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Ricci tensors on unit tangent sphere bundles over 4-dimensional Riemannian manifolds

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Abstract. For a 4-dimensional Riemannian manifold (M, g) let T_1M be its unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Then we prove that the Ricci operator S and the structure operator ϕ commute i.e., $S\phi = \phi S$ (anti-commute i.e., $S\phi + \phi S = 2k\phi$, respectively) if and only if (M, g) is of constant sectional curvature 1 or 2 ((M, g) is of constant sectional curvature, respectively).

1. Introduction

It is intriguing to study the interplay between Riemannian manifolds and their unit tangent sphere bundles. In particular, we are interested in the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ of a unit tangent sphere bundle T_1M over a given Riemannian manifold (M, g). As a classical result, Tashiro ([13]) proved that $(T_1M, \eta, \bar{g})$ is a K-contact manifold (i.e., the Reeb vector field ξ is a Killing vector field) if and only if (M, g) has constant sectional curvature 1.

Boeckx and Vanhecke ([4]) proved that T_1M is Einstein, that is $\bar{\rho} = \alpha\bar{g}$ if and only if (M, g) is 2-dimensional and is locally isometric to the Euclidean plane or the unit sphere, where $\bar{\rho}$ denotes the Ricci curvature tensor of T_1M and α is a function of T_1M. In [6], for a 4-dimensional Riemannian manifold M it was proved that T_1M is η-Einstein, that is $\bar{\rho} = \alpha\bar{g} + \beta\eta \otimes \eta$ if and only if M is of constant sectional curvature 1 or 2, where α, β are functions of T_1M. Later, Park and Sekigawa ([9]) generalized the result for higher dimensional cases. In fact, they proved that T_1M is η-Einstein if and only if (M^n, g) is of constant sectional curvature 1 or $n-2$, where $\dim M = n$. After all, we are aware that (η-)Einstein condition is too strong to impose on T_1M. This

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motivates us to consider geometry of T_1M under some weaker restrictions. Namely, in Section 3 we prove the following theorems.

Theorem 1. *Let $M = (M, g)$ be a 4-dimensional Riemannian manifold and let T_1M be the unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ over M. Then the Ricci operator S and the structure operator ϕ of T_1M commute i.e., $S\phi = \phi S$ if and only if (M, g) is of constant sectional curvature 1 or 2.*

From Theorem 1 we find that the commutativity condition $S\phi = \phi S$ is already reduced to η-Einstein condition at least for lower (≤ 4) dimensional base manifolds. Next, we prove

Theorem 2. *Let $M = (M, g)$ be a 4-dimensional Riemannian manifold and let T_1M be the unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ over M. Then the Ricci operator S and the structure operator ϕ of T_1M anti-commute i.e., $S\phi + \phi S = 2k\phi$, where k is a function of T_1M if and only if (M, g) is a space of constant sectional curvature.*

The unit tangent sphere bundle T_1M treated in this paper has a so-called *H-contact structure*, which means that the Reeb vector field ξ is a harmonic vector field. Indeed, Perrone ([10]) proved that a contact metric manifold is H-contact if and only if ξ is an eigenvector of the Ricci operator S, that is, $S\xi = \alpha\xi$ for some function α. For 2- or 3-dimensional Riemannian manifolds M, Boeckx and Vanhecke ([3]) proved that the standard contact metric structure of T_1M is H-contact if and only if M is of constant curvature. Recently, for 4-dimensional Riemannian manifolds M, Chun, Park and Sekigawa ([8]) proved the necessary and sufficient condition for T_1M to admit an H-contact structure is that M is a 2-stein manifold, that is, an Einstein manifold satisfying $\sum_{i,j}^{n} (R_{uiuj})^2 = \mu(p)|u|^2$ for all $u \in T_pM$, $p \in M$, where $R_{uiuj} = g(R(u, e_i)u, e_j)$, $|u|^2 = g(u, u)$ and μ is a real-valued function on M. In a continuing work [7] they generalized their result for higher dimensional Einstein manifolds. And they showed that the base manifolds of H-contact unit tangent sphere bundle include many Einstein spaces other than two-point homogeneous spaces.

2. The unit tangent sphere bundle

We start by reviewing some fundamental facts on contact metric manifolds. We refer to [1] for more details. All manifolds are assumed to be connected and of class C^∞. A $(2n-1)$-dimensional manifold $\overline{M}$ is said to be a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^{n-1} \neq 0$ everywhere on $\overline{M}$, where the exponent denotes the $(n-1)$-th exterior power of the exterior derivative $d\eta$ of η. We call such η a *contact form* of $\overline{M}$. It is well known that for a contact form η, there exists a unique vector field ξ, which is called the *characteristic vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, \overline{X}) = 0$ for any vector field $\overline{X}$ on $\overline{M}$. A Riemannian metric $\bar{g}$ on $\overline{M}$ is an associated metric to a contact form η if there exists a $(1,1)$-tensor field ϕ satisfying

$$\eta(\overline{X}) = \bar{g}(\overline{X}, \xi), \qquad d\eta(\overline{X}, \overline{Y}) = \bar{g}(\overline{X}, \phi\overline{Y}), \qquad \phi^2\overline{X} = -\overline{X} + \eta(\overline{X})\xi, \quad (1)$$

where $\overline{X}$ and $\overline{Y}$ are vector fields on $\overline{M}$. From (1) it follows that

$$\phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \bar{g}(\phi\overline{X}, \phi\overline{Y}) = \bar{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}).$$

A Riemannian manifold $\overline{M}$ equipped with structure tensors $(\eta, \bar{g}, \phi, \xi)$ satisfying (1) is said to be a *contact metric manifold*.

Let (M, g) be an n-dimensional Riemannian manifold and ∇ the associated Levi-Civita connection. Its Riemann curvature tensor R is defined by $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ for all vector fields X, Y and Z on M. The tangent bundle over (M, g) is denoted by TM and consists of pairs (p, u), where p is a point in M and u a tangent vector to M at p. The mapping $\pi : TM \to M$, $\pi(p, u) = p$, is the natural projection from TM onto M. For a vector field X on M, its *vertical lift* X^v on TM is the vector field defined by $X^v\omega = \omega(X) \circ \pi$, where ω is a 1-form on M. For the Levi-Civita connection ∇ on M, the *horizontal lift* X^h of X is defined by $X^h\omega = \nabla_X\omega$. The tangent bundle TM can be endowed in a natural way with a Riemannian metric $\tilde{g}$, the so-called *Sasaki metric*, depending only on the Riemannian metric g on M. It is determined by

$$\tilde{g}(X^h, Y^h) = \tilde{g}(X^v, Y^v) = g(X, Y) \circ \pi, \qquad \tilde{g}(X^h, Y^v) = 0$$

for all vector fields X and Y on M. Also, TM admits an almost complex structure tensor J defined by $JX^h = X^v$ and $JX^v = -X^h$. Then $\tilde{g}$ is a Hermitian metric for the almost complex structure J.

The unit tangent sphere bundle $\bar{\pi} : T_1M \to M$ is a hypersurface of TM given by $g_p(u, u) = 1$. Note that $\bar{\pi} = \pi \circ i$, where i is the immersion of T_1M into TM. A unit normal vector field $N = u^v$ to T_1M is given by the vertical lift of u for (p, u). The horizontal lift of a vector is tangent to T_1M, but the vertical lift of a vector is not tangent to T_1M in general. So, we define the *tangential lift* of X to $(p, u) \in T_1M$ by

$$X^t_{(p,u)} = (X - g(X, u)u)^v.$$

Clearly, the tangent space $T_{(p,u)}T_1M$ is spanned by vectors of the form X^h and X^t, where $X \in T_pM$.

We now define the standard contact metric structure of the unit tangent sphere bundle T_1M over a Riemannian manifold (M, g). The metric g' on

T_1M is induced from the Sasaki metric $\tilde{g}$ on TM. Using the almost complex structure J on TM, we define a unit vector field ξ', a 1-form η' and a $(1,1)$-tensor field ϕ' on T_1M by

$$\xi' = -JN, \qquad \phi' = J - \eta' \otimes N.$$

Since $g'(\overline{X}, \phi'\overline{Y}) = 2d\eta'(\overline{X}, \overline{Y})$, (η', g', ϕ', ξ') is not a contact metric structure. If we rescale this structure by

$$\xi = 2\xi', \qquad \eta = \frac{1}{2}\eta', \qquad \phi = \phi', \qquad \bar{g} = \frac{1}{4}g',$$

we get the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Here the tensor ϕ is explicitly given by

$$\phi X^t = -X^h + \frac{1}{2}g(X,u)\xi, \qquad \phi X^h = X^t, \tag{2}$$

where X and Y are vector fields on M. From now on, we consider $T_1M = (T_1M, \eta, \bar{g}, \phi, \xi)$ with the standard contact metric structure.

The Levi-Civita connection $\overline{\nabla}$ of T_1M is described by

$$\begin{aligned}
\overline{\nabla}_{X^t} Y^t &= -g(Y,u)X^t, \\
\overline{\nabla}_{X^t} Y^h &= \frac{1}{2}(R(u,X)Y)^h, \\
\overline{\nabla}_{X^h} Y^t &= (\nabla_X Y)^t + \frac{1}{2}(R(u,Y)X)^h, \\
\overline{\nabla}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{1}{2}(R(X,Y)u)^t
\end{aligned} \tag{3}$$

for all vector fields X and Y on M.

Also the Riemann curvature tensor $\overline{R}$ of T_1M is given by

$$\begin{aligned}
\overline{R}(X^t, Y^t)Z^t &= -(g(X,Z) - g(X,u)g(Z,u))Y^t \\
&\quad + (g(Y,Z) - g(Y,u)g(Z,u))X^t, \\
\overline{R}(X^t, Y^t)Z^h &= \{R(X - g(X,u)u, Y - g(Y,u)u)Z\}^h \\
&\quad + \frac{1}{4}\{[R(u,X), R(u,Y)]Z\}^h, \\
\overline{R}(X^h, Y^t)Z^t &= -\frac{1}{2}\{R(Y - g(Y,u)u, Z - g(Z,u)u)X\}^h \\
&\quad - \frac{1}{4}\{R(u,Y)R(u,Z)X\}^h,
\end{aligned}$$

$$\begin{aligned}
\bar{R}(X^h, Y^t)Z^h &= \frac{1}{2}\{R(X,Z)(Y-g(Y,u)u)\}^t - \frac{1}{4}\{R(X,R(u,Y)Z)u\}^t \\
&\quad + \frac{1}{2}\{(\nabla_X R)(u,Y)Z\}^h, \\
\bar{R}(X^h, Y^h)Z^t &= \{R(X,Y)(Z-g(Z,u)u)\}^t \\
&\quad + \frac{1}{4}\{R(Y,R(u,Z)X)u - R(X,R(u,Z)Y)u\}^t \\
&\quad + \frac{1}{2}\{(\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X\}^h, \\
\bar{R}(X^h, Y^h)Z^h &= (R(X,Y)Z)^h + \frac{1}{2}\{R(u,R(X,Y)u)Z\}^h \\
&\quad - \frac{1}{4}\{R(u,R(Y,Z)u)X - R(u,R(X,Z)u)Y\}^h \\
&\quad + \frac{1}{2}\{(\nabla_Z R)(X,Y)u\}^t \qquad (4)
\end{aligned}$$

for all vector fields X, Y and Z on M.

Next, to calculate the Ricci tensor $\bar{\rho}$ of T_1M at the point $(p,u) \in T_1M$, let $e_1,\dots,e_n$ be an orthonormal basis of T_pM. Then $\bar{\rho}$ is given by

$$\begin{aligned}
\bar{\rho}(X^t, Y^t) &= (n-2)(g(X,Y) - g(X,u)g(Y,u)) \\
&\quad + \frac{1}{4}\sum_{i=1}^{n} g(R(u,X)e_i, R(u,Y)e_i), \\
\bar{\rho}(X^t, Y^h) &= \frac{1}{2}((\nabla_u \rho)(X,Y) - (\nabla_X \rho)(u,Y)), \\
\bar{\rho}(X^h, Y^h) &= \rho(X,Y) - \frac{1}{2}\sum_{i=1}^{n} g(R(u,e_i)X, R(u,e_i)Y), \qquad (5)
\end{aligned}$$

where ρ denotes the Ricci curvature tensor of M. We can refer to [2, 5] for the formulas (3)–(5).

3. Proofs of Theorems

Proof of Theorem 1. Suppose that the unit tangent sphere bundle T_1M over an n-dimensional Riemannian manifold M satisfies the condition $S\phi = \phi S$ for the Ricci operator S and the structure tensor field ϕ on T_1M. Then from (2) and (5), we have

$$
\begin{aligned}
0 &= \bar{g}(S\phi X^t, Y^t) - \bar{g}(\phi S X^t, Y^t) \\
&= \bar{\rho}(\phi X^t, Y^t) + \bar{\rho}(X^t, \phi Y^t) \\
&= 2(\nabla_u \rho)(X, Y) - (\nabla_X \rho)(u, Y) - (\nabla_Y \rho)(u, X) \\
&\quad - g(X, u)\{(\nabla_u \rho)(Y, u) - (\nabla_Y \rho)(u, u)\} \\
&\quad - g(Y, u)\{(\nabla_u \rho)(X, u) - (\nabla_X \rho)(u, u)\}, \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \bar{g}(S\phi X^h, Y^t) - \bar{g}(\phi S X^h, Y^t) \\
&= \bar{\rho}(\phi X^h, Y^t) + \bar{\rho}(X^h, \phi Y^t) \\
&= (n-2)(g(X, Y) - g(X, u)g(Y, u)) + \frac{1}{4}\sum_{i=1}^{n} g(R(u, X)e_i, R(u, Y)e_i) \\
&\quad - \rho(X, Y) + \frac{1}{2}\sum_{i=1}^{n} g(R(u, e_i)X, R(u, e_i)Y) \\
&\quad + g(Y, u)\left\{\rho(X, u) - \frac{1}{2}\sum_{i=1}^{n} g(R(u, e_i)u, R(u, e_i)X)\right\}, \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \bar{g}(S\phi X^h, Y^h) - \bar{g}(\phi S X^h, Y^h) \\
&= \bar{\rho}(\phi X^h, Y^h) + \bar{\rho}(X^h, \phi Y^h) \\
&= 2(\nabla_u \rho)(X, Y) - (\nabla_X \rho)(u, Y) - (\nabla_Y \rho)(u, X). \qquad (8)
\end{aligned}
$$

Thus T_1M satisfies the condition $S\phi = \phi S$ if and only if (M, g) satisfies (6)–(8).

In (7) we put $X = e_a$, $Y = e_b$, $u = e_c$. Then we have

$$
\begin{aligned}
&(n-2)(\delta_{ab} - \delta_{ac}\delta_{bc}) + \frac{1}{4}\sum_{i,j=1}^{n} R_{caij}R_{cbij} - \rho_{ab} + \frac{1}{2}\sum_{i,j=1}^{n} R_{ciaj}R_{cibj} \\
&\quad + \delta_{bc}\left(\rho_{ac} - \frac{1}{2}\sum_{i,j=1}^{n} R_{ciaj}R_{cicj}\right) = 0, \qquad (9)
\end{aligned}
$$

where δ_{ab} denotes the Kronecker's delta, $R_{abcd} = g(R(e_a, e_b)e_c, e_d)$ and $\rho_{ab} = \rho(e_a, e_b)$. For $a = b \neq c$ in (9), we get

$$
4\rho_{aa} = 4(n-2) + \sum_{i,j=1}^{n} R_{caij}^2 + 2\sum_{i,j=1}^{n} R_{ciaj}^2 = 0. \qquad (10)
$$

In particular, from the assumption $S\phi = \phi S$ we easily see that T_1M satisfies $S\xi = \alpha\xi$, that is, it has an H-contact structure. We suppose that $n = 4$.

Then, owing to a result in [8], M is 2-stein. Now, since M is Einstein i.e., $\rho = \gamma g$ (γ is a function on M), we may choose an orthonormal basis $\{e_i\}_{i=1}^4$ (known as the Singer-Thorpe basis) at each point $p \in M$ such that

$$\begin{cases} R_{1212} = R_{3434} = \lambda_1, \quad R_{1313} = R_{2424} = \lambda_2, \quad R_{1414} = R_{2323} = \lambda_3, \\ R_{1234} = \mu_1, \quad R_{1342} = \mu_2, \quad R_{1423} = \mu_3, \\ R_{ijkl} = 0 \text{ whenever just three of the indices } i, j, k, l \text{ are distinct (cf. [12]).} \end{cases} \tag{11}$$

Note that

$$\mu_1 + \mu_2 + \mu_3 = 0 \tag{12}$$

by the first Bianchi identity and

$$\lambda_1 + \lambda_2 + \lambda_3 = -\frac{\tau}{4}, \tag{13}$$

where τ is the scalar curvature of M.

It is also known that a 4-dimensional Einstein manifold M is 2-stein if and only if

$$\mu_1 = \lambda_1 + \frac{\tau}{12}, \qquad \mu_2 = \lambda_2 + \frac{\tau}{12}, \qquad \mu_3 = \lambda_3 + \frac{\tau}{12} \tag{14}$$

or

$$-\mu_1 = \lambda_1 + \frac{\tau}{12}, \qquad -\mu_2 = \lambda_2 + \frac{\tau}{12}, \qquad -\mu_3 = \lambda_3 + \frac{\tau}{12} \tag{15}$$

holds for any Singer-Thorpe basis $\{e_i\}_{i=1}^4$ at each point $p \in M$ (cf. [11]).

On the other hand, if we put $a = b = 1$, $c = 2$ and $a = b = 3$, $c = 4$ in (10), then, using (11), we have

$$4\gamma = 8 + 4\lambda_1^2 + 2(\mu_1^2 + \mu_2^2 + \mu_3^2). \tag{16}$$

Similarly, put $a = b = 1$, $c = 3$ and $a = b = 2$, $c = 4$ in (10) to have

$$4\gamma = 8 + 4\lambda_2^2 + 2(\mu_1^2 + \mu_2^2 + \mu_3^2). \tag{17}$$

For $a = b = 1$, $c = 4$ and $a = b = 2$, $c = 3$ in (10), we have

$$4\gamma = 8 + 4\lambda_3^2 + 2(\mu_1^2 + \mu_2^2 + \mu_3^2). \tag{18}$$

From (16)–(18), we get

$$\lambda_1^2 = \lambda_2^2 = \lambda_3^2. \tag{19}$$

Then, from (12), (13), (14) and (19) we obtain the following four cases.

(i) $\lambda_1 = \lambda_2 = \lambda_3 = -\frac{\tau}{12}$ and $\mu_1 = \mu_2 = \mu_3 = 0$,
(ii) $\lambda_1 = \lambda_2 = -\frac{\tau}{4}$, $\lambda_3 = \frac{\tau}{4}$ and $\mu_1 = \mu_2 = -\frac{\tau}{6}$, $\mu_3 = \frac{\tau}{3}$,
(iii) $\lambda_1 = \lambda_3 = -\frac{\tau}{4}$, $\lambda_2 = \frac{\tau}{4}$ and $\mu_1 = \mu_3 = -\frac{\tau}{6}$, $\mu_2 = \frac{\tau}{3}$,
(iv) $\lambda_2 = \lambda_3 = -\frac{\tau}{4}$, $\lambda_1 = \frac{\tau}{4}$ and $\mu_2 = \mu_3 = -\frac{\tau}{6}$, $\mu_1 = \frac{\tau}{3}$.

In the case (i), we get from (17)

$$(\tau - 12)(\tau - 24) = 0.$$

Therefore M is of constant sectional curvature 1 or 2. Conversely, we easily check that such a space satisfies (6)–(8). In the other cases (ii)–(iv), we get from (17)

$$7\tau^2 - 12\tau + 96 = 0,$$

which can not occur. This completes the proof of Theorem 1. □

Proof of Theorem 2. Suppose that the unit tangent sphere bundle T_1M over an n-dimensional Riemannian manifold M satisfies the condition $S\phi + \phi S = 2k\phi$. Then, at first we can easily find that T_1M satisfies $S\xi = \alpha\xi$. From (2) and (5), we have

$$\begin{aligned} 0 &= \bar{g}(S\phi X^t + \phi S X^t - 2k\phi X^t, Y^t) \\ &= \bar{\rho}(\phi X^t, Y^t) - \bar{\rho}(X^t, \phi Y^t) - 2k\bar{g}(\phi X^t, Y^t) \\ &= \frac{1}{2}\{(\nabla_Y \rho)(u, X) - (\nabla_X \rho)(u, Y) + g(X, u)((\nabla_u \rho)(Y, u) - (\nabla_Y \rho)(u, u)) \\ &\qquad - g(Y, u)((\nabla_u \rho)(X, u) - (\nabla_X \rho)(u, u))\}, \end{aligned} \tag{20}$$

$$\begin{aligned} 0 &= \bar{g}(S\phi X^h + \phi S X^h - 2k\phi X^h, Y^t) \\ &= \bar{\rho}(\phi X^h, Y^t) - \bar{\rho}(X^h, \phi Y^t) - 2k\bar{g}(\phi X^h, Y^t) \\ &= (n - 2 - 2k)(g(X, Y) - g(X, u)g(Y, u)) + \frac{1}{4}\sum_{i=1}^{n} g(R(u, X)e_i, R(u, Y)e_i) \\ &\quad + \rho(X, Y) - \frac{1}{2}\sum_{i=1}^{n} g(R(u, e_i)X, R(u, e_i)Y) \\ &\quad - g(Y, u)\left\{\rho(X, u) - \frac{1}{2}\sum_{i=1}^{n} g(R(u, e_i)u, R(u, e_i)X)\right\}, \end{aligned} \tag{21}$$

$$\begin{aligned} 0 &= \bar{g}(S\phi X^h + \phi S X^h - 2k\phi X^h, Y^h) \\ &= \bar{\rho}(\phi X^h, y^h) - \bar{\rho}(X^h, \phi y^h) - 2k\bar{g}(\phi X^h, Y^h) \\ &= \frac{1}{2}\{(\nabla_Y \rho)(u, X) - (\nabla_X \rho)(u, Y)\}. \end{aligned} \tag{22}$$

Thus T_1M satisfies the condition $S\phi + \phi S = 2k\phi$ if and only if (M, g) satisfies (20)–(22). In (21), we put $X = e_a$, $Y = e_b$, $u = e_c$. Then we have

$$(n-2-2k)(\delta_{ab} - \delta_{ac}\delta_{bc}) + \frac{1}{4}\sum_{i,j=1}^{n} R_{caij}R_{cbij}$$
$$+ \rho_{ab} - \frac{1}{2}\sum_{i,j=1}^{n} R_{ciaj}R_{cibj} - \delta_{bc}\left(\rho_{ac} - \frac{1}{2}\sum_{i,j=1}^{n} R_{ciaj}R_{cibj}\right) = 0. \tag{23}$$

For $a = b \neq c$ in (23), we get

$$(n-2-2k) + \frac{1}{4}\sum_{i,j=1}^{n} R_{caij}^2 + \rho_{aa} - \frac{1}{2}\sum_{i,j=1}^{n} R_{ciaj}^2 = 0. \tag{24}$$

Now we suppose that $n = 4$. Since our T_1M is an H-contact manifold, M is a 2-stein manifold. In a similar way as in the proof of Theorem 1, for $a = b = 1$, $c = 2$ and $a = b = 3$, $c = 4$ in (24), we have

$$2\gamma = -\mu_1^2 + \mu_2^2 + \mu_3^2 - 4(1-k), \tag{25}$$

where γ is the function defined in the proof of Theorem 1. For $a = b = 1$, $c = 3$ and $a = b = 2$, $c = 4$ in (24), we have

$$2\gamma = \mu_1^2 + \mu_2^2 - \mu_3^2 - 4(1-k). \tag{26}$$

For $a = b = 1$, $c = 4$ and $a = b = 2$, $c = 3$ in (24), we have

$$2\gamma = \mu_1^2 - \mu_2^2 + \mu_3^2 - 4(1-k). \tag{27}$$

From (25)–(27), we get

$$\mu_1^2 = \mu_2^2 = \mu_3^2. \tag{28}$$

From (12) and (28), we have

$$\mu_1 = \mu_2 = \mu_3 = 0. \tag{29}$$

Hence, from (14) or (15), we have

$$\lambda_1 = \lambda_2 = \lambda_3 = -\frac{\tau}{12},$$

that is, M is a space of constant sectional curvature $\frac{\tau}{12}$. Moreover, we find that $\gamma = \frac{\tau}{4}$ and then from (25) we get $k - 1 = \frac{\tau}{8}$. Conversely, we suppose that M is a space of constant sectional curvature c and $k = 1 + \frac{\tau}{8}$. Then we first get $\rho(X, Y) = 3cg(X, Y)$, $\tau = 12c$, and $k = 1 + \frac{3}{2}c$. Moreover, we easily check that T_1M satisfies (20) and (22). For checking (21), we compute

$$\sum_{i=1}^{n} g(R(u, X)e_i, R(u, Y)e_i) = 2c^2(g(X, Y) - g(X, u)g(Y, u)),$$

$$\sum_{i=1}^{n} g(R(u, e_i)X, R(u, e_i)Y) = c^2(g(X, Y) + (n-2)g(X, u)g(Y, u)),$$

$$\sum_{i=1}^{n} g(R(u, e_i)u, R(u, e_i)X) = c^2(n-1)g(X, u).$$

After all, we can see that T_1M satisfies (21). This completes the proof of Theorem 2. □

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