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Confluence of general Schlesinger systems and Twistor theory

To the memory of Professor Kenjiro Okubo

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ABSTRACT. We give a description of confluence for the general Schlesinger systems (GSS) from the view point of twistor theory. GSS is a system of nonlinear differential equations on the Grassmannian manifold $G_{2,N}(\mathbf{C})$ which is obtained, for any partition λ of N, as the integrability condition of a connection ∇_λ on $\mathbf{P}^1 \times G_{2,N}$ constructed using the twistor-theoretic point of view and is known to describe isomonodromic deformation of linear differential equations on the projective space $\mathbf{P}^1$. For a pair of partitions λ, μ of N such that μ is obtained from λ by making two parts into one parts and leaving other parts unchanged, we construct the limit process $\nabla_\lambda \to \nabla_\mu$ and as a result the confluence for GSS.

1. Introduction

In the study of nonlinear differential equations in the complex domain, Painlevé equations and their generalizations form an important class in the sense that they define new special functions and play important roles in various research fields of mathematics and theoretical physics. Historically, P. Painlevé and B. Gambier [1, 9] classified equations of the form

$$q'' = R(t, q, q'), \qquad R \in \mathbf{C}(t, q, q')$$

having no movable branch point and, as a result, they obtained six equations $P_I, \dots, P_{VI}$ called Painlevé equations. It is known that the Painlevé equations are also obtained from the isomonodromic deformations of systems of linear differential equation on $\mathbf{P}^1$ with regular and/or irregular singular points, and from this view point they are widely generalized. For example, for P_{VI}, we consider the isomonodromic deformation of a Fuchsian system of rank 2:

$$\frac{dy}{d\zeta} = \left(\frac{A_1(t)}{\zeta} + \frac{A_2(t)}{\zeta - 1} + \frac{A_3(t)}{\zeta - t}\right) y, \qquad A_i(t) \in M_2(\mathbf{C}) \tag{1.1}$$

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with 4 regular singular points $\zeta = 0, 1, t, \infty$. The linear equation which controls the dependence of y on the parameter t is

$$\frac{\partial y}{\partial t} = -\frac{A_3(t)}{\zeta - t} y. \tag{1.2}$$

The equations (1.1) and (1.2) can be written as

$$dy = (A_1 d \log \zeta + A_2 d \log(\zeta - 1) + A_3 d \log(\zeta - t)) y \tag{1.3}$$

and its integrability condition gives the system of nonlinear equations

$$\frac{dA_1}{dt} = \frac{[A_3, A_1]}{t}, \qquad \frac{dA_2}{dt} = \frac{[A_3, A_2]}{t-1}, \qquad \frac{dA_3}{dt} = -\frac{[A_3, A_1]}{t} - \frac{[A_3, A_2]}{t-1}, \tag{1.4}$$

which is a particular case of Schlesinger system [2, 10]. It is explained in [2] that if we define $q(t)$ from a solution $(A_1(t), A_2(t), A_3(t))$ of (1.4) by

$$q = \frac{t(A_1)_{12}}{(t+1)(A_1)_{12} + t(A_2)_{12} + (A_3)_{12}},$$

where $(A_j)_{12}$ is the $(1,2)$-entry of A_j, then $q(t)$ satisfies P_{VI}. The situation for the other Painlevé equations is similar.

The Painlevé equations $P_I, \dots, P_V$ can be obtained from P_{VI} by certain limit process called degeneration (or confluence). The degeneration scheme is expressed as

$$\begin{array}{ccccccc} & & & P_{III} & & & \\ & & \nearrow & & \searrow & & \\ P_{VI} \longrightarrow P_V & & & & & P_{II} \longrightarrow P_I. & \\ & & \searrow & & \nearrow & & \\ & & & P_{IV} & & & \end{array} \tag{1.5}$$

This degeneration for P_J is induced from the confluence of singularities for the linear systems which are deformed isomonodromically. Hence we can associate the above diagram with the diagram consisting of partitions of 4 which encode the nature of singular points of linear systems:

$$\begin{array}{ccccccc} & & & (2,2) & & & \\ & & \nearrow & & \searrow & & \\ (1,1,1,1) \longrightarrow (2,1,1) & & & & & (4) & \\ & & \searrow & & \nearrow & & \\ & & & (3,1) & & & \end{array} \tag{1.6}$$

As for the lack of corresponding part for P_I in the above diagram, we make a comment at the end of this paragraph. In the above diagram, a partition

$(2,1,1)$ means, for example, that the linear system corresponding to P_V has 3 singular points in $\mathbf{P}^1$, two of them are regular singular points and one is an irregular singular point of Poincaré rank 1. In fact, it is given by

$$\frac{dy}{d\eta} = \left(B_1(s) + \frac{B_2(s)}{\eta} + \frac{B_3(s)}{\eta - s}\right) y \tag{1.7}$$

where $\eta = 0, s$ are regular singular points and $\eta = \infty$ is an irregular singular point, and the dependence on s in the isomonodromic deformation is controlled by

$$\frac{\partial y}{\partial s} = -\frac{B_3(s)}{\eta - s} y. \tag{1.8}$$

Note that (1.7) and (1.8) can be written as

$$dy = (B_1\, d\eta + B_2 d\,\log(\eta) + B_3 d\,\log(\eta - s))y \tag{1.9}$$

and the integrability condition gives the degenerated Schlesinger system corresponding to P_V. The arrow $(1,1,1,1) \to (2,1,1)$ in the diagram (1.6) means the system (1.9) is obtained from (1.3) by the confluence of singularity $\zeta = 1$, $\infty \to \eta = \infty$. The explicit form of this process will be given in Section 4. Notice that there is no partition corresponding to P_I in the diagram (1.6). This comes from the fact that the linear differential equation, which gives P_I by isomonodromic deformation, has only one singular point $\zeta = \infty$ where we need functions and power series of $\zeta^{-1/2}$ to obtain the formal fundamental system of solutions. This situation is different from the other Painlevé equations and the degeneration $P_{II} \to P_I$ should be treated separately.

The purpose of this paper is to give this process in a more general situation, namely for the systems analogous to (1.3) or (1.9) corresponding to arbitrary partitions of integer N. To describe these systems, we use the viewpoint of twistor theory due to Mason and Woodhouse [6, 7, 8]. In their theory, a partition λ of N implies a maximal abelian subgroup H_λ of $\mathrm{GL}_N(\mathbf{C})$ which is obtained as a centralizer of regular element of $\mathrm{GL}_N(\mathbf{C})$ indexed by the partition λ, see Section 2. We remark that the same group appeared in the theory of general hypergeometric functions on the Grassmannian manifold [5].

This paper is organized as follows. We review the result of [4] about the general Schlesinger system or the corresponding isomonodromic deformation in Section 2. In Section 3, we construct the process of confluence for the isomonodromic deformation and prove the main theorem. In the last section, we discuss the confluence process for Painlevé equations as examples to illustrate the theorem.

2. General Schlesinger system

We give in this section the definition of general Schlesinger systems. See also [4].

2.1. Maximal abelian subgroup. Let $G = \mathrm{GL}_N(\mathbf{C})$ be the complex general linear group of $N \times N$ matrices. For $g \in G$, let $\mathrm{Ad}_g : G \to G$ be defined by $a \mapsto \mathrm{Ad}_g(a) = gag^{-1}$, which gives the adjoint action of G on itself. Denote the orbit of $a \in G$ by $O(a) = \{\mathrm{Ad}_g(a) \mid g \in G\}$ and the centralizer of $a \in G$ by $Z_G(a) = \{g \in G \mid \mathrm{Ad}_g(a) = a\}$. We know that both $O(a)$ and $Z_G(a)$ are complex manifolds and $\dim_{\mathbf{C}} G = \dim_{\mathbf{C}} O(a) + \dim_{\mathbf{C}} Z_G(a)$.

Definition 2.1. *An element* $a \in G$ *is said to be regular if* $\dim O(a)$ *is maximum, in other words,* $\dim Z_G(a)$ *is minimum.*

It is seen that $\dim Z_G(a) = N$ if a is a regular element and that $a \in G$ is a regular element iff the Jordan cells of the Jordan normal form of a have distinct eigenvalues, i.e., for some partition $\lambda = (n_1, \dots, n_\ell)$ of N, a is conjugate to

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_\ell \end{pmatrix}, \qquad A_k = \begin{pmatrix} a_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & a_k \end{pmatrix} \tag{2.1}$$

with distinct $a_1, \dots, a_\ell \in \mathbf{C}$, where $A_k \in \mathrm{GL}_{n_k}(\mathbf{C})$. We call such element a a regular element of type λ.

What we concern is the groups obtained as centralizers of regular elements, which are given explicitly as follows. When $a \in G$ itself is the Jordan normal form as in (2.1), then

$$Z_G(a) = \left\{ \begin{pmatrix} h^{(1)} & & \\ & \ddots & \\ & & h^{(\ell)} \end{pmatrix} \middle| \, h^{(k)} \in J(n_k) \right\}, \tag{2.2}$$

where $J(n)$ is an abelian subgroup of $\mathrm{GL}_n(\mathbf{C})$ of the form

$$J(n) = \left\{ h = \begin{pmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & h_1 \\ & & & h_0 \end{pmatrix} \middle| \, h_0 \neq 0 \right\} \tag{2.3}$$

called n-dimensional *Jordan group*. We also write $h \in J(n)$ as

$$h = h_0 I + h_1 \Lambda + \cdots + h_{n-1} \Lambda^{n-1}$$

using the shift matrix $\Lambda = (\delta_{i+1,j})_{0 \le i,j < n}$ of size n. The group $Z_G(a)$, which is isomorphic to the product group $J(n_1) \times \cdots \times J(n_\ell)$, and is irrelevant to the eigenvalues of a, will be denoted as H_λ so as to emphasize that the group is determined by the partition λ.

Let $\mathfrak{j}(n)$ and $\mathfrak{h}_\lambda$ be the Lie algebras of $J(n)$ and H_λ, respectively:

$$\mathfrak{j}(n) = \{\xi = \xi_0 I + \xi_1 \Lambda + \cdots + \xi_{n-1}\Lambda^{n-1} \mid \xi_i \in \mathbf{C}\} \simeq \mathbf{C}^n$$

and

$$\mathfrak{h}_\lambda = \left\{ \left. \begin{pmatrix} \xi^{(1)} & & \\ & \ddots & \\ & & \xi^{(\ell)} \end{pmatrix} \right| \xi^{(k)} \in \mathfrak{j}(n_k) \right\} \simeq \mathfrak{j}(n_1) \oplus \cdots \oplus \mathfrak{j}(n_\ell).$$

In order to make explicit the relation between H_λ and its Lie algebra $\mathfrak{h}_\lambda$, we introduce the following functions.

Definition 2.2. *Let T be an indeterminate. Define the functions $\theta_m(x)$ of $x = (x_0, x_1, \ldots)$ by*

$$\log(x_0 + x_1 T + x_2 T^2 + \cdots) = \sum_{m=0}^{\infty} \theta_m(x) T^m. \tag{2.4}$$

We see that $\theta_0 = \log x_0$ and, for $m \ge 1$,

$$\theta_m(x) = \sum (-1)^{k_1+\cdots+k_m-1} \frac{(k_1 + \cdots + k_m - 1)!}{k_1! \ldots k_m!} \left(\frac{x_1}{x_0}\right)^{k_1} \cdots \left(\frac{x_m}{x_0}\right)^{k_m}, \tag{2.5}$$

where the sum is taken over all $(k_1, \ldots, k_m) \in \mathbf{Z}_{\ge 0}^m$ satisfying $k_1 + 2k_2 + \cdots + mk_m = m$.

For example, first few of them are

$$\theta_0(x) = \log x_0,$$

$$\theta_1(x) = \frac{x_1}{x_0},$$

$$\theta_2(x) = \frac{x_2}{x_0} - \frac{1}{2}\left(\frac{x_1}{x_0}\right)^2,$$

$$\theta_3(x) = \frac{x_3}{x_0} - \left(\frac{x_1}{x_0}\right)\left(\frac{x_2}{x_0}\right) + \frac{1}{3}\left(\frac{x_1}{x_0}\right)^3,$$

$$\theta_4(x) = \frac{x_4}{x_0} - \frac{1}{2}\left\{\left(\frac{x_2}{x_0}\right)^2 + 2\left(\frac{x_1}{x_0}\right)\left(\frac{x_3}{x_0}\right)\right\} + \left(\frac{x_1}{x_0}\right)^2\left(\frac{x_2}{x_0}\right) - \frac{1}{4}\left(\frac{x_1}{x_0}\right)^4.$$

From these explicit form we see that $\theta_m(x)$, $m \geq 1$, has a pole along $x_0 = 0$ of order m and is a weighted homogeneous polynomial of $x_1/x_0, \dots, x_m/x_0$ of weight m when the weight of x_i is set to be i.

Let $\tilde{J}(n)$ be the universal covering group of $J(n)$. Then we see that $\log : \tilde{J}(n) \to \mathfrak{j}(n)$ defined by

$$h \mapsto \log h = \begin{pmatrix} \theta_0(h) & \theta_1(h) & \cdots & \theta_{n-1}(h) \\ & \ddots & \ddots & \vdots \\ & & \ddots & \theta_1(h) \\ & & & \theta_0(h) \end{pmatrix}$$

gives a biholomorphic map.

2.2. General Schlesinger systems. Let $\mathbf{P}^{N-1}$ be the $(N-1)$-dimensional complex projective space which we call the twistor space. Let $x = (x_0, \dots, x_{N-1})$ be the homogeneous coordinates of $\mathbf{P}^{N-1}$ and $[x]$ denote the point of $\mathbf{P}^{N-1}$ with the homogeneous coordinates x. Define the right action of H_λ on $\mathbf{P}^{N-1}$ by

$$\mathbf{P}^{N-1} \times H_\lambda \to \mathbf{P}^{N-1}, \qquad ([x], h) \mapsto [xh]. \tag{2.6}$$

If we write the homogeneous coordinate x block-wise as

$$x = (x^{(1)}, \dots, x^{(\ell)}), \qquad x^{(k)} = (x_0^{(k)}, \dots, x_{n_k-1}^{(k)}) \tag{2.7}$$

according as the partition $\lambda = (n_1, \dots, n_\ell)$, then the action of $h = (h^{(1)}, \dots, h^{(\ell)}) \in H_\lambda$ is written as

$$[xh] = [x^{(1)}h^{(1)}, \dots, x^{(\ell)}h^{(\ell)}].$$

We prepare the space whose elements parametrize lines in the twistor space $\mathbf{P}^{N-1}$ and define the action of H_λ on this space. Given a matrix $z \in \mathrm{M}_{2,N}(\mathbf{C})$, we write z block-wise as

$$z = (z^{(1)}, \dots, z^{(\ell)}), \qquad z^{(k)} = (z_0^{(k)}, \dots, z_{n_k-1}^{(k)}) \in \mathrm{M}_{2,n_k}(\mathbf{C}),$$

where $z_i^{(k)}$ is a two dimensional column vector. Define an open subset Z_λ of $\mathrm{M}_{2,N}(\mathbf{C})$ by

$$Z_\lambda = \left\{ z \in \mathrm{M}_{2,N}(\mathbf{C}) \;\middle|\; \begin{array}{ll} \det(z_0^{(k)}, z_1^{(k)}) \neq 0 & (n_k \geq 2), \\ \det(z_0^{(k)}, z_0^{(l)}) \neq 0 & (k \neq l) \end{array} \right\}.$$

It is seen that the map $M_{2,N}(\mathbf{C}) \times H_\lambda \ni (z,h) \mapsto zh \in M_{2,N}(\mathbf{C})$ defines an action of H_λ on Z_λ, see [5].

Let $\Phi : \mathbf{P}^1 \times Z_\lambda \to \mathbf{P}^{N-1}$ be the holomorphic map

$$([\vec{\zeta}], z) \mapsto [\vec{\zeta} z] = [\vec{\zeta} z^{(1)}, \dots, \vec{\zeta} z^{(\ell)}], \tag{2.8}$$

where $\vec{\zeta} = (1, \zeta)$ and ζ denotes the affine coordinate of $\mathbf{P}^1$.

Theorem 2.3 ([4, 7]). *Let $U \subset \mathbf{P}^{N-1}$ be an open set containing a projective line and let $\pi : E \to U$ be a holomorphic vector bundle on U of rank r. Assume that*

(i) *U is invariant by the action of H_λ on $\mathbf{P}^{N-1}$ defined by* (2.6),

(ii) *E is trivial on any projective line contained in U,*

(iii) *the action of H_λ on U can be lifted to E.*

Then the infinitesimal action of H_λ on U gives a flat connection $\tilde{\nabla}_\lambda$ on E and the induced connection $\nabla_\lambda = \Phi^* \tilde{\nabla}_\lambda$ on $\Phi^* E$ is locally written as $\nabla_\lambda = d - \omega_\lambda \wedge$, where

$$\omega_\lambda = \sum_{k=1}^{\ell} \sum_{\alpha=0}^{n_k-1} A_\alpha^{(k)}(z) d\theta_\alpha(\vec{\zeta} z^{(k)}), \qquad \sum_{k=1}^{\ell} A_0^{(k)}(z) = 0.$$

The integrability of the connection ∇_λ gives the isomonodromic deformation of a system of linear differential equation

$$\frac{dy}{d\zeta} = \left(\sum_{k=1}^{\ell} \sum_{\alpha=0}^{n_k-1} A_\alpha^{(k)}(z) \frac{d\theta_\alpha(\vec{\zeta} z^{(k)})}{d\zeta} \right) y \tag{2.9}$$

with unknown vector $y \in \mathbf{C}^r$.

Remark 2.4. (i) *$d\theta_j(\vec{\zeta} z^{(k)})/d\zeta$, as a function of ζ, has a pole $\zeta = -z_{00}^{(k)}/z_{10}^{(k)}$ of order $j+1$, and hence the equation* (2.9) *has ℓ singular points of Poincaré rank $n_1 - 1, \dots, n_\ell - 1$. When these ℓ points are in a finite plane, $\zeta = \infty$ is not a singular point of* (2.9) *because of $\sum_{k=1}^{\ell} A_0^{(k)}(z) = 0$.*

(ii) *By the action* (2.6) *of H_λ, the twistor space $\mathbf{P}^{N-1}$ is expressed as a union of orbits. There is an open dense orbit $O(a)$ passing through $[a] \in \mathbf{P}^{N-1}$, where*

$$a = (a^{(1)}, \dots, a^{(\ell)}) \in \mathbf{C}^N, \qquad a^{(k)} = (1, 0, \dots, 0) \in \mathbf{C}^{n_k} \tag{2.10}$$

and there are codimension 1 *orbits $O(b_j)$, $j = 1, \dots, \ell$, where $b_j = (b_j^{(1)}, \dots, b_j^{(\ell)})$ with*

$$b_j^{(k)} = (1, 0, \dots, 0), \qquad (k \neq j), \qquad b_j^{(j)} = (0, 1, 0, \dots, 0).$$

When $U = O(a) \cup O(b_1) \cup \cdots \cup O(b_\ell)$, *the set* Z_λ *is the space which parametrizes all the projective lines contained in* U.

DEFINITION 2.5. *The system of nonlinear differential equations for* $A_\alpha^{(k)}$ *obtained as the complete integrability condition of the connection* ∇_λ *is called the general Schlesinger system (GSS) of type* λ.

3. Confluence

In this section we construct a process of confluence of the connections ∇_λ given in Theorem 2.3. This construction is a concrete realization of adherence relations among strata of a natural stratification in the space of regular elements G_{reg} of $G = \mathrm{GL}_N(\mathbf{C})$. So we describe first the adherence relation among strata.

3.1. Stratification of the set of regular elements. Let $\mathscr{P}_N$ denote the set of partitions of N. Then we have the decomposition of G_{reg} as

$$G_{reg} = \bigsqcup_{\lambda \in \mathscr{P}_N} G_\lambda \tag{3.1}$$

where G_λ is the set of regular elements of type λ.

DEFINITION 3.1. *Let* $\lambda, \mu \in \mathscr{P}_N$. μ *is* said to be adjacent *to* λ *when* μ *is obtained from* λ *by making two parts of* λ *into one parts and leaving the other parts unchanged. In this case we denote it as* $\lambda \to \mu$.

EXAMPLE 3.2. *In the set of* $\mathscr{P}_4$, *the adjacency is described as in* (1.6).

DEFINITION 3.3. *Let* $\mu \in \mathscr{P}_N$ *be obtained from* $\lambda \in \mathscr{P}_N$ *by successive chains of adjacent partitions, namely, there are* $\lambda_1, \dots, \lambda_p \in \mathscr{P}_N$ *such that* $\lambda = \lambda_1 \to \lambda_2 \to \cdots \to \lambda_p = \mu$. *In this case we write* $\mu < \lambda$.

The relations $<$ defines a partial order in the set $\mathscr{P}_N$. We notice a well-known fact that (3.1) defines a stratification of G_{reg} in the sense that each G_λ is a complex manifold of dimension $N^2 - N + \ell(\lambda)$ and we have

$$\bar{G}_\lambda = \bigcup_{\mu \leq \lambda} G_\mu,$$

where $\ell(\lambda)$ denotes the number of parts of λ and $\bar{G}_\lambda$ denotes the closure of G_λ in G_{reg} with respect to the usual topology of G_{reg}. What we want to do is to construct, for $\lambda, \mu \in \mathscr{P}_N$ such that $\lambda \to \mu$, the confluence $\nabla_\lambda \to \nabla_\mu$ explicitly.

The first step is to give explicit realization of the adjacency. Namely, for $a \in G_\mu$, we construct $a(\varepsilon) \in G_\lambda$ depending holomorphically on $\varepsilon \in \mathbf{C}^*$ in some neighbourhood of 0 such that $\lim_{\varepsilon \to 0} a(\varepsilon) = a$. Before entering the general situation, we explain this step by a simple example.

Example 3.4. *Consider the case* $G = \mathrm{GL}_2(\mathbf{C})$. *Only partitions of* 2 *are* $\lambda = (1,1)$ *and* $\mu = (2)$ *and we have* $\lambda \to \mu$. *Let* $a = \begin{pmatrix} \alpha & 1 \\ & \alpha \end{pmatrix} \in G_\mu$ *be given. Put* $g(\varepsilon) = \begin{pmatrix} 1 & 1 \\ & \varepsilon \end{pmatrix}$. *Then we define* $a(\varepsilon) \in G_\lambda$ *depending holomorphically on* $\varepsilon \in \mathbf{C}^*$:

$$\begin{pmatrix} \alpha & 1 \\ & \alpha \end{pmatrix} \to (\alpha, 1) \to (\alpha, 1) g(\varepsilon) = (\alpha, \alpha + \varepsilon) \to$$

$$\begin{pmatrix} \alpha & \\ & \alpha + \varepsilon \end{pmatrix} \to g(\varepsilon) \begin{pmatrix} \alpha & \\ & \alpha + \varepsilon \end{pmatrix} g(\varepsilon)^{-1} =: a(\varepsilon),$$

where the vector $(\alpha, 1)$ *is constructed from* $\begin{pmatrix} \alpha & 1 \\ & \alpha \end{pmatrix}$ *by arraying the element in the main diagonal, and then that in the upper subdiagonal. Computation shows*

$$a(\varepsilon) = \begin{pmatrix} \alpha & 1 \\ & \alpha + \varepsilon \end{pmatrix}. \qquad \lim_{\varepsilon \to 0} a(\varepsilon) = a.$$

Since we are considering the situation $\lambda \to \mu$, namely μ is obtained from λ by making some two parts of λ into one parts, our construction reduces to the case where $\lambda, \mu \in \mathscr{P}_N$ are of the form $\lambda = (p, q)$ and $\mu = (N)$.

Define $g(\varepsilon) \in \mathrm{M}_N(\mathbf{C})$ by

$$g(\varepsilon) = \begin{pmatrix} I_p & g_1(\varepsilon) \\ 0 & g_2(\varepsilon) \end{pmatrix}$$

where $g_1(\varepsilon) \in \mathrm{M}_{p,q}(\mathbf{C})$, $g_2(\varepsilon) \in \mathrm{M}_q(\mathbf{C})$ are given by

$$\begin{pmatrix} g_1(\varepsilon) \\ g_2(\varepsilon) \end{pmatrix} = D_N(\varepsilon) \begin{pmatrix} \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{q-1} \\ \binom{1}{0} & \binom{1}{1} & \cdots & \binom{1}{q-1} \\ \vdots & \vdots & & \vdots \\ \binom{p+q-1}{0} & \binom{p+q-1}{1} & \cdots & \binom{p+q-1}{q-1} \end{pmatrix} D_q(\varepsilon)^{-1},$$

$D_m(\varepsilon)$ denoting $\mathrm{diag}(1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{m-1})$ and $\binom{i}{j}$ denoting the binomial coefficient which is equal to 0 when $i < j$ by usual convention. Then we have

$$g(\varepsilon)=\left(\begin{array}{ccc|ccc} 1 & & & \binom{0}{0} & \cdots & \binom{0}{q-1}\varepsilon^{-q+1} \\ & \ddots & & \vdots & & \vdots \\ & & 1 & \binom{p-1}{0}\varepsilon^{p-1} & \cdots & \binom{p-1}{q-1}\varepsilon^{p-q} \\ \hline & & & \binom{p}{0}\varepsilon^{p} & \cdots & \binom{p}{q-1}\varepsilon^{p-q+1} \\ & & & \vdots & & \vdots \\ & & & \binom{p+q-1}{0}\varepsilon^{p+q-1} & \cdots & \binom{p+q-1}{q-1}\varepsilon^{p} \end{array}\right).$$

It is seen from the expression of $g_1(\varepsilon)$, $g_2(\varepsilon)$ that $\det g(\varepsilon)=\varepsilon^{pq}$. Hence $\mathbf{C}^*\ni\varepsilon\mapsto g(\varepsilon)\in \mathrm{GL}_N(\mathbf{C})$ is a holomorphic map. Take

$$a=\begin{pmatrix} \alpha & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha \end{pmatrix}\in G_{(N)}.$$

Taking account of $(\alpha,1,0,\dots,0)g(\varepsilon)=(\overbrace{\alpha,1,0,\dots,0}^{p},\overbrace{\alpha+\varepsilon,1,0,\dots,0}^{q})$, we put

$$a(\varepsilon)=g(\varepsilon)\begin{pmatrix} \alpha I_p+\Lambda_p & \\ & (\alpha+\varepsilon)I_q+\Lambda_q \end{pmatrix}g(\varepsilon)^{-1}\in G_{(p,q)}$$

for $\varepsilon\in\mathbf{C}^*$, where $\Lambda_p=(\delta_{i+1,j})_{0\le i,j<p}$ is the shift matrix of size p. Then we can show that $\lim_{\varepsilon\to 0}a(\varepsilon)=a$, see [3].

3.2. Confluence of the connections. Let $\lambda,\mu\in\mathscr{P}_N$ be given by $\lambda=(p,q)$ and $\mu=(N)$. Using the above $g(\varepsilon)$, we construct the confluence $\nabla_\lambda\to\nabla_\mu$. Suppose we are given the connection ∇_μ described in Theorem 2.3. Write the connection form ω_μ of ∇_μ as

$$\omega_\mu=\sum_{0\le j<N}B_j(w)d\theta_j(\vec{\zeta}w),\qquad w\in Z_\mu.$$

We construct $\nabla_\lambda(\varepsilon):=d-\omega_\lambda\wedge$ with the connection form $\omega_\lambda(\varepsilon)$. Consider a change of variables

$$z=w\cdot g(\varepsilon) \tag{3.2}$$

and a change of gauge potentials

$$A=B\cdot({}^t g(\varepsilon)^{-1}\otimes I_r), \tag{3.3}$$

where $A = (A_0^{(1)}, \dots, A_{p-1}^{(1)}, A_0^{(2)}, \dots, A_{q-1}^{(2)}) \in \mathfrak{gl}_r(\mathbf{C})^N$ and $B = (B_0, \dots, B_{N-1}) \in \mathfrak{gl}_r(\mathbf{C})^N$. Since z and the gauge potentials A depend on ε by (3.2) and (3.3), we denote them as

$$z(\varepsilon) = (z^{(1)}(\varepsilon), z^{(2)}(\varepsilon)),$$

$$A(\varepsilon) = (A_0^{(1)}(\varepsilon), \dots, A_{p-1}^{(1)}(\varepsilon), A_0^{(2)}(\varepsilon), \dots, A_{q-1}^{(2)}(\varepsilon)).$$

Put

$$\omega_\lambda(\varepsilon) = \omega_\lambda^{(1)}(\varepsilon) + \omega_\lambda^{(2)}(\varepsilon) \tag{3.4}$$

$$= \sum_{0 \le j < p} A_j^{(1)}(\varepsilon) d\theta_j(\vec{\zeta} z^{(1)}(\varepsilon)) + \sum_{0 \le j < q} A_j^{(2)}(\varepsilon) d\theta_j(\vec{\zeta} z^{(2)}(\varepsilon)). \tag{3.5}$$

Note that, for $w \in Z_\mu$, we have $z(\varepsilon) \in Z_\lambda$ for any $\varepsilon \in \mathbf{C}^*$ in a neighbourhood of 0.

THEOREM 3.5 (Confluence). *If we put*

$$\omega_\mu = \sum_{0 \le j < N} B_j(w) d\theta_j(\vec{\zeta} w), \qquad w \in Z_\mu,$$

then

$$\omega_\lambda(\varepsilon) = \omega_\mu + O(\varepsilon).$$

Hence,

$$\lim_{\varepsilon \to 0} \omega_\lambda(\varepsilon) = \omega_\mu.$$

3.3. Proof of Theorem 3.5. We need the following lemma.

LEMMA 3.6 [3]. *Let $x = (x_0, x_1, \dots)$ and let $y(x,t) = (y_0(x,t), y_1(x,t), \dots)$ be a sequence of formal power series of t defined by*

$$y_j(x,t) = \sum_{k \ge 0} \binom{j+k}{k} x_{j+k} t^k, \qquad j \ge 0.$$

Then we have

$$\theta_j(y_0(x,t), y_1(x,t), \dots) = \sum_{k \ge 0} \binom{j+k}{k} \theta_{j+k}(x_0, x_1, \dots) t^k, \qquad j \ge 0,$$

where θ_j are functions defined by (2.4).

By the relation (3.2), $z(\varepsilon)$ is written as

$$z_j^{(1)}(\varepsilon) = w_j, \qquad 0 \le j < p,$$

$$z_j^{(2)}(\varepsilon) = \sum_{k \ge 0} \binom{j+k}{k} w_{j+k}\varepsilon^k, \qquad 0 \le j < q.$$

Note also that the relation (3.3) is written as

$$A_j^{(1)}(\varepsilon) + \sum_{0 \le k < q} \binom{j}{k} A_k^{(2)}(\varepsilon)\varepsilon^{j-k} = B_j, \qquad 0 \le j < p, \tag{3.6}$$

$$\sum_{0 \le k < q} \binom{j}{k} A_k^{(2)}(\varepsilon)\varepsilon^{j-k} = B_j, \qquad p \le j < N. \tag{3.7}$$

Then $\omega_\lambda^{(2)}(\varepsilon)$ of (3.4) is written as

$$\begin{aligned}
\omega_\lambda^{(2)}(\varepsilon) &= \sum_{0 \le j < q} A_j^{(2)}(\varepsilon) d\theta_j(\vec{\zeta} z^{(2)}(\varepsilon)) \\
&= \sum_{0 \le j < q} A_j^{(2)}(\varepsilon) d\theta_j\left(\vec{\zeta} \sum_{k \ge 0} \binom{j+k}{k} w_{j+k}\varepsilon^k\right) \\
&= \sum_{0 \le j < q} A_j^{(2)}(\varepsilon) d\theta_j(y_0(\vec{\zeta} w, \varepsilon), y_1(\vec{\zeta} w, \varepsilon), \dots) \\
&= \sum_{0 \le j < q} A_j^{(2)}(\varepsilon) \sum_{k \ge 0} \binom{j+k}{k} d\theta_{j+k}(\vec{\zeta} w)\varepsilon^k.
\end{aligned}$$

Here we used Lemma 3.6 in the last equality. Then, using this and the identities (3.6), (3.7), we have

$$\begin{aligned}
\omega_\lambda(\varepsilon) &= \sum_{0 \le j < p} \left(A_j^{(1)}(\varepsilon) + \sum_{0 \le k < q} \binom{j}{k} A_k^{(2)}(\varepsilon)\varepsilon^{j-k} \right) d\theta_j(\vec{\zeta} w) \\
&\quad + \sum_{p \le j < N} \left(\sum_{0 \le k < q} \binom{j}{k} A_k^{(2)}(\varepsilon)\varepsilon^{j-k} \right) d\theta_j(\vec{\zeta} w) + O(\varepsilon) \\
&= \sum_{0 \le j < N} B_j \, d\theta_j(\vec{\zeta} w) + O(\varepsilon) \\
&= \omega_\mu + O(\varepsilon).
\end{aligned}$$

Thus we proved Theorem 3.5.

4. Examples

Using Theorem 2.3, we have obtained [4] the general Schlesinger systems which give Painlevé equations P_J $(J = II, \dots, VI)$ in particular cases $r = 2$, $N = 4$ through reduction of the systems using first integrals. We give in this section the process of confluence for these GSS.

4.1. GSS for Painlevé equations. At first we list up, for each Painlevé equation, the following data:

(1) a partition λ of 4 which specifies the abelian group $H_\lambda \subset \mathrm{GL}_4(\mathbf{C})$,

(2) the subspace X_λ of Z_λ which is a realization of $\mathrm{GL}_2(\mathbf{C})\backslash Z_\lambda/H_\lambda$ and parametrizes lines in an invariant open subset U of the twistor space $\mathbf{P}^3$,

(3) the connection form ω of the flat connection $\nabla_\lambda = d - \omega\wedge$,

(4) the GSS equivalent to the Painlevé equation obtained as the zero-curvature condition of ∇_λ (see [8] for the equivalence).

Note that in each of the following cases, the invariant open subset $U \subset \mathbf{P}^3$ in Theorem 2.3 is a union of the open dense orbit and the orbits of codimension one.

4.1.1. *Painlevé P_{VI}.*

(1) $\lambda = (1,1,1,1)$, $H_\lambda = \left\{ \begin{pmatrix} h_0 & & & \\ & h_1 & & \\ & & h_2 & \\ & & & h_3 \end{pmatrix} \right\}$.

(2) $X_\lambda = \left\{ z = \begin{pmatrix} 1 & -1 & 0 & -t \\ 0 & 1 & 1 & 1 \end{pmatrix} \middle| \, t \neq 0, 1, \infty \right\}$.

(3) $\omega = A_1 \dfrac{d\zeta}{\zeta - 1} + A_2 \dfrac{d\zeta}{\zeta} + A_3 \dfrac{d\zeta - dt}{\zeta - t}$ with $A_0 + A_1 + A_2 + A_3 = 0$.

(4)

$$\frac{dA_1}{dt} = \frac{[A_3, A_1]}{t-1}, \qquad \frac{dA_2}{dt} = \frac{[A_3, A_2]}{t}, \qquad \frac{dA_3}{dt} = -\frac{[A_3, A_1]}{t-1} - \frac{[A_3, A_2]}{t}. \quad (4.1)$$

4.1.2. *Painlevé P_V.*

(1) $\lambda = (2,1,1)$, $H_\lambda = \left\{ \begin{pmatrix} h_0 & h_1 & & \\ & h_0 & & \\ & & h_2 & \\ & & & h_3 \end{pmatrix} \right\}$.

(2) $X_\lambda = \left\{ z = \begin{pmatrix} 1 & 0 & 0 & -t \\ 0 & 1 & 1 & 1 \end{pmatrix} \middle| \, t \neq 0, \infty \right\}$.

(3) $\omega = A_1\, d\zeta + A_2 \dfrac{d\zeta}{\zeta} + A_3 \dfrac{d\zeta - dt}{\zeta - t}$ with $A_0 + A_2 + A_3 = 0$.

(4)

$$\frac{dA_1}{dt} = 0, \qquad \frac{dA_2}{dt} = \frac{[A_3, A_2]}{t}, \qquad \frac{dA_3}{dt} = [A_1, A_3] - \frac{[A_3, A_2]}{t}. \tag{4.2}$$

4.1.3. *Painlevé P_{IV}.*

(1) $\lambda = (3,1),\ H_\lambda = \left\{ \begin{pmatrix} h_0 & h_1 & h_2 & \\ & h_0 & h_1 & \\ & & h_0 & \\ & & & h_3 \end{pmatrix} \right\}.$

(2) $X_\lambda = \left\{ z = \begin{pmatrix} 1 & 0 & 0 & -t \\ 0 & 1 & 0 & 1 \end{pmatrix} \,\middle|\, t \neq \infty \right\}.$

(3) $\omega = A_1\, d\zeta - A_2 \zeta\, d\zeta + A_3 \dfrac{d\zeta - dt}{\zeta - t}$ with $A_0 + A_3 = 0$.

(4)

$$\frac{dA_1}{dt} = [A_3, A_2], \qquad \frac{dA_2}{dt} = 0, \qquad \frac{dA_3}{dt} = [A_1 - tA_2, A_3]. \tag{4.3}$$

4.1.4. *Painlevé P_{III}.*

(1) $\lambda = (2,2),\ H_\lambda = \left\{ \begin{pmatrix} h_0 & h_1 & & \\ & h_0 & & \\ & & h_2 & h_3 \\ & & & h_2 \end{pmatrix} \right\}.$

(2) $X_\lambda = \left\{ z = \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 1 & 0 \end{pmatrix} \,\middle|\, t \neq 0, \infty \right\}.$

(3) $\omega = A_1\, d\zeta + A_2 \dfrac{d\zeta}{\zeta} + A_3 d\left(\dfrac{t}{\zeta}\right)$ with $A_0 + A_2 = 0$.

(4)

$$\frac{dA_1}{dt} = 0, \qquad \frac{dA_2}{dt} = [A_3, A_1], \qquad \frac{dA_3}{dt} = \frac{[A_2, A_3]}{t}. \tag{4.4}$$

4.1.5. *Painlevé P_{II}.*

(1) $\lambda = (4),\ H_\lambda = \left\{ \begin{pmatrix} h_0 & h_1 & h_2 & h_3 \\ & h_0 & h_1 & h_2 \\ & & h_0 & h_1 \\ & & & h_0 \end{pmatrix} \right\}.$

(2) $X_\lambda = \left\{ z = \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \,\middle|\, t \neq \infty \right\}.$

(3) $\omega = A_1\, d\zeta + A_2 d\left(t - \frac{1}{2}\zeta^2\right) + A_3 d\left(-\zeta t + \frac{1}{3}\zeta^3\right)$ with $A_0 = 0$.

(4)

$$\frac{dA_1}{dt} = [A_2, A_1 - tA_3], \qquad \frac{dA_2}{dt} = [A_3, A_1], \qquad \frac{dA_3}{dt} = 0. \tag{4.5}$$

In the subsequent subsections, we use the following notations. In the case $\lambda \to \mu, \lambda, \mu \in \mathscr{P}_4$, we denote a point of X_λ as z, the variable parameter in z as t, the connection form of ∇_λ as ω, the coordinate of $\mathbf{P}^1$ describing the lines in the forms ω as ζ and the vector consisting of potentials in the form ω as $A = (A_0, A_1, A_2, A_3)$. Correspondingly, for the partition μ, we use the symbols w, s, $\tilde{\omega}$, η and $B = (B_0, B_1, B_2, B_3)$.

4.2. From P_{VI} to P_V. In this case, the partitions are $\lambda = (1,1,1,1) \to \mu = (2,1,1)$. For $w = \begin{pmatrix} 1 & 0 & 0 & -s \\ 0 & 1 & 1 & 1 \end{pmatrix} \in X_\mu$, put $z(\varepsilon) = wg(\varepsilon) \in X_\lambda$:

$$z(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & -s \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & & \\ & \varepsilon & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & -s \\ 0 & \varepsilon & 1 & 1 \end{pmatrix}$$

and consider the change of potentials $A(\varepsilon) = B \cdot ({}^t g(\varepsilon)^{-1} \otimes I_2)$. Explicitly we have

$$A_0(\varepsilon) = B_0 - \varepsilon^{-1} B_1, \qquad A_1(\varepsilon) = \varepsilon^{-1} B_1, \qquad A_2(\varepsilon) = B_2, \qquad A_3(\varepsilon) = B_3. \tag{4.6}$$

Then we consider the connection form $\omega(\varepsilon)$ defined by

$$\begin{aligned} \omega(\varepsilon) &= \sum_{0 \le j \le 3} A_j(\varepsilon) d \log(\vec{\eta} z_j(\varepsilon)) \\ &= \varepsilon^{-1} B_1 d \log(1 + \varepsilon\eta) + B_2 d \log(\eta) + B_3 d \log(\eta - s) \\ &= B_1\, d\eta + B_2 \frac{d\eta}{\eta} + B_3 \frac{d\eta - ds}{\eta - s} + O(\varepsilon). \end{aligned}$$

Hence we have $\lim_{\varepsilon \to 0} \omega(\varepsilon) = \tilde{\omega}$.

To derive the confluence on the level of nonlinear equations, we try to transform $z(\varepsilon)$ to the normal form of elements in X_λ by the action of $\mathrm{GL}_2(\mathbf{C}) \times H_\lambda$:

$$\begin{aligned}
\vec{\eta}z(\varepsilon) &= \vec{\eta}\begin{pmatrix} 1 & 1 & 0 & -s \\ 0 & \varepsilon & 1 & 1 \end{pmatrix} \\
&= \vec{\eta}\begin{pmatrix} 1 & \\ & -\varepsilon \end{pmatrix}\begin{pmatrix} 1 & \\ & -\varepsilon^{-1} \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & -s \\ 0 & \varepsilon & 1 & 1 \end{pmatrix} \\
&= (1, -\varepsilon\eta)\begin{pmatrix} 1 & 1 & 0 & -s \\ 0 & -1 & -\varepsilon^{-1} & -\varepsilon^{-1} \end{pmatrix} \\
&= (1, -\varepsilon\eta)\begin{pmatrix} 1 & -1 & 0 & -(-\varepsilon s) \\ 0 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -\varepsilon^{-1} & \\ & & & -\varepsilon^{-1} \end{pmatrix}.
\end{aligned}$$

This computation implies that the form $\omega(\varepsilon)$ can be obtained from ω by the change of variables $\zeta = -\varepsilon\eta$, $t = -\varepsilon s$ and the change of potentials (4.6). From this observation, we can conclude that the system (4.2) can be obtained from (4.1) by the change of variable $t = -\varepsilon s$ and the change of potentials (4.6).

4.3. From P_V to P_{IV}. In this case, the partitions are $\lambda = (2,1,1) \to \mu = (3,1)$. For $w = \begin{pmatrix} 1 & 0 & 0 & -s \\ 0 & 1 & 0 & 1 \end{pmatrix} \in X_\mu$, put $z(\varepsilon) = wg(\varepsilon) \in X_\lambda$:

$$z(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & -s \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & 1 & \\ & 1 & \varepsilon & \\ & & \varepsilon^2 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & -s \\ 0 & 1 & \varepsilon & 1 \end{pmatrix}$$

and consider the change of potentials $A(\varepsilon) = B \cdot ({}^t g(\varepsilon)^{-1} \otimes I_2)$. Explicitly we have

$$\begin{aligned}
&A_0(\varepsilon) = B_0 - \varepsilon^{-2}B_2, \qquad A_1(\varepsilon) = B_1 - \varepsilon^{-1}B_2, \\
&A_2(\varepsilon) = \varepsilon^{-2}B_2, \qquad A_3(\varepsilon) = B_3.
\end{aligned} \tag{4.7}$$

Then we consider the connection form $\omega(\varepsilon)$ defined by using $z(\varepsilon)$. Then

$$\begin{aligned}
\omega(\varepsilon) &= (B_1 - \varepsilon^{-1}B_2)d\eta + \varepsilon^{-2}B_2 d\log(1+\varepsilon\eta) + B_3 d\log(\eta - s) \\
&= B_1\,d\eta - B_2\eta\,d\eta + B_3\frac{d\eta - ds}{\eta - s} + O(\varepsilon) \\
&\to \tilde{\omega}.
\end{aligned}$$

To derive the confluence on the level of nonlinear equations, we transform $z(\varepsilon)$ to the normal form of elements in X_λ by the action of $\mathrm{GL}_2(\mathbf{C}) \times H_\lambda$:

$$
\begin{aligned}
\vec{\eta}z(\varepsilon) &= \vec{\eta}\begin{pmatrix} 1 & 0 & 1 & -s \\ 0 & 1 & \varepsilon & 1 \end{pmatrix} \\
&= \vec{\eta}\begin{pmatrix} 1 & 1 \\ & \varepsilon \end{pmatrix}\begin{pmatrix} 1 & -\varepsilon^{-1} & 0 & -s-\varepsilon^{-1} \\ 0 & \varepsilon^{-1} & 1 & \varepsilon^{-1} \end{pmatrix} \\
&= \vec{\eta}\begin{pmatrix} 1 & 1 \\ & \varepsilon \end{pmatrix}\begin{pmatrix} 1 & 0 \\ & \varepsilon^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -s-\varepsilon^{-1} \\ 0 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\varepsilon^{-1} & & \\ & 1 & & \\ & & \varepsilon & \\ & & & 1 \end{pmatrix} \\
&= (1, \varepsilon^{-1}+\eta)\begin{pmatrix} 1 & 0 & 0 & -(s+\varepsilon^{-1}) \\ 0 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\varepsilon^{-1} & & \\ & 1 & & \\ & & \varepsilon & \\ & & & 1 \end{pmatrix}.
\end{aligned}
$$

This computation implies that the form $\omega(\varepsilon)$ can be obtained from ω by the change of variables $\zeta = \eta + \varepsilon^{-1}$, $t = s + \varepsilon^{-1}$ and the change of potentials (4.7). Then we can conclude that the system (4.3) can be obtained from (4.2) by the change of variable $t = s + \varepsilon^{-1}$ and the change of potentials (4.7).

4.4. From P_V to P_{III}. This is the case where $\lambda = (2,1,1) \to \mu = (2,2)$. For $w = \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 1 & 0 \end{pmatrix} \in X_\mu$, put $z(\varepsilon) = wg(\varepsilon) \in X_\lambda$:

$$
z(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & \varepsilon \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \varepsilon s \\ 0 & 1 & 1 & 1 \end{pmatrix}
$$

and consider the change of potentials $A(\varepsilon) = B \cdot ({}^t g(\varepsilon)^{-1} \otimes I_2)$. Explicitly we have

$$
A_0(\varepsilon) = B_0, \qquad A_1(\varepsilon) = B_1, \qquad A_2(\varepsilon) = B_2 - \varepsilon^{-1}B_3, \qquad A_3(\varepsilon) = \varepsilon^{-1}B_3. \quad (4.8)
$$

Then we consider the connection form $\omega(\varepsilon)$ defined by using $z(\varepsilon)$. Then

$$
\begin{aligned}
\omega(\varepsilon) &= B_1\, d\eta + (B_2 - \varepsilon^{-1}B_3)d\log\eta + \varepsilon^{-1}B_3 d\log(\eta + \varepsilon s) \\
&= B_1\, d\eta + B_2\frac{d\eta}{\eta} + B_3 d\left(\frac{s}{\eta}\right) + O(\varepsilon) \\
&\to \tilde{\omega}.
\end{aligned}
$$

Here we used

$$d\log(\eta+\varepsilon s)=\frac{d\eta}{\eta}+\varepsilon d\left(\frac{s}{\eta}\right)+O(\varepsilon^2).$$

We derive the confluence on the level of nonlinear equations. Since $z(\varepsilon)$ is already of the normal form in X_λ, it is only necessary to make a change of parameter $t=-\varepsilon s$ and a change of potentials (4.8). Then we obtain the system (4.4) from (4.2) if we take a limit $\varepsilon\to 0$.

4.5. From P_{IV} to P_{II}. In this case, the partitions are $\lambda=(3,1)\to\mu=(4)$. For $w=\begin{pmatrix}1&0&s&0\\0&1&0&0\end{pmatrix}\in X_\mu$, put $z(\varepsilon)=wg(\varepsilon)\in X_\lambda$:

$$z(\varepsilon)=\begin{pmatrix}1&0&s&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}1&&0&1\\&1&&\varepsilon\\&&1&\varepsilon^2\\&&&\varepsilon^3\end{pmatrix}=\begin{pmatrix}1&0&s&1+\varepsilon^2 s\\0&1&0&\varepsilon\end{pmatrix}$$

and consider the change of potentials $A(\varepsilon)=B\cdot({}^tg(\varepsilon)^{-1}\otimes I_2)$. Explicitly we have

$$\begin{aligned}&A_0(\varepsilon)=B_0-\varepsilon^{-3}B_3, \qquad A_1(\varepsilon)=B_1-\varepsilon^{-2}B_3,\\&A_2(\varepsilon)=B_2-\varepsilon^{-1}B_3, \qquad A_3(\varepsilon)=\varepsilon^{-3}B_3.\end{aligned}\tag{4.9}$$

Then we consider the connection form $\omega(\varepsilon)$ defined by using $z(\varepsilon)$. Then

$$\begin{aligned}\omega(\varepsilon)&=(B_1-\varepsilon^{-2}B_3)d\eta+(B_2-\varepsilon^{-1}B_3)d\left(s-\frac{1}{2}\eta^2\right)+\varepsilon^{-3}B_3 d\log(1+\varepsilon\eta+\varepsilon^2 s)\\&=B_1\,d\eta+B_2 d\left(s-\frac{1}{2}\eta^2\right)+B_3 d\left(\frac{1}{3}\eta^3-s\eta\right)+O(\varepsilon)\\&\to\tilde{\omega}.\end{aligned}$$

Here we used

$$\log(1+\varepsilon\eta+\varepsilon^2 s)=\eta\varepsilon+\left(s-\frac{1}{2}\eta^2\right)\varepsilon^2+\left(\frac{1}{3}\eta^3-s\eta\right)\varepsilon^3+O(\varepsilon^4).$$

To derive the confluence on the level of nonlinear equations, we transform $z(\varepsilon)$ to the normal form of elements in X_λ by the action of $\mathrm{GL}_2(\mathbf{C})\times H_\lambda$. We have

$$\vec{\eta}z(\varepsilon) = \vec{\eta}\begin{pmatrix} 1 & 0 & s+h_2 & h_3(1+\varepsilon^2 s) \\ 0 & 1 & 0 & h_3\varepsilon \end{pmatrix}\begin{pmatrix} 1 & 0 & h_2 & \\ & 1 & 0 & \\ & & 1 & \\ & & & h_3 \end{pmatrix}^{-1}$$

$$= \vec{\eta}\begin{pmatrix} 1 & 0 & 0 & -(-\varepsilon^{-1}-\varepsilon s) \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & h_2 & \\ & 1 & 0 & \\ & & 1 & \\ & & & h_3 \end{pmatrix}^{-1}.$$

Here we determined h_2, h_3 as $h_2 = -s$, $h_3 = \varepsilon^{-1}$. Put $h = \begin{pmatrix} 1 & 0 & h_2 \\ & 1 & 0 \\ & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & s \\ & 1 & 0 \\ & & 1 \end{pmatrix}$. The above computation implies that the change of parameter should be $t = -\varepsilon^{-1} - \varepsilon s$. So, to obtain the form $\omega(\varepsilon)$ from ω, first we modify ω as $\omega + A_2\, d\theta_2(h) = \omega - \varepsilon^{-1}A_2\, dt$, and then make a change of parameter $t = -\varepsilon^{-1} - \varepsilon s$ and of potentials (4.9). From this observation, we can conclude that the system (4.5) can be obtained from (4.3) as follows. First we modify (4.3) as

$$\frac{dA_1}{dt} = [A_3, A_2] + \varepsilon^{-1}[A_1, A_2], \qquad \frac{dA_2}{dt} = 0, \qquad \frac{dA_3}{dt} = [A_1 - tA_2, A_3] + \varepsilon^{-1}[A_3, A_2]$$

according as the modification of ω. Then the change of variable $t = -\varepsilon^{-1} - \varepsilon s$ and of potentials (4.9) together with the limit $\varepsilon \to 0$ gives the system (4.5).

4.6. From P_{III} to P_{II}. In this case, the partitions are $\lambda = (2,2) \to \mu = (4)$. For $w = \begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in X_\mu$, put $z(\varepsilon) = wg(\varepsilon) \in X_\lambda$:

$$z(\varepsilon) = \begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & & 1 & \\ & 1 & \varepsilon & 1 \\ & & \varepsilon^2 & 2\varepsilon \\ & & \varepsilon^3 & 3\varepsilon^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1+\varepsilon^2 s & 2\varepsilon s \\ 0 & 1 & \varepsilon & 1 \end{pmatrix}$$

and consider the change of potentials $A(\varepsilon) = B \cdot ({}^t g(\varepsilon)^{-1} \otimes I_2)$. Explicitly we have

$$A_0(\varepsilon) + A_2(\varepsilon) = B_0, \qquad A_1(\varepsilon) = B_1 - 2\varepsilon^{-1}B_2 + \varepsilon^{-2}B_3, \tag{4.10}$$

$$A_2(\varepsilon) = 3\varepsilon^{-2}B_2 - 2\varepsilon^{-3}B_3, \qquad A_3(\varepsilon) = -\varepsilon^{-1}B_2 + \varepsilon^{-2}B_3. \tag{4.11}$$

Then we consider the connection form $\omega(\varepsilon)$ defined by using $z(\varepsilon)$. Then we can check that

$$\begin{aligned}\omega(\varepsilon) &= A_1(\varepsilon)d\eta + A_2(\varepsilon)d\log(1+\varepsilon\eta+\varepsilon^2 s) + A_3(\varepsilon)d\left(\frac{\eta+2\varepsilon s}{1+\varepsilon\eta+\varepsilon^2 s}\right)\\ &= B_1\, d\eta + B_2 d\left(s-\frac{1}{2}\eta^2\right) + B_3 d\left(\frac{1}{3}\eta^3 - s\eta\right) + O(\varepsilon)\\ &\to \tilde{\omega}.\end{aligned}$$

The confluence on the level of nonlinear equations can be carried out in a similar way as in the case $P_{IV} \to P_{II}$.

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