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Topological properties of some flat Lorentzian manifolds of low cohomogeneity

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ABSTRACT. We give a topological description of orbit spaces and orbits of some flat Lorentzian G-manifolds.

1. Introduction

One of the important approaches to differential geometry is the differential geometry of G-manifolds, that is, a manifold M with a group of diffeomorphisms. Of particular importance is the situation when M is a Riemannian or semi-Riemannian manifold and G is a closed and connected subgroup of $\mathrm{Iso}(M)$, the Lie group of all isometries of M. When the maximum dimension of the orbits of the action of G on M is $\dim M - k$, then the orbit space $G\backslash M$ is a topological space of dimension k, and the action is said to be of cohomogeneity k. Throughout this paper, we use the symbol $G(x)$ as the G-orbit in M through a point $x \in M$. If $k = 0$ and M is a connected Riemannian manifold, then there exists $x \in M$ such that $\dim G(x) = \dim M$. Since $G(x)$ is a submanifold without boundary then it is an open submanifold of M, and since G is closed in $\mathrm{Iso}(M)$ then $G(x)$ is closed in M. Thus, we get from connectivity of M that $G(x) = M$. So, G acts transitively on M and M is a homogeneous G-manifold. If M is a homogeneous flat Riemannian manifold then it is diffeomorphic to $\mathbf{R}^{n_1} \times T^{n_2}$, $n_1 + n_2 = \dim M$ [12]. If M is a connected cohomogeneity one flat Riemannian G-manifold, the orbit space is homeomorphic to one of the spaces S^1, $\mathbf{R}$ or $[0, 1)$ [4]. The orbits of cohomogeneity one complete and connected flat Riemannian manifolds are studied in [10]. There is a characterization of orbits and orbit spaces of connected cohomogeneity two flat Riemannian manifolds in the series of papers [7, 8, 9]. In the present paper, we give similar results for some flat Lorentzian G-manifolds. Properness of the actions on Riemannian manifolds, plays an important role in the study of orbits and orbit spaces. In the semi-Riemannian

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case, this properness condition fails, so the situation is much more complicated. There are some results about homogeneous flat Lorentzian manifolds in [13]. Also, there are some interesting algebraic results about cohomogeneity one Lorentzian G-manifolds (see [1]). But characterization of orbits and orbit spaces of Lorentzian G-manifolds of low cohomogeneity is in general an open problem. In the Theorems 4, 5 and 6 of the present paper, we study the orbits and orbit spaces of flat cohomogeneity k, $k = 1, 2, 3$, Lorentzian G-manifolds under some conditions on G.

2. Preliminaries

Throughout the following, M will denote a connected semi-Riemannian manifold and we will write $\mathrm{Coh}(G, M) = k$ if M is of cohomogeneity k under the action of a subgroup G of the isometry group $\mathrm{Iso}(M)$. The fixed point set of the action of G on M will be denoted by

$$M^G = \{x \in M : G(x) = x\}.$$

Collection of all orbits $G \backslash M = \{G(x) : x \in M\}$ endowed with the quotient topology is called the orbit space.

We will write $A = B$ if A and B are isomorphic groups or homeomorphic topological spaces.

$\mathrm{Iso}\,\mathbf{R}^n$ will denote the isometry group of $\mathbf{R}^n$ under the scalar product

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = x_1 y_1 + \cdots + x_n y_n.$$

L^{n+1} will denote the Minkowsky space of dimension $n+1$, that is $\mathbf{R}^{n+1}$ endowed with the usual Lorentzian scalar product

$$\langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i, \qquad x = (x_0, \dots, x_n),\ y = (y_0, \dots, y_n).$$

Let $SO(n,1)$ be the special isometry group of L^{n+1}. We denote by $SO_0(n,1)$ the identity component of $SO(n,1)$. Let G be a subgroup of the isometries of L^{n+1}. The action of G on L^{n+1} is said to be irreducible if G does not leave invariant any proper subspace of L^{n+1} and the action is called weakly irreducible if any G-invariant proper subspace has a degenerate induced metric.

Lemma 1 ([10]). *Let G be a compact and connected subgroup of* $\mathrm{Iso}(\mathbf{R}^n)$ *and* $\mathrm{Coh}(G, \mathbf{R}^n) = 1$. *Then*

$$G \backslash \mathbf{R}^n = [0, +\infty).$$

There is a unique zero dimensional orbit and the other orbits are isometric to $S^{n-1}(r)$, $r > 0$.

Lemma 2. *If G is a compact and connected subgroup of* $\mathrm{Iso}(\mathbf{R}^n)$ *such that* $\mathrm{Coh}(G, \mathbf{R}^n) = 2$, *then* $G \backslash \mathbf{R}^n = [0, +\infty) \times \mathbf{R}$ *and one of the following is true:*

(1) $(\mathbf{R}^n)^G$ *is a one point set and the orbits which are not zero dimensional are homogeneous hypersurfaces of spheres.*

(2) $(\mathbf{R}^n)^G$ *is isometric to* $\mathbf{R}$ *and the orbits which are not zero dimensional are isometric to* $S^{n-2}(r)$, $r > 0$.

Proof. (1) is proved in [9]. For (2) see the proof of Theorem 2 in [7]. □

Theorem 1 ([5]). *If G is a connected subgroup of $SO(n, 1)$ and the action of G on L^{n+1} is irreducible then $G = SO_0(n, 1)$.*

Lemma 3. *Let m, k be non-negative integers and G be a closed and connected subgroup of $SO_0(m, 1) \times O(k)$ such that the projection of G on $SO_0(m, 1)$ acts irreducibly on L^{m+1}. Then there is a closed and connected subgroup H of $O(k)$ such that $G = SO_0(m, 1) \times H$.*

Proof. The proof is as like as the proof of Lemma 2.1 in [2], which we rewrite for facility. We have $G \subset \{(g, h) : g \in SO_o(m, 1), h \in O(k)\}$. Put

$$G_1 = \{g : (g, h) \in G \text{ for some } h \in O(k)\}$$

and

$$H = \{h : (g, h) \in G \text{ for some } g \in SO_o(m, 1)\}.$$

Since $O(\mathbf{R}^k)$ is compact then G_1 is isomorphic to the non-compact semi-simple Levi factor of G, so there is a homomorphism $\rho : G_1 \to O(k)$ such that $\{(g, \rho(g)) : g \in G_1\}$ is isomorphic to G_1. But there is no non-trivial homomorphism from a non-compact semi-simple Lie group to a compact group, so ρ must be trivial. Then $G_1 \times \{I\} \subset G$. But, the action of G_1 on L^{m+1} is irreducible. Then by Theorem 1, $G_1 = SO_0(m, 1)$ and $SO_0(m, 1) \times \{I\} \subset G$. Now, it is easy to show that $G = SO_0(m, 1) \times H$. □

A vector $v \in L^{n+1}$ is called eigenvector for $G \subset SO(n, 1)$ if v is eigenvector for all $g \in G$.

Remark 1. *Let G be a connected subgroup of $SO(n, 1)$. If G does not have null eigenvector, then by Theorem* 1.3 *in* [5], *there is a proper G-invariant Lorentzian subspace in L^{n+1}. Let m be the minimum non-negative integer number with the property that there is a G-invariant $(m + 1)$-dimensional Lorentzian subspace W of L^{n+1}. The action of G on W is irreducible. It is because, if not, then from minimality of m, the action must be weakly irreducible, so there must be a null eigenvector.*

3. Results

Theorem 2. *If G is a closed and connected subgroup of $SO(n,1)$ without null eigenvector, then either $G = SO_0(n,1)$ or there is a non-negative integer $m < n$ and a closed and connected subgroup H of the isometries of $\mathbf{R}^{n-m}$ such that $G = SO_0(m,1) \times H$.*

Proof. If the action of G on L^{n+1} is irreducible then by Theorem 1, $G = SO_0(n,1)$. Suppose that the action of G is not irreducible. Then by Remark 1, there is a G-invariant Lorentzian vector subspace W of L^{n+1} such that the action of G on W is irreducible. Without lose of generality we can assume that $W = L^{m+1}$, $m < n$. Consider L^{n+1} as the product $L^{n+1} = L^{m+1} \times \mathbf{R}^{n-m}$. Since G leaves invariant L^{m+1} and $\mathbf{R}^{n-m}$ then G can be considered as a subgroup of $SO_o(m,1) \times O(n-m)$. Then, by Lemma 3, there is a subgroup H of $O(n-m)$ such that $G = SO_0(m,1) \times H$. □

Theorem 3. *Let G be a closed and connected subgroup of $SO(n,1)$, which does not have null eigenvector and suppose that G acts by cohomogeneity k on L^{n+1}. Then, $k > 0$ and the following assertions are true:*

(a) *If $k = 1$ then $G = SO_0(n,1)$.*

(b) *If $k = 2$ then there is a non-negative integer m such that $G \backslash L^{n+1}$ is homeomorphic to $SO_{m,1} \backslash L^{m+1} \times [0, +\infty)$.*

(c) *If $k = 3$ then there is a non-negative integer m such that $G \backslash L^{n+1}$ is homeomorphic to $SO_0(m,1) \backslash L^{m+1} \times [0, +\infty) \times \mathbf{R}$.*

Proof. By Theorem 2, either $G = SO_0(n,1)$ or there is a non-negative integer m and a connected subgroup H of $O(n-m)$ such that $G \backslash L^{n+1} = SO_0(m,1) \backslash L^{m+1} \times H \backslash \mathbf{R}^{n-m}$. H is closed in $O(n-m)$ so it is compact. Since the action of $SO_0(m,1)$ on L^{m+1} is of cohomogeneity one, then we get the results from the Lemmas 1 and 2. □

Remark 2. *If M is a semi-Riemannian manifold and G is a connected subgroup of* $\mathrm{Iso}(M)$*, and if $\tilde{M}$ is the universal semi-Riemannian covering manifold of M with the covering map $\kappa : \tilde{M} \to M$, then there is a connected covering $\hat{G}$ of G such that $\hat{G}$ acts isometrically on $\tilde{M}$,* $\mathrm{Coh}(G, M) = \mathrm{Coh}(\hat{G}, \tilde{M})$*, and the following assertions are true:*

(1) *Each deck transformation δ of the covering $\kappa : \tilde{M} \to M$ maps $\hat{G}$-orbits on to $\hat{G}$-orbits.*

(2) *If $x \in M$ and $\tilde{x} \in \tilde{M}$ such that $\kappa(\tilde{x}) = x$, then $\kappa(\hat{G}(\tilde{x})) = G(x)$.*

(3) *If G has a fixed point in M then $\hat{G} = G$ and $(\tilde{M})^G = \kappa^{-1}(M^G)$.*

(4) *Following* (3)*, if G has only one fixed point then $\tilde{M} = M$.*

Proof. The group $\hat{G}$ can be defined in the same way in [4, page 63], and the proofs of (1), (2) and (3) are as like as the proof of Theorem 9.1 in

[4, page 64]. For the proof of (4) note that if x_0 is a fixed point of G in M then by (3), $\kappa^{-1}(x_0)$ is a set consisting of fixed points of the action of $\hat{G}$ on $\tilde{M}$. By assumption of (4), $\kappa^{-1}(x_0)$ must be a one point set. So, κ is one to one and $\tilde{M} = M$. □

If $G \subset \mathrm{Iso}(M)$ and γ is a null curve in M such that $G(\gamma) = \gamma$ then γ is called a null G-curve.

Theorem 4. *Let M be a flat Lorentzian manifold which is of cohomogeneity one under the action of a closed and connected Lie subgroup G of isometries. Let us assume that there exists no null G-curve and $M^G \neq \emptyset$. Then $M = L^{n+1}$ and $G = SO_0(n, 1)$.*

Proof. L^{n+1} is the universal covering of M. According to Remark 2, let $\hat{G}$ be the connected covering of G, which acts by cohomogeneity one on L^{n+1}. Since $M^G \neq \emptyset$ then by Remark 2 (3), $(L^{n+1})^{\hat{G}} \neq \emptyset$. Without lose of generality we can assume that the origin of L^{n+1} is a fixed point of $\hat{G}$, so $\hat{G}$ can be considered as a connected subgroup of $SO_0(n, 1)$. Since there is no null G-curve in M then $\hat{G}$ does not have null eigenvector. The action of $\hat{G}$ on L^{n+1} is of cohomogeneity one. Then by Theorem 3 (a), $\hat{G} = SO_0(n, 1)$ and $\kappa^{-1}(M^G)$ is a one point set. So, by Remark 2 (4), $M = L^{n+1}$ and $G = \hat{G} = SO_0(n+1)$. □

Remark 3. *Following Remark 2, if $\tilde{M}^{\hat{G}}$ is diffeomorphic to $\mathbf{R}$ then $\pi_1(M) = Z$.*

Proof. Let Δ be the deck transformation group related to the covering map $\kappa : \tilde{M} \to M$. Each member of Δ maps $\hat{G}$-orbits of $\tilde{M}$ on to $\hat{G}$-orbits. Thus, by dimensional reasons, if $\delta \in \Delta$ then $\delta(\tilde{M}^{\hat{G}}) = \tilde{M}^{\hat{G}}$. Then, Δ can be viewed as a discrete subgroup of $(\mathbf{R}, +)$, so it is isomorphic to $(Z, +)$. □

Theorem 5. *Let M be a flat Lorentzian manifold which is of cohomogeneity two under the action of a closed and connected subgroup G of isometries and let us assume that there exists no null G-curve and $M^G \neq \emptyset$. Then one of the following is true:*

(a) *$M = L^{n+1}$ and there is a non-negative integer $m < n$ and a connected and closed subgroup H of $SO(n-m)$ such that $G = SO_0(m, 1) \times H$. There is one zero dimensional orbit and the other orbits are isometric to D or $D \times S^{n-m-1}(r)$, where D is a $SO_0(m, 1)$-orbit in L^{m+1}, $r > 0$.*

(b) *$M = Z \backslash L^{n+1}$. $M^G = Z \backslash \mathbf{R} = S^1$. The orbits which are not zero dimensional are covered by $S^{n-1}(r)$, $r > 0$.*

Proof. Following Remark 2, let $\hat{G}$ be the connected covering of G which acts by cohomogeneity two on L^{n+1}, the universal covering of M. Since

$\mathrm{Coh}(\hat{G}, L^{n+1}) = 2$ then $\hat{G}$ is not isomorphic to $SO_0(n,1)$, so by Theorem 2, there is a non-negative integer m and a closed and connected subgroup H of $O(n-m)$ such that $\hat{G} = SO_0(m,1) \times H$. Consider L^{n+1} as the product $L^{n+1} = L^{m+1} \times \mathbf{R}^{n-m}$. Since the action of $SO(m,1)$ on L^{m+1} is of cohomogeneity one then the action of H on $\mathbf{R}^{n-m}$ must be of cohomogeneity one. Now, consider two cases $m > 0$ and $m = 0$ separately.

If $m > 0$ then the origin of L^{m+1} is the unique fixed point of the action of $SO_0(m,1)$ and by Lemma 1, the action of H on $\mathbf{R}^{n-m}$ has a unique fixed point. So, $\hat{G}$ has a unique fixed point in L^{n+1}. Then by Remark 2 (4), $M = L^{n+1}$ and $G = \hat{G} = SO_0(m+1) \times H$. By Lemma 1, non-zero dimensional H-orbits of $\mathbf{R}^{n-m}$ are isometric to $S^{n-m-1}(r)$, $r > 0$. Thus we get part (a).

If $m = 0$ then $\hat{G} = \{I\} \times H$, where I is the identity map on L^1. Thus, $(L^{n+1})^{\hat{G}}$ is diffeomorphic to $\mathbf{R}$ and by Remark 3, $\Delta = Z$. So $M = Z \backslash L^{n+1}$ and $M^G = Z \backslash \mathbf{R} = S^1$. By Remark 2 (3), $G = \hat{G} = \{I\} \times H \simeq H$. Thus, $\hat{G}$-orbits are isometric to H-orbits, so by Lemma 1, G-orbits which are not zero dimensional are covered by $S^{n-1}(r)$, $r > 0$. This is part (b). □

Theorem 6. *Let M be a flat Lorentzian manifold which is of cohomogeneity three under the action of a closed and connected subgroup G of isometries and let us assume that there is no null G-orbit and $M^G \neq \emptyset$. Then one of the following is true:*

(a) *$M = L^{n+1}$, there is a non-negative integer m such that the orbits of positive dimension are isometric to $D \times E$, where D is a $SO_0(m,1)$-orbit in L^{m+1} and E is a homogeneous hypersurface of $S^{n-m-1}(r)$, $r > 0$. M^G is a one point set.*

(b) *$M = Z \backslash L^{n+1}$, there is a non-negative integer m such that the orbits of positive dimension are covered by $D \times S^{n-m-2}(r)$, where D is a $SO_0(m,1)$-orbit in L^{m+1}. M^G is diffeomorphic to S^1.*

(c) *$M = Z \backslash L^{n+1}$. Each orbit is covered by a homogeneous hypersurface of $S^{n-1}(r)$, $r > 0$, and $M^G = S^1$.*

(d) *$M = \Delta \backslash L^{n+1}$, where Δ is a discrete subgroup of the isometries of $\mathbf{R}^2$. $M^G = S^1 \times \mathbf{R}$ or T^2, and positive dimensional orbits are covered by $S^{n-2}(r)$, $r > 0$.*

Proof. In the same way as the proof of previous theorems, there is a non-negative integer m such that $\hat{M} = L^{n+1}$ and $\hat{G} = SO_0(m,1) \times H$, where H is a connected and closed subgroup of $SO(n-m)$. Since the action of $SO_0(m,1)$ on L^{m+1} is of cohomogeneity one then the action of H on $\mathbf{R}^{n-m}$ is of cohomogeneity two. We study two cases $m > 0$ and $m = 0$ separately.

Case 1: $m > 0$.

By Lemma 2, either $(\mathbf{R}^{n-m})^H$ is a one point set or it is diffeomorphic to $\mathbf{R}$. If $(\mathbf{R}^{n-m})^H$ is a one point set then $(L^{n+1})^{\hat{G}}$ must be a one point set, so by Remark 2, $M = L^{n+1}$ and $G = \hat{G} = SO_0(m,1) \times H$. By Lemma 2, all H-orbits of $\mathbf{R}^{n-m}$ which have positive dimensions are included in spheres of $\mathbf{R}^{n-m}$. Then the G-orbits of positive dimension are isometric to the product of $SO_0(m,1)$-orbits of L^{m+1} and homogeneous hypersurfaces of $S^{n-m-1}(r)$, $r > 0$. This is part (a). If $(\mathbf{R}^{n-m})^H$ is diffeomorphic to $\mathbf{R}$ then $(L^{n+1})^G = \{o\} \times \mathbf{R} \simeq \mathbf{R}$, so $\Delta = Z$ and $M = Z \backslash L^{n+1}$ and $M^G = Z \backslash \mathbf{R} = S^1$. By Lemma 2, H-orbits of positive dimension in $\mathbf{R}^{n-m}$ are isometric to $S^{n-m-2}(r)$, $r > 0$, so G-orbits of positive dimension in M are covered by the product of $SO_0(m,1)$-orbits of L^{m+1} and the spheres $S^{n-m-2}(r)$. Thus, we get part (b).

Case 2: $m = 0$.

As like as the proof of Theorem 5, $G = \{I\} \times H$, where H is a closed and connected subgroup of $SO(n)$. By Lemma 2, $(L^{n+1})^{\hat{G}} = \mathbf{R} \times \{o\} \simeq \mathbf{R}$ or $(L^{n+1})^{\hat{G}} = \mathbf{R} \times \mathbf{R} = \mathbf{R}^2$. Since each $\delta \in \Delta$ maps $\hat{G}$ orbits on to $\hat{G}$-orbits then by dimensional reasons $\Delta(L^{n+1})^{\hat{G}} = (L^{n+1})^{\hat{G}}$. So, Δ can be considered as a discrete group acting on $\mathbf{R}$ or $\mathbf{R}^2$. In the first case we get that $M = Z \backslash L^{n+1}$, and positive dimensional orbits are covered by H-orbits of $\mathbf{R}^n$ which are homogeneous hypersurfaces of $S^{n-1}(r)$, $r > 0$. This is part (c). In the second case, since Δ is a discrete subgroup of the isometries of $\mathbf{R}^2$. $M^G = \Delta \backslash \mathbf{R}^2 = S^1 \times \mathbf{R}$ or $M^G = T^2$. G-orbits of positive dimension are covered by $S^{n-2}(r)$, $r > 0$. So, we get part (d). □

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