$  webs with the following local relations:

(Loop relation)

$$\bigcirc = \frac{[2][7][12]}{[4][6]}$$

(Monogon relations)

$$\text{loop with top edge} = 0, \quad \text{loop with bottom edge} = 0$$

(Digon relations)

$$\text{digon with top edge} = -\frac{[3][8]}{[4]} \mid, \quad \text{digon with bottom edge} = -[2][3] \parallel$$

(Triangle relations)

$$\text{triangle with top edge} = \frac{[6]}{[2]} \text{Y-junction}, \quad \text{triangle with bottom edge} = 0, \quad \text{triangle with double edge} = \frac{[3]^2[4][6]}{[2][12]} \text{Y-junction with double edge}$$

(Double edge elimination)

$$\text{double edge} = -\frac{[3]}{[2]} \left( + \frac{[3][4][6]}{[2]^2[12]} \text{arc} + \frac{1}{[2]} \text{Y-junction} + \frac{[3]}{[2]} \text{Y-junction} \right)$$

Using the above relations, we obtain the following additional relations.

PROPOSITION 1. (*Loop relation*)

$$\bigcirc \bigcirc = \frac{[7][8][15]}{[3][4][5]}$$

(Monogon relation)

$$\text{Monogon} = 0$$

(Digon relations)

$$\text{Digon 1} = 0, \quad \text{Digon 2} = -\frac{[6][8][15]}{[5][12]} \quad |, \quad \text{Digon 3} = -\frac{[2]^2[12][18]}{[3]^2[4][9]} \quad ||$$

(Triangle relations)

$$\begin{aligned} \text{Triangle 1} &= -[3] \text{Triangle 2}, & \text{Triangle 3} &= -\frac{[4][6][15]}{[5][12]} \text{Triangle 4} \\ \text{Triangle 5} &= -\frac{[3][4][6](q^2 - 2 + q^{-2})}{[12]} \text{Triangle 6}, & \text{Triangle 7} &= \frac{[6][18]}{[3][9]} \text{Triangle 8} \\ \text{Triangle 9} &= -\frac{[2][12](q^8 - q^2 + 1 - q^{-2} + q^{-8})}{[3][6]} \text{Triangle 10} \end{aligned}$$

(Square relations)

$$\begin{aligned} \text{Square 1} &= [3] \left( + [3] \text{Square 2} - \frac{[4]}{[2]} \text{Square 3} - \frac{[4]}{[2]} \text{Square 4} \right) \\ \text{Square 5} &= \text{Square 6} \\ &= \frac{[3][7]}{[2]} \left( + \frac{[3]}{[2]} \text{Square 7} \right. \\ &\quad \left. - \frac{[4][6](q^6 - q^2 - 1 - q^{-2} + q^{-6})}{[2]^2[12]} \text{Square 8} + \frac{[7]}{[2]} \text{Square 9} \right) \\ \text{Square 11} &= \frac{[3][4][6](q^{14} + q^8 + 2q^4 - q^2 + 1 - q^{-2} + 2q^{-4} + q^{-8} + q^{-14})}{[2][12]} \left( \right. \\ &\quad \left. + \frac{[3][4]^2[6]^2(q^6 - 2q^4 + q^2 + 1 + q^{-2} - 2q^{-4} + q^{-6})}{[2]^2[12]^2} \text{Square 12} \right. \\ &\quad \left. - \frac{[4][6](q^4 - 2q^2 + 1 - 2q^{-2} + q^{-4})}{[2][12]} \text{Square 13} \right. \\ &\quad \left. - \frac{[4][6]S}{[2][12]} \text{Square 14} \right) \end{aligned}$$

where  $S = q^{12} + q^{10} + q^6 - q^4 + q^2 - 1 + q^{-2} - q^{-4} + q^{-6} + q^{-10} + q^{-12}$ 

$$\begin{aligned} \text{Square 15} &= \text{Square 16} + \text{Square 17} \\ \text{Square 18} &= \frac{[3][4][6]}{[12]} \text{Square 19} + \frac{[3][4][6]}{[12]} \text{Square 20} + \text{Square 21} \end{aligned}$$

$$\begin{aligned}
 \text{Diagram 1} &= \frac{[4][6]^2}{[2][3][12]} \text{Diagram 2} - [4] \text{Diagram 3} \\
 \text{Diagram 4} &= \frac{[3][4][6][10]}{[5][12]} \text{Diagram 5} + \frac{[3][4][6]}{[2][12]} \text{Diagram 6} + \frac{[4][6]^2}{[2][3][12]} \text{Diagram 7} \\
 \text{Diagram 8} &= \frac{[2][18]}{[3][9]} \text{Diagram 9} + \frac{[2][18]}{[3][9]} \text{Diagram 10} \\
 \text{Diagram 11} &= -\text{Diagram 12} - \text{Diagram 13} - \frac{[12]}{[3][6]} \text{Diagram 14} \\
 \text{Diagram 15} &= \frac{[2][18]}{[3][9]} \text{Diagram 16} + \frac{(q^{10} + q^8 - q^2 - 1 - q^{-2} + q^{-8} + q^{-10})}{[3]} \text{Diagram 17} \\
 &\quad - \frac{[2][5][12][18]}{[3]^2[4][6][9]} \text{Diagram 18} \\
 \text{Diagram 19} &= \frac{[2]^4[5][12]^2(q^2 - 2 + q^{-2})}{[3]^2[4]^2[6]^2} \left( \text{Diagram 20} \right) \left( \text{Diagram 21} + \text{Diagram 22} \right) \\
 &\quad - \frac{(q^6 - q^4 - 1 - q^{-4} + q^{-6})[2]^2[12]}{[3][4][6]} \left( \text{Diagram 23} + \text{Diagram 24} \right) + \frac{[2]^3[5][12]^4}{[3]^4[4]^3[6]^4} \text{Diagram 25} \\
 \text{Diagram 26} &= \frac{(q^{-2} - 2 + q^2)[2]^5[5]^2[12]^3[18]}{[3]^4[4]^3[6]^3[9]} \left( \text{Diagram 27} - \frac{[2]^5[5][12]^3[16][18]}{[3]^4[4]^3[6]^3[8][9]} \text{Diagram 28} \right) \\
 &\quad + \frac{[2]^3[12]^2}{[3]^2[4]^3[6]^2} \left( \frac{[3][16]}{[8]} - 2 \right) \left( \frac{[2][18]}{[9]} - [3] \right) \text{Diagram 29} \\
 &\quad - \frac{[2]^3[5][12]^2 L}{[3]^3[4]^2[6]^2} \text{Diagram 30} \\
 &\quad - \frac{[2][12](q^{-10} + q^{-6} - 3 + q^6 + q^{10})}{[6][3]^2} \text{Diagram 31}
 \end{aligned}$$

where

$$L = q^{-15} - q^{-13} - q^{-9} - 2q^{-5} + q^{-3} + q^{-1} + q + q^3 - 2q^5 - q^9 - q^{13} + q^{15}.$$

(Pentagon relation)

$$\begin{aligned}
 \text{Diagram 32} &= \left( \text{Diagram 33} + \text{Diagram 34} + \text{Diagram 35} + \text{Diagram 36} + \text{Diagram 37} \right) \\
 &\quad - \left( \text{Diagram 38} + \text{Diagram 39} + \text{Diagram 40} + \text{Diagram 41} + \text{Diagram 42} \right) \\
 \text{Diagram 43} &= \frac{[2][3]}{[4][5]} \left( \text{Diagram 44} + \text{Diagram 45} + \text{Diagram 46} + \text{Diagram 47} + \text{Diagram 48} \right) \\
 &\quad + \frac{[12][2]^3 P_1}{[6][4]^2[3][5]} \left( \text{Diagram 49} + \text{Diagram 50} + \text{Diagram 51} + \text{Diagram 52} + \text{Diagram 53} \right)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{[2]^3[12]^2 P_2}{[4]^3[3]^2[6]^2} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} + \begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \end{array} + \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} + \begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \end{array} \right) \\
& - \frac{[2]^6[8][12]^5}{[3]^5[4]^6[6]^5} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}
\end{aligned}$$

where

$$P_1 = \frac{2}{q^8} + \frac{1}{q^6} + \frac{1}{q^4} - \frac{2}{q^2} - 2q^2 + q^4 + q^6 + 2q^8$$

$$P_2 = \frac{1}{q^8} - \frac{1}{q^6} - \frac{2}{q^4} + 2 - 2q^4 - q^6 + q^8.$$

PROOF (Sketch of proof). Applying the relation (Double edge elimination) or its rearrangement

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \left( -\frac{[4][6]}{[2][12]} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - \frac{1}{[3]} \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} + \frac{[2]}{[3]} \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} \right)$$

to the left-hand side of an identity in this proposition and using the relations in Definition 1, we obtain the identity. If we can not apply the elimination or the rearrangement to the left-hand side, we first create single edges on the web by using the relations

$$\parallel = -\frac{1}{[2][3]} \begin{array}{c} \circ \\ \parallel \end{array} \quad \text{and} \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \frac{[2][12]}{[3]^2[4][6]} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}$$

and apply the elimination or its rearrangement.

For example, the first digon relation in this proposition is obtained as follows. First, create single edges on the  $G_2$  web:

$$\begin{array}{c} \circ \\ \parallel \end{array} = -\frac{1}{[2][3]} \begin{array}{c} \circ \\ \circ \end{array}$$

By applying (Double edge elimination) to the right-hand side of the above identity and using monogon, digon and triangle relations in Definition 1, we obtain the following identity:

$$\begin{aligned}
-\frac{1}{[2][3]} \begin{array}{c} \circ \\ \circ \end{array} &= -\frac{1}{[2][3]} \left( -\frac{[3]}{[2]} \begin{array}{c} \circ \\ \parallel \end{array} + \frac{[3][4][6]}{[2]^2[12]} \begin{array}{c} \circ \\ \circ \end{array} + \frac{1}{[2]} \begin{array}{c} \circ \\ \circ \end{array} + \frac{[3]}{[2]} \begin{array}{c} \circ \\ \parallel \end{array} \right) \\
&= -\frac{1}{[2][3]} \left( -\frac{[3]}{[2]} + 0 + \frac{1}{[2]} \left( -\frac{[3][8]}{[4]} + \frac{[3]}{[2]} \frac{[6]}{[2]} \right) \right) \begin{array}{c} \circ \\ \circ \end{array} = 0. \quad \square
\end{aligned}$$

### 3. Web space $W_{G_2}$ and invariant space of representation

In this section, we define a  $G_2$  web space  $W_{G_2}$ , which is a  $\mathbb{C}$ -vector space spanned by  $G_2$  web diagrams ( $G_2$  webs embedded in a unit disk), where  $G_2$  web diagrams are defined as follows.

Let  $D$  be a closed unit disk in  $\mathbb{R}^2$  with a fixed base point  $*$  on the boundary  $\partial D$ . A  $G_2$  web diagram is the image of an embedding of a  $G_2$  web  $P$  in  $D$  such that every univalent of  $P$  lies in  $\partial D \setminus \{*\}$ . We do not consider a  $G_2$  web which can not be embedded in the disk  $D$ .

For a given  $G_2$  web diagram  $W$ , put the number 1 at each intersection of single edges of  $W$  with  $\partial D$  and put the number 2 at each intersection on double edges of  $W$  with  $\partial D$ . A coloring of  $W$  is defined to be the sequence obtained by reading numbers 1 and 2 on  $\partial D$  clockwise from the base point  $*$ . If  $W$  has no univalent, a coloring of  $W$  is defined to be the empty sequence  $\emptyset$ . Denote by  $s(W)$  the coloring of  $W$ .

For example, the colorings of  $G_2$  web diagrams in Figure 3 are given by  $s(W_1) = (1, 1, 1, 1)$ ,  $s(W_2) = (2, 1, 1)$ ,  $s(W_3) = (1, 1, 2, 1)$ ,  $s(W_4) = (1, 2, 2, 1, 1)$ .

Two  $G_2$  web diagrams  $W_1$  and  $W_2$  are isotopic if there exists a base point-preserving isotopy of  $D$  which moves  $W_1$  to  $W_2$ .

Hereafter we fix a base point as  $G_2$  web diagrams in Figure 3 and omit the boundary of the unit disk.

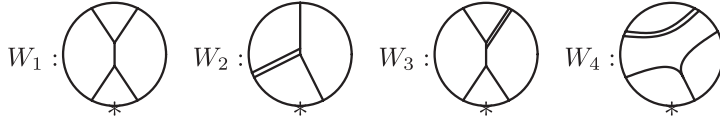


Fig. 3.  $G_2$  web diagrams

Write

$$S := \{s = (s_1, s_2, \dots, s_n) \mid n \geq 1, s_i \in \{1, 2\} \ (i = 1, 2, \dots, n)\} \cup \{\emptyset\}.$$

We define a  $G_2$  web space  $W_{G_2}(s)$  for  $s \in S$  by a  $\mathbb{C}$ -vector space spanned by the isotopy classes of  $G_2$  web diagrams with the coloring  $s$ , modulo the relations in Definition 1.

REMARK 1. The collection of the web spaces  $\{W_{G_2}(s)\}_{s \in S}$  has the spider structure in the sense of Kuperberg [6, Section 3]:

(Join)

$$\mu_{s,t} : W_{G_2}(s) \times W_{G_2}(t) \rightarrow W_{G_2}(st)$$

(Rotation)

$$\rho_{s,t} : W_{G_2}(st) \rightarrow W_{G_2}(ts)$$

(Stitch)

$$\sigma_{sst} : \mathbf{W}_{G_2}(sst) \rightarrow \mathbf{W}_{G_2}(t).$$

For  $s = (s_1, s_2, \dots, s_n) \in S$ , let  $V_s$  be the tensor representation of  $G_2$  quantum group  $V_{\varpi_{s_1}} \otimes V_{\varpi_{s_2}} \otimes \cdots \otimes V_{\varpi_{s_n}}$ , where  $V_{\varpi_{s_i}}$  is the  $s_i$ -th fundamental representation ( $i = 1, \dots, n$ ).

The following theorem is due to [6, Theorem 6.10].

**THEOREM 1** ([6]). *The vector spaces  $\mathbf{W}_{G_2}(s)$  and the invariant space  $\text{Inv}(V_s)$  have the same dimension.*

**PROOF.** Replacing numbers 2 in the coloring  $s$  into  $[1, 1]$ , we obtain a clasp sequence  $C$  (see [6]). Since the web space  $\mathbf{W}_{G_2}(s)$  and the clasp web space  $\mathbf{W}_{G_2}(C)$  have the same dimension, we obtain the theorem.  $\square$

We denote by  $\mathbf{B}(s)$  a basis of the vector space  $\mathbf{W}_{G_2}(s)$ , called a  $G_2$  web basis.

**EXAMPLE 1.** For  $s = (1, 1, 1, 1)$ ,  $(1, 2, 1, 2)$  and  $(2, 2, 2, 2)$ , we have a  $G_2$  web basis  $\mathbf{B}(s)$ .

$$\begin{aligned} \mathbf{B}(1, 1, 1, 1) &= \left\{ \begin{array}{c} \text{)} \\ \text{(} \\ \text{)} \\ \text{(} \end{array} \right\} \\ \mathbf{B}(1, 2, 1, 2) &= \left\{ \begin{array}{c} \text{)} \\ \text{(} \\ \text{)} \\ \text{(} \end{array} \right\} \\ \mathbf{B}(2, 2, 2, 2) &= \left\{ \begin{array}{c} \text{)} \\ \text{(} \\ \text{)} \\ \text{(} \end{array} \right\} \end{aligned}$$

#### 4. Crossing formula in the $G_2$ web space

Let  $\#(s)$  be the length of a sequence  $s \in S$ , and define

$$S[n] := \{s \in S \mid \#(s) = n\}.$$

We define an action of the braid group

$$B_n = \left\langle b_i \ (1 \leq i \leq n-1) \ \middle| \ \begin{array}{l} b_i b_j = b_j b_i \quad (|i-j| > 1), \\ b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \quad (1 \leq i \leq n-2) \end{array} \right\rangle$$

on the collection of the web spaces

$$\{\mathbf{W}_{G_2}(s)\}_{s \in S[n]}.$$

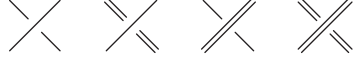
For each  $s = (s_1, s_2, \dots, s_n) \in S[n]$ , we define an action of the braid group  $B_n$  on the representation  $V_s$ , where  $V_s$  is the tensor representation  $V_{\varpi_{s_1}} \otimes V_{\varpi_{s_2}} \otimes \cdots \otimes$



$V_{\varpi_{s_i}}$ , by setting  $\rho_s(b_i)$  to be the invertible intertwiner composed of  $R$ -matrix

$$\text{Id}_{V_{\varpi_{s_1}} \otimes \cdots \otimes V_{\varpi_{s_{i-1}}}} \otimes R_{s_i s_{i+1}} \otimes \text{Id}_{V_{\varpi_{s_{i+2}}} \otimes \cdots \otimes V_{\varpi_{s_n}}} \in \text{Hom}_{U_q(G_2)}(V_s, V_{\sigma_i(s)}),$$

where  $\sigma_i$  is the transposition between  $i$ -th and  $(i+1)$ -th entries. We represent the  $R$ -matrices  $R_{11}$ ,  $R_{12}$ ,  $R_{21}$  and  $R_{22}$  by the following crossing diagrams



and represent the inverse  $R_{11}^{-1}$ ,  $R_{12}^{-1}$ ,  $R_{21}^{-1}$  and  $R_{22}^{-1}$  by the diagram obtained by operating  $\frac{\pi}{2}$ -rotation on the above crossing diagrams.

The vector space  $\text{Hom}_{U_q(G_2)}(V_s, V_{s'})$  ( $s, s' \in S$ ) is isomorphic to the invariant space  $\text{Inv}(V_{w(s)} \otimes V_{s'})$ , where  $w(s)$  is the sequence obtained by reversing the order of the elements in the sequence  $s$ . By Theorem 1, each of the above crossing diagrams has a description as a linear sum of the  $G_2$  web diagrams.

**THEOREM 2.** *The four types of crossings corresponding to the  $R$ -matrices have the following descriptions in the  $G_2$  web diagrams:*

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \frac{q^3}{[2]} \left( + \frac{q^{-3}}{[2]} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \frac{q^{-1}}{[2]} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + \frac{q}{[2]} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) \quad (1)$$

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \frac{q^3}{[3]} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + \frac{q^{-3}}{[3]} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + \frac{1}{[2][3]} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \quad (2)$$

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \frac{q^3}{[3]} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + \frac{q^{-3}}{[3]} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + \frac{1}{[2][3]} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \quad (3)$$

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \frac{(q^{10} - q^6 - q^4)[4][6]}{[2][12]} \left( + \frac{(q^{-10} - q^{-6} - q^{-4})[4][6]}{[2][12]} \begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \\ + \frac{q^{-3}[3][4]^2[6]^2}{[2]^2[12]^2} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + \frac{q^3[3][4]^2[6]^2}{[2]^2[12]^2} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + \frac{1}{[3]} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \quad (4)$$

**PROOF.** The crossing diagram on the left-hand side of Identity (1) corresponds to the  $R$ -matrix  $R_{11}$  in  $\text{End}_{U_q(G_2)}(V_{(1,1)})$ . The vector space  $\text{End}_{U_q(G_2)}(V_{(1,1)})$  is isomorphic to the invariant space  $\text{Inv}(V_{(1,1,1,1)})$  and, by Theorem 1, is isomorphic to the web space  $W_{G_2}(1,1,1,1)$ . Therefore, the crossing diagram corresponding to  $R_{11}$  is expressed by a linear sum of  $G_2$  web basis  $B(1,1,1,1)$ . That is, Identity (1) is the expression of the crossing diagram by  $G_2$  web diagrams. The crossing formula about the  $R$ -matrix  $R_{11}^{-1}$  is the identity obtained by operating the  $\frac{\pi}{2}$ -rotation on each diagram in Identity (1). Other Identities (2), (3) and (4) are also the expressions of the crossing diagrams corresponding to the  $R$ -matrices  $R_{12}$ ,  $R_{21}$  and  $R_{22}$  by  $G_2$  web diagrams.

The 2nd and 3rd Reidemeister moves are corresponding to  $R$ -matrix invertibility and the Yang-Baxter equation. We have to check that Identities (1), (2), (3) and (4) are well-defined. That is, we need to check that crossing diagrams related by the 2nd and 3rd Reidemeister moves are identical as elements of the  $G_2$  web space, using Identities (1), (2), (3) and (4).

For crossing diagrams with only single edges, it is enough to prove the following identities by the 2nd and 3rd Reidemeister moves with only single edges:

$$(R2) \quad \left( \text{crossing} = \right) \left( \text{ , } \quad (R3) \quad \left( \text{crossing} = \text{crossing} \right) \right)$$

Other identities by the 2nd and 3rd Reidemeister moves including double edge can be obtained by Identities (R2), (R3) and relations in Definition 1.

Proof of Identity (R2): By Identity (1), the left-hand side of Identity (R2) is equal to

$$\begin{aligned} & \frac{q^6}{[2]^2} \text{diagram} + \frac{1}{[2]^2} \text{diagram} + \frac{q^2}{[2]^2} \text{diagram} + \frac{q^4}{[2]^2} \text{diagram} \\ & + \frac{1}{[2]^2} \text{diagram} + \frac{q^{-6}}{[2]^2} \text{diagram} + \frac{q^{-4}}{[2]^2} \text{diagram} + \frac{q^{-2}}{[2]^2} \text{diagram} \\ & + \frac{q^2}{[2]^2} \text{diagram} + \frac{q^{-4}}{[2]^2} \text{diagram} + \frac{q^{-2}}{[2]^2} \text{diagram} + \frac{1}{[2]^2} \text{diagram} \\ & + \frac{q^4}{[2]^2} \text{diagram} + \frac{q^{-2}}{[2]^2} \text{diagram} + \frac{1}{[2]^2} \text{diagram} + \frac{q^2}{[2]^2} \text{diagram}. \end{aligned} \quad (5)$$

Using relations in Definition 1 and Proposition 1, we have the following identities:

$$\begin{aligned} & \text{diagram} = \frac{[2][7][12]}{[4][6]} \text{diagram}, \quad \text{diagram} = \text{diagram} = -\frac{[3][8]}{[4]} \text{diagram}, \\ & \text{diagram} = \text{diagram} = 0, \quad \text{diagram} = \text{diagram} = \frac{[6]}{[2]} \text{diagram}, \quad \text{diagram} = -\frac{[3][8]}{[4]} \text{diagram}, \\ & \text{diagram} = [3] \left( +[3] \text{diagram} - \frac{[4]}{[2]} \text{diagram} - \frac{[4]}{[2]} \text{diagram} \right) \end{aligned}$$

By these identities, we find that the linear sum (5) is equal to the right-hand side of Identity (R2).

The following lemma is helpful to prove other identities by Reidemeister moves.

LEMMA 1. *We have the following identities:*

$$\begin{array}{lll}
 \text{(Fp1)} & \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}, & \text{(Fp2)} & \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} = \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array}, & \text{(Fp3)} & \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} = \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array}, \\
 \text{(Fn1)} & \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array} = \begin{array}{c} \text{Diagram 15} \\ \text{Diagram 16} \end{array}, & \text{(Fn2)} & \begin{array}{c} \text{Diagram 17} \\ \text{Diagram 18} \end{array} = \begin{array}{c} \text{Diagram 19} \\ \text{Diagram 20} \end{array}, & \text{(Fn3)} & \begin{array}{c} \text{Diagram 21} \\ \text{Diagram 22} \end{array} = \begin{array}{c} \text{Diagram 23} \\ \text{Diagram 24} \end{array}.
 \end{array}$$

This lemma is proved in Appendix A.

Proof of Identity (R3): By Identity (1), we have the following identity.

$$\begin{array}{c} \text{Diagram 25} \end{array} = \frac{q^3}{[2]} \begin{array}{c} \text{Diagram 26} \end{array} + \frac{q^{-3}}{[2]} \begin{array}{c} \text{Diagram 27} \end{array} + \frac{q^{-1}}{[2]} \begin{array}{c} \text{Diagram 28} \end{array} + \frac{q}{[2]} \begin{array}{c} \text{Diagram 29} \end{array}$$

By Identity (R2) and Lemma (1) (Fp1), the right-hand side is equal to the following:

$$\frac{q^3}{[2]} \begin{array}{c} \text{Diagram 30} \end{array} + \frac{q^{-3}}{[2]} \begin{array}{c} \text{Diagram 31} \end{array} + \frac{q^{-1}}{[2]} \begin{array}{c} \text{Diagram 32} \end{array} + \frac{q}{[2]} \begin{array}{c} \text{Diagram 33} \end{array}$$

By Identity (1), this linear sum is equal to the right-hand side of Identity (R3).

The invariance of crossing diagrams including double edges, as elements of the  $G_2$  web space, by the Reidemeister moves can be proved by using Identities (R2) and (R3), Lemma 1 and the following digon relation

$$\begin{array}{c} \text{Diagram 34} \end{array} = -[2][3] \begin{array}{c} \text{Diagram 35} \end{array}.$$

Here, we prove the following identity in the  $G_2$  web space, corresponding to the invertibility of  $R_{22}$  in  $\text{End}_{U_q(G_2)}(V_{(2,2)})$ :

$$\begin{array}{c} \text{Diagram 36} \end{array} = \begin{array}{c} \text{Diagram 37} \end{array} \begin{array}{c} \text{Diagram 38} \end{array} \quad (6)$$

By the digon relation, the left-hand side is equal to

$$\begin{array}{c} \text{Diagram 39} \end{array} = \frac{1}{[2]^2[3]^2} \begin{array}{c} \text{Diagram 40} \end{array}$$

and, by Lemma 1 (Fp2), Identity (R2) and the digon relation, this is equal to

$$\frac{1}{[2]^2[3]^2} \begin{array}{c} \text{Diagram 41} \end{array} = \frac{1}{[2]^2[3]^2} \begin{array}{c} \text{Diagram 42} \end{array} \begin{array}{c} \text{Diagram 43} \end{array} = \begin{array}{c} \text{Diagram 44} \end{array} \begin{array}{c} \text{Diagram 45} \end{array} = \begin{array}{c} \text{Diagram 46} \end{array} \begin{array}{c} \text{Diagram 47} \end{array}$$

Proofs for the identities corresponding to the remaining Reidemeister moves containing double edges can be done in a similar way.  $\square$

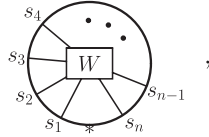
### 5. Braid action on the $G_2$ web space $W_{G_2}$

Using the crossing formulas in Theorem 2, we can define an action of the braid group  $B_n$  on  $W_{G_2}[n] = \bigoplus_{s \in S[n]} W_{G_2}(s)$ .

The action of the  $i$ -th generators  $b_i$  and  $b_i^{-1}$  of  $B_n$  ( $i = 1, \dots, n-1$ ) is defined as follows. For a  $G_2$  web diagram  $W \in W_{G_2}(s) \subset W_{G_2}[n]$ ,  $b_i(W)$  is the element of  $W_{G_2}(\sigma_i(s))$ , where  $\sigma_i$  is the transposition of  $i$ -th and  $(i+1)$ -th entries, obtained from  $W$  by gluing the  $(s_i, s_{i+1})$ -boundary of  $W$  and the positive crossing (as the  $s_i$ -univalent of  $W$  connects to the over arc of the crossing). Similarly,  $b_i^{-1}(W)$  is the element of  $W_{G_2}(\sigma_i(s))$  obtained from  $W$  by gluing the  $(s_i, s_{i+1})$ -boundary of  $W$  and the negative crossing (as the  $s_i$ -univalent of  $W$  connects to the under arc of the crossing). Then, we replace the obtained knotted diagram into the linear sum of  $G_2$  web diagrams by the formulas in Theorem 2.

In other words, we regard the action of generators  $b_i$  and  $b_i^{-1}$  as positive and negative crossings and univalents on the unit disc with a hole in Figure 4.

For a  $G_2$  web diagram  $W$  diagrammatically denoted by



the action of the generators  $b_i$  (resp.  $b_i^{-1}$ ) amounts to putting the diagram  $W$  into the hole of the diagram of  $b_i$  (resp.  $b_i^{-1}$ ) in Figure 4 and gluing these diagrams.

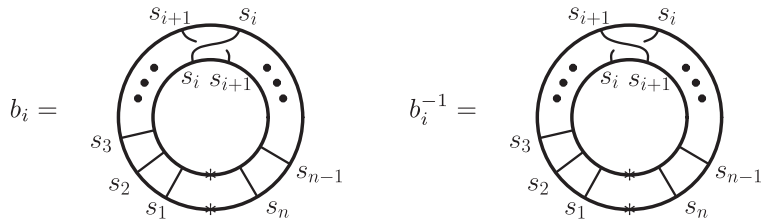


Fig. 4. Diagrammatic description of generator action

For example, the  $B_5$  action on the  $G_2$  web space  $W_{G_2}(1, 2, 2, 1, 1)$  is described as follows. To the  $G_2$  web diagram  $W_4$  in Figure 3, the actions of  $b_1, b_4, b_4^{-1} \in B_5$  are given by:

$$\begin{aligned}
 b_1 \left( \text{web diagram} \right) &= \left( \text{web diagram} \right) = \frac{q^3}{[3]} \left( \text{web diagram} \right) + \frac{q^{-3}}{[3]} \left( \text{web diagram} \right) + \frac{1}{[2][3]} \left( \text{web diagram} \right) \\
 b_4 \left( \text{web diagram} \right) &= \left( \text{web diagram} \right) = \frac{q^3}{[2]} \left( \text{web diagram} \right) + \frac{q^{-3}}{[2]} \left( \text{web diagram} \right) + \frac{q^{-1}}{[2]} \left( \text{web diagram} \right) + \frac{q}{[2]} \left( \text{web diagram} \right) \\
 &= \left( \frac{q^3}{[2]} - \frac{q^{-1}[3][8]}{[2][4]} + \frac{q[6]}{[2]^2} \right) \left( \text{web diagram} \right) = -q^{-6} \left( \text{web diagram} \right) \\
 b_4^{-1} \left( \text{web diagram} \right) &= \left( \text{web diagram} \right) = \frac{q^3}{[2]} \left( \text{web diagram} \right) + \frac{q^{-3}}{[2]} \left( \text{web diagram} \right) + \frac{q^{-1}}{[2]} \left( \text{web diagram} \right) + \frac{q}{[2]} \left( \text{web diagram} \right) \\
 &= \left( \frac{q^{-3}}{[2]} - \frac{q[3][8]}{[2][4]} + \frac{q^{-1}[6]}{[2]^2} \right) \left( \text{web diagram} \right) = -q^6 \left( \text{web diagram} \right)
 \end{aligned}$$

## 6. Relation to idempotents and $R$ -matrix of other irreducible representations

In this section, we describe a relation between  $G_2$  web diagrams and idempotents in the hom set  $\text{Hom}_{U_q(G_2)}(V_{\varpi_i} \otimes V_{\varpi_j}, V_{\varpi_j} \otimes V_{\varpi_i})$ , where  $\varpi_i$  and  $\varpi_j$  are fundamental weights. Using the idempotents, we construct the crossing formulas for the  $R$ -matrices associated to other irreducible representations.

Let  $P_{11}[\varpi]$  be the idempotent in  $\text{End}_{U_q(G_2)}(V_{\varpi_1}^{\otimes 2})$  which factors through the irreducible representation with the highest weight  $\varpi$ . Note that the idempotents satisfy

$$P_{11}[\varpi]P_{11}[\varpi'] = \delta_{\varpi, \varpi'} P_{11}[\varpi].$$

By Theorem 1,  $\text{End}_{U_q(G_2)}(V_{\varpi_1}^{\otimes 2})$  is isomorphic to the web space  $W_{G_2}(1, 1, 1, 1)$ . Therefore, we have the following identities which express idempotents  $P_{11}[\varpi]$  by linear sums of  $G_2$  web diagrams:

$$\begin{aligned}
 P_{11}[2\varpi_1] &= \left( + \frac{[4]}{[3][8]} \text{web diagram} + \frac{1}{[2][3]} \text{web diagram} - \frac{[4][6]}{[2][7][12]} \text{web diagram} \right) \\
 P_{11}[\varpi_1] &= -\frac{[4]}{[3][8]} \text{web diagram} \\
 P_{11}[\varpi_2] &= -\frac{1}{[2][3]} \text{web diagram} \\
 P_{11}[0] &= \frac{[4][6]}{[2][7][12]} \text{web diagram}
 \end{aligned}$$

By these identities and Identity (1), the  $R$ -matrix  $R_{11}$  is expressed by a linear sum of idempotents  $P_{11}[\varpi]$  as follows:

$$R_{11} = q^2 P_{11}[2\varpi_1] - q^{-6} P_{11}[\varpi_1] - P_{11}[\varpi_2] + q^{-12} P_{11}[0].$$

(This identity can be found in [7, Sec. 8.1.1].)

Let  $P_{ij}[\varpi]$ ,  $i, j \in \{1, 2\}$ , be the idempotent in  $\text{Hom}_{U_q(G_2)}(V_{\varpi_i} \otimes V_{\varpi_j}, V_{\varpi_j} \otimes V_{\varpi_i})$  which factors through the representation  $V_{\varpi}$ . Note that the idempotents satisfy

$$P_{12}[\varpi]P_{21}[\varpi']P_{12}[\varpi] = \delta_{\varpi, \varpi'}P_{12}[\varpi], \quad P_{22}[\varpi]P_{22}[\varpi'] = \delta_{\varpi, \varpi'}P_{22}[\varpi].$$

We have the following identities which express the idempotents  $P_{ij}[\varpi]$  by linear sums of  $G_2$  web diagrams:

$$\begin{aligned} P_{12}[\varpi_1 + \varpi_2] &= \frac{1}{[3]} \text{web}_1 + \frac{[5](q^8 + q^2 - 1 + q^{-2} + q^{-8})}{[7][15]} \text{web}_2 + \frac{[4]}{[2][3][7]} \text{web}_3 \\ P_{12}[2\varpi_1] &= \frac{1}{[2][7]} \text{web}_4 + \frac{[3][4]}{[2][7][8]} \text{web}_5 \\ P_{12}[\varpi_1] &= -\frac{[5][12]}{[6][8][15]} \text{web}_6 \\ P_{22}[2\varpi_2] &= \frac{[3][4][5](q^2 - 2 + q^{-2})}{[12]} \left( -\frac{[3]^2[4][5][14]}{[7][8][12][15]} \text{web}_7 \right. \\ &\quad \left. + \frac{[3]^2[4]^2[6][9]([4][14] - [7])}{[2]^2[7][8][12]^2[18]} \text{web}_8 + \frac{[3]^2[4]^2[6]}{[2]^2[12]^2} \text{web}_9 \right. \\ &\quad \left. + \frac{[5]}{[6][8]} \text{web}_{10} \right) \\ P_{22}[3\varpi_1] &= \frac{[3][4]}{[12]} \left( +\frac{[2][3][4]^2[5]}{[8][10][12]} \text{web}_{11} - \frac{[3]^4[4]^2[5]}{[2]^2[10][12]^2} \text{web}_{12} \right. \\ &\quad \left. - \frac{[3]^2[4]^2[6]}{[2]^2[12]^2} \text{web}_{13} - \frac{[4][5]}{[2][6][10]} \text{web}_{14} \right) \\ P_{22}[2\varpi_1] &= -\frac{[2][3]^2[4][5][6]}{[7][8][10][12]} \text{web}_{15} + \frac{[3]^3[4]^2[5][6]^2}{[2]^3[8][10][12]^2} \text{web}_{16} + \frac{[5]}{[8][10]} \text{web}_{17} \\ P_{22}[\varpi_2] &= -\frac{[3]^2[4][9]}{[2]^2[12][18]} \text{web}_{18} \\ P_{22}[0] &= \frac{[3][4][5]}{[7][8][15]} \text{web}_{19} \end{aligned}$$

The idempotent  $P_{21}[\varpi]$  is equal to the linear sum obtained by operating (left-right) symmetry on each diagram of the right-hand side of  $P_{12}[\varpi]$ . In other words,

$$P_{21}[\varpi] = R_{21}P_{12}[\varpi]R_{12}^{-1}.$$

The  $R$ -matrices  $R_{12} \in \text{Hom}_{U_q(G_2)}(V_{\varpi_1} \otimes V_{\varpi_2}, V_{\varpi_2} \otimes V_{\varpi_1})$ ,  $R_{21} \in \text{Hom}_{U_q(G_2)}(V_{\varpi_2} \otimes V_{\varpi_1}, V_{\varpi_1} \otimes V_{\varpi_2})$  and  $R_{22} \in \text{End}_{U_q(G_2)}(V_{\varpi_2}^{\otimes 2})$  are expressed by the following linear sums of idempotents.

$$\begin{aligned}
 R_{12} &= q^3 P_{12}[\varpi_1 + \varpi_2] + q^{-4} P_{12}[2\varpi_1] - q^{-12} P_{12}[\varpi_1] \\
 R_{21} &= q^3 P_{21}[\varpi_1 + \varpi_2] + q^{-4} P_{21}[2\varpi_1] - q^{-12} P_{21}[\varpi_1] \\
 R_{22} &= q^6 P_{22}[2\varpi_2] - P_{22}[3\varpi_1] + q^{-10} P_{22}[2\varpi_1] - q^{-12} P_{22}[\varpi_2] + q^{-24} P_{22}[0],
 \end{aligned}$$

Moreover, using the identities for the idempotents, we obtain crossing formulas for the  $R$ -matrices associated to other irreducible representations. For example, using the identity for  $P_{11}[2\varpi_1]$ , we obtain the following formulas for the  $R$ -matrices in  $\text{Hom}_{U_q(G_2)}(V_{2\varpi_1} \otimes V_{\varpi_1}, V_{\varpi_1} \otimes V_{2\varpi_1})$  and  $\text{Hom}_{U_q(G_2)}(V_{2\varpi_1} \otimes V_{\varpi_2}, V_{\varpi_1} \otimes V_{2\varpi_2})$

$$\begin{aligned}
 2\varpi_1 \begin{array}{c} \diagup \\ \diagdown \end{array} &= \begin{array}{c} \diagup \\ \diagdown \end{array} - \frac{[4][6]}{[2][7][12]} \begin{array}{c} \diagup \\ \diagdown \end{array} \cup + \frac{[4]}{[3][8]} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} + \frac{1}{[2][3]} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \\
 2\varpi_1 \begin{array}{c} \parallel \\ \parallel \end{array} &= \begin{array}{c} \parallel \\ \parallel \end{array} - \frac{[4][6]}{[2][7][12]} \begin{array}{c} \parallel \\ \parallel \end{array} \cup + \frac{[4]}{[3][8]} \begin{array}{c} \parallel \\ \parallel \end{array} \begin{array}{c} \parallel \\ \parallel \end{array} + \frac{1}{[2][3]} \begin{array}{c} \parallel \\ \parallel \end{array} \begin{array}{c} \parallel \\ \parallel \end{array}
 \end{aligned}$$

We also have a crossing formula which expresses the following crossing corresponding to the  $R$ -matrix in  $\text{End}_{U_q(G_2)}(V_{2\varpi_1}^{\otimes 2})$

$$2\varpi_1 \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array}$$

by a linear sum of 16 diagrams. Similarly, we have crossing formulas with colorings  $\varpi_1 + \varpi_2$ ,  $2\varpi_2$  and  $3\varpi_1$  by using the idempotents  $P_{12}[\varpi_1 + \varpi_2]$ ,  $P_{22}[2\varpi_2]$  and  $P_{22}[3\varpi_1]$ .

An open problem is to construct the idempotents which factor through other irreducible representations as linear sums of  $G_2$  web diagrams. If this problem is solved, we can explicitly construct crossing formulas for the  $R$ -matrix associated to other irreducible representations of  $U_q(G_2)$  as above.

### 7. $G_2$ quantum invariant of generalized twist link

We can obtain the following evaluations of positive and negative crossings curls (diagrams in Reidemeister move 1) by using the crossing formulas (1) and (4) in Theorem 2.

$$\begin{array}{cc}
 \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} = q^{12} \quad | \quad \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} = q^{-12} \quad | \\
 \begin{array}{c} \parallel \\ \parallel \end{array} \begin{array}{c} \parallel \\ \parallel \end{array} = q^{24} \quad || \quad \begin{array}{c} \parallel \\ \parallel \end{array} \begin{array}{c} \parallel \\ \parallel \end{array} = q^{-24} \quad ||
 \end{array}$$

Therefore, to obtain  $G_2$  quantum invariant of an oriented link, we need to normalize the crossing formulas in Theorem 2.

Let  $L$  be an oriented link with  $k$  components  $(L_1, L_2, \dots, L_k)$ , and let  $D = (D_1, D_2, \dots, D_k)$  be an unoriented link diagram of  $L$ . Using the crossing formulas, we define the polynomial evaluation for a link diagram  $D$ , denoted by  $\langle D \rangle_{(\varpi_{i_1}, \varpi_{i_2}, \dots, \varpi_{i_k})}$ ,  $i_j \in \{1, 2\}$  and  $j = 1, \dots, k$ , as follows: First, replace each component  $D_j$  with the double line of  $G_2$  web diagram if  $\varpi_{i_j} = \varpi_2$ . (We regard  $D_j$  as the single line of  $G_2$  web diagram if  $\varpi_{i_j} = \varpi_1$ .) Next, apply the crossing formulas in Theorem 2 to all crossings of the replaced diagram of  $D$ . The polynomial  $\langle D \rangle_{(\varpi_{i_1}, \varpi_{i_2}, \dots, \varpi_{i_k})}$  is defined to be the polynomial which is the evaluation of the above linear sum of  $G_2$  web diagrams by using the relations in Definition 1 and Proposition 1.

**THEOREM 3.** *For an oriented link  $L$ ,*

$$(q^{-12})^{\omega_{11}(D)} (q^{-24})^{\omega_{22}(D)} \langle D \rangle_{(\varpi_{i_1}, \varpi_{i_2}, \dots, \varpi_{i_k})}$$

*is a link invariant of  $L$ , where  $D$  is a link diagram of  $L$  and  $\omega_{11}(D)$  (resp.  $\omega_{22}(D)$ ) is the number of positive crossings of single edge on  $D$  minus the number of negative crossings of single edge (resp. the number of positive crossings of double edge minus the number of negative crossings of double edge).*

The link invariant is Reshetikhin-Turaev's quantum link invariant associated to the  $U_q(G_2)$  fundamental representations, called  $G_2$  quantum invariant for short. Denote by  $P_{(\varpi_{i_1}, \varpi_{i_2}, \dots, \varpi_{i_k})}(L)$  the  $G_2$  quantum invariant of an oriented link  $L$ .

In the following, we determine the  $G_2$  quantum invariant of the generalized twist link  $TW(m, n)$  in Figure 5. The box of  $TW(m, n)$  is the tangle diagram with  $n$ -crossing in Figure 6. Denote by  $Cr(n)$  the box illustrated in Figure 6.

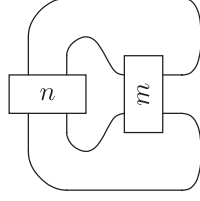
The evaluation  $\langle Cr(n) \rangle_{(\varpi_1, \varpi_1)}$  is given by the following formula:

$$\begin{aligned} \langle Cr(n) \rangle_{(\varpi_1, \varpi_1)} &= (q^2 P_{11}[2\varpi_1] - q^{-6} P_{11}[\varpi_1] - P_{11}[\varpi_2] + q^{-12} P_{11}[0])^n \\ &= q^{2n} \left( + A_{11}^{(n)} \begin{array}{c} \frown \\ \smile \end{array} + B_{11}^{(n)} \begin{array}{c} \frown \\ \smile \\ \frown \\ \smile \end{array} + C_{11}^{(n)} \begin{array}{c} \frown \\ \smile \\ \frown \\ \smile \\ \frown \\ \smile \end{array} \right), \end{aligned}$$

where

$$\begin{aligned} A_{11}^{(n)} &= \frac{[4][6]}{[2][7][12]} (-q^{2n} + q^{-12n}), & B_{11}^{(n)} &= \frac{[4]}{[3][8]} (q^{2n} - (-q^{-6})^n), \\ C_{11}^{(n)} &= \frac{1}{[2][3]} (q^{2n} - (-1)^n). \end{aligned}$$



Fig. 5. Generalized twist link  $TW(m, n)$ 

$$Cr(n) := \begin{array}{c} \text{---} \\ | \\ \boxed{n} \\ | \\ \text{---} \end{array} = \begin{cases} \left. \begin{array}{c} \text{---} \\ \diagdown \diagup \\ \text{---} \\ \vdots \\ \text{---} \\ \diagdown \diagup \\ \text{---} \end{array} \right\} n \text{ crossings} & \text{if } n \geq 0 \\ \left. \begin{array}{c} \text{---} \\ \diagup \diagdown \\ \text{---} \\ \vdots \\ \text{---} \\ \diagup \diagdown \\ \text{---} \end{array} \right\} -n \text{ crossings} & \text{if } n < 0 \end{cases}$$

Fig. 6.  $n$ -crossing

Using the above evaluation of  $\langle Cr(n) \rangle_{(\varpi_1, \varpi_1)}$ , we obtain the following:

$$\begin{aligned} P_{(\varpi_1, \varpi_1)}(TW(m, n)) &= q^{-12\omega(TW(m, n))} \langle TW(m, n) \rangle_{(\varpi_1, \varpi_1)} \\ &= q^{-12\omega(TW(m, n))} \left\{ q^{2n-12m} \frac{[2][7][12]}{[4][6]} \right. \\ &\quad + A_{11}^{(n)} \frac{[2][7]}{[4]} \left( q^{2m} \frac{[2][7][12]^2}{[4][6]^2} + A_{11}^{(m)} \frac{[12]}{[6]} - B_{11}^{(m)} \frac{[3][8][12]}{[4][6]} - C_{11}^{(m)} \frac{[8][15]}{[5]} \right) \\ &\quad + B_{11}^{(n)} \frac{[3][7][8]}{[4]} \left( -A_{11}^{(m)} \frac{[2][12]}{[4][6]} - B_{11}^{(m)} \frac{[12]}{[4]} + C_{11}^{(m)} \frac{[2][15]}{[5]} \right) \\ &\quad \left. + C_{11}^{(n)} \frac{[2][7][8][15]}{[5]} \left( -A_{11}^{(m)} \frac{1}{[4]} + B_{11}^{(m)} \frac{[3]}{[4]} + C_{11}^{(m)} \frac{[3][6]}{[12]} (q^2 - 2 + q^{-2}) \right) \right\}. \end{aligned}$$

Similarly, when  $m$  and  $n$  are odd integers, we obtain the following:

$$\begin{aligned} P_{(\varpi_1, \varpi_2)}(TW(m, n)) &= \langle TW(m, n) \rangle_{(\omega_1, \omega_2)} \\ &= A_{12}^{(n)} \frac{[2][6][7][8][15]}{[5][12]} \left( A_{12}^{(m)} [3](q^2 - 2 + q^{-2}) - B_{12}^{(m)} \frac{[3][15]}{[5]} + C_{12}^{(m)} \frac{[8][15]}{[4][5]} \right) \\ &\quad - B_{12}^{(n)} \frac{[2][3][6][7][8][15]}{[5][12]} \left( A_{12}^{(m)} \frac{[15]}{[5]} + B_{12}^{(m)} [3]T - C_{12}^{(m)} \frac{[15]}{[5]} \right) \\ &\quad + C_{12}^{(n)} \frac{[2][6][7][8][15]}{[5][12]} \left( A_{12}^{(m)} \frac{[8][15]}{[4][5]} - B_{12}^{(m)} \frac{[3][15]}{[5]} + C_{12}^{(m)} [3](q^2 - 2 + q^{-2}) \right), \end{aligned}$$

where

$$\begin{aligned} T &= q^{-12} - q^{-8} + 2q^{-6} + q^{-4} - q^{-2} + 1 - q^2 + q^4 + 2q^6 - q^8 + q^{12}, \\ A_{12}^{(m)} &= -(-q^{-12})^m \frac{[5][12]}{[6][8][15]} + q^{-4m} \frac{[3][4]}{[2][7][8]} + q^{3m} \frac{[3][5]([4][14] + [7])}{[7]^2[15]}, \\ B_{12}^{(m)} &= q^{-4m} \frac{1}{[2][7]} + q^{3m} \frac{[4]}{[2][3][7]}, \quad C_{12}^{(m)} = q^{3m} \frac{1}{[3]}. \end{aligned}$$

When  $m, n \in \mathbb{Z}$ , we obtain the following:

$$\begin{aligned} &P_{(\varpi_2, \varpi_2)}(TW(m, n)) \\ &= q^{-24\omega(TW(m, n))} \langle TW(m, n) \rangle_{(\omega_2, \omega_2)} := q^{-24\omega(TW(m, n))} \left\{ A_{22}^{(n)} q^{-24m} \frac{[7][8][15]}{[3][4][5]} \right. \\ &\quad + B_{22}^{(n)} \frac{[7][8][15]}{[4][5]} \left( A_{22}^{(m)} \frac{[7][8][15]}{[3]^2[4][5]} + B_{22}^{(m)} \frac{1}{[3]} - C_{22}^{(m)} \frac{[2]^2[12][18]}{[3]^3[4][9]} + E_{22}^{(m)} \frac{[2][6][8][15]}{[5][12]} \right) \\ &\quad - C_{22}^{(n)} \frac{[2][7][8][15][18]}{[3]^2[4][5][9]} \left( B_{22}^{(m)} \frac{[2][12]}{[3][4]} + C_{22}^{(m)} \frac{[2]^2[12]^2([3][18] - [2][9])}{[3]^2[4]^2[6][9]} \right. \\ &\quad \quad \left. + D_{22}^{(m)} \frac{[2]^3[12]^2[18]}{[3]^3[4]^2[9]} - E_{22}^{(m)} \frac{[6]^2[18]}{[9]} \right) \\ &\quad - D_{22}^{(n)} \frac{[2][7][8][15][18]}{[3]^2[4][5][9]} \left( A_{22}^{(m)} \frac{[2][12]}{[3][4]} + C_{22}^{(m)} \frac{[2]^3[12]^2[18]}{[3]^3[4]^2[9]} \right. \\ &\quad \quad \left. + D_{22}^{(m)} \frac{[2]^2[12]^2([3][18] - [2][9])}{[3]^2[4]^2[6][9]} - E_{22}^{(m)} \frac{[6]^2[18]}{[9]} \right) \\ &\quad \left. + E_{22}^{(n)} \frac{[2][6][7][8][15]}{[5]} \left( A_{22}^{(m)} \frac{[8][15]}{[4][5][12]} + B_{22}^{(m)} \frac{[8][15]}{[4][5][12]} - C_{22}^{(m)} \frac{[6][18]^2}{[3]^2[4][9]^2} \right. \right. \\ &\quad \quad \left. \left. - D_{22}^{(m)} \frac{[6][18]^2}{[3]^2[4][9]^2} + E_{22}^{(m)} \frac{[3][4][6]^2[7][8][15]}{[5][12]^2} U \right) \right\}, \end{aligned}$$

where

$$\begin{aligned} U &= \frac{[35]}{[7]} + 2 \frac{[3][25]}{[5]} + \frac{[8][21]}{[4][7]} + 2 \frac{[2][8][10]}{[4][5]} + 3, \\ A_{22}^{(m)} &= \frac{[3][4]}{[12]} (q^{6m} [5] (q^2 - 2 + q^{-2}) + (-1)^m), \\ B_{22}^{(m)} &= \frac{[3][4][5]}{[8]} \left( -q^{6m} \frac{[3][14]}{[7][12][15]} + (-1)^m \frac{[2][4]}{[10][12]} \right. \\ &\quad \left. - q^{-10m} \frac{[2][3][6]}{[7][10][12]} + q^{-24m} \frac{1}{[7][15]} \right), \\ C_{22}^{(m)} &= \frac{[3]^2[4]}{[2]^2[12]} \left( q^{6m} \frac{[4][6][9]([4][14] - [7])}{[7][8][12][18]} - (-1)^m \frac{[3]^2[4][5]}{[10][12]} \right. \\ &\quad \left. + q^{-10m} \frac{[3][4][5][6]^2}{[2][8][10][12]} - (-q^{-12})^m \frac{[9]}{[18]} \right), \\ D_{22}^{(m)} &= (q^{6m} - (-1)^m) \frac{[3]^2[4]^2[6]}{[2]^2[12]^2}, \\ E_{22}^{(m)} &= q^{6m} \frac{[5]}{[6][8]} - (-1)^m \frac{[4][5]}{[2][6][10]} + q^{-10m} \frac{[5]}{[8][10]}. \end{aligned}$$

- REMARK 2. (1) If either (i)  $m$  and  $n$  are even or (ii)  $m$  is even and  $n$  is odd, then  $\omega(TW(m, n)) = m - n$ . If  $m$  is odd and  $n$  is even, then  $\omega(TW(m, n)) = n - m$ . If  $n$  and  $m$  are odd,  $TW(m, n)$  is a link. Therefore the number  $\omega(TW(m, n))$  is  $n - m$  or  $m - n$ .
- (2) Since  $TW(0, 0)$  is the 2-component trivial link,  $P_{(\omega_1, \omega_1)}(TW(0, 0)) = \frac{[2]^2[7]^2[12]^2}{[4]^2[6]^2}$  and  $P_{(\omega_2, \omega_2)}(TW(0, 0)) = \frac{[7]^2[8]^2[15]^2}{[3]^2[4]^2[5]^2}$ .
- (3) Since  $TW(m, 0)$  and  $TW(0, m)$  ( $m \leq -1$ ,  $1 \leq m$ ) are the trivial knot, we have  $P_{(\omega_1, \omega_1)}(TW(m, 0)) = P_{(\omega_1, \omega_1)}(TW(0, m)) = \frac{[2][7][12]}{[4][6]}$  and  $P_{(\omega_2, \omega_2)}(TW(m, 0)) = P_{(\omega_2, \omega_2)}(TW(0, m)) = \frac{[7][8][15]}{[3][4][5]}$ .
- (4) Since  $TW(-1, n-1)$ ,  $TW(n-1, -1)$ ,  $TW(1, n+1)$  and  $TW(n+1, 1)$  are the  $(2, n)$ -torus link, we find these  $G_2$  link invariant associated to the fundamental representations are the same evaluation.
- (5) By the up-down symmetry of the generalized twist link  $TW(m, n)$ , we have  $P_{(\omega_2, \omega_1)}(TW(m, n)) = P_{(\omega_1, \omega_2)}(TW(m, n))$ .

### Appendix A. Proof of Lemma 1

Here, we give proofs of Lemma 1 (Fp1), (Fp2) and (Fp3). (The proofs of Lemma 1 (Fn1), (Fn2) and (Fn3) are similar.)

Proof of Lemma 1 (Fp1): By Identity (1), the left-hand side of (Fp1) is equal to

$$\begin{aligned}
& \frac{q^6}{[2]^2} \text{diagram}_1 + \frac{1}{[2]^2} \text{diagram}_2 + \frac{q^2}{[2]^2} \text{diagram}_3 + \frac{q^4}{[2]^2} \text{diagram}_4 + \frac{1}{[2]^2} \text{diagram}_5 + \frac{q^{-6}}{[2]^2} \text{diagram}_6 \\
& + \frac{q^{-4}}{[2]^2} \text{diagram}_7 + \frac{q^{-2}}{[2]^2} \text{diagram}_8 + \frac{q^2}{[2]^2} \text{diagram}_9 + \frac{q^{-4}}{[2]^2} \text{diagram}_{10} + \frac{q^{-2}}{[2]^2} \text{diagram}_{11} + \frac{1}{[2]^2} \text{diagram}_{12} \\
& + \frac{q^4}{[2]^2} \text{diagram}_{13} + \frac{q^{-2}}{[2]^2} \text{diagram}_{14} + \frac{1}{[2]^2} \text{diagram}_{15} + \frac{q^2}{[2]^2} \text{diagram}_{16}. \tag{7}
\end{aligned}$$

Using relations in Section 2, we have the following identities:

$$\begin{aligned}
& \text{diagram}_1 = 0, \quad \text{diagram}_2 = -\frac{[3][8]}{[4]} \text{diagram}_3, \quad \text{diagram}_5 = -\frac{[3][8]}{[4]} \text{diagram}_4, \\
& \text{diagram}_7 = \frac{[6]}{[2]} \text{diagram}_8, \quad \text{diagram}_9 = \frac{[6]}{[2]} \text{diagram}_{10}, \quad \text{diagram}_{11} = \frac{[6]}{[2]} \text{diagram}_{12}, \\
& \text{diagram}_{13} = [3] \text{diagram}_{14} + [3] \text{diagram}_{15} - \frac{[4]}{[2]} \text{diagram}_{16} - \frac{[4]}{[2]} \text{diagram}_{17}, \\
& \text{diagram}_{18} = [3] \text{diagram}_{19} + [3] \text{diagram}_{20} - \frac{[4]}{[2]} \text{diagram}_{21} - \frac{[4]}{[2]} \text{diagram}_{22},
\end{aligned}$$

$$\begin{aligned} \text{Diagram} &= \text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3 + \text{Diagram}_4 + \text{Diagram}_5 \\ &\quad - (\text{Diagram}_6 + \text{Diagram}_7 + \text{Diagram}_8 + \text{Diagram}_9 + \text{Diagram}_{10}). \end{aligned}$$

By these identities, the linear sum (7) is equal to

$$\frac{q^3}{[2]} \text{Diagram}_1 + \frac{q^{-3}}{[2]} \text{Diagram}_2 + \frac{q^{-1}}{[2]} \text{Diagram}_3 + \frac{q}{[2]} \text{Diagram}_4.$$

We see, by Identity (1), that this is equal to the right-hand side of Identity (Fp1).

Proof of Lemma 1 (Fp2): By Identity (1), the left-hand side of (Fp2) is equal to

$$\begin{aligned} &\frac{q^6}{[2]^2} \text{Diagram}_1 + \frac{1}{[2]^2} \text{Diagram}_2 + \frac{q^2}{[2]^2} \text{Diagram}_3 + \frac{q^4}{[2]^2} \text{Diagram}_4 + \frac{1}{[2]^2} \text{Diagram}_5 + \frac{q^{-6}}{[2]^2} \text{Diagram}_6 \\ &+ \frac{q^{-4}}{[2]^2} \text{Diagram}_7 + \frac{q^{-2}}{[2]^2} \text{Diagram}_8 + \frac{q^2}{[2]^2} \text{Diagram}_9 + \frac{q^{-4}}{[2]^2} \text{Diagram}_{10} + \frac{q^{-2}}{[2]^2} \text{Diagram}_{11} + \frac{1}{[2]^2} \text{Diagram}_{12} \\ &+ \frac{q^4}{[2]^2} \text{Diagram}_{13} + \frac{q^{-2}}{[2]^2} \text{Diagram}_{14} + \frac{1}{[2]^2} \text{Diagram}_{15} + \frac{q^2}{[2]^2} \text{Diagram}_{16}. \end{aligned} \quad (8)$$

Using relations in Section 2, we have the following identities:

$$\begin{aligned} \text{Diagram}_1 &= 0, \text{Diagram}_2 = 0, \text{Diagram}_3 = 0, \\ \text{Diagram}_4 &= -[3] \text{Diagram}_5, \text{Diagram}_6 = -[3] \text{Diagram}_7, \text{Diagram}_8 = -[3] \text{Diagram}_9, \\ \text{Diagram}_{10} &= \text{Diagram}_{11} + \text{Diagram}_{12}, \text{Diagram}_{13} = \text{Diagram}_{14} + \text{Diagram}_{15}, \\ \text{Diagram}_{16} &= \text{Diagram}_{17} - \frac{[4][6]}{[2][12]} \text{Diagram}_{18} - \frac{1}{[3]} \text{Diagram}_{19} + \frac{[2]}{[3]} \text{Diagram}_{20} \\ &= \text{Diagram}_{17} + \frac{[3][4][6]}{[2][12]} \text{Diagram}_{18} - \frac{1}{[3]} (\text{Diagram}_{19} + \text{Diagram}_{20}) + \frac{[2]}{[3]} \text{Diagram}_{21}. \end{aligned}$$

By these identities, the linear sum (8) is equal to

$$\begin{aligned} &-\frac{q^3}{[2]} \text{Diagram}_1 + \frac{(q^{-3} + q^3)[4][6]}{[2]^2[12]} \text{Diagram}_2 + \frac{q^3}{[2][3]} \text{Diagram}_3 + \frac{q^3}{[2]} \text{Diagram}_4 \\ &- \frac{q^{-3}}{[2]} \text{Diagram}_5 + \frac{q^{-3}}{[2]} \text{Diagram}_6 + \frac{q^{-3}}{[2][3]} \text{Diagram}_7 + \frac{1}{[2][3]} \text{Diagram}_8. \end{aligned}$$

By Relation (Double edge elimination) in Definition 1, we have the following identities:

$$\begin{aligned} \text{Diagram 1} &= [3] \text{Diagram 2} - \frac{[3][4][6]}{[2][12]} \text{Diagram 3} - [3] \text{Diagram 4} + [2] \text{Diagram 5}, \\ \text{Diagram 6} &= [3] \text{Diagram 7} - \frac{[3][4][6]}{[2][12]} \text{Diagram 8} - [3] \text{Diagram 9} + [2] \text{Diagram 10}. \end{aligned}$$

Therefore, the linear sum (8) is equal to

$$\frac{q^3}{[3]} \text{Diagram 11} + \frac{q^{-3}}{[3]} \text{Diagram 12} + \frac{1}{[2][3]} \text{Diagram 13}.$$

We see, by Identity (3), that this is equal to the right-hand side of Identity (Fp2).

**Proof of Lemma 1 (Fp3):** By Identity (2), the left-hand side of (Fp3) is equal to

$$\begin{aligned} &\frac{q^6}{[3]^2} \text{Diagram 14} + \frac{1}{[3]^2} \text{Diagram 15} + \frac{q^3}{[2][3]^2} \text{Diagram 16} + \frac{1}{[3]^2} \text{Diagram 17} + \frac{q^{-6}}{[3]^2} \text{Diagram 18} \\ &+ \frac{q^{-3}}{[2][3]^2} \text{Diagram 19} + \frac{q^3}{[2][3]^2} \text{Diagram 20} + \frac{q^{-3}}{[2][3]^2} \text{Diagram 21} + \frac{1}{[2]^2[3]^2} \text{Diagram 22} \end{aligned} \quad (9)$$

Using relations in Section 2, we have the following identities:

$$\begin{aligned} \text{Diagram 23} &= -\frac{[3][4][6](q^{-2} - 2 + q^2)}{[12]} \text{Diagram 24} \\ \text{Diagram 25} &= \frac{[3][4][6][10]}{[5][12]} \text{Diagram 26} + \frac{[3][4][6]}{[2][12]} \text{Diagram 27} + \frac{[4][6]^2}{[2][3][12]} \text{Diagram 28} \\ \text{Diagram 29} &= \frac{[3][4][6][10]}{[5][12]} \text{Diagram 30} + \frac{[3][4][6]}{[2][12]} \text{Diagram 31} + \frac{[4][6]^2}{[2][3][12]} \text{Diagram 32} \\ \text{Diagram 33} &= -\frac{[3][4]^2[6]}{[12]} \text{Diagram 34} - \frac{[3][4]^2[6]}{[12]} \text{Diagram 35} - \left( \frac{[4][6]^2}{[2][12]} + [4] \right) \text{Diagram 36} \\ \text{Diagram 37} &= -\frac{[3][4]^2[6]}{[12]} \text{Diagram 38} - \frac{[3][4]^2[6]}{[12]} \text{Diagram 39} - \left( \frac{[4][6]^2}{[2][12]} + [4] \right) \text{Diagram 40} \\ \text{Diagram 41} &= [3] \text{Diagram 42} - \frac{[3]^2[4]^2[6]^3}{[2]^2[12]^2} \text{Diagram 43} + \frac{[3][4][6][10]}{[2][5][12]} \text{Diagram 44} - [4] \text{Diagram 45} \\ \text{Diagram 46} &= -2 \frac{[4][6]}{[12]} \text{Diagram 47} - \frac{[3][4]^2[6]}{[12]} \text{Diagram 48} + \frac{[2][3][4][6]}{[12]} \text{Diagram 49} \\ &+ [2] \text{Diagram 50} + \frac{[3]^3[4]^2[6]^2}{[2][12]^2} \text{Diagram 51} + \frac{[3]^3[4]^2[6]^2}{[2][12]^2} \text{Diagram 52} \end{aligned}$$

$$\begin{aligned}
& \text{Diagram 1} = -2 \frac{[4][6]}{[12]} \text{Diagram 2} - \frac{[3][4]^2[6]}{[12]} \text{Diagram 3} + \frac{[2][3][4][6]}{[12]} \text{Diagram 4} \\
& \quad + [2] \text{Diagram 5} + \frac{[3]^3[4]^2[6]^2}{[2][12]^2} \text{Diagram 6} + \frac{[3]^3[4]^2[6]^2}{[2][12]^2} \text{Diagram 7} \\
& \text{Diagram 8} = + \frac{[2][3][4][6]}{[12]} \text{Diagram 9} + \frac{[2][3][4][6]}{[12]} \text{Diagram 10} + \frac{[2][3][4][6]}{[12]} \text{Diagram 11} \\
& \quad + \frac{[2][3][4][6]}{[12]} \text{Diagram 12} + 2 \frac{[2][4][6]^2}{[3][12]} \text{Diagram 13} + [2]^3 \text{Diagram 14}
\end{aligned}$$

Therefore, the linear sum (9) is equal to

$$\begin{aligned}
& \frac{[4][6](q^{10} - q^6 - q^4)}{[2][12]} \text{Diagram 15} + \frac{[4][6](q^{-10} - q^{-6} - q^{-4})}{[2][12]} \text{Diagram 16} \\
& + \frac{q^{-3}[3][4]^2[6]^2}{[2]^2[12]^2} \text{Diagram 17} + \frac{q^3[3][4]^2[6]^2}{[2]^2[12]^2} \text{Diagram 18} + \frac{1}{[3]} \text{Diagram 19}
\end{aligned}$$

We see, by Identity (4), that this is equal to the right-hand side of Identity (Fp3).

### Acknowledgements

We thank Greg Kuperberg for comments on the earlier version of our paper. We also thank Takahiro Hayashi, Tomoki Nakanishi, Soichi Okada and Kenichi Shimizu for comments on our work and Makoto Sakuma for carefully reading the manuscript. The second author thanks Yoshiyuki Kimura and Jun Murakami for helpful discussions.

### References

- [1] V. G. Drinfel'd, Quantum groups, In Proceedings of the International Congress of Mathematicians, Berkeley, California, (1987), 798–820.
- [2] J. Huang and C. Zhu, Weyl's construction and tensor power decomposition for  $G_2$ , Proc. Amer. Math. Soc., **127** (1999), 925–934.
- [3] M. Jimbo, A  $q$ -difference analogue of  $U_q(\mathfrak{g})$  and the Yang-Baxter equation, Lett. Math. Phys., **10** (1985), 63–69.
- [4] L. H. Kauffman, An invariant of regular isotopy, Trans. Amer. Math. Soc., **318** (1990), 417–471.
- [5] G. Kuperberg, The quantum  $G_2$  link invariant, International Journal of Mathematics, **5** (1994), 61–85.
- [6] G. Kuperberg, Spiders for rank 2 Lie algebras, Comm. Math. Phys., **180** (1996), 109–151.
- [7] G. I. Lehrer and R. B. Zhang, Strongly multiplicity free modules for Lie algebras and quantum groups, J. Algebra, **306** (2006), 138–174.

- [8] S. Morrison, E. Peters and N. Snyder, Knot polynomial identities and quantum group coincidences, *Quantum Topol.*, **2** (2011), 101–156.
- [9] N. Yu. Reshetikhin and V. G. Turaev, Ribbon graphs and their invariants derived from quantum groups, *Comm. Math. Phys.*, **127** (1990), 1–26.
- [10] T. Sakamoto, Link invariant of  $G_2$  quantum group and fundamental representations (in Japanese), Master thesis, Nagoya University, (2015).
- [11] G. E. Schwarz, Invariant theory of  $G_2$  and  $Spin_7$ , *Comment. Math. Helv.*, **63** (1988), 624–663.

*Takuro Sakamoto*

*Graduate School of Mathematics, Nagoya University*  
*Current address: Furocho Chikusaku Nagoya 464-8602, Japan*  
*E-mail: ml3029e@math.nagoya-u.ac.jp*

*Yasuyoshi Yonezawa*

*Institute for Advanced Research, Nagoya University*  
*Graduate School of Mathematics, Nagoya University*  
*Current address: Furocho Chikusaku Nagoya 464-8602, Japan*  
*E-mail: yasuyoshi.yonezawa@math.nagoya-u.ac.jp*