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Spherical means of super-polyharmonic functions in the unit ball

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ABSTRACT. For a super-polyharmonic function u on the unit ball satisfying a growth condition on spherical means, we study a growth property of the Riesz measure of u near the boundary.

1. Introduction and statement of result

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. We use the notation $B(x,r)$ to denote the open ball centered at x with radius r, whose boundary is written as $S(x,r) = \partial B(x,r)$. In particular, $\mathbf{B}$ denotes the unit ball $B(0,1)$.

For a Borel measurable function u on $S(0,r)$, letting dS denote the surface area measure on $S(0,r)$, we define the spherical mean over $S(0,r)$ by

$$M(u,r) = \frac{1}{\omega_n r^{n-1}} \int_{S(0,r)} u(x)dS(x) = \fint_{S(0,r)} u(x)dS(x),$$

where ω_n denotes the surface area of the unit sphere $S(0,1)$.

Let m be a positive integer. Consider the Riesz kernel of order $2m$ defined by

$$\mathscr{R}_{2m}(x) = \begin{cases} \alpha_{n,m}(-1)^{(2m-n)/2}|x|^{2m-n}\log(1/|x|) & \text{if } 2m-n \text{ is an even} \\ & \text{nonnegative integer,} \\ \alpha_{n,m}(-1)^{\max\{0,(2m-n+1)/2\}}|x|^{2m-n} & \text{otherwise,} \end{cases}$$

where $\alpha_{n,m}$ is a positive constant so chosen that $(-\Delta)^m\mathscr{R}_{2m}$ is the Dirac measure at the origin.

We say that a locally integrable function u on $\mathbf{B}$ is super-polyharmonic of order m in $\mathbf{B}$ if

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(1) $(-\Delta)^m u$ is a nonnegative measure on $\mathbf{B}$, that is,

$$\int_{\mathbf{B}} u(x)(-\Delta)^m \varphi(x)dx \geq 0 \qquad \text{for all nonnegative } \varphi \in C_0^\infty(\mathbf{B});$$

(2) u is lower semicontinuous in $\mathbf{B}$;

(3) every point of $\mathbf{B}$ is a Lebesgue point of u

(see [4] and [3]); $(-\Delta)^m u$ is referred to as the Riesz measure of u and denoted by μ_u.

Let u be super-polyharmonic of order m on $\mathbf{B}$ with the associated Riesz measure μ_u. If $0 < R < 1$, then u is represented as

$$u(x) = \int_{B(0,R)} \mathscr{R}_{2m}(x-y)d\mu_u(y) + h_R(x) \tag{1}$$

for $x \in B(0,R)$, where h_R is a polyharmonic function on $B(0,R)$. This is referred to as the Riesz decomposition (see e.g. Armitage-Gardiner [1], Axler-Bourdon-Ramey [2], Hayman-Kennedy [5] and Mizuta [6]). With the aid of the Riesz decomposition, one can obtain a kind of the Poisson-Jensen formula, which assures a representation of $M(u,r)$ by use of the Riesz measure of u (see Lemma 1 below).

Our first aim in this note is to prove the following.

Theorem 1. *Let h be a nonincreasing function on $(0,1)$ such that $\lim_{r\to+0} h(r) = \infty$ and let $h_0 \geq 0$. Suppose that for all $0 < b < 1$, there exists a constant $A > 0$ such that*

$$h(br) \leq b^{-h_0}h(r) + A \tag{2}$$

for all $r \in (0,1)$. Let u be super-polyharmonic of order m in $\mathbf{B}$ and $\mu_u = (-\Delta)^m u$. Suppose

$$M((-1)^m u, r) \leq A_1 + A_2 h(1-r) \tag{3}$$

for $r \in (0,1)$, where $A_1, A_2 > 0$ are positive constants. Then

(1)
$$\limsup_{r\to 1-0} (1-r)^{2m-1}h(1-r)^{-1}\mu_u(B(0,r)) \leq \frac{(2m-2)!\omega_n}{(2m-1)^{2m-2}}\left(1+\frac{2m-1}{h_0}\right)^{h_0+2m-1} h_0^{2m-1}A_2.$$

(2) *If in addition h satisfies*

$$\liminf_{r\to 1-0} h(1-r)^{-1}\int_{1/2}^{r}(r-t)^{2m-2}(1-t)^{-2m+1}h(1-t)dt \geq h_0^{-1}, \tag{4}$$

then

$$\liminf_{r\to 1-0}(1-r)^{2m-1}h(1-r)^{-1}\mu_u(B(0,r)) \le (2m-2)!\omega_n h_0 A_2. \quad (5)$$

Note here that

$$\frac{(2m-2)!\omega_n}{(2m-1)^{2m-2}}\left(1+\frac{2m-1}{h_0}\right)^{h_0+2m-1} h_0^{2m-1}A_2 \ge (2m-2)!\omega_n h_0 A_2.$$

This gives an extension of a result by Supper ([7, Corollary 1 and Theorem 2]), who treated subharmonic functions u on $\mathbf{B}$ satisfying

$$u(x) \le A(1-|x|)^{-\gamma}.$$

2. Fundamental lemma on spherical means

Since $\Delta^k \mathcal{R}_{2m}(x)$ is radial, we write

$$\Delta^k \mathcal{R}_{2m}(r) = \Delta^k \mathcal{R}_{2m}(x)$$

when $r = |x|$.

Lemma 1. *Let $0 < r_0 < 1$. If u is super-polyharmonic of order m in* $\mathbf{B}$, *then there exist constants b_j (depending on r_0) such that*

$$M(u,r) = \int_{B(0,r)\setminus B(0,r_0)}\left(\sum_{j=0}^{m-1} a_j|y|^{2j}\Delta^j\mathcal{R}_{2m}(r) - \sum_{j=0}^{m-1} a_j r^{2j}\Delta^j\mathcal{R}_{2m}(y)\right)d\mu_u(y)$$

$$+\int_{B(0,r_0)}\sum_{j=0}^{m-1} a_j|y|^{2j}\Delta^j\mathcal{R}_{2m}(r)d\mu_u(y) + \sum_{j=0}^{m-1} b_j r^{2j}$$

for $r_0 < r < 1$, where $a_0 = 1$ and

$$a_j = \frac{1}{2^j j! n(n+2)\dots(n+2j-2)}$$

for $j = 1, 2, \dots, m-1$.

Proof. Let u be super-polyharmonic of order m on $\mathbf{B}$ and $0 < r_0 < R < 1$. As mentioned in (1), we have

$$u(x) = \int_{B(0,R)} \mathcal{R}_{2m}(x-y)d\mu_u(y) + h_R(x),$$

for $x \in B(0,R)$, where h_R is a polyharmonic function on $B(0,R)$. Then we see that

$$u(x) = \int_{B(0,R)\setminus B(0,r_0)} \left(\mathscr{R}_{2m}(x-y) - \sum_{j=0}^{m-1} a_j |x|^{2j} \Delta^j \mathscr{R}_{2m}(y) \right) d\mu_u(y)$$

$$+ \int_{B(0,r_0)} \mathscr{R}_{2m}(x-y) d\mu_u(y) + H_R(x),$$

for $x \in B(0,R)$, where H_R is a polyharmonic function on $B(0,R)$ defined by

$$H_R(x) = \sum_{j=0}^{m-1} a_j \left(\int_{B(0,R)\setminus B(0,r_0)} \Delta^j \mathscr{R}_{2m}(y) d\mu_u(y) \right) |x|^{2j} + h_R(x).$$

If $r_0 < r < R$, then

$$M(u,r) = \int_{B(0,R)\setminus B(0,r_0)} \left(⨍_{S(0,r)} \left(\mathscr{R}_{2m}(x-y) - \sum_{j=0}^{m-1} a_j |x|^{2j} \Delta^j \mathscr{R}_{2m}(y) \right) dS(x) \right) d\mu_u(y)$$

$$+ \int_{B(0,r_0)} \left(⨍_{S(0,r)} \mathscr{R}_{2m}(x-y) dS(x) \right) d\mu_u(y) + M(H_R, r)$$

$$= \int_{B(0,r)\setminus B(0,r_0)} \left(\sum_{j=0}^{m-1} a_j |y|^{2j} \Delta^j \mathscr{R}_{2m}(r) - \sum_{j=0}^{m-1} a_j r^{2j} \Delta^j \mathscr{R}_{2m}(y) \right) d\mu_u(y)$$

$$+ \int_{B(0,r_0)} \sum_{j=0}^{m-1} a_j |y|^{2j} \Delta^j \mathscr{R}_{2m}(r) d\mu_u(y) + \sum_{j=0}^{m-1} a_j \Delta^j H_R(0) r^{2j}.$$

This implies that

$$\sum_{j=0}^{m-1} a_j \Delta^j H_{R_1}(0) r^{2j} = \sum_{j=0}^{m-1} a_j \Delta^j H_{R_2}(0) r^{2j}$$

whenever $r_0 < r < R_1 < R_2$, so that $a_j \Delta^j H_R(0)$ does not depend on R, and hence it is a constant b_j (depending on r_0).

Set

$$g_m(t,r) = \sum_{j=0}^{m-1} a_j t^{2j} \Delta^j \mathscr{R}_{2m}(r) - \sum_{j=0}^{m-1} a_j r^{2j} \Delta^j \mathscr{R}_{2m}(t).$$

Remark 1. Let u be super-polyharmonic of order m on $\mathbf{B}$ and $\mu_u = (-\Delta)^m u$. By Lemma 1 and integration by parts, we have

$$
\begin{aligned}
M(u,r) &= \int_{B(0,r)\setminus B(0,r_0)} g_m(|y|,r)\,d\mu_u(y) + O(1) \\
&= \int_{r_0}^{r} g_m(t,r)\,d\mu_u(B(0,t)) + O(1) \\
&= \int_{r_0}^{r} \left(-\frac{\partial}{\partial t} g_m(t,r)\right) \mu_u(B(0,t))\,dt + O(1)
\end{aligned}
$$

as $r \to 1-0$.

Lemma 2. *The following hold:*

(1) $(-1)^m g_m(t,r)$ *is positive and decreasing as a function of* t *in* $(0,r)$.

(2) $(-1)^{m-1}\dfrac{\partial}{\partial t} g_m(t,r) \ge \dfrac{r^{1-n}}{(2m-2)!\omega_n}(r-t)^{2m-2}$ *for* $0 < t < r$.

Proof. For fixed $r > 0$, set $g_m(t) = g_m(t,r)$. We prove this lemma by induction on m. In case $m = 1$, we have

$$
g_1(t) = \begin{cases} \alpha_{2,1}\log(t/r) & \text{if } n = 2, \\ \alpha_{n,1}(r^{2-n} - t^{2-n}) & \text{if } n \ge 3, \end{cases}
$$

where $\alpha_{2,1} = \omega_2^{-1}$ and $\alpha_{n,1} = \omega_n^{-1}(n-2)^{-1}$. Hence (1) and (2) hold for $m = 1$.

Suppose that (1) and (2) hold for $m-1$ when $m \ge 2$. By the assumption on induction and $g_{m-1}(r) = 0$, we have

$$
(-1)^{m-1} g_{m-1}(t) \ge \int_t^r \frac{r^{1-n}}{(2m-4)!\omega_n}(r-\rho)^{2m-4}\,d\rho = \frac{r^{1-n}}{(2m-3)!\omega_n}(r-t)^{2m-3} \tag{6}
$$

for $0 < t < r$. Noting that

$$
\Delta g_m(t) = -g_{m-1}(t)
$$

and

$$
\Delta g_m(t) = g_m''(t) + \frac{n-1}{t} g_m'(t) = t^{1-n}(t^{n-1} g_m'(t))',
$$

we have

$$
\begin{aligned}
(-1)^m g_m(t) &= (-1)^m \int_t^r s^{1-n}\left(\int_s^r (\rho^{n-1} g_m'(\rho))'\,d\rho\right) ds \\
&= \int_t^r s^{1-n}\left(\int_s^r \rho^{n-1}(-1)^{m-1} g_{m-1}(\rho)\,d\rho\right) ds.
\end{aligned}
$$

Hence (1) holds.

On the other hand, noting that

$$(-1)^m g_m'(t) = -t^{1-n} \int_t^r \rho^{n-1} (-1)^{m-1} g_{m-1}(\rho) d\rho,$$

we have by (6)

$$\begin{aligned}(-1)^{m-1} g_m'(t) &= t^{1-n} \int_t^r \rho^{n-1} (-1)^{m-1} g_{m-1}(\rho) d\rho \\ &\geq t^{1-n} \int_t^r \rho^{n-1} \frac{r^{1-n}}{(2m-3)!\omega_n} (r-\rho)^{2m-3} d\rho \\ &\geq \frac{r^{1-n}}{(2m-3)!\omega_n} \int_t^r (r-\rho)^{2m-3} d\rho \\ &= \frac{r^{1-n}}{(2m-2)!\omega_n} (r-t)^{2m-2},\end{aligned}$$

which implies (2). Thus the lemma is obtained.

3. Proof of Theorem 1

First we show assertion (1). By Remark 1, we have

$$M((-1)^m u, r) = \int_{r_0}^r \left((-1)^{m-1} \frac{\partial}{\partial t} g_m(t,r) \right) \mu_u(B(0,t)) dt + O(1) \qquad \text{as } r \to 1-0.$$

For $a > 0$, we find by Lemma 2 (2)

$$\begin{aligned}&M((-1)^m u, r) \\ &\quad \geq \int_{r-a(1-r)}^r \left((-1)^{m-1} \frac{\partial}{\partial t} g_m(t,r) \right) \mu_u(B(0,t)) dt + O(1) \\ &\quad \geq \mu_u(B(0, r-a(1-r))) \int_{r-a(1-r)}^r \left((-1)^{m-1} \frac{\partial}{\partial t} g_m(t,r) \right) dt + O(1) \\ &\quad \geq \mu_u(B(0, r-a(1-r))) \int_{r-a(1-r)}^r \left(\frac{r^{1-n}}{(2m-2)!\omega_n} (r-t)^{2m-2} \right) dt + O(1) \\ &\quad = \frac{r^{1-n}}{(2m-1)!\omega_n} a^{2m-1} (1-r)^{2m-1} \mu_u(B(0, r-a(1-r))) + O(1)\end{aligned}$$

when $r - a(1-r) > r_0$, so that

$$\limsup_{r \to 1-0} (1-r)^{2m-1} h(1-r)^{-1} \mu_u(B(0, r-a(1-r))) \leq (2m-1)!\omega_n a^{-2m+1} A_2$$

by (3). By change of variable $t = r - a(1-r)$, we obtain by (2)

$$\limsup_{t\to 1-0}(1-t)^{2m-1}h(1-t)^{-1}\mu_u(B(0,t)) \le (2m-1)!\omega_n a^{-2m+1}(1+a)^{h_0+2m-1}A_2.$$

Now, since $a^{-2m+1}(1+a)^{h_0+2m-1}$ attains its minimum at $a = \frac{2m-1}{h_0}$, we obtain the result.

Next, we show assertion (2). By Remark 1 and Lemma 2 (2), we have

$$M((-1)^m u, r) \ge \frac{1}{(2m-2)!\omega_n}\int_{r_0}^{r}(r-t)^{2m-2}\mu_u(B(0,t))dt + O(1) \qquad \text{as } r \to 1-0.$$

If there exist constants $A' > (2m-2)!\omega_n h_0 A_2$ and $r_0 > 0$ such that $\mu_u(B(0,t)) > A'(1-t)^{-2m+1}h(1-t)$ for all $r_0 < t < 1$, then

$$\frac{1}{(2m-2)!\omega_n}h(1-r)^{-1}\int_{r_0}^{r}(r-t)^{2m-2}\mu_u(B(0,t))dt$$
$$> \frac{A'}{(2m-2)!\omega_n}h(1-r)^{-1}\int_{r_0}^{r}(r-t)^{2m-2}(1-t)^{-2m+1}h(1-t)dt,$$

which gives by (4)

$$\liminf_{r\to 1-0}\ h(1-r)^{-1}M((-1)^m u, r) \ge \frac{A'}{(2m-2)!\omega_n}h_0^{-1} > A_2.$$

Thus a contradiction follows from (3).

4. Corollaries

In this section, we introduce some consequences of Theorem 1.

Corollary 1. *Let u be super-polyharmonic in* **B** *and $\mu_u = (-\Delta)^m u$. Suppose*

$$M((-1)^m u, r) \le \left(\log\frac{e}{1-r}\right)^{\gamma}$$

for $r \in (0,1)$, where $\gamma > 0$ is a positive constant. Then

(i) $\displaystyle\limsup_{r\to 1-0}(1-r)^{2m-1}\left(\log\frac{e}{1-r}\right)^{-\gamma}\mu_u(B(0,r)) \le (2m-1)!\omega_n$; *and*

(ii) $\displaystyle\liminf_{r\to 1-0}(1-r)^{2m-1}\left(\log\frac{e}{1-r}\right)^{-\gamma}\mu_u(B(0,r)) = 0.$

Proof. First, we show statement (i). Let $h_1 > 0$. For all $0 < b < 1$, we can find a constant $A' > 0$ such that

$$\left(\log\frac{e}{br}\right)^{\gamma} \le b^{-h_1}\left(\log\frac{e}{r}\right)^{\gamma} + A' \tag{7}$$

whenever $r \in (0,1)$. Applying Theorem 1 with $A_1 = 0$, $A_2 = 1$, $h(r) = (\log(e/r))^{\gamma}$, $A = A'$ and $h_0 = h_1$, we obtain

$$\limsup_{r\to 1-0} (1-r)^{2m-1}\left(\log\frac{e}{1-r}\right)^{-\gamma}\mu_u(B(0,r))$$

$$\le \frac{(2m-2)!\omega_n}{(2m-1)^{2m-2}}\left(1+\frac{2m-1}{h_1}\right)^{h_1+2m-1} h_1^{2m-1}$$

$$\le \frac{(2m-2)!\omega_n}{(2m-1)^{2m-2}}\left(1+\frac{2m-1}{h_1}\right)^{h_1} (h_1+2m-1)^{2m-1},$$

which tends to $(2m-1)!\omega_n$ as $h_1 \to 0$.

Next, we show statement (ii). First note that

$$\lim_{r\to 1-0}\left(\log\frac{e}{1-r}\right)^{-\gamma}\int_{1/2}^{r}(r-t)^{2m-2}(1-t)^{-2m+1}\left(\log\frac{e}{1-t}\right)^{\gamma}dt = \infty.$$

Applying Theorem 1 with $A_1 = 0$, $A_2 = 1$, $h(r) = (\log(e/r))^{\gamma}$, $A = A'$ and $h_0 = h_1$, we have

$$\liminf_{r\to 1-0} (1-r)^{2m-1}\left(\log\frac{e}{1-r}\right)^{-\gamma}\mu_u(B(0,r)) \le (2m-2)!\omega_n h_1,$$

which tends to 0 as $h_1 \to 0$.

COROLLARY 2. *Let u be super-polyharmonic in* **B** *and $\mu_u = (-\Delta)^m u$. Suppose*

$$M((-1)^m u, r) \le (1-r)^{-\gamma}$$

for $r \in (0,1)$, where $\gamma > 0$ is a positive constant. Then

(i)

$$\limsup_{r\to 1-0} (1-r)^{\gamma+2m-1}\mu_u(B(0,r)) \le \frac{(2m-2)!\omega_n}{(2m-1)^{2m-2}}\left(1+\frac{2m-1}{\gamma}\right)^{\gamma+2m-1}\gamma^{2m-1};$$

and

(ii) $\liminf_{r\to 1-0} (1-r)^{\gamma+2m-1}\mu_u(B(0,r)) \le \omega_n \gamma_m$, *where* $\gamma_m = (\gamma+2m-2)(\gamma+2m-3)\dots\gamma$.

For a proof, apply Theorem 1 with $h(r) = r^{-\gamma}$.

In the superharmonic case, Corollary 2 is reduced to the following:

COROLLARY 3. *Let u be superharmonic in* **B** *and $\mu_u = -\Delta u$. Suppose*

$$M(-u, r) \le (1-r)^{-\gamma}$$

for $r \in (0,1)$, *where* $\gamma > 0$ *is a positive constant. Then*

(i) $\limsup_{r \to 1-0} (1-r)^{\gamma+1} \mu_u(B(0,r)) \le \omega_n \left(1 + \frac{1}{\gamma}\right)^{\gamma+1} \gamma$; *and*

(ii) $\liminf_{r \to 1-0} (1-r)^{\gamma+2m-1} \mu_u(B(0,r)) \le \omega_n \gamma$.

5. Best possibility of Theorem 1 for $m = 1$

Here we discuss the best possibility of "lim sup" and "lim inf" in Theorem 1 for $m = 1$.

Example 1. For $a > 1$ and $\gamma > 0$, we can find a measure μ satisfying

(i) $\limsup_{r \to 1-0} (1-r)^{\gamma+1} \mu(B(0,r)) = 1$,

(ii) $\liminf_{r \to 1-0} (1-r)^{\gamma+1} \mu(B(0,r)) = a^{-\gamma-1}$ and

(iii) $\limsup_{r \to 1-0} (1-r)^{\gamma} \int_0^r \mu(B(0,t))dt = \gamma^{-1} \left[\left\{ \frac{(a-1)}{a(a^\gamma - 1)} + 1 \right\} \frac{\gamma}{1+\gamma} \right]^{1+\gamma}$.

Set $a_n = 1 - a^{-n}$ and $b_n = a^{n(\gamma+1)}$. Define $\mu = \sum_{n=1}^{\infty} (b_n - b_{n-1}) \delta_{x_n}$, where $x_n = (a_n, 0, \dots, 0) \in \mathbf{B}$ and $b_0 = 0$.

For $a_n < r \le a_{n+1}$, note that

$$\mu(B(0,r)) = b_n$$

and

$$\begin{aligned} \int_0^r \mu(B(0,t))dt &= \sum_{j=1}^{n-1} b_j(a_{j+1} - a_j) + (r - a_n) b_n \\ &= \sum_{j=1}^{n-1} \frac{a-1}{a} a^{j\gamma} + (r - a_n) b_n \\ &= C_n + (r - a_n) b_n, \end{aligned}$$

where $C_n = \frac{(a-1)}{a(a^\gamma - 1)}(a^{n\gamma} - a^\gamma)$. Hence we have

$$(1-r)^\gamma \int_0^r \mu(B(0,t))dt = \{C_n + (1 - a_n) b_n\}(1-r)^\gamma - b_n(1-r)^{1+\gamma}$$

which attains the maximum at

$$-\{C_n + (1 - a_n) b_n\}\gamma + b_n(1+\gamma)(1-r) = 0,$$

or

$$1 - r = \{C_n / b_n + (1 - a_n)\} \frac{\gamma}{1+\gamma}.$$

Here, note that $a_n < r \le a_{n+1}$ for sufficiently large n since

$$\frac{1+\gamma-\gamma a}{\gamma a} < \frac{a-1}{a(a^\gamma - 1)} < \frac{1}{\gamma}.$$

Hence

$$\max_{a_n < r \le a_{n+1}} (1-r)^\gamma \int_0^r \mu(B(0,t))dt = b_n \gamma^{-1} \left[\{C_n/b_n + (1-a_n)\} \frac{\gamma}{1+\gamma} \right]^{1+\gamma}$$

for sufficiently large n. Since the right hand term in the above equality is increasing on n, the above equality gives

$$\limsup_{r \to 1-0} (1-r)^\gamma \int_0^r \mu(B(0,t))dt = \gamma^{-1} \left[\left\{ \frac{(a-1)}{a(a^\gamma - 1)} + 1 \right\} \frac{\gamma}{1+\gamma} \right]^{1+\gamma}.$$

Further, we have

$$\limsup_{r \to 1-0} (1-r)^{\gamma+1} \mu(B(0,r)) = 1$$

and

$$\liminf_{r \to 1-0} (1-r)^{\gamma+1} \mu(B(0,r)) = a^{-\gamma-1},$$

as required.

Now, we show that Theorem 1 is best possible for $m = 1$. Let μ be as in Example 1. For $0 < A < 1$ and $\gamma > 0$, find $a > 1$ such that

$$\left\{ \frac{(a-1)}{a(a^\gamma - 1)} + 1 \right\}^{1+\gamma} = A^{-1}.$$

If we set $\nu = \omega_n \left(1 + \frac{1}{\gamma}\right)^{1+\gamma} \gamma A \mu$, then

(1′) $\limsup_{r \to 1-0} (1-r)^{\gamma+1} \nu(B(0,r)) = \omega_n \left(1 + \frac{1}{\gamma}\right)^{1+\gamma} \gamma A$;

(2′) $\liminf_{r \to 1-0} (1-r)^{\gamma+1} \nu(B(0,r)) = a^{-\gamma-1} \omega_n \left(1 + \frac{1}{\gamma}\right)^{1+\gamma} \gamma A$; and

(3) $\limsup_{r \to 1-0} (1-r)^\gamma \int_0^r \nu(B(0,t))dt = \omega_n$.

As a superharmonic function u whose Riesz measure is ν, we may consider the potential

$$u(x) = \int_{\mathbf{B}} K_{1,L}(x,y) d\nu(y);$$

see [3] for the definition of $K_{1,L}(x,y)$. With the aid of Remark 1, (3) gives

(4) $\limsup_{r \to 1-0} (1-r)^\gamma M(-u,r) \le 1$.

By $(1')$ and (4), if we let $A \to 1$, then we see that (i) of Corollary 3 is best possible. Further, by $(2')$ and (4), if we let $a \to 1$ (and hence $A^{-1} \to \left(\frac{1}{\gamma}+1\right)^{1+\gamma}$), then we see that (ii) of Corollary 3 is best possible.

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