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The number of paperfolding curves in a covering of the plane

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Abstract. This paper completes our previous one in the same journal (vol. 42, pp. 37–75). Let $\mathscr{C}$ be a covering of the plane by disjoint complete folding curves which satisfies the local isomorphism property. We show that $\mathscr{C}$ is locally isomorphic to an essentially unique covering generated by an ∞-folding curve. We prove that $\mathscr{C}$ necessarily consists of 1, 2, 3, 4 or 6 curves. We give examples for each case; the last one is realized if and only if $\mathscr{C}$ is generated by the alternating folding curve or one of its successive antiderivatives. We also extend the results of our previous paper to another class of paperfolding curves introduced by M. Dekking.

1. Definitions, context and main results

In order to simplify some notations, we identify $\mathbf{R}^2$ with $\mathbf{C}$ and $\mathbf{Z}^2$ with the set $\mathbf{Z}+i\mathbf{Z}$ of Gaussian integers. We denote by $\mathbf{N}^*$ the set of strictly positive integers.

Concerning folding sequences and folding curves, we use the definitions, the notations and the results of [4].

We consider sequences $(a_k)_{1\leq k\leq n-1}$, $(a_k)_{k\in\mathbf{N}^*}$, $(a_k)_{k\in\mathbf{Z}}$ with $a_k=\pm 1$ for each k and associated *curves* $(C_k)_{1\leq k\leq n}$, $(C_k)_{k\in\mathbf{N}^*}$, $(C_k)_{k\in\mathbf{Z}}$ such that:

(a) each *segment* C_k is an oriented interval $[x_k, x_k+\varepsilon_k]$ with $x_k\in\mathbf{Z}+i\mathbf{Z}$ and $\varepsilon_k\in\{1,-1,i,-i\}$;

(b) if C_k and C_{k+1} exist, then $x_k+\varepsilon_k=x_{k+1}$ and $\varepsilon_{k+1}=i^{a_k}\varepsilon_k$; moreover the curve is "rounded" in x_{k+1} so that it does not pass through that point.

We call $(C_k)_{k\in\mathbf{Z}}$ a *complete curve*. For each $S=(a_k)_{1\leq k\leq n-1}$ (resp. $(a_k)_{k\in\mathbf{N}^*}$), we write $\overline{S}=(-a_{n-k})_{1\leq k\leq n-1}$ (resp. $(-a_{-k})_{k\in-\mathbf{N}^*}$).

We say that a set of curves $\mathscr{C}$ *covers* the plane (resp. the square $S=\{x+u+iv \mid u,v\in[0,m]\}$ with $x\in\mathbf{Z}+i\mathbf{Z}$ and $m\in\mathbf{N}^*$) if each nonoriented interval $[z,z+\varepsilon]\subset\mathbf{C}$ (resp. $[z,z+\varepsilon]\subset S$) with $z\in\mathbf{Z}+i\mathbf{Z}$ and $\varepsilon\in\{1,i\}$ is the support of exactly one segment of one curve of $\mathscr{C}$. A set of complete curves which covers the plane will be called a *covering*.

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Two sets of curves are considered isomorphic if they are equivalent up to translation. The notion of local isomorphism is defined in [4, p. 58]: two coverings $\mathscr{C}$, $\mathscr{D}$ are *locally isomorphic* if each bounded fragment of $\mathscr{C}$ (resp. $\mathscr{D}$) is isomorphic to a fragment of $\mathscr{D}$ (resp. $\mathscr{C}$); any covering $\mathscr{C}$ satisfies the *local isomorphism property* if each bounded fragment of $\mathscr{C}$ has copies everywhere in $\mathscr{C}$.

For each $n \in \mathbf{N}^*$, we consider *n-folding sequences* $(a_k)_{1 \le k \le 2^n - 1}$ and associated *n-folding curves* $(C_k)_{1 \le k \le 2^n}$, obtained by folding n times a strip of paper in two, each time possibly to the left or to the right, then unfolding it with right angles. These curves, rounded as it is mentioned above, are self-avoiding. We also consider ∞*-folding sequences* $(a_k)_{k \in \mathbf{N}^*}$, where each $(a_1, \dots, a_{2^n - 1})$ is an n-folding sequence, and associated ∞*-folding curves*.

These two types of folding sequences and curves have been considered by various authors (see for instance [1] and [3]).

In [4], we introduced *complete folding sequences* $(a_k)_{k \in \mathbf{Z}}$, where each tuple $(a_{k+1}, \dots, a_{k+l})$ is a subsequence of an n-folding sequence for an integer n, and associated *complete folding curves*. For each ∞-folding sequence S, $(\overline{S}, +1, S)$ and $(\overline{S}, -1, S)$ are complete folding sequences.

One motivation for introducing them came from two *plane filling* properties which are mentioned by various authors:

First, by [4, Th. 3.1], for each $m \in \mathbf{N}^*$, there exists $n \in \mathbf{N}^*$ such that each n-folding curve covers a square $[x, x+m] \times [y, y+m]$ for some $(x, y) \in \mathbf{Z}^2$. It follows that each ∞-folding or complete folding curve covers arbitrarily large squares.

Second, by [4, Th. 3.15], for each ∞-folding curve C associated to an ∞-folding sequence S, the 4 curves obtained from C by rotations of angles 0, $\pi/2$, π, $3\pi/2$ around its origin are disjoint. They can be connected in two different ways in order to form 2 complete folding curves, both associated to $(\overline{S}, +1, S)$ or both associated to $(\overline{S}, -1, S)$.

These 2 curves form a covering, except for C belonging to a particular class. By [4, Th. 3.15], if C belongs to that class, then the 2 curves can be completed in a unique way to form a covering by 6 curves. The 4 other curves are not associated to sequences of the form $(\overline{T}, +1, T)$ or $(\overline{T}, -1, T)$.

All these coverings satisfy the local isomorphism property. According to Theorem 4 below, any covering which satisfies the local isomorphism property is locally isomorphic to one of them.

More generally, it follows from [4, Th. 3.10] that each complete folding curve can be completed in an essentially unique way into a covering which satisfies the local isomorphism property. By Theorem 3 below, in such a covering, each bounded fragment appears with a well determined density.

By [4, Th. 3.12], any such covering contains at most 6 curves. In the last part of Section 2, we show that it can actually contain 1, 2, 3 or 4 curves. We also prove that it cannot contain 5 curves and that it contains 6 curves only in the particular case described above. We also consider the following question: If $\mathscr{C}$ is such a covering, for which integers n does there exist a covering by n curves which is locally isomorphic to $\mathscr{C}$?

In [2], M. Dekking considers another notion of folding sequence. He calls a *folding string* any sequence $(a_1, \ldots, a_{m-1})$ in $\{-1, +1\}$. He introduces the *folding convolution* $S * T$ of two folding strings S, T. For each folding string S, he defines the sequences S^{*n} with $S^{*1} = S$ and $S^{*(n+1)} = S^{*n} * S$ for each $n \in \mathbf{N}^*$. Then he considers $S^{*\infty} = \bigcup_{n \in \mathbf{N}^*} S^{*n}$.

He gives characterizations of the folding strings S which satisfy (1) (resp. (1) and (2), resp. (1) and (2) and (3)) for the properties (1), (2), (3) below:

(1) $S^{*\infty}$ is self-avoiding;

(2) each curve associated to $S^{*\infty}$ covers arbitrarily large squares;

(3) S is *perfect* in the sense that, for each curve C associated to $S^{*\infty}$, the plane is covered by the 4 curves obtained from C by rotations of angles 0, $\pi/2$, π, $3\pi/2$ around its origin.

Many examples of ∞-folding sequences are actually constructed in that way, including the positive folding sequence associated to the dragon curve (see [4, Example 3.13]) and the alternating folding sequence (see [4, Example 3.14]). Some of them are used in the present paper.

Some results similar to those of [4] are also true for folding sequences and curves in Dekking's sense:

For each folding string $S = (a_1, \ldots, a_{m-1})$, we call a *complete S-folding sequence* any sequence $T = (b_k)_{k \in \mathbf{Z}}$ such that $S^{*\infty}$ contains a copy of each $(b_{k+1}, \ldots, b_{k+l})$. A *complete S-folding curve* is a curve associated to such a sequence.

It follows from the properties of the convolution $*$ that $(\overline{S^{*\infty}}, +1, S^{*\infty})$ and $(\overline{S^{*\infty}}, -1, S^{*\infty})$ are complete S-folding sequences, that each complete S-folding sequence satisfies the local isomorphism property and that any two such sequences are locally isomorphic.

The following results are true for each folding string $S = (a_1, \ldots, a_{m-1})$ which satisfies the properties (1), (2) above. Their proofs are similar to [4]:

Any complete S-folding curve C is self-avoiding and covers arbitrarily large squares. We have a *derivation* on C such that m consecutive segments of C are replaced with one segment. The derivative C' of C is also a complete S-folding curve.

Each complete S-folding curve can be completed, in an essentially unique way, into a covering by such curves which satisfies the local isomorphism property. Moreover, all the coverings obtained in that way from complete

S-folding curves are locally isomorphic. It would be interesting to determine the number of curves which can appear in such a covering.

The 4 curves considered in the property (3) described above are disjoint. They can be connected in two different ways in order to form 2 complete S-folding curves associated to the string $(\overline{S^{*\infty}}, +1, S^{*\infty})$, or 2 complete S-folding curves associated to the string $(\overline{S^{*\infty}}, -1, S^{*\infty})$. If S satisfies (3), then these 2 curves form a covering which satisfies the local isomorphism property.

If S does not satisfy (3), then the 2 curves do not form a covering, but they can be completed in a unique way into a covering which satisfies the local isomorphism property. It follows from Theorem 2 below that this covering contains exactly 6 curves.

A simple example of that situation is obtained with $S = (+1, -1, -1)$. Then $S^{*\infty}$ is the alternating folding sequence. It follows from [4, Th. 3.15] that all coverings by 6 folding curves are obtained from ∞-folding sequences $R * S^{*\infty}$ with R finite.

Many other examples exist for folding curves in the sense of Dekking. One of them is given in [2, Fig. 18] with $S = (+1, -1, +1, +1, -1, -1, -1, +1, -1)$.

2. Detailed results and proofs

First we introduce some notions which will be useful both for classical folding curves and for folding curves in the sense of Dekking.

For each curve C, we denote by $\alpha(C)$ the initial point of the first segment of C and $\beta(C)$ the terminal point of the last segment of C, if they exist.

From now on, we consider sets $\mathcal{C}$ of disjoint self-avoiding curves such that, for each endpoint $z = x + iy \in \mathbf{Z} + i\mathbf{Z}$ of a segment of a curve of $\mathcal{C}$, one of the two following possibilities is realized, depending on the parity of $x + y$:

(a) the oriented segments of curves of $\mathcal{C}$ which have z as an endpoint are among $[z, z+1]$, $[z, z-1]$, $[z+i, z]$, $[z-i, z]$;

(b) they are among $[z+1, z]$, $[z-1, z]$, $[z, z+i]$, $[z, z-i]$.

We note that this property is necessarily true if $\mathcal{C}$ consists of one curve. It follows that it is also true if $\mathcal{C}$ is a covering which satisfies the local isomorphism property and if each curve in $\mathcal{C}$ covers arbitrarily large squares.

For any such sets $\mathcal{C}$, $\mathcal{D}$, we write $\mathcal{C} \simeq \mathcal{D}$ if there exists a translation τ such that $\tau(\mathcal{C}) = \mathcal{D}$. We write $\mathcal{C} < \mathcal{D}$ if $\mathcal{C} \neq \mathcal{D}$, if each curve of $\mathcal{C}$ is contained in a curve of $\mathcal{D}$, and if any consecutive segments of a curve of $\mathcal{D}$ which belong to curves of $\mathcal{C}$ are consecutive in one of them. We write $\mathcal{C} \ll \mathcal{D}$, if $\mathcal{C} < \mathcal{D}$ and if, for each segment A of a curve of $\mathcal{C}$, the 6 segments which form two squares with A belong to curves of $\mathcal{D}$.

If $\mathscr{C}$ and $\mathscr{D}$ are coverings, we say that a map Δ from the set of segments of $\mathscr{C}$ to the set of segments of $\mathscr{D}$ is a *derivation* if:

(a) there exists a sequence $S = (a_1, \dots, a_{m-1})$ in $\{+1, -1\}$ such that, for each segment A of a curve of $\mathscr{D}$, $\Delta^{-1}(A)$ is a subcurve of a curve of $\mathscr{C}$ associated to S or to $\bar{S}$, depending on the parity of $x + y$ where $\alpha(A) = x + iy$;

(b) for any consecutive segments A, B of a curve of $\mathscr{D}$, $\Delta^{-1}(A)$ and $\Delta^{-1}(B)$ are consecutive subcurves of a curve of $\mathscr{C}$.

(c) there exists a direct similitude σ such that $\sigma(\alpha(A)) = \alpha(\Delta^{-1}(A))$ and $\sigma(\beta(A)) = \beta(\Delta^{-1}(A))$ for each segment A of a curve of $\mathscr{D}$.

The composition of two derivations is a derivation. If $\mathscr{C}$ is a covering by folding curves which satisfies the local isomorphism property, then, by [4, Prop. 3.3], the derivation Δ defined in [4] satisfies the properties above with $m = 2$ and $\Delta(\mathscr{C})$ is a covering by folding curves which satisfies the local isomorphism property. For each folding string $S = (a_1, \dots, a_{m-1})$ which satisfies the properties (1), (2) considered in Section 1, we have a derivation Δ associated to S on each covering $\mathscr{C}$ by S-folding curves which satisfies the local isomorphism property and $\Delta(\mathscr{C})$ is a covering by S-folding curves which satisfies the local isomorphism property. In both cases, the n-th derivation Δ^n is defined on $\mathscr{C}$ for each $n \in \mathbf{N}$.

For each covering $\mathscr{C}$, each set of curves $\mathscr{F} < \mathscr{C}$, each derivation $\Delta : \mathscr{C} \to \mathscr{C}$ and each $n \in \mathbf{N}$ such that Δ^n is defined on $\mathscr{C}$, we denote by $\Delta^{-n}(\mathscr{F})$ the union of the sets $(\Delta^n)^{-1}(A)$ for A a segment of a curve of $\mathscr{F}$.

Proposition 1. *Consider a covering $\mathscr{C}$ with a derivation $\Delta : \mathscr{C} \to \mathscr{C}$, a set of curves $\mathscr{F} < \mathscr{C}$ and a translation τ such that $\tau(\mathscr{F}) \ll \Delta^{-1}(\mathscr{F})$. For each $n \in \mathbf{N}$, denote by τ_n the translation such that $\tau_n(\Delta^{-n}(\mathscr{F})) = \Delta^{-n}(\tau(\mathscr{F})) \subset \Delta^{-n-1}(\mathscr{F})$. Then the inductive limit of the sets $\Delta^{-n}(\mathscr{F})$ relative to the translations τ_n is a covering $\mathscr{D}$ with the same number of curves as $\mathscr{F}$. Moreover, $\mathscr{D}$ is locally isomorphic to $\mathscr{C}$ if $\mathscr{C}$ satisfies the local isomorphism property.*

Proof. It suffices to prove that $\tau_n(\Delta^{-n}(\mathscr{F})) \ll \Delta^{-n-1}(\mathscr{F})$ for each $n \in \mathbf{N}^*$. We show that, for each segment S of $\Delta^{-n}(\mathscr{F})$, the segments $S_1, \dots, S_6$ which form two squares with $\tau_n(S)$ all belong to $\Delta^{-n-1}(\mathscr{F})$.

Write $U_0 = V_0 = \tau(\Delta^n(S))$. Consider the segments $U_1, U_2, U_3, V_1, V_2, V_3 \in \Delta^{-1}(\mathscr{F})$ such that $U_0, \dots, U_4$ (resp. $V_0, \dots, V_4$) are consecutive segments of a square.

Consider the closed curve $A = \Delta^{-n}(U_0) \cup \dots \cup \Delta^{-n}(U_3)$ and the closed region P limited by A. Note that, for each $i \in \{0, \dots, 3\}$, the last segment of $\Delta^{-n}(U_i)$ and the first segment of $\Delta^{-n}(U_{i+1})$ form a right angle directed to the exterior of P (here we identify $3 + 1$ with 0). As $\mathscr{C}$ is a covering and $\Delta^{-n}(U_0), \dots, \Delta^{-n}(U_3)$ are subcurves of curves of $\mathscr{C}$, it follows that no curve of $\mathscr{C}$ can cross the frontier A of P. Consequently, the interior of P contains no segment.

We prove in the same way that the interior of the closed region Q limited by $B = \Delta^{-n}(V_0) \cup \cdots \cup \Delta^{-n}(V_3)$ contains no segment. As $S_1, \ldots, S_6$ are necessarily contained in $P \cup Q$, it follows that they belong to $A \cup B \subset \Delta^{-n-1}(\mathscr{F})$. ∎

Concerning folding curves in the sense of Dekking, we have:

THEOREM 2. *Let S be a folding string which satisfies the properties* (1), (2) *of Section* 1, *but not the property* (3). *Let $\mathscr{C}$ be a covering by S-folding curves which satisfies the local isomorphism property and contains a curve C associated to $(\overline{S^{*\infty}}, \mp 1, S^{*\infty})$. Then $\mathscr{C}$ contains exactly* 6 *curves.*

PROOF. We consider a covering $\mathscr{D}$ and a derivation $\Delta : \mathscr{C} \to \mathscr{D}$ associated to S. We have $\Delta(C) \cong C$. Consequently, we can suppose $\Delta(C) = C$, which implies $\mathscr{D} = \mathscr{C}$ since C can be extended into a unique covering which satisfies the local isomorphism property. We can also suppose without restricting the generality that S begins with $+1$ and that the two first segments of C are $[0, 1]$ and $[1, 1+i]$.

It follows from [2, Th. 5] and its proof that $\mathscr{C}$ contains at least 6 curves: Two of them including C are associated to $(\overline{S^{*\infty}}, \varepsilon, S^{*\infty})$ with $\varepsilon = \mp 1$. Each of these 2 curves contains 2 of the segments $[0, 1]$, $[0, -1]$, $[i, 0]$, $[-i, 0]$. On the other hand, they do not contain the segments $[1+i, i]$, $[-1, -1+i]$, $[-1-i, -i]$, $[1, 1-i]$, which necessarily belong to 4 other curves.

We are going to prove that $\mathscr{F} \ll \Delta^{-2}(\mathscr{F})$ for the set of 6 disjoint curves $\mathscr{F} < \mathscr{C}$ which consists of the 8 segments $[0, 1]$, $[0, -1]$, $[i, 0]$, $[-i, 0]$, $[1+i, i]$, $[-1, -1+i]$, $[-1-i, -i]$, $[1, 1-i]$. Then it follows from Proposition 1 applied to the derivation Δ^2 that $\bigcup_{n \in \mathbf{N}} \Delta^{-2n}(\mathscr{F}) \subset \mathscr{C}$ is a covering by 6 curves, and therefore $\mathscr{C}$ contains exactly 6 curves.

By symmetry, it suffices to show that the 6 segments which form 2 squares with $[1+i, i]$ belong to $\Delta^{-2}(\mathscr{F})$. Our hypotheses imply $[0, 1] \in \Delta^{-1}([0, 1])$, $[1, 1+i] \in \Delta^{-1}([0, 1])$, $[i, 0] \in \Delta^{-1}([i, 0])$. They also imply $[1+i, i] \in \Delta^{-1}([1+i, i])$ since $[1+i, i]$ does not belong to $\Delta^{-1}([0, 1])$, $\Delta^{-1}([1, 1+i])$ and $\Delta^{-1}([i, 0])$ which are contained in the 2 curves of $\mathscr{C}$ associated to $(\overline{S^{*\infty}}, \varepsilon, S^{*\infty})$.

As Δ satisfies the property (c) of the definition of derivations, there exists $z \in (\mathbf{Z} + i\mathbf{Z}) - \{0, 1, -1, i, -i\}$ such that $z = \beta(\Delta^{-1}([0, 1])) = \alpha(\Delta^{-1}([1, 1+i]))$, $(1+i)z = \beta(\Delta^{-1}([1, 1+i])) = \alpha(\Delta^{-1}([1+i, i]))$ and $iz = \beta(\Delta^{-1}([1+i, i])) = \alpha(\Delta^{-1}([i, 0]))$. It follows that $\alpha(\Delta^{-1}([1+i, i])) \neq 1+i$ and $\beta(\Delta^{-1}([1+i, i])) \neq i$. Consequently, $\Delta^{-1}([1+i, i])$ contains $[1+2i, 1+i]$ and $[i, 2i]$.

It follows that $[i, 0]$, $[0, 1]$, $[1, 1+i]$, $[1+2i, 1+i]$, $[i, 2i]$ belong to $\Delta^{-1}(\mathscr{F})$. Now it suffices to show that $[2i, 1+2i] \in \Delta^{-2}([1+i, i])$.

We observe that $\Delta^{-1}([1+2i, 1+i])$ and $\Delta^{-1}([i, 2i])$ necessarily have a common vertex since $\Delta^{-1}([0,1])$ and $\Delta^{-1}([1+i, i])$ have the common vertex $1+i$. As $\mathscr{C}$ is a covering, it follows that $[2i, 1+2i] \in \Delta^{-1}([1+2i, 1+i]) \cup \Delta^{-1}([1+i, i]) \cup \Delta^{-1}([i, 2i]) \subset \Delta^{-2}([1+i, i])$. ■

From now on, we consider a covering $\mathscr{C}$ by folding curves which satisfies the local isomorphism property. We do not mention the orientation of the curves when it is not necessary.

The definition of the sets $E_n(\mathscr{C})$ for $n \in \mathbf{N} \cup \{\infty\}$ and $F_n(\mathscr{C})$ for $n \in \mathbf{N}$ is given in [4]. Their existence follows from [4, Prop. 3.3]. When there is no ambiguity, we write E_n and F_n instead of $E_n(\mathscr{C})$ and $F_n(\mathscr{C})$.

For each $n \in \mathbf{N}$ and each $z \in E_n$, the 4 nonoriented subcurves of curves of $\mathscr{C}$ with endpoint z and length 2^n are all obtained from one of them by successive rotations of center z and angle $\pi/2$.

We say that $z = x + iy \in \mathbf{Z} + i\mathbf{Z}$ is *even* (resp. *odd*) if $x + y$ is even (resp. odd).

For each $n \in \mathbf{N}$ and each $u \in \mathbf{Z} + i\mathbf{Z}$, we have:

$$F_{2n} = \{u + 2^n v \mid v \in \mathbf{Z} + i\mathbf{Z} \text{ even}\} \qquad \text{if } u \in F_{2n};$$

$$E_{2n+1} = \{u + 2^n v \mid v \in \mathbf{Z} + i\mathbf{Z} \text{ even}\} \qquad \text{if } u \in E_{2n+1};$$

$$F_{2n+1} = \{u + 2^{n+1} v \mid v \in \mathbf{Z} + i\mathbf{Z}\} \qquad \text{if } u \in F_{2n+1};$$

$$E_{2n+2} = \{u + 2^{n+1} v \mid v \in \mathbf{Z} + i\mathbf{Z}\} \qquad \text{if } u \in E_{2n+2}.$$

For each $u \in \mathbf{Z} + i\mathbf{Z}$, the translation $\tau_u : v \to u + v$ preserves (resp. inverses) the orientation of the segments of curves of $\mathscr{C}$ if u is even (resp. odd).

For $n \in \mathbf{N}$ and $u \in \mathbf{Z} + i\mathbf{Z}$ even, we have $\tau_{2^n u}(F_{2n}) = F_{2n}$. For each $v \in F_{2n}$, the connections between the 4 segments which have v as an endpoint are preserved by $\tau_{2^n u}$ if and only if $u \in 2(\mathbf{Z} + i\mathbf{Z})$.

For $n \in \mathbf{N}$ and $u \in \mathbf{Z} + i\mathbf{Z}$, we have $\tau_{2^{n+1} u}(F_{2n+1}) = F_{2n+1}$. For each $v \in F_{2n+1}$, the connections between the 4 segments which have v as an endpoint are preserved by $\tau_{2^{n+1} u}$ if and only if u is even.

It follows that, for each $n \in \mathbf{N}$, each $u \in \mathbf{Z} + i\mathbf{Z}$ and each set of curves $\mathscr{B} < \mathscr{C}$, we have $\tau_{2^{n+1} u}(\mathscr{B}) < \mathscr{C}$ if $\mathscr{B}$ contains no pair of consecutive segments with a common vertex in E_{2n+1}.

We denote by $\mathbf{R}_+$ (resp. $\mathbf{R}_+^*$) the set of non-negative (resp. strictly positive) real numbers. For each set of curves $\mathscr{B} < \mathscr{C}$ which is bounded in $\mathbf{R}^2$, we say that $\mathscr{B}$ has the *density* $d \in \mathbf{R}_+$ in $\mathscr{C}$ if, for each $\varepsilon \in \mathbf{R}_+^*$, there exists $r \in \mathbf{R}_+^*$ such that, for each $s \in \mathbf{R}_+^*$ with $s \geq r$ and each $z \in \mathbf{C}$,

$$s^2 d(1 - \varepsilon) < |\{\mathscr{F} < \mathscr{C} \mid \mathscr{F} \cong \mathscr{B} \text{ and } \mathscr{F} \subset \Sigma_s(z)\}| < s^2 d(1 + \varepsilon),$$

where $\Sigma_s(z) = \{z + x + iy \mid x, y \in [0, s]\}$. The density of $\mathscr{B}$ in $\mathscr{C}$ is uniquely determined.

THEOREM 3. *Each bounded set of curves $\mathscr{B} < \mathscr{C}$ has a density $d > 0$ in $\mathscr{C}$.*

REMARK. *As the definition of density is local, $\mathscr{B}$ has the same density in any covering $\mathscr{D}$ which is locally isomorphic to $\mathscr{C}$.*

PROOF. First we give a lower bound for d. As $\mathscr{C}$ satisfies the local isomorphism property and $|E_\infty| \leq 1$, there exists a copy of $\mathscr{B}$ with no vertex in E_∞. Let n be the smallest integer such that some $\mathscr{A} < \mathscr{C}$ with $\mathscr{A} \cong \mathscr{B}$ has no vertex in E_{2n+1}. Then we have $\tau_{2^{n+1}z}(\mathscr{A}) < \mathscr{C}$ for each $z \in \mathbf{Z} + i\mathbf{Z}$, and therefore $d \geq 1/2^{2n+2}$.

Now we prove that d exists. For each $u \in \mathbf{C}$ and each $s \in \mathbf{R}_+^*$, we consider $E_s(u) = \{\mathscr{F} < \mathscr{C} \mid \mathscr{F} \cong \mathscr{B}$ and $\mathscr{F} \subset \Sigma_s(u)\}$. We show that, for large s and for $u, v \in \mathbf{C}$, $|E_s(v)| - |E_s(u)|$ is small compared to s^2.

We consider two integers r, s such that s is large compared to 2^r and 2^r is large compared to the size of $\mathscr{B}$. There exists $z \in \mathbf{Z} + i\mathbf{Z}$ such that $v - u - 2^{r+1}z = x + iy$ with $\sup\{|x|, |y|\} \leq 2^r$. We have $|E_s(u)| \leq |E_s(v)| + m + n$ where

$$m = |\{\mathscr{F} < \mathscr{C} \mid \mathscr{F} \cong \mathscr{B}, \mathscr{F} \subset \Sigma_s(u) \text{ and } \tau_{2^{r+1}z}(\mathscr{F}) \not\subseteq \Sigma_s(v)\}|, \qquad \text{and}$$

$$n = |\{\mathscr{F} < \mathscr{C} \mid \mathscr{F} \cong \mathscr{B}, \mathscr{F} \subset \Sigma_s(u) \text{ and } \tau_{2^{r+1}z}(\mathscr{F}) \not< \mathscr{C}\}|.$$

The integer m is small compared to s^2 since s is large compared to 2^r. The integer n is small compared to s^2 because 2^r is large compared to the size of $\mathscr{B}$, and because we have $\tau_{2^{r+1}z}(\mathscr{F}) < \mathscr{C}$ if $\mathscr{F} < \mathscr{C}$ contains no pair of consecutive segments with a common vertex in E_{2r+1}.

It follows that $|E_s(u)| - |E_s(v)|$ is small compared to s^2 if it is positive. The same result is true for $|E_s(v)| - |E_s(u)|$. ■

From now on, we do not use the identification of $\mathbf{R}^2$ with $\mathbf{C}$.

THEOREM 4. *$\mathscr{C}$ is locally isomorphic to a covering generated by a curve associated to an ∞-folding sequence.*

PROOF. For $n \in \mathbf{N}$ and $(u, v) \in E_n(\mathscr{C})$, we denote by $\mathscr{C}_n(u, v)$ the set of curves obtained from $\mathscr{C}$ by keeping only the segments contained in $[u - 2^n, u + 2^n] \times [v - 2^n, v + 2^n]$. For $m < n$, $\mathscr{C}_m(u, v)$ is the restriction of $\mathscr{C}_n(u, v)$ to $[u - 2^m, u + 2^m] \times [v - 2^m, v + 2^m]$.

By König's lemma, there exists a sequence $(X_n)_{n \in \mathbf{N}} \in \prod_{n \in \mathbf{N}} E_n(\mathscr{C})$ such that, for $m < n$, the translation $X_m \to X_n$ induces an embedding of $\mathscr{C}_m(X_m)$ in $\mathscr{C}_n(X_n)$. As $\mathscr{C}$ satisfies the local isomorphism property, the inductive limit of

$(\mathscr{C}_n(X_n))_{n\in\mathbf{N}}$ relative to these embeddings is a covering $\mathscr{D}$ which is locally isomorphic to $\mathscr{C}$ and satisfies the local isomorphism property. The image X of the elements X_n in $\mathscr{D}$ belongs to $E_\infty(\mathscr{D})$. Each of the two halves of curves of $\mathscr{D}$ which start at X is associated to an ∞-folding sequence. ∎

REMARK. *According to [4, Th. 3.10], the covering $\mathscr{D}$ given by Theorem 4 is essentially unique: two such coverings only differ by a translation or/and a change in the connections at the E_∞ point.*

For each $(x, y) \in \mathbf{Z}^2$, the *unit square* $[x, x+1] \times [y, y+1]$ is essentially contained in one of the connected components of $\mathbf{R}^2 - \mathscr{C}$, but each of its 4 vertices can belong to that component or to another one. We say that two unit squares S, T are *connected* if they have exactly 1 common vertex X and if X and their centers belong to the same component.

For each $X \in \mathbf{Z}^2$, we say that X satisfies the condition (P) if $X \in E_2$ and if each unit square with the vertex X is connected to 2 unit squares without the vertex X.

LEMMA 5. *There exists $A \in \mathbf{Z}^2$ such that $\{X \in \mathbf{Z}^2 \mid X$ satisfies $(P)\} = \{A + r(2, -2) + s(2, 2) \mid r, s \in \mathbf{Z}\}$.*

PROOF. For each $X \in E_2$, the 4 nonoriented subcurves of curves of $\mathscr{C}$ with endpoint X and length 4 are all obtained from one of them by successive rotations of center X and angle $\pi/2$. Consequently, for each of them, X satisfies the condition (P) if and only if the second and the third segment starting from X are obtained from the first one by turning left then right, or right then left.

It follows that each $X \in E_2$ satisfies P if and only if $X + (2, 0)$ (resp. $X + (0, 2)$) does not satisfy P. ∎

Notation. We denote by O the point $(0, 0) \in \mathbf{R}^2$.

THEOREM 6. *One of the two following properties is true:*

(1) *$\mathscr{C}$ consists of 1, 2, 3 or 4 curves;*

(2) *$\mathscr{C}$ consists of 6 curves and $\mathscr{C}$ is generated by a curve associated to the alternating folding sequence or to one of its primitives.*

PROOF. By [4, Th. 3.15], if $E_\infty(\mathscr{C}) \neq \varnothing$, then $\mathscr{C}$ consists of 2 curves or the property (2) above is true. It remains to be proved that, if $E_\infty(\mathscr{C}) = \varnothing$, then $\mathscr{C}$ consists of at most 4 curves.

For each $X \in \mathbf{R}^2$ and each curve D, we denote by $\delta(X, D)$ the minimum distance between X and a vertex of D. In the proof of [4, Th. 3.12], we saw that there exist $k \in \mathbf{N}$ and $X \in \mathbf{R}^2$ such that $\delta(X, D) < 1.16$ for each D in the k-th derivative $\mathscr{C}^{(k)}$ of $\mathscr{C}$. Moreover, $\mathscr{C}$ and $\mathscr{C}^{(k)}$ have the same number of

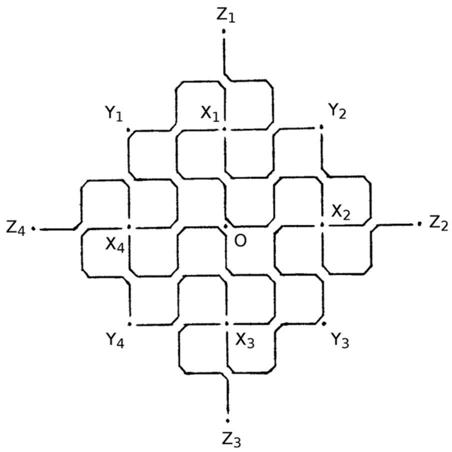

Fig. 1A

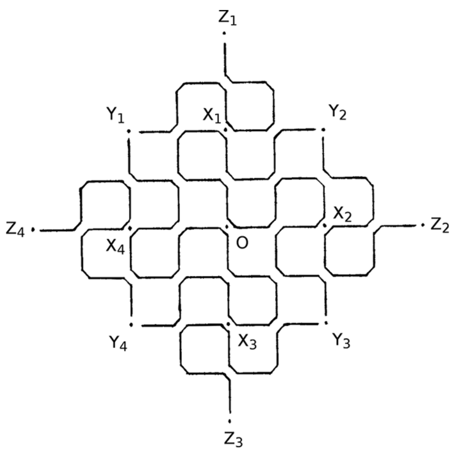

Fig. 1B

curves and $E_\infty(\mathscr{C}) = \varnothing$ implies $E_\infty(\mathscr{C}^{(k)}) = \varnothing$. Consequently, we can replace $\mathscr{C}$ with $\mathscr{C}^{(k)}$, and therefore suppose for the remainder of the proof that there exists $(x, y) \in \mathbf{R}^2$ such that $\delta((x, y), C) < 1.16$ for each $C \in \mathscr{C}$.

Now we apply Lemma 5 to $\mathscr{C}$. There exists $A \in \mathbf{Z}^2$ such that $\{B \in \mathbf{Z}^2 \mid P(B)\} = A + \mathbf{Z}(2, -2) + \mathbf{Z}(2, 2)$, and therefore $(c, d) \in \mathbf{Z}^2$ such that $P(c, d)$ and $|x - c| + |y - d| \leq 2$.

For each $n \in \mathbf{N}$, we consider the images (c_n, d_n) and (x_n, y_n) of (c, d) and (x, y) in $\mathscr{C}^{(n)}$. We have $|x_n - c_n| + |y_n - d_n| \leq 2$. In the proof of [4, Th. 3.12], we saw that $\delta((x, y), C) < 1.16$ for each $C \in \mathscr{C}$ implies $\delta((x_n, y_n), D) < 1.16$ for each $D \in \mathscr{C}^{(n)}$.

As $E_\infty(\mathscr{C}) = \varnothing$, there exists a maximal integer k such that (c_k, d_k) satisfies the condition (P) for $\mathscr{C}^{(k)}$. Replacing $\mathscr{C}$ with $\mathscr{C}^{(k)}$ if necessary, we can assume $k = 0$. We can also assume $(c_0, d_0) = (c_1, d_1) = \mathrm{O}$. Then we have two cases:

(a) $\mathrm{O} \in F_2(\mathscr{C})$;

(b) $\mathrm{O} \in E_3(\mathscr{C})$ and, in the derived covering $\mathscr{C}'$, each unit square with vertex O is connected to exactly one unit square without vertex O.

Figures 1A and 1B represent these two cases. Whatever the case, there are 2 possible dispositions for the subcurves of length 4 with endpoint O of the curves of $\mathscr{C}$. We only consider one of them since the other one is equivalent modulo a symmetry. Similarly, we only consider one of the 2 possible choices for the connections in O. We note that the ball $B((x, y), 1.16)$ is contained in the interior of the square (Z_1, Z_2, Z_3, Z_4) because (x, y) belongs to the square (X_1, X_2, X_3, X_4).

In Figure 1A, the connections in Y_1, Y_3 are imposed by the connections in O, since the property $\mathrm{O} \in F_2(\mathscr{C})$ implies $X_1, X_2, X_3, X_4 \in E_3(\mathscr{C})$. Because of the existence of connections in X_1, X_2, X_3, X_4, all the subcurves represented are contained in at most 4 curves of $\mathscr{C}$. As no other curve of $\mathscr{C}$ can reach the vertices in $B((x,y),1.16)$, $\mathscr{C}$ contains at most 4 curves.

In Figure 1B, the connections in X_1, X_2, X_3, X_4 are imposed since, in $\mathscr{C}'$, each unit square with vertex O is connected to only one unit square without vertex O. Consequently, $\mathscr{C}$ contains at most 2 curves with segments in the interior of the square (Y_1, Y_2, Y_3, Y_4). As at most 2 other curves of $\mathscr{C}$ can reach the vertices in $B((x,y),1.16)$, it follows that $\mathscr{C}$ contains at most 4 curves. ∎

By [4, Th. 3.2, Cor. 3.6 and Th. 3.7], $\mathscr{C}$ is locally isomorphic to a covering by 1 curve. We also have:

Proposition 7. *$\mathscr{C}$ is locally isomorphic to a covering by 2 curves.*

Proof. By Theorem 4, we can suppose that $\mathscr{C}$ is generated by a curve associated to an ∞-folding sequence S. Then, by [4, Th. 3.15], $\mathscr{C}$ itself consists of 2 curves except if S is the alternating folding sequence or one of its primitives. So we can suppose for the remainder of the proof that there exists $k \in \mathbf{N}$ such that $S^{(k)}$ is the alternating folding sequence T. Then $\mathscr{C}^{(k)}$ is generated by a curve associated to T.

If there exists a covering $\mathscr{D}$ by 2 curves which is locally isomorphic to $\mathscr{C}^{(k)}$, then there exists a k-th primitive $\mathscr{E}$ of $\mathscr{D}$ which is locally isomorphic to $\mathscr{C}$, and $\mathscr{E}$ also consists of 2 curves. So we can suppose $k = 0$. Then $\mathscr{C}$ is the covering shown in [4, Fig. 8].

For each $n \in \mathbf{N}^*$ and each $r \in \mathbf{Z}$, two bounded subcurves of distinct curves of $\mathscr{C}$ form a covering $\mathscr{C}_{n,r}$, in the sense given in the proof of [4, Example 3.8], of the triangle $T_{n,r} = ((0,(2r)2^n),(2^n,(2r+1)2^n),(-2^n,(2r+1)2^n))$. For each $s \in \mathbf{Z}$, the translation $X \to X + (0, s2^{n+1})$ induces an isomorphism from $\mathscr{C}_{n,r}$ to $\mathscr{C}_{n,r+s}$.

Now, for each $n \in \mathbf{N}^*$, we consider the embedding $\pi_n : \mathscr{C}_{2n,0} \to \mathscr{C}_{2n,1} \subset \mathscr{C}_{2n+2,0}$ induced by the translation $\tau_n : X \to X + (0, 2^{2n+1})$. We observe that $\bigcup_{n\in\mathbf{N}} \tau_1^{-1}(\ldots(\tau_n^{-1}(T_{2n+2,0}))\ldots) = \mathbf{R}^2$. It follows that the inductive limit of $(\mathscr{C}_{2n,0})_{n\in\mathbf{N}^*}$ relative to the embeddings π_n is a covering by 2 curves which is locally isomorphic to $\mathscr{C}$. ∎

We do not know presently for which coverings $\mathscr{C}$ there exists a covering $\mathscr{D}$ consisting of 3 or 4 curves which is locally isomorphic to $\mathscr{C}$. However, we are going to give examples of the two situations. Here, for any sets of curves $\mathscr{F}$, $\mathscr{G}$, we say that $\mathscr{F}$ is *interior* to $\mathscr{G}$ if $\mathscr{F} \ll \mathscr{G}$.

Example 8. *Let $\mathscr{C}$ be a covering generated by a dragon curve associated to the ∞-folding sequence $(a_k)_{k\in\mathbf{N}^*}$ with $a_{2^n} = +1$ for each $n \in \mathbf{N}$. Then $\mathscr{C}$ is*

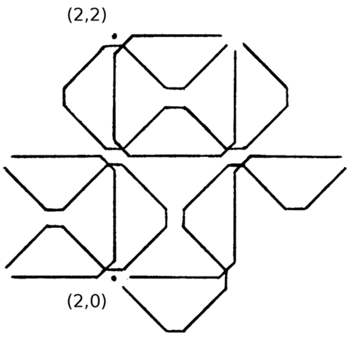

Fig. 2A

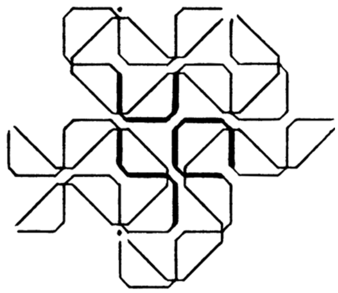

Fig. 2B

locally isomorphic to two coverings by 3 *curves, one where each pair of curves has common vertices and one with two curves separated by the third one.*

PROOF. For the first covering, we apply Proposition 1 to the set of curves $\mathscr{D}$ shown in Figure 2A with horizontal and vertical segments. It is embedded in $\mathscr{C}$ since it appears in [4, Fig. 7] between $(2,0)$ and $(2,2)$.

We consider the first primitive of $\mathscr{D}$, shown in Figures 2A and 2B with diagonal segments, and the second primitive, shown in Figure 2B with horizontal and vertical segments. They are also embedded in $\mathscr{C}$. By Figure 2B, the second primitive contains the image of $\mathscr{D}$ under a rotation by $-\pi/2$; we note that this image is not interior to it because the condition is not satisfied for one of the segments, but it is satisfied at the following step. Repeating this process 3 more times, we obtain a copy of $\mathscr{D}$ which is interior to the 8-th primitive of $\mathscr{D}$.

For the second covering, we apply Proposition 1 to the set of curves $\mathscr{E}$ shown in Figure 3A (page 13) with horizontal and vertical segments. It is embedded in $\mathscr{C}$ since it appears in [4, Fig. 7] between O and $(0,3)$.

We consider the first primitive of $\mathscr{E}$, shown in Figures 3A and 3B, and the second primitive, shown in Figure 3B. They are also embedded in $\mathscr{C}$. By Figure 3B, an image of $\mathscr{E}$ under a rotation by π is interior to the second primitive of $\mathscr{E}$. Repeating this process 1 more time, we obtain a copy of $\mathscr{E}$ which is interior to the 4-th primitive of $\mathscr{E}$. ∎

EXAMPLE 9. *Let* $\mathscr{C}$ *be a covering generated by a curve associated to the* ∞*-folding sequence* $(a_k)_{k\in\mathbf{N}^*}$ *with* $a_{2^{4n}} = a_{2^{4n+1}} = +1$ *and* $a_{2^{4n+2}} = a_{2^{4n+3}} = -1$ *for each* $n \in \mathbf{N}$. *Then* $\mathscr{C}$ *is locally isomorphic to a covering by* 4 *curves.*

PROOF. The covering $\mathscr{C}$ is represented in Figure 4 (page 13). We apply Proposition 1 to the set of curves $\mathscr{D}$ shown in Figure 5A (page 14) with

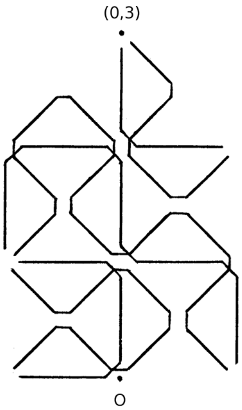

Fig. 3A

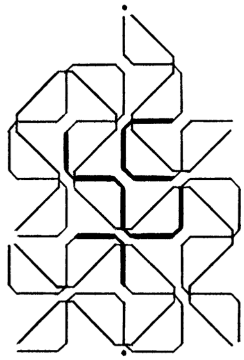

Fig. 3B

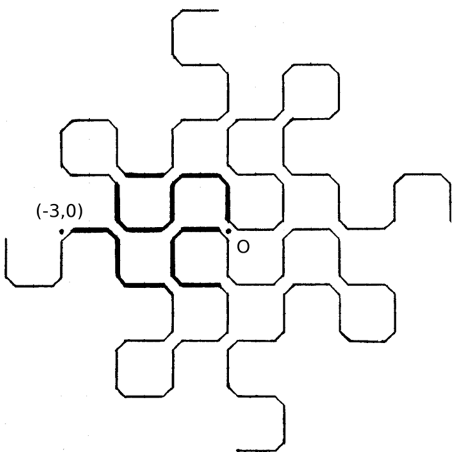

Fig. 4

horizontal and vertical segments. It is embedded in $\mathscr{C}$ since it appears in Figure 4 between $(-3,0)$ and O.

We consider the first primitive of $\mathscr{D}$, shown in Figures 5A and 5B, and the second primitive, shown in Figure 5B. The second primitive is embedded in the covering generated by a curve associated to the ∞-folding sequence $(a_k)_{k\in\mathbf{N}^*}$ with $a_{2^{4n}} = a_{2^{4n+1}} = -1$ and $a_{2^{4n+2}} = a_{2^{4n+3}} = +1$ for each $n\in\mathbf{N}$. According to Figure 5B, it contains a copy of the image of $\mathscr{D}$ under the reflection about the y-axis; we note that this copy is not interior to it because the condition is not satisfied for one of the segments, but it is satisfied at the following step.

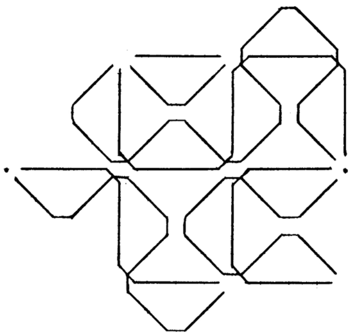

Fig. 5A

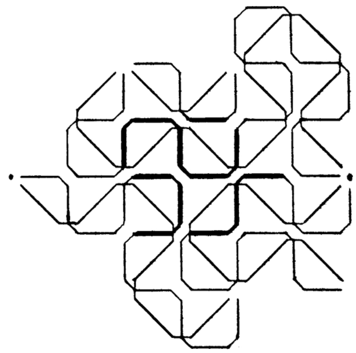

Fig. 5B

Then, by applying a process which is the image of the previous one under the reflection about the y-axis, we obtain a copy of $\mathscr{D}$ which is interior to the 4-th primitive of $\mathscr{D}$. ■

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References

[1] D. Davis and D. E. Knuth, Number representations and dragon curves I and II, J. Recreational Math. 3 (1970), 66–81 and 133–149.

[2] M. Dekking, Paperfolding morphisms, planefilling curves, and fractal tiles, Theoret. Comput. Sci. 414 (2012), 20–37.

[3] M. Dekking, M. Mendès France and A. Van Der Poorten, Folds, Math. Intelligencer 4 (1982), 130–138.

[4] F. Oger, Paperfolding sequences, paperfolding curves and local isomorphism, Hiroshima Math. Journal 42 (2012), 37–75.

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