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# The boundary of a fibered face of the magic 3-manifold and the asymptotic behavior of minimal pseudo-Anosov dilatations

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**Abstract.** Let $\delta_{g,n}$ be the minimal dilatation of pseudo-Anosovs defined on an orientable surface of genus $g$ with $n$ punctures. It is proved by Tsai that for any fixed $g \geq 2$, there exists a constant $c_g$ depending on $g$ such that

$$\frac{1}{c_g} \cdot \frac{\log n}{n} < \log \delta_{g,n} < c_g \cdot \frac{\log n}{n} \qquad \text{for any } n \geq 3.$$

This means that the logarithm of the minimal dilatation $\log \delta_{g,n}$ is on the order of $\log n/n$. We prove that if $2g+1$ is relatively prime to $s$ or $s+1$ for each $0 \leq s \leq g$, then

$$\limsup_{n \to \infty} \frac{n(\log \delta_{g,n})}{\log n} \leq 2$$

holds. In particular, if $2g+1$ is prime, then the above inequality on $\delta_{g,n}$ holds. Our examples of pseudo-Anosovs $\phi$'s which provide the upper bound above have the following property: The mapping torus $M_\phi$ of $\phi$ is a single hyperbolic 3-manifold $N$ called the magic manifold, or the fibration of $M_\phi$ comes from a fibration of $N$ by Dehn filling cusps along the boundary slopes of a fiber.

## 1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus $g$ with $n$ punctures and $\mathrm{Mod}(\Sigma)$ the mapping class group of $\Sigma$. By Thurston's classification theorem of surface automorphisms, elements of $\mathrm{Mod}(\Sigma)$ are either periodic, reducible, or pseudo-Anosov, see [20]. Pseudo-Anosov mapping classes have rich dynamical properties. The hyperbolization theorem by Thurston [21] relates the dynamics of pseudo-Anosovs and the geometry of hyperbolic fibered 3-manifolds. The theorem asserts that $\phi \in \mathrm{Mod}(\Sigma)$ is pseudo-Anosov if and

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only if the mapping torus $M_\phi$ of $\phi$ admits a complete hyperbolic metric of finite volume.

Each pseudo-Anosov element $\phi \in \mathrm{Mod}(\Sigma)$ has a representative $\Phi : \Sigma \to \Sigma$ called a pseudo-Anosov homeomorphism. Such a homeomorphism is equipped with a constant $\lambda = \lambda(\Phi) > 1$ called the *dilatation* of $\Phi$. If we let $\mathrm{ent}(\Phi)$ be the *topological entropy* of $\Phi$, then the equality $\mathrm{ent}(\Phi) = \log \lambda(\Phi)$ holds. Moreover $\mathrm{ent}(\Phi)$ attains the minimal entropy among all homeomorphisms which are isotopic to $\Phi$, see [3, Exposé 10]. The *dilatation* $\lambda(\phi)$ of $\phi$ is defined to be $\lambda(\Phi)$. We call the quantities $\mathrm{ent}(\phi) = \log \lambda(\phi)$ and $\mathrm{Ent}(\phi) = |\chi(\Sigma)| \log \lambda(\phi)$ the *entropy* and *normalized entropy* of $\phi$ respectively, where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$.

If we fix $\Sigma$, the set of dilatations of pseudo-Anosovs defined on $\Sigma$ is a closed discrete subset of $\mathbf{R}$, see [7] for example. In particular there exists a minimum. We denote by $\delta(\Sigma) > 1$, the minimal dilatation of pseudo-Anosov elements in $\mathrm{Mod}(\Sigma)$. The minimal dilatations are determined in only a few cases. (See for example [9] which is a survey on minimal pseudo-Anosov dilatations.)

Let us set $\delta_{g,n} = \delta(\Sigma_{g,n})$ and $\delta_g = \delta_{g,0}$. We write $A \asymp B$ if there exists a universal constant $c$ such that $A/c < B < cA$. Penner proved in [17] that $\log \delta_g \asymp \frac{1}{g}$. This work by Penner was a starting point for the study of the asymptotic behavior of the minimal dilatations on surfaces varying topology. Later it was proved by Hironaka-Kin [6] that $\log \delta_{0,n} \asymp \frac{1}{n}$, and by Tsai [22] that $\log \delta_{1,n} \asymp \frac{1}{n}$. See also Valdivia [23]. The following theorem of Tsai is in contrast with the cases of genera 0 and 1.

THEOREM 1.1 ([22]). *For any fixed $g \geq 2$, there exists a constant $c_g$ depending on $g$ such that*

$$\frac{1}{c_g} \cdot \frac{\log n}{n} < \log \delta_{g,n} < c_g \cdot \frac{\log n}{n} \qquad \textit{for any } n \geq 3.$$

*In particular for any fixed $g \geq 2$, we have*

$$\log \delta_{g,n} \asymp \frac{\log n}{n}.$$

The following question is due to Tsai.

QUESTION 1.2. *What is the optimal constant $c_g$ in Theorem 1.1?*

One can also ask the following.

QUESTION 1.3. *Given $g \geq 2$, does $\lim_{n \to \infty} \dfrac{n(\log \delta_{g,n})}{\log n}$ exist? What is its value?*

This is an analogous question, posed by McMullen, which is asking whether $\lim_{g\to\infty} g \log \delta_g$ exists or not, see [15]. Toward Questions 1.2 and 1.3, we prove the following.

Theorem 1.4. *Given $g \geq 2$, there exists a sequence $\{n_i\}_{i=0}^{\infty}$ with $n_i \to \infty$ such that*

$$\limsup_{i\to\infty} \frac{n_i \log \delta_{g,n_i}}{\log n_i} \leq 2.$$

Theorem 1.4 improves the previous upper bound on $\log \delta_{g,n}$ by Tsai. In fact for any $g \geq 2$, Tsai's examples in [22] yield the upper bound $\limsup_{n\to\infty} \frac{n(\log \delta_{g,n})}{\log n} \leq 2(2g+1)$, which is proved by a similar computation in the proof of Theorem 1.4. As a corollary of Theorem 1.4, we have the following.

Corollary 1.5. *Given $g \geq 2$, the following set*

$$\left\{\frac{n}{\log n} \cdot \mathrm{ent}(\phi) \mid \phi \in \mathrm{Mod}(\Sigma_{g,n}) \text{ is pseudo-Anosov, } n \geq 1\right\}$$

*has an accumulation point* 2.

To state other results which are related to Questions 1.2 and 1.3, we define a polynomial $B_{(g,p)}(t)$ for nonnegative integers $g$ and $p$:

$$B_{(g,p)}(t) = t^{2p+1}(t^{2g+1} - 1) + 1 - 2t^{p+g+1} - t^{2g+1}.$$

We shall see that there exists a unique real root $r_{(g,p)}$ greater than 1 of $B_{(g,p)}(t)$, and these satisfy

$$\lim_{p\to\infty} \frac{p \log r_{(g,p)}}{\log p} = 1$$

(Lemma 4.1). The root $r_{(g,p)}$ gives the following upper bound.

Theorem 1.6. *For $g \geq 2$ and $p \geq 0$, suppose that* $\gcd(2g+1, p+g+1) = 1$. *Then*

$$\delta_{g,2p+i} \leq r_{(g,p)} \qquad \text{for each } i \in \{1,2,3,4\}.$$

If $g$ satisfies $(*)$ in the next Theorem 1.7, then one can take the sequence $\{n_i\}_{i=0}^{\infty}$ in Theorem 1.4 to be the sequence $\{n\}_{n=1}^{\infty}$ of natural numbers.

Theorem 1.7. *Suppose that $g \geq 2$ satisfies*

$(*)$ $\gcd(2g+1, s) = 1$ *or* $\gcd(2g+1, s+1) = 1$ *for each* $0 \leq s \leq g$.

*Then*

$$\limsup_{n\to\infty} \frac{n(\log \delta_{g,n})}{\log n} \le 2.$$

For example, $(*)$ holds for $g = 4$ since 9 is relatively prime to $1, 2, 4$ and 5; $(*)$ does not hold for $g = 7$ because $\gcd(15, 5) = 5$ and $\gcd(15, 6) = 3$. We point out that infinitely many $g$'s satisfy $(*)$. In fact if $2g + 1$ is prime, then $2g + 1$ is relatively prime to $s'$ for each $1 \le s' \le g + 1$.

Corollary 1.8. *If $2g + 1$ is prime for $g \ge 2$, then*

$$\limsup_{n\to\infty} \frac{n(\log \delta_{g,n})}{\log n} \le 2.$$

Remark 1.9. One can simplify $(*)$ in Theorem 1.7, since $2g + 1$ is relative prime to $1, 2$ and $g$. In the case $g \ge 5$, $(*)$ is equivalent to

$(**)$ $\gcd(2g+1, s) = 1$ or $\gcd(2g+1, s+1) = 1$ or each $3 \le s \le g - 2$.

Our results are proved by using the theory of fibered faces of hyperbolic and fibered 3-manifolds $M$, developed by Thurston [19], Fried [4], Matsumoto [14] and McMullen [15], see Section 2. We focus on a fibered face of a particular hyperbolic fibered 3-manifold, called the *magic manifold* $N$. This manifold is the exterior of the 3 chain link $\mathscr{C}_3$, see Figure 1. Our examples of pseudo-Anosovs $\phi$'s which provide the upper bounds in Theorems 1.4, 1.6 and 1.7 have the following property: The mapping torus $M_\phi$ of $\phi$ is homeomorphic to $N$, or the fibration of $M_\phi$ comes from a fibration of $N$ by Dehn filling cusps along the boundary slopes of a fiber. An explicit construction of these examples is given by the first author, see [8, Example 4.8].

We turn to hyperbolic volumes of hyperbolic 3-manifolds. The set of volumes of hyperbolic 3-manifolds is a well-ordered closed subset in $\mathbf{R}$ of order

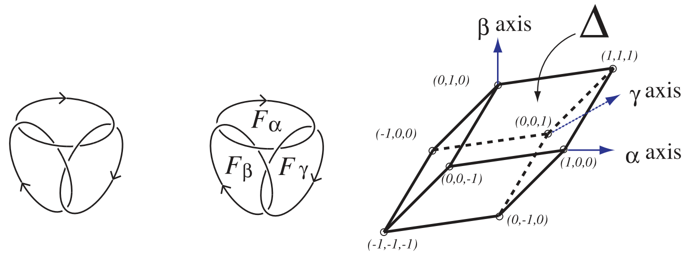


**Fig. 1.** (left) 3 chain link $\mathscr{C}_3$. (center) $F_\alpha$, $F_\beta$, $F_\gamma$. (right) Thurston norm ball $U_N$. (fibered face $\Delta$ is indicated.)

type $\omega^\omega$, see [18]. In particular if we fix a surface $\Sigma$, then there exists a minimum among volumes of hyperbolic $\Sigma$-bundles over the circle. The proofs of Theorems 1.4, 1.7 immediately imply the following.

Proposition 1.10. *For each $g \geq 2$, there exists a sequence $\{n_i\}_{i=0}^{\infty}$ with $n_i \to \infty$ such that the minimal volume of $\Sigma_{g,n_i}$-bundles over the circle is less than or equal to* $\mathrm{vol}(N) \approx 5.3334$, *the volume of the magic manifold $N$. In particular, for any $g \geq 2$ satisfying $(*)$ and any $n \geq 3$, the minimal volume of $\Sigma_{g,n}$-bundles over the circle is less than or equal to* $\mathrm{vol}(N)$.

We close the introduction by asking

Question 1.11 (cf. Theorems 1.4 and 1.7). *Does* $\limsup_{n\to\infty} \frac{n(\log \delta_{g,n})}{\log n} \leq 2$ *hold for all fixed $g \geq 2$?*

## 2. The Thurston norm and fibered 3-manifolds

Let $M$ be an oriented hyperbolic 3-manifold with boundary $\partial M$ (possibly $\partial M = \emptyset$). We recall the Thurston norm $\|\cdot\| : H_2(M, \partial M; \mathbf{R}) \to \mathbf{R}$. Let $F$ be a finite union of oriented, connected surfaces. We define $\chi_-(F)$ to be

$$\chi_-(F) = \sum_{F_i \subset F} \max\{0, -\chi(F_i)\},$$

where $F_i$'s are the connected components of $F$ and $\chi(F_i)$ is the Euler characteristic of $F_i$. The Thurston norm $\|\cdot\|$ is defined for an integral class $a \in H_2(M, \partial M; \mathbf{Z})$ by

$$\|a\| = \min_F \{\chi_-(F) \mid a = [F]\},$$

where the minimum ranges over all oriented surfaces $F$ embedded in $M$. A surface $F$ which realizes this minimum is called a *minimal representative* of $a$, denoted by $F_a$. Then $\|\cdot\|$ defined on integral classes admits a unique continuous extension $\|\cdot\| : H_2(M, \partial M; \mathbf{R}) \to \mathbf{R}$ which is linear on the ray through the origin. The unit ball $U_M$ with respect to the Thurston norm is a compact, convex polyhedron. See [19] for more details.

Suppose that $M$ is a surface bundle over the circle and let $F$ be its fiber. The fibration determines a cohomology class $a^* \in H^1(M; \mathbf{Z})$, and hence a homology class $a \in H_2(M, \partial M; \mathbf{Z})$ by Poincaré duality. Thurston proved in [19] that there exists a top dimensional face $\Omega$ on $\partial U_M$ such that $a = [F]$ is an integral class of $int(C_\Omega)$, where $C_\Omega$ is the cone over $\Omega$ with the origin and $int(C_\Omega)$ is its interior. Moreover the minimal representative $F_a$ for any integral class $a$ in $int(C_\Omega)$ becomes a fiber of the fibration associated to $a$. Such a face $\Omega$ is called a *fibered face*, and an integral class $a \in int(C_\Omega)$ is called a *fibered* *class*. This work of Thurston tells us that if $M$ has second Betti number greater than 1, then $M$ provides infinitely many pseudo-Anosov monodromies defined on surfaces with variable topology.

The set of integral and rational classes of $int(C_\Omega)$ are denoted by $int(C_\Omega(\mathbf{Z}))$ and $int(C_\Omega(\mathbf{Q}))$ respectively. When $a \in int(C_\Omega(\mathbf{Z}))$ is primitive, the associated fibration on $M$ has a connected fiber represented by $F_a$. Let $\Phi_a : F_a \to F_a$ be the monodromy. Since $M$ is hyperbolic, $\phi_a = [\Phi_a]$ is pseudo-Anosov. The *dilatation* $\lambda(a)$ and *entropy* $\mathrm{ent}(a) = \log \lambda(a)$ are defined as the dilatation and entropy of $\phi_a$ respectively. The entropy defined on primitive fibered classes can be extended to rational classes by homogeneity. It is shown by Fried in [4] that $\frac{1}{\mathrm{ent}} : int(C_\Omega(\mathbf{Q})) \to \mathbf{R}$ is concave, and in particular $\mathrm{ent} : int(C_\Omega(\mathbf{Q})) \to \mathbf{R}$ admits a unique continuous extension

$$\mathrm{ent} : int(C_\Omega) \to \mathbf{R}.$$

Moreover Fried proved that the restriction

$$\mathrm{ent}|_{int(\Omega)} (= \mathrm{Ent}|_{int(\Omega)}) : int(\Omega) \to \mathbf{R}$$

on the open face $int(\Omega)$ has the property that $\mathrm{ent}(a)$ goes to $\infty$ as $a \in int(\Omega)$ goes to a point on $\partial\Omega$. Thus we have a continuous function

$$\mathrm{Ent} = \|\cdot\|\, \mathrm{ent}(\cdot) : int(C_\Omega) \to \mathbf{R}$$

which is constant on each ray in $int(C_\Omega)$ through the origin.

These properties give us the following observation: Fix a hyperbolic fibered 3-manifold $M$ with a fibered face $\Omega$ as above. For any compact set $\mathscr{D} \subset int(\Omega)$, there exists a constant $C = C_{\mathscr{D}} > 0$ satisfying the following. Let $a \in int(C_\Omega)$ be any integral class of $H_2(M, \partial M; \mathbf{Z})$. The normalized entropy $\mathrm{Ent}(a) (= \mathrm{Ent}(\phi_a))$ is bounded by $C$ from above whenever $\bar{a} \in \mathscr{D}$, where $\bar{a}$ is the projection of $a$ into $int(\Omega)$.

This observation enables us to investigate the following asymptotic behaviors of minimal dilatations.

(1) $\limsup_{n\to\infty} n \log \delta_{0,n} \le 2 \log(2+\sqrt{3})$, see [6, 11].

(2) $\limsup_{n\to\infty} n \log \delta_{1,n} \le 2 \log \lambda_0$, where $\lambda_0 \approx 2.2966$ is the largest real root of $t^4 - 2t^3 - 2t + 1$, see [10].

(3) $g \log \delta_g \le \log\left(\frac{3+\sqrt{5}}{2}\right)$, see [2, Appendix] and [5, 1, 12].

We note that for fixed $g \ge 2$, different methods for investigating the asymptotic behavior of $\delta_{g,n}$ varying $n$ are necessary. Theorem 1.1 says that there exists no constant $C > 0$, independent of $n$ so that $|\chi(\Sigma_{g,n})| \log \delta_{g,n} < C$. Thus if, for fixed $g \ge 2$, there exists a sequence of fibered classes $\{a_i\}$ with $a_i \in int(C_\Omega) \cap H_2(M, \partial M; \mathbf{Z})$ such that the fiber of the fibration associated to $a_i$ is a surface of genus $g$ having $n_i$ boundary components with $n_i \to \infty$, then the accumulation points of the sequence of projective classes $\{\bar{a}_i\}$ must lie on the boundary of $\Omega$. To prove Theorems 1.4, 1.6 and 1.7, we pay special attention to the magic manifold $N$. In Section 4.3, we choose such a sequence of fibered classes $\{a_i\}$ of $N$ carefully. We analyze the asymptotic behavior of $\lambda(a_i)$'s by using a technique given in Section 3.

The Teichmüller polynomial, developed by McMullen [15] is a certain element $\Theta_\Omega$ (associated to the fibered face $\Omega$) in the group ring $\mathbf{Z}G$, where $G = H_1(M;\mathbf{Z})/\text{torsion}$, i.e, $\Theta_\Omega$ is a finite sum

$$\Theta_\Omega = \sum_{g \in G} c_g g,$$

where $c_g$ is an integer. For every fibered class $a \in int(C_\Omega)$, the *specialization* of $\Theta_\Omega$ at the cohomology class $a^* \in H^1(M;\mathbf{Z})$ is defined by

$$\Theta_\Omega^{(a^*)}(t) = \sum_{g \in G} c_g t^{a^*(g)}$$

which is a polynomial with a variable $t$. It is a result in [15] that for all fibered class $a \in int(C_\Omega)$, the dilatation $\lambda(a)$ is equal to the largest real root of $\Theta_\Omega^{(a^*)}(t)$.

## 3. Roots of polynomials

This section concerns the asymptotic behavior of roots of families of polynomials. Let

$$g(t) = a_n t^{b_n} + a_{n-1} t^{b_{n-1}} + \cdots + a_1 t^{b_1} + a_0$$

be a polynomial with real coefficients $a_0, a_1, \ldots, a_n$ $(a_1, a_2, \ldots, a_n \neq 0)$, where $g(t)$ is arranged in the order of descending powers of $t$. Let $\mathfrak{D}(g)$ be the number of variations in signs of the coefficients $a_n, a_{n-1}, \ldots, a_0$. For example if $g(t) = +t^4 + t^3 - 2t^2 + t - 1$, then $\mathfrak{D}(g) = 3$; if $h(t) = +t^4 + t^3 - 2t^2 + t + 1$, then $\mathfrak{D}(h) = 2$. Descartes's rule of signs (see [24]) says that the number of positive real roots of $g(t)$ (counted with multiplicities) is equal to either $\mathfrak{D}(g)$ or less than $\mathfrak{D}(g)$ by an even integer.

Lemma 3.1. *Let $r \geq 0$, $s > 0$ and $u > 0$ be integers. Let*

$$\begin{aligned} P_m(t) &= t^{2m+r}(t^s - 1) + 1 - Q(t)t^m - t^u \\ &= t^{2m+r+s} - t^{2m+r} - Q(t)t^m - t^u + 1 \end{aligned}$$

*be a polynomial for each $m \in \mathbf{N}$, where $Q(t)$ is a polynomial whose coefficients are positive integers. ($Q(t)$ could be a positive constant.)*

(1) *Suppose that* $t^{2m+r+s}$ *is the leading term of* $P_m(t)$. *Then* $P_m(t)$ *has a unique real root* $\lambda_m$ *greater than* 1.

(2) *Given* $0 < c_1 < 1$ *and* $c_2 > 1$, *we have*

$$m^{c_1/m} < \lambda_m < m^{c_2/m} \qquad \text{for } m \text{ large.}$$

*In particular*

$$\lim_{m \to \infty} \frac{m \log \lambda_m}{\log m} = 1.$$

(3) *For any real numbers* $q \neq 0$ *and* $v$, *we have*

$$\lim_{m \to \infty} \frac{(qm + v) \log \lambda_m}{\log(qm + v)} = q.$$

Proof. (1) Under the assumption on $P_m(t)$, we have $\mathfrak{D}(P_m) = 2$. By Descartes's rule of signs, the number of positive real roots of $P_m(t)$ is either 2 or 0. Since $P_m(0) = 1$ and $P_m(1) = -Q(1) < 0$, the number of positive real roots of $P_m(t)$ is exactly 2. Because $P_m(t)$ goes to $\infty$ as $t$ does, $P_m(t)$ has a unique real root $\lambda_m > 1$.

(2) We have

$$P_m(t)t^{-(2m+r)} = t^s - 1 + t^{-(2m+r)} - Q(t)t^{-(m+r)} - t^{-(2m+r-u)}.$$

We define $f_m(t)$ and $g_m(t)$ such that $P_m(t)t^{-(2m+r)} = f_m(t) + g_m(t)$ as follows.

$$f_m(t) = t^s - 1 + t^{-(2m+r)}, \qquad \text{and}$$

$$g_m(t) = Q(t)t^{-(m+r)} + t^{-(2m+r-u)}.$$

We let $t = m^{c/m}$ for $c > 0$. Then

$$\begin{aligned} f_m(m^{c/m}) &= (m^{c/m})^s - 1 + (m^{c/m})^{-(2m+r)} \\ &= ((e^{\log m})^{c/m})^s - 1 + m^{-c(2+r/m)} \\ &= e^{(sc \log m)/m} - 1 + m^{-c(2+r/m)}. \end{aligned}$$

By Maclaurin expansion of $e^{(sc \log m)/m}$, we have

$$e^{(sc \log m)/m} = 1 + \frac{sc \log m}{m} + R_2,$$

where

$$R_2 = \frac{e^w}{2} \left( \frac{sc \log m}{m} \right)^2 \qquad \text{for some } 0 < w < \frac{sc \log m}{m}.$$

Since $\frac{sc \log m}{m}$ goes to 0 as $m$ goes to $\infty$, we may assume that $\frac{e^w}{2} < B$ for some constant $B > 0$. Then

$$\begin{aligned} f_m(m^{c/m}) &= \frac{sc \log m}{m} + R_2 + \frac{m^{1-c(2+r/m)}}{m} \\ &< \frac{sc \log m}{m} + B\left(\frac{sc \log m}{m}\right)^2 + \frac{m^{1-c(2+r/m)}}{m} \\ &= \frac{sc \log m}{m} + Bs^2c^2\left(\frac{\log m}{m}\right)^2 + \frac{m^{1-c(2+r/m)}}{m} \\ &< \frac{sc \log m}{m} + Bs^2c^2\left(\frac{\log m}{m}\right) + \frac{m^{1-c(2+r/m)}}{m} \\ &= \frac{(sc + Bs^2c^2) \log m + m^{1-c(2+r/m)}}{m}. \end{aligned}$$

(The last inequality comes from $0 < \frac{\log m}{m} < 1$ for $m$ large.) Thus

$$f_m(m^{c/m}) < \frac{(sc + Bs^2c^2) \log m + m^{1-c(2+r/m)}}{m}. \tag{1}$$

The first equality $f_m(m^{c/m}) = \frac{sc \log m}{m} + R_2 + \frac{m^{1-c(2+r/m)}}{m}$ above together with $R_2 > 0$ and $\frac{m^{1-c(2+r/m)}}{m} > 0$ tells us that

$$f_m(m^{c/m}) > \frac{sc \log m}{m}. \tag{2}$$

Recall that all coefficients of $Q(t)$ (appeared in $P_m(t)$) are positive integers. If we write $Q(t) = \sum_{j=0}^{\ell} a_j t^j$, where $a_j \geq 0$, then

$$\begin{aligned} g_m(m^{c/m}) &= Q(m^{c/m})m^{-c(1+r/m)} + m^{-c(2+r/m-u/m)} \\ &= \left(\sum_{j=0}^{\ell} a_j m^{-c(1+r/m-j/m)}\right) + m^{-c(2+r/m-u/m)}. \end{aligned}$$

Thus we obtain

$$g_m(m^{c/m}) = \frac{\left(\sum_{j=0}^{\ell} a_j m^{1-c(1+r/m-j/m)}\right) + m^{1-c(2+r/m-u/m)}}{m}. \tag{3}$$

For the proof of the claim (1), it is enough to prove that for $0 < c_1 < 1$ and $c_2 > 1$, we have $f_m(m^{c_1/m}) < g_m(m^{c_1/m})$ and $f_m(m^{c_2/m}) > g_m(m^{c_2/m})$ for $m$ large.

First, suppose that $0 < c < \frac{1}{2}$. Let us consider how the following four terms grow.

$$\log m,\ m^{1-c(2+r/m)},\ m^{1-c(1+r/m-j/m)} \text{ and } m^{1-c(2+r/m-u/m)}. \tag{4}$$

The first two terms appear in (1), and the last two are coming from (3). All four terms go to $\infty$ as $m$ does, since the last three terms have the positive powers of $m$. Note that for any $C > 0$, we have $\log m < m^C$ for $m$ large. Keeping this in mind, we observe that among the four terms in (4), $m^{1-c(1+r/m-j/m)}$ is dominant. This is because

$$1 - c\left(1 + \frac{r}{m} - \frac{j}{m}\right) > 1 - c\left(2 + \frac{r}{m} - \frac{u}{m}\right) \geq 1 - c\left(2 + \frac{r}{m}\right)$$

for $m$ large. These imply that $f_m(m^{c/m}) < g_m(m^{c/m})$ holds for $m$ large, since $m^{1-c(1+r/m-j/m)}$ appears in the numerator of $g_m(m^{c/m})$, see (3).

Next, we suppose that $\frac{1}{2} \leq c < 1$. We can check that $m^{1-c(1+r/m-j/m)}$ is still dominant among the four in (4). (The second and fourth terms are bounded as $m$ goes to $\infty$.) Therefore we still have $f_m(m^{c/m}) < g_m(m^{c/m})$ for $m$ large.

Finally we suppose that $c > 1$. Clearly, the last three terms in (4) go to 0 as $m$ goes to $\infty$. Thus the numerator of $g_m(m^{c/m})$, see (3), goes to 0 as $m$ tends to $\infty$. On the other hand, $f_m(m^{c/m}) > \frac{sc \log m}{m}$ holds (see (2)), and hence the numerator of

$$\frac{sc \log m + mR_2 + m^{1-c(2+r/m)}}{m} (= f_m(m^{c/m}))$$

goes to $\infty$ as $m$ does. Thus $f_m(m^{c/m}) > g_m(m^{c/m})$ for $m$ large. This completes the proof of the first part of the claim (2).

Taking logarithms on both sides of $m^{c_1/m} < \lambda_m < m^{c_2/m}$ yields

$$c_1 < \frac{m \log \lambda_m}{\log m} < c_2 \qquad \text{for } m \text{ large.}$$

Since $0 < c_1 < 1$ and $c_2 > 1$ are arbitrary, we have the desired limit. This completes the proof of the second half of the claim (2).

(3) By the claim (2),

$$\frac{c_1 \log m}{m} < \log \lambda_m < \frac{c_2 \log m}{m} \qquad \text{for } m \text{ large.}$$

Let us set $n = qm + v$. We substitute $m = \frac{n-v}{q}$ above:

$$\frac{c_1 \log\left(\frac{n-v}{q}\right)}{\frac{n-v}{q}} < \log \lambda_m < \frac{c_2 \log\left(\frac{n-v}{q}\right)}{\frac{n-v}{q}}.$$

Hence

$$\frac{qc_1(\log(n-v)-\log q)}{n-v} < \log \lambda_m < \frac{qc_2(\log(n-v)-\log q)}{n-v}.$$

We multiply all sides above by $\frac{n}{\log n} > 0$ (for $n$ large). Then

$$\frac{qc_1 n(\log(n-v)-\log q)}{(n-v)\log n} < \frac{n(\log \lambda_m)}{\log n} < \frac{qc_2 n(\log(n-v)-\log q)}{(n-v)\log n}.$$

Note that $\frac{n(\log(n-v)-\log q)}{(n-v)\log n}$ goes to 1 as $n$ (and hence $m$) goes to $\infty$. Since $0 < c_1 < 1$ and $c_2 > 1$ are arbitrary, it follows that

$$\lim_{m\to\infty} \frac{n(\log \lambda_m)}{\log n} = \lim_{m\to\infty} \frac{(qm+v)\log \lambda_m}{\log(qm+v)} = q. \qquad \square$$

## 4. The magic 3-manifold $N$

Monodromies of fibrations on $N$ have been studied in [10, 11, 12]. In Sections 4.1 and 4.2, we recall some results which tell us that the topology of fibered classes $a$ and the actual value of $\lambda(a)$. In Section 4.3, we find a family of fibered classes $a_{(g,p)}$ of $N$ with two variables $g$ and $p$, and we shall prove that it is a suitable family to prove theorems in Section 1 (cf. Remark 4.4).

Recall that $\Sigma_{g,n}$ is an orientable surface of genus $g$ with $n$ punctures. Abusing the notation, we sometimes denote by $\Sigma_{g,n}$, an orientable surface of genus $g$ with $n$ boundary components.

**4.1. Fibered face $\Delta$.** Let $K_\alpha$, $K_\beta$ and $K_\gamma$ be the components of the 3 chain link $\mathscr{C}_3$. They bound the oriented disks $F_\alpha$, $F_\beta$ and $F_\gamma$ with 2 holes, see Figure 1. Let $\alpha = [F_\alpha]$, $\beta = [F_\beta]$, $\gamma = [F_\gamma] \in H_2(N, \partial N; \mathbf{Z})$. The set $\{\alpha, \beta, \gamma\}$ is a basis of $H_2(N, \partial N; \mathbf{Z})$. Figure 1 illustrates the Thurston norm ball $U_N$ for $N$ which is the parallelepiped with vertices $\pm\alpha$, $\pm\beta$, $\pm\gamma$, $\pm(\alpha+\beta+\gamma)$ ([19, Example 3 in Section 2]). Because of the symmetry of $\mathscr{C}_3$, every top dimensional face of $U_N$ is a fibered face.

We denote a class $x\alpha + y\beta + z\gamma \in H_2(N, \partial N; \mathbf{R})$ by $(x, y, z)$. We pick a fibered face $\Delta$ with vertices $\alpha = (1,0,0)$, $\alpha+\beta+\gamma = (1,1,1)$, $\beta = (0,1,0)$ and $-\gamma = (0,0,-1)$, see Figure 1. The open face $int(\Delta)$ is written by

$$int(\Delta) = \{(X, Y, Z) \mid X + Y - Z = 1, X > 0, Y > 0, X > Z, Y > Z\}.$$

A class $a = (x, y, z) \in H_2(N, \partial N; \mathbf{R})$ is an element of $int(C_\Delta)$ if and only if $x > 0$, $y > 0$, $x > z$ and $y > z$. In this case, we have $\|a\| = x + y - z$.

Let $a = (x, y, z)$ be a fibered class in $int(C_\Delta)$. The minimal representative of this class is denoted by $F_a$ or $F_{(x,y,z)}$. We recall some formula which tells us that the number of the boundary components of $F_a$. We denote the tori $\partial\mathcal{N}(K_\alpha)$, $\partial\mathcal{N}(K_\beta)$, $\partial\mathcal{N}(K_\gamma)$ by $T_\alpha$, $T_\beta$, $T_\gamma$ respectively, where $\mathcal{N}(K)$ is a regular neighborhood of a knot $K$ in $S^3$. Let us set $\partial_\alpha F_{(x,y,z)} = \partial F_{(x,y,z)} \cap T_\alpha$ which consists of the parallel simple closed curves on $T_\alpha$. We define the subsets $\partial_\beta F_{(x,y,z)}, \partial_\gamma F_{(x,y,z)} \subset \partial F_{(x,y,z)}$ in the same manner. By [11, Lemma 3.1], the number of the boundary components

$$\#(\partial F_{(x,y,z)}) = \#(\partial_\alpha F_{(x,y,z)}) + \#(\partial_\beta F_{(x,y,z)}) + \#(\partial_\gamma F_{(x,y,z)})$$

is given by

$$\#(\partial F_{(x,y,z)}) = \gcd(x, y+z) + \gcd(y, z+x) + \gcd(z, x+y) \tag{5}$$

where $\#(\partial_\alpha F_{(x,y,z)}) = \gcd(x, y+z)$, $\#(\partial_\beta F_{(x,y,z)}) = \gcd(y, z+x)$, $\#(\partial_\gamma F_{(x,y,z)}) = \gcd(z, x+y)$ and $\gcd(0, w)$ is defined by $|w|$.

**4.2. Dilatations and stable foliations of fibered classes $a$'s.** The Teichmüller polynomial associated to the fibered face $\Delta$ is computed in [11, Section 3.2], and it tells us that the dilatation $\lambda_{(x,y,z)}$ of a fibered class $(x, y, z) \in int(C_\Delta)$ is the largest real root of

$$f_{(x,y,z)}(t) = t^{x+y-z} - t^x - t^y - t^{x-z} - t^{y-z} + 1,$$

see [11, Theorem 3.1]. (In fact, $\lambda_{(x,y,z)}$ is a unique real root greater than 1 of $f_{(x,y,z)}(t)$ by Descartes's rule of signs.)

Let $\Phi_{(x,y,z)} : F_{(x,y,z)} \to F_{(x,y,z)}$ be the monodromy of the fibration associated to a primitive class $(x, y, z) \in int(C_\Delta)$. Let $\mathcal{F}_{(x,y,z)}$ be the stable foliation of the pseudo-Anosov $\Phi_{(x,y,z)}$. The components of $\partial_\alpha F_{(x,y,z)}$ (resp. $\partial_\beta F_{(x,y,z)}$, $\partial_\gamma F_{(x,y,z)}$) are permuted cyclically by $\Phi_{(x,y,z)}$. In particular the number of prongs of $\mathcal{F}_{(x,y,z)}$ at a component of $\partial_\alpha F_{(x,y,z)}$ (resp. $\partial_\beta F_{(x,y,z)}$, $\partial_\gamma F_{(x,y,z)}$) is independent of the choice of the component. By [12, Proposition 3.3], the stable foliation $\mathcal{F}_{(x,y,z)}$ has the following properties.

- Each component of $\partial_\alpha F_{(x,y,z)}$ has $x/\gcd(x, y+z)$ prongs.
- Each component of $\partial_\beta F_{(x,y,z)}$ has $y/\gcd(y, x+z)$ prongs.
- Each component of $\partial_\gamma F_{(x,y,z)}$ has $(x+y-2z)/\gcd(z, x+y)$ prongs.
- $\mathcal{F}_{(x,y,z)}$ does not have singularities in the interior of $F_{(x,y,z)}$.

**4.3. Proofs of theorems.** Let $\mathfrak{a} = (1,1,0)$ and $\mathfrak{b} = (0,1,1)$. For $g \geq 0$ and $p \geq 0$, define a fibered class $a_{(g,p)}$ as follows.

$$a_{(g,p)} = (p+g+1)\mathfrak{a} + (p-g)\mathfrak{b} = (p+g+1, 2p+1, p-g) \in int(C_\Delta).$$

The class $a_{(g,p)}$ is primitive if and only if $2g+1$ and $p+g+1$ are relatively prime. One can check the identity

$$B_{(g,p)}(t) = f_{(p+g+1,2p+1,p-g)}(t)$$

(see Section 1 for the definition of $B_{(g,p)}(t)$). We denote by $r_{(g,p)}$, the dilatation $\lambda(a_{(g,p)})$ of the fibered class $a_{g,p}$. (Thus the dilatation $r_{(g,p)} = \lambda(a_{(g,p)})$ of $a_{(g,p)}$ is a unique real root of $B_{(g,p)}(t)$ which is greater than 1, see Section 4.2.)

Lemma 4.1. *We fix $g \geq 0$. Given $0 < c_1 < 1$ and $c_2 > 1$, we have*

$$p^{c_1/p} < r_{(g,p)} < p^{c_2/p} \qquad \textit{for } p \textit{ large}.$$

*In particular*

$$\lim_{p\to\infty} \frac{p \log r_{(g,p)}}{\log p} = 1.$$

Proof. Apply Lemma 3.1 to the polynomial $B_{(g,p)}(t)$. □

Lemma 4.2. *Suppose that $a_{(g,p)}$ is primitive. The minimal representative $F_{a_{(g,p)}}$ is a surface of genus $g$ with $2p+4$ boundary components, and the stable foliation $\mathscr{F}_{a_{(g,p)}}$ has the following properties. If $p+g$ is odd (resp. even), then $\#(\partial_\alpha F_{a_{(g,p)}}) = 2$ (resp. 1) and $\#(\partial_\gamma F_{a_{(g,p)}}) = 1$ (resp. 2). A component of $\partial_\alpha F_{a_{(g,p)}}$ has $\frac{p+g+1}{2}$ prongs (resp. $(p+g+1)$ prongs), and a component of $\partial_\gamma F_{a_{(g,p)}}$ has $(p+3g+2)$ prongs (resp. $\frac{p+3g+2}{2}$ prongs).*

Proof. By (5), we have that $\#(\partial_\beta F_{a_{(g,p)}}) = 2p+1$. We have

$$\#(\partial_\alpha F_{a_{(g,p)}}) = \gcd(p+g+1, 3p-g+1) = \gcd(p+g+1, 2(2g+1)).$$

Since $a_{(g,p)}$ is primitive, $p+g+1$ and $2g+1$ must be relatively prime. Hence $\#(\partial_\alpha F_{a_{(g,p)}}) = 1$ (resp. 2) if $p+g$ is even (resp. odd). Let us compute $\#(\partial_\gamma F_{a_{(g,p)}})$. We have

$$\#(\partial_\gamma F_{a_{(g,p)}}) = \gcd(3p+g+2, p-g) = \gcd(2(2g+1), p-g).$$

Since $\gcd(2g+1, p-g) = \gcd(2g+1, p+g+1) = 1$, we have that $\#(\partial_\gamma F_{a_{(g,p)}}) = 2$ (resp. 1) if $p-g$ is even (resp. odd), equivalently $p+g$ is even (resp. odd). The genus of $F_{a_{(g,p)}}$ is computed from the identities $\|a_{(g,p)}\| (= |\chi(F_{a_{(g,p)}})|) = 2p+2g+2$ and $\#(\partial F_{a_{(g,p)}}) = 2p+4$.

The singularity data of $\mathscr{F}_{a_{(g,p)}}$ is obtained from the formula at the end of Section 4.2. □

By Lemma 4.2, it is straightforward to prove the following.

Lemma 4.3. *Suppose that* $a_{(g,p)}$ *is primitive. Then* $(g,p) \notin \{(0,0),(0,1),(1,0)\}$ *if and only if* $\mathscr{F}_{a_{(g,p)}}$ *does not have a* 1 *prong on each component of* $\partial_\alpha F_{a_{(g,p)}} \cup \partial_\gamma F_{a_{(g,p)}}$. *In particular if* $g \geq 2$ *and* $p \geq 0$, *then* $\mathscr{F}_{a_{(g,p)}}$ *does not have a* 1 *prong on each component of* $\partial_\alpha F_{a_{(g,p)}} \cup \partial_\gamma F_{a_{(g,p)}}$.

We are now ready to prove theorems in Section 1.

Proof of Theorem 1.4. There exists a sequence of primitive fibered classes $\{a_{(g,p_i)}\}_{i=0}^{\infty}$ with $p_i \to \infty$. (In fact, if we take $p_i = (g+1)+(2g+1)i$, then $2g+1$ and $p_i+g+1$ are relatively prime. Hence $a_{(g,p_i)}$ is primitive.) Then $N$ is a $\Sigma_{g,2p_i+4}$-bundle over the circle whose monodromy of the fibration has the dilatation $r_{(g,p_i)}$. Therefore $\delta_{g,2p_i+4} \leq r_{(g,p_i)}$. If we set $n_i = 2p_i+4$, then

$$\frac{n_i \log \delta_{g,n_i}}{\log n_i} \leq \frac{n_i \log r_{(g,p_i)}}{\log n_i} = \frac{(2p_i+4) r_{(g,p_i)}}{\log(2p_i+4)}.$$

The right hand side goes to 2 as $i$ goes to $\infty$, see Lemmas 3.1(3) and 4.1. This completes the proof. □

Proof of Theorem 1.6. The monodromy $\Phi_{a_{(g,p)}}$ of the fibration associated to the primitive fibered class $a_{(g,p)}$ is defined on the surface of genus $g$ with $2p+4$ boundary components. It has the dilatation $r_{(g,p)}$, and hence $\delta_{g,2p+4} \leq r_{(g,p)}$.

Now let us prove $\delta_{g,2p+1} \leq r_{(g,p)}$. The fibration associated to $a_{(g,p)}$ extends naturally to a fibration on the manifold obtained from $N$ by Dehn filling two cusps specified by the tori $T_\alpha$ and $T_\gamma$ along the boundary slopes of the fiber. Then $\Phi_{a_{(g,p)}} : F_{a_{(g,p)}} \to F_{a_{(g,p)}}$ extends to the monodromy $\hat{\Phi} : \hat{F} \to \hat{F}$ of the extended fibration, where the extended fiber $\hat{F}$ is obtained from $F_{a_{(g,p)}}$ by filling each disk bounded by each component of $\partial_\alpha F_{a_{(g,p)}} \cup \partial_\gamma F_{a_{(g,p)}}$. Thus $\hat{F}$ has the genus $g$ with $2p+1$ boundary components, see Lemma 4.2. By Lemma 4.3, $\mathscr{F}_{a_{(g,p)}}$ does not have 1 prong at each component of $\partial_\alpha F_{a_{(g,p)}} \cup \partial_\gamma F_{a_{(g,p)}}$. Hence $\mathscr{F}_{a_{(g,p)}}$ extends canonically to the stable foliation $\hat{\mathscr{F}}$ of $\hat{\Phi}$. Therefore $\hat{\phi} = [\hat{\Phi}]$ is pseudo-Anosov with the same dilatation as $\Phi_{a_{(g,p)}}$. This implies that $\delta_{g,2p+1} \leq r_{(g,p)}$.

The proofs of the rest of the bounds $\delta_{g,2p+2} \leq r_{(g,p)}$ and $\delta_{g,2p+3} \leq r_{(g,p)}$ are similar. In fact, the extended fiber of the fibration on the manifold obtained from $N$ by Dehn filling a cusp specified by $T_\alpha$ or $T_\gamma$ along the boundary slope of the fiber has the genus $g$ with $2p+2$ or $2p+3$ boundary components, see Lemma 4.2. Lemma 4.3 ensures that the extended monodromy is pseudo-Anosov with the same dilatation as $\Phi_{a_{(g,p)}}$. □

Proof of Theorem 1.7. By Theorem 1.6 together with the assumption $(*)$ in Theorem 1.7, we have that for any $p \geq 0$ and for $j \in \{3, 4\}$,

$$\delta_{g, 2p+j} \leq r_{(g,p)} \qquad \text{or} \qquad \delta_{g, 2p+j} \leq r_{(g,p+1)}.$$

Thus

$$\frac{(2p+j) \log \delta_{g, 2p+j}}{\log(2p+j)} \leq \frac{(2p+j) \log r_{(g,p)}}{\log(2p+j)} \qquad \text{or}$$

$$\frac{(2p+j) \log \delta_{g, 2p+j}}{\log(2p+j)} \leq \frac{(2p+j) \log r_{(g,p+1)}}{\log(2p+j)}.$$

By Lemma 3.1, it is easy to see that the both right hand sides in the above two inequalities go to 2 as $p$ goes to $\infty$. Thus

$$\limsup_{p \to \infty} \frac{(2p+j) \log \delta_{g, 2p+j}}{\log(2p+j)} \leq 2.$$

Since this holds for $j \in \{3, 4\}$, the proof is done. □

Proof of Proposition 1.10. We prove the claim in the second half. (The proof in the first half is similar.) If $g \geq 2$ satisfies $(*)$, then for any $p \geq 0$ there exist a $\Sigma_{g, 2p+3}$-bundle and a $\Sigma_{g, 2p+4}$-bundle over the circle obtained from $N$, see proof of Theorem 1.7. More precisely such a bundle is homeomorphic to $N$ or it is obtained from $N$ by Dehn filling cusps along the boundary slopes of the fiber. Thus Proposition 1.10 holds from the result which says that the hyperbolic volume decreases after Dehn filling, see [16, 18]. □

Remark 4.4. To address Question 1.3, we explored fibered classes of the magic manifold whose dilatations have a suitable asymptotic behavior. We found a family of primitive fibered classes $a_{(g,p)}$ by computer. By Lemma 4.2, most of the components of $\partial F_{a_{(g,p)}}$ lie on the torus $T_\beta$. The pseudo-Anosov stable foliation associated to $a_{(g,p)}$ has the property that each component of $\partial_\beta F_{a_{(g,p)}}$ has 1 prong. The striking property of $a_{(g,p)}$ is that the slope of the components of $\partial_\beta F_{a_{(g,p)}}$ is exactly equal to $-1$. Moreover, for any fixed $g$, the projective class $\bar{a}_{(g,p)}$ goes to a single point $\left(\frac{1}{2}, 1, \frac{1}{2}\right) \in \partial \Delta$ as $p$ goes to $\infty$. It is proved by Martelli and Petronio [13] that the manifold $N(-1)$ obtained from $N$ by Dehn filling a cusp along the boundary slope $-1$ is not hyperbolic. The property that each component of $\partial_\beta F_{a_{(g,p)}}$ has 1 prong can also be seen from the fact hat $N(-1)$ is a non hyperbolic manifold.

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