Hiroshima Math. J.
45 (2015), 137–146

On non-periodic 3-Archimedean tilings with 6-fold rotational symmetry

Naoko Kinoshita and Kazushi Komatsu

(Received June 14, 2013)
(Revised July 15, 2014)

Abstract. The purpose of this article is to construct a family of uncountably many non-periodic 3-Archimedean tilings with 6-fold rotational symmetry, which admit three types of vertex configurations by regular triangles and squares.

1. Introduction

In 1982, a quasicrystal with 5-fold rotational symmetry was discovered by Shechtman et al. (published in 1984, [14]). Its model is a non-periodic tiling with 5-fold rotational symmetry, called a Penrose tiling ([11], [12]), constructed using the substitution rule of tiles which replaces tiles by unions of tiles. In addition, there is an Ammann-Beenker tiling with 8-fold rotational symmetry ([2], [3]) and a Danzer tiling with 7-fold rotational symmetry ([10]) constructed by the substitution rule of tiles. In [6], the second author and his collaborators studied the procedure for constructing non-periodic tilings with rotational symmetry under the substitution rule of tiles.

We recall basic definitions concerning a tiling following [4]. A *tiling* is a set of non-overlapping polygons with the property that their union is the Euclidean plane. Here polygons are said to be *non-overlapping* if their interiors are pairwise disjoint. A *non-periodic* tiling is one that admits no nontrivial translations to itself. A *patch* is a set of finitely many non-overlapping polygons with the property that their union is a topological disk (cf. [4, p. 19]). Each polygon of a tiling (or patch) is called a *tile*. Moreover, we say that a tiling (resp. patch) by polygons is *edge-to-edge* if each pair of tiles in the tiling (resp. patch) intersects along a common edge, at a common vertex, or not at all (cf. [4, p. 58]). In this paper, we assume that tilings (or patches) are edge-to-edge. For a point x in the Euclidean plane, a *vertex configuration* (of x) is a patch $P = \{T_\alpha\}_{\alpha \in A}$ such that x is a vertex of every tile T_α and that x is contained in the interior of $\bigcup_{\alpha \in A} T_\alpha$. A tiling by regular polygons is said to be *k-Archimedean* if its vertex configurations belong to k congruence classes.

2010 *Mathematics Subject Classification*. Primary 52C23; Secondary 52C20.
Key words and phrases. non-periodic tiling, rotational symmetry.

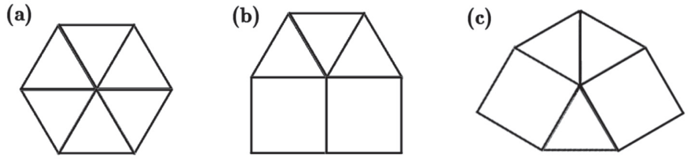

Fig. 1. (a) 3^6 (beehive-shaped) vertex configuration, (b) $(4^2, 3^3)$ (house-shaped) vertex configuration, (c) $(3, 4, 3^2, 4)$ (tent-shaped) vertex configuration

The (1-)Archimedean tiling by regular triangles and squares is well-known and is periodic (for instance, see [4, p. 63]).

In 2007, a polymeric quasicrystal was discovered ([5]). It is modeled by a non-periodic 3-Archimedean tiling with the three vertex configurations shown in Figure 1. In models of quasicrystals, we often see 3-Archimedean tilings with these three vertex configurations (Figure 1), which are not periodic ([5], [9], [13]).

In [8], we constructed uncountably many non-periodic 3-Archimedean tilings with these three vertex configurations (Figure 1) by using the substitution rule of patches which replaces patches by other patches. Unfortunately, there is no tiling with rotational symmetry in this family.

We call the following procedures for laying tiles "ringed expansion": First, vertex configurations are given. Starting from a patch P_0, we then attach a vertex configuration on a vertex in the boundary of P_0. We then attach a vertex configuration on the next vertex counterclockwise, repeatedly. If we can attach vertex configurations on all vertices, we get a larger connected patch P_1. If a similar expansion can be repeated ad infinitum, we get a tiling with given vertex configurations. The ringed expansion works fine in the construction of k-Archimedean tiling with rotational symmetry.

In this article, we propose a new method for constructing non-periodic tilings with rotational symmetry by using ringed expansion:

Theorem 1. *There exists a family of uncountably many non-periodic* 3-*Archimedean tilings by regular triangles and squares which have* 6-*fold rotational symmetry.*

In order to prove Theorem 1, we use representation by words to describe procedures for laying tiles, which is called the substitution rule of boundary words. This idea was introduced by Prof. Shigeki Akiyama for tilings in the hyperbolic plane ([1]).

2. Proof of Theorem 1

We use the three vertex configurations shown in Figure 1. The symbol 3 or 4 denotes a vertex of a regular triangle or square, respectively (Figure 2).

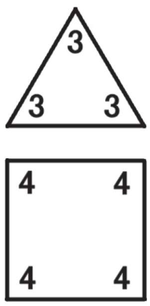

Fig. 2. Tiles with symbols 3, 4

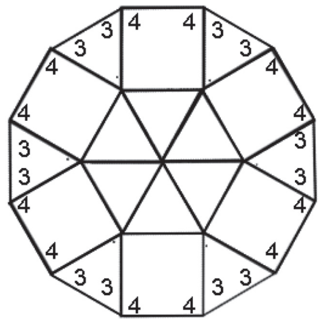

Fig. 3. A boundary word

We prepare 10 symbols 4, $\overline{33}$, $\overline{34}$, $\overline{43}$, $\overline{44}$, $\overline{333}$, $\overline{334}$, $\overline{433}$, $\overline{434}$, $\overline{3333}$, where 4 denotes a vertex of degree 2 of angle $\pi/2$ and $\overline{a_1 \dots a_\ell}$ $(a_i = 3, 4)$ denotes a vertex of degree $\ell + 1$ in the boundary of a patch which consists of corners of angles $\theta_1, \dots, \theta_\ell$ in this order, where $\theta_i = \pi/2$ or $\pi/3$ according to whether $a_i = 3$ or 4. For simplicity, we use the notation $\overline{33} = \overline{3^2}$, $\overline{44} = \overline{4^2}$, $\overline{333} = \overline{3^3}$, $\overline{334} = \overline{3^2 4}$, $\overline{433} = \overline{43^2}$, and $\overline{3333} = \overline{3^4}$. For a patch P, let $w(P)$ be a cyclic word obtained by reading the symbols of vertices of P in the counterclockwise direction along the edges. We call it a *boundary word* of P. Note that $w(P)$ is well-defined up to cyclic permutation. For example, for the patch P in Figure 3, the boundary word $w(P)$ is given by $\overline{43}\,\overline{34}\,\overline{43}\,\overline{34}\,\overline{43}\,\overline{34}\,\overline{43}\,\overline{34}\,\overline{43}\,\overline{34}\,\overline{43}\,\overline{34}$. For simplicity, we use the notation $\overline{43}\,\overline{34}\,\overline{43}\,\overline{34}\,\overline{43}\,\overline{34}\,\overline{43}\,\overline{34}\,\overline{43}\,\overline{34}\,\overline{43}\,\overline{34} = (\overline{43}\,\overline{34})^6$.

When we attach a vertex configuration on a vertex in the boundary of a patch, the symbol at the vertex of the boundary is replaced by a subword of the boundary of the larger patch. We call such a replacement a *substitution* of boundary words. For example, as in Figure 4, we can construct a new patch by attaching a vertex configuration (c) on the vertex for a given patch P and its vertex with the symbol $\overline{33}$ (here, the shaded portion is a part of P). Let $b[s]$ (resp. $h[s]$ or $t[s]$) denote a substitution given by attaching a vertex configuration (a) (resp. (b) or (c)) on a vertex with a symbol s. Then, in Figure 4, the symbol at the vertex of the boundary is replaced by $t[\overline{33}] : \overline{33} \to \overline{43}\,\overline{34}$.

We use the following substitution rules:

$$
\begin{array}{lll}
b[\overline{333}] : \overline{333} \to \overline{33}\,\overline{33}, & b[\overline{3333}] : \overline{3333} \to \overline{33}, & h[\overline{44}] : \overline{44} \to \overline{33}\,\overline{33}, \\
h[\overline{333}] : \overline{333} \to \overline{44}, & t[4] : \bar{4} \to \overline{34}\;4\;\overline{43}\,\overline{33}, & t[4] : \bar{4} \to \overline{33}\,\overline{34}\;4\;\overline{43}, \\
t[\overline{33}] : \overline{33} \to \overline{43}\,\overline{34}, & t[\overline{34}] : \overline{34} \to \overline{43}\,\overline{33}, & t[\overline{34}] : \overline{34} \to \overline{34}\;4\;\overline{43}, \\
t[\overline{43}] : \overline{43} \to \overline{33}\,\overline{34}, & t[\overline{43}] : \overline{43} \to \overline{34}\;4\;\overline{43}, & t[\overline{334}] : \overline{334} \to \overline{43}, \\
t[\overline{433}] : \overline{433} \to \overline{34}, & t[\overline{434}] : \overline{434} \to \overline{33}. &
\end{array}
$$

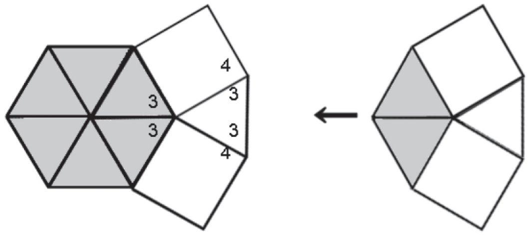

Fig. 4. A substitution of a boundary word

Note that we might have different substitution rules even when the same vertex configuration is attached on a vertex with the same symbol.

We explain how the boundary word of a patch changes when we apply the substitutions successively to adjoining vertices. For example, in Figure 5, the two symbol $\overline{44}$'s at the vertex of the boundary of the original patch are replaced by the substitution $h[\overline{44}] : \overline{44} \to \overline{33}\,\overline{33}$. Then, the symbol $\overline{33}$ is doubly assigned in the vertex at the center above. If more than one symbols $s_1, \dots, s_n$ are assigned to a vertex by successive substitutions, then we temporarily assign the symbol $s_1 \cdot \dots \cdot s_n$ to the vertex. To identify this temporary symbol with a subword of a boundary word, we need the relations that $\overline{\alpha 3} \cdot \overline{3\beta} = \overline{\alpha 3\beta}$, $\overline{\alpha_1 \dots \alpha_k 3} \cdot \overline{3\beta_1 \dots \beta_\ell} = \overline{\alpha_1 \dots \alpha_k 3\beta_1 \dots \beta_\ell}$. In fact, we use the following relations: $\overline{33} \cdot \overline{33} = \overline{333}$ (Figure 5), $\overline{43} \cdot \overline{33} = \overline{433}$, $\overline{33} \cdot \overline{34} = \overline{334}$, $\overline{43} \cdot \overline{34} = \overline{434}$, $\overline{33} \cdot \overline{33} \cdot \overline{33} = \overline{3333}$, and so on.

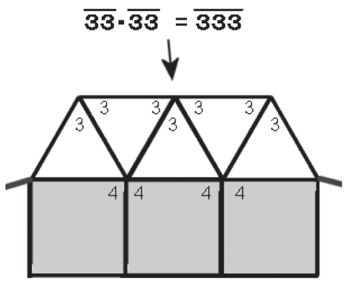

Fig. 5. The relation $\overline{33} \cdot \overline{33} = \overline{333}$

Let P_0 be the vertex configuration (a) in Figure 1. In the following, we construct an infinite sequence of patches P_n ($n \in \mathbf{N}$) starting from the patch P_0 by repeatedly applying ringed expansions.

Up to Step 4 of the ringed expansion, the patch is expanded by using the following five substitutions:

$$h[\overline{44}] : \overline{44} \to \overline{33}\ \overline{33},$$

$$h[\overline{333}] : \overline{333} \to \overline{44},$$

$$t[\overline{33}] : \overline{33} \to \overline{43}\ \overline{34},$$

$$t[\overline{34}] : \overline{34} \to \overline{43}\ \overline{33},$$

$$t[\overline{43}] : \overline{43} \to \overline{33}\ \overline{34}.$$

In Step 4, the boundary word $(\overline{3^3}\ \overline{3^3}\ \overline{34}\ \overline{4^2}\ \overline{43})^6$ of P_4 (Figure 6) is a cycle in the directed graph I (Figure 7).

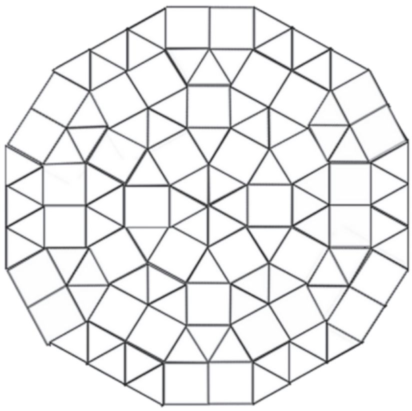

Fig. 6. Step 4 patch P_4

When we expand the patch P_n $(n \geq 4)$ to P_{n+1}, we apply one of the five operations I-1, I-2, II, III and IV, described below.

Operation I-1. Suppose that $w(P_n)$ is a cycle in the directed graph I (Figure 7). Then Operation I-1 denotes the ringed expansion described as below. We apply the substitution $\overline{333} \to \overline{33}\ \overline{33}$ for all vertices with the symbol $\overline{333}$ in the boundary of P_n. And we expand the patch P_n to P'_{n+1} by using the following five substitutions:

$$b[\overline{333}] : \overline{333} \to \overline{33}\ \overline{33},$$

$$b[\overline{3333}] : \overline{3333} \to \overline{33},$$

$$h[\overline{44}] : \overline{44} \to \overline{33}\ \overline{33},$$

$$t[\overline{34}] : \overline{34} \to \overline{34}\ 4_1\ \overline{43},$$

$$t[\overline{43}] : \overline{43} \to \overline{34}\ 4_2\ \overline{43},$$

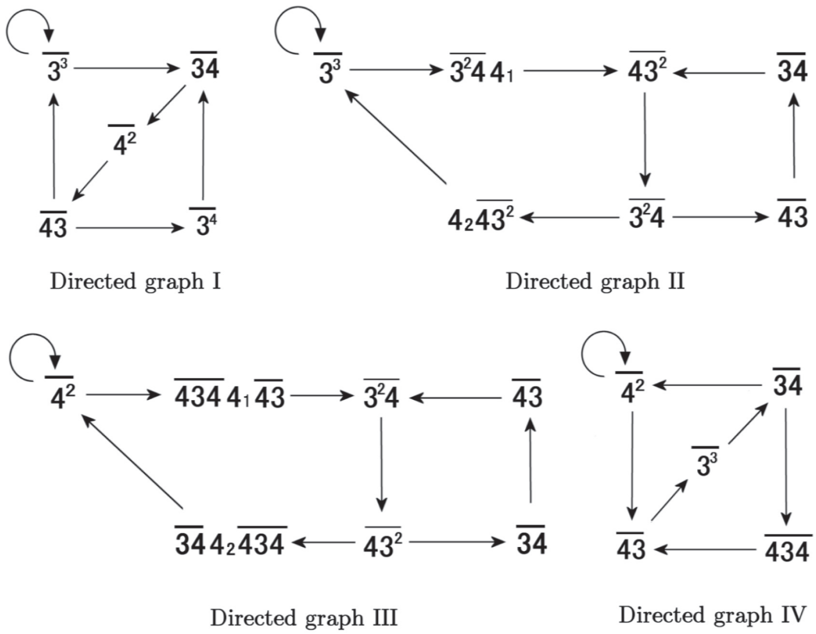

Fig. 7. Directed graphs I, II, III, IV

where 4_k $(k = 1, 2)$ denotes the symbol 4. We add the suffixes 1, 2 because 4_1 and 4_2 play different roles in the next step. 4_1 and 4_2 use replacements by different substitutions in the directed graph II.

If the symbol $\overline{4334}$ appears in $w(P'_{n+1})$ and the ringed expansion proceeds, we need more symbols, substitutions and relations. To remedy this situation, we partially expand P'_{n+1} to P_{n+1} by $S : 4\,\overline{4334}\,4 \to \overline{43}\,\overline{34}$. As a result, the patch P_n is expanded to P_{n+1} by using the above five substitutions and S. Note that $w(P_{n+1})$ is a cycle in the directed graph II (Figure 7).

Operation I-2. Suppose that $w(P_n)$ is a cycle in the directed graph I (Figure 7). Then Operation I-2 denotes the ringed expansion described as below. We apply the substitution $\overline{333} \to \overline{44}$ for all vertices with the symbol $\overline{333}$ in the boundary of P_n. And we expand the patch P_n to P_{n+1} by using the following five substitutions:

$$h[\overline{44}] : \overline{44} \to \overline{33}\,\overline{33},$$

$$h[\overline{333}] : \overline{333} \to \overline{44},$$

$$t[\overline{33}] : \overline{33} \to \overline{43}\,\overline{34},$$

$$t[\overline{34}] : \overline{34} \to \overline{43}\,\overline{33},$$

$$t[\overline{43}] : \overline{43} \to \overline{33}\,\overline{34}.$$

Note that $w(P_{n+1})$ is a cycle in the directed graph IV (Figure 7).

Operation II. Suppose that $w(P_n)$ is a cycle in the directed graph II (Figure 7). Then Operation II denotes the ringed expansion described as below. We apply the following seven substitutions:

$$h[\overline{333}] : \overline{333} \to \overline{44},$$

$$t[\bar{4}_1] : \bar{4}_1 \to \overline{34}\, 4_1\, \overline{43}\, \overline{33},$$

$$t[\bar{4}_2] : \bar{4}_2 \to \overline{33}\, \overline{34}\, 4_2\, \overline{43},$$

$$t[\overline{34}] : \overline{34} \to \overline{43}\, \overline{33},$$

$$t[\overline{43}] : \overline{43} \to \overline{33}\, \overline{34},$$

$$t[\overline{334}] : \overline{334} \to \overline{43},$$

$$t[\overline{433}] : \overline{433} \to \overline{34}.$$

Note that $w(P_{n+1})$ is a cycle in the directed graph III (Figure 7).

Operation III. Suppose that $w(P_n)$ is a cycle in the directed graph III (Figure 7). Then Operation III denotes the ringed expansion described as below. We apply the following eight substitutions:

$$h[\overline{44}] : \overline{44} \to \overline{33}\, \overline{33},$$

$$t[4_1] : 4_1 \to \overline{34}\, 4\, \overline{43}\, \overline{33},$$

$$t[4_2] : 4_2 \to \overline{33}\, \overline{34}\, 4\, \overline{43},$$

$$t[\overline{34}] : \overline{34} \to \overline{43}\, \overline{33},$$

$$t[\overline{43}] : \overline{43} \to \overline{33}\, \overline{34},$$

$$t[\overline{334}] : \overline{334} \to \overline{43},$$

$$t[\overline{433}] : \overline{433} \to \overline{34},$$

$$t[\overline{434}] : \overline{434} \to \overline{33}.$$

Note that $\overline{33} \cdot \overline{33} \cdot \overline{34} = \overline{3334}$ and $\overline{43} \cdot \overline{33} \cdot \overline{33} = \overline{4333}$. If the subword $4\, \overline{4333}\, (\overline{333})^k\, \overline{3334}\, 4$ appears in $w(P'_{n+1})$ and the ringed expansion proceeds, we need more symbols, substitutions, and relations. To remedy this situation, we partially expand P'_{n+1} to P_{n+1} by $S_k : 4\, \overline{4333}\, (\overline{333})^k\, \overline{3334}\, 4 \to \overline{44}\, (\overline{44})^k\, \overline{44}$. As a result, the patch P_n is expanded to P_{n+1} by using the above five substitutions and S_k. Note that $w(P_{n+1})$ is a cycle in the directed graph IV (Figure 7).

Operation IV. Suppose that $w(P_n)$ is a cycle in the directed graph IV (Figure 7). Then Operation IV denotes the ringed expansion described as below. We apply the following five substitutions:

$$h[\overline{44}] : \overline{44} \to \overline{33}\,\overline{33},$$

$$h[\overline{333}] : \overline{333} \to \overline{44},$$

$$t[\overline{34}] : \overline{34} \to \overline{43}\,\overline{33},$$

$$t[\overline{43}] : \overline{43} \to \overline{33}\,\overline{34},$$

$$t[\overline{434}] : \overline{434} \to \overline{33}.$$

Note that $w(P_{n+1})$ is a cycle in the directed graph I (Figure 7).

The oriented labeled graph in Figure 8 illustrates relation among the five operations described in the above, where Op.I-1, for example, denotes the Operation I-1. Note we can choose Op.I-1 or Op.I-2 as we like when $w(P_n)$ is a cycle in the directed graph I (Figure 7). For a given infinite edge path in the oriented graph starting from the vertex I, we can construct an infinite sequence $\{P_n\}$ of ringed expansions of patches starting from the patch P_4 in Figure 6 by successively applying the operations indicated by the edge path.

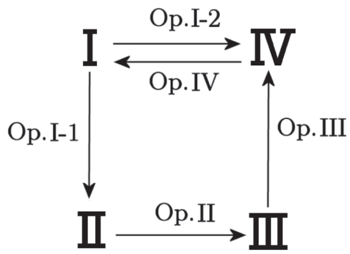

Fig. 8. Upper-level directed graph

We show that the tiling determined by $\{P_n\}$ has only one rotational symmetry. To this end, we look at blocks in the boundary layer of P_n consisting of three or more consecutive squares. Let b_n be the maximum of the lengths of such blocks contained in P_n. Then b_n grows as n becomes bigger. Hence, the tiling admits only one rotational center, and so it is non-periodic.

The loop $\mathrm{I} \to \mathrm{II} \to \mathrm{III} \to \mathrm{IV} \to \mathrm{I}$ doesn't change b_n, whereas the loop $\mathrm{I} \to \mathrm{IV} \to \mathrm{I}$ increases b_n by one. There is a one-to-one correspondence between the set of tilings by our construction, and the set of increasing sequences $1, 2, k_3, k_4, k_5, \ldots$ $(2 \le k_3 \le k_4 \le k_5 \le \cdots)$ of positive integers. For example, a tiling in Figure 7 corresponds to an increasing sequence $1, 2, 2, 3, 3, 4, \ldots$. This

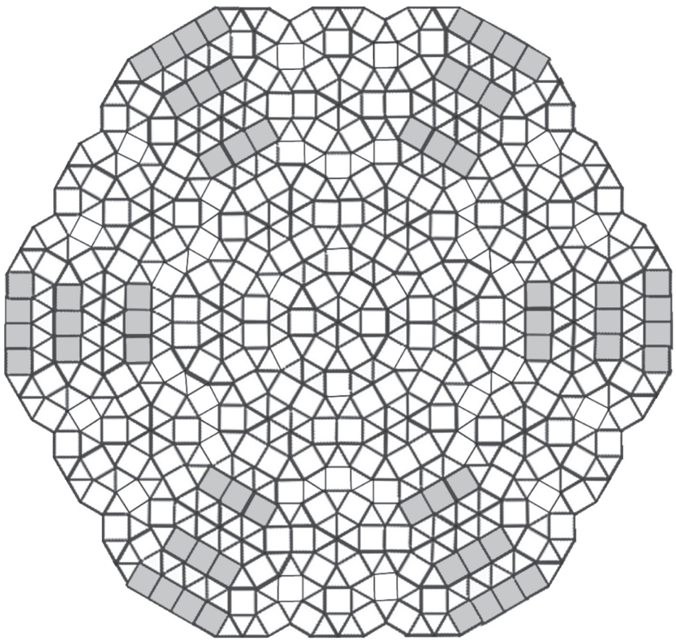

Fig. 9. A tiling by ringed expansion

set of increasing sequences is clearly uncountable. Hence we have uncountably many number of tilings up to isomorphism, and our proof of Theorem 1 is completed.

REMARK 1. In [7], the first author tried to extend the scheme of substitution rules, and to handle partial expansions used in our proof in a unified scheme of substitution rules.

Acknowledgement

The authors would like to express their sincere gratitude to Prof. Shigeki Akiyama for his great idea, the substitution rule of the boundary word, and to the referee for the valuable comments. The authors would like to express the deepest appreciation to the editor for valuable advices.

References

[1] S. Akiyama, Uncountably many non-periodic Ahara tilings, Private communication.
[2] R. Ammann, B. Grünbaum and G. C. Shephard, Aperiodic tiles, Discrete Comput. Geom. **8** (1992), 1–25.

[3] F. P. M. Beenker, Algebraic theory of non-periodic tilings of the plane by two simple building blocks: a square and a rhombus, TH-Repert82-WSK04, Eindhoven University of Technology, 1982.
[4] B. Grünbaum and G. C. Shephard, Tilings and Patterns, W. H. Freeman and Company, New York, 1987.
[5] K. Hayashida, T. Dotera, A. Takano and Y. Matsushita, Polymeric Quasicrystal: Mesoscopic Quasicrystalline Tiling in ABC Star Polymers, Phys. Rev. Lett. **98** (2007), 195502.
[6] K. Kato, K. Komatsu, F. Nakano, K. Nomakuchi and M. Yamauchi, Remarks on 2-dimensional quasiperiodic tilings with rotational symmetries, Hiroshima Math. J. **38** (2008), 385–395.
[7] N. Kinoshita and K. Komatsu, Non-periodic Archimedean-like tilings with 6-fold rotational symmetry, Talk in Topology of tiling spaces and related topics, RIMS cooperative research meeting, Kyoto Univ., 3 Oct.–5 Oct. (2012).
[8] N. Kinoshita and K. Komatsu, An example of a quasiperiodic 3-Archimedean tiling by regular triangles and squares, Kochi J. of Math. **9** (2014), 121–125.
[9] J. Mikhael, J. Roth, L. Helden and C. Bechinger, Archimedean-like tiling on decagonal quasicrystalline surfaces, Nature **454** (2008), 501.
[10] K.-P. Nischke and L. Danzer, A construction of inflation rules based on n-fold symmetry, Discrete Comput. Geom. **15** (1996), 221–236.
[11] R. Penrose, The rôle of aesthetics in pure and applied mathematical research, Bull. Inst. Math. Appl. **10** (1974), 266–271.
[12] R. Penrose, Pentaplexity, Math. Intelligencer **2** (1979), 32–37.
[13] S. C. Glotzer and A. S. Keys, A tale of two tilings, News and Views: Nature **454** (2008), 420–421.
[14] D. Shechtman, I. Blech, D. Gratias and J. W. Cahn, Metallic phase with long-range orientational order and no translational symmetry, Physical Review Letters **53** (1984), 1951–1953.

Naoko Kinoshita
Department of Mathematics
Graduate School of Science
Kobe University
Kobe 657-8501*, Japan*
E-mail: nkinost@math.kobe-u.ac.jp

Kazushi Komatsu
Department of Mathematics
Faculty of Science
Kochi University
Kochi 780-8520*, Japan*
E-mail: komatsu@kochi-u.ac.jp