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On the minimality of the corresponding submanifolds to four-dimensional solvsolitons

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ABSTRACT. In our previous study, the author and Tamaru proved that a left-invariant Riemannian metric on a three-dimensional simply-connected solvable Lie group is a solvsoliton if and only if the corresponding submanifold is minimal. In this paper, we study the minimality of the corresponding submanifolds to solvsolitons on four-dimensional cases. In four-dimensional nilpotent cases, we prove that a left-invariant Riemannian metric is a nilsoliton if and only if the corresponding submanifold is minimal. On the other hand, there exists a four-dimensional simply-connected solvable Lie group so that the above correspondence dose not hold. More precisely, there exists a solvsoliton whose corresponding submanifold is not minimal, and a left-invariant Riemannian metric which is not solvsoliton and whose corresponding submanifold is minimal.

1. Introduction

A left-invariant Riemannian metric $\langle , \rangle$ on a simply-connected solvable Lie group G is called a *solvsoliton* if the following holds for some $c \in \mathbf{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$:

$$\mathrm{Ric}_{\langle , \rangle} = cI + D.$$

Here $\mathrm{Ric}_{\langle , \rangle}$ is the Ricci operator of $\langle , \rangle$, $\mathfrak{g}$ is the Lie algebra of G, and $\mathrm{Der}(\mathfrak{g})$ is the algebra of derivations of $\mathfrak{g}$. When G is nilpotent, a solvsoliton on G is called a *nilsoliton*. Solvsolitons have been introduced by Lauret [9, 14]. Solvsolitons have been studied very actively and played a key role in the study of homogeneous Ricci solitons (See, for instance, [4, 5, 6, 7, 9, 12, 13, 14, 15, 18, 19]). In particular, every solvsoliton on a simply-connected solvable Lie group is a Ricci soliton ([14]), and every left-invariant Ricci soliton on a solvable Lie group is isometric to a solvsoliton ([7]).

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Let n be the dimension of G. Note that the set of all left-invariant Riemannian metrics on G can be naturally identify with the set of all inner products on $\mathfrak{g}$. We define

$$\tilde{\mathfrak{M}} := \{\langle\, , \rangle \mid \text{an inner product on } \mathfrak{g}\} \cong \mathrm{GL}_n(\mathbf{R})/\mathrm{O}(n), \tag{1.1}$$

and an equivalence relation "isometric up to scaling" on $\tilde{\mathfrak{M}}$. For any inner product, we call its equivalence class the *corresponding submanifold*. As we see in Section 2, the corresponding submanifolds are $\mathbf{R}^\times \mathrm{Aut}(\mathfrak{g})$-homogeneous submanifolds of the noncompact Riemannian symmetric space $\mathrm{GL}_n(\mathbf{R})/\mathrm{O}(n)$. Since solvsolitons are preserved by the action of $\mathbf{R}^\times \mathrm{Aut}(\mathfrak{g})$, it would be natural to ask the following question.

QUESTION 1. *Is it possible to characterize solvsolitons by properties of the corresponding submanifolds?*

The author and Tamaru ([4]) proved that the answer to Question 1 is affirmative in the case of three-dimensional simply-connected solvable Lie groups. More precisely, we proved that a left-invariant Riemannian metric on a three-dimensional simply-connected solvable Lie group is a solvsoliton if and only if the corresponding submanifold is minimal. This result makes us interested in the minimality of the corresponding submanifolds.

The aim of this paper is to study the following question:

QUESTION 2. *Is it true that a left-invariant Riemannian metric is a solvsoliton if and only if the corresponding submanifold is minimal?*

In this paper, we examine the minimality of the corresponding submanifolds to solvsolitons on four-dimensional simply-connected solvable Lie groups. As a result, we show that the answer to Question 2 is affirmative in four-dimensional nilpotent cases.

THEOREM 1.1. *A left-invariant Riemannian metric on a four-dimensional simply-connected nilpotent Lie group is a nilsoliton if and only if the corresponding submanifold is minimal.*

On the other hand, we construct examples which show that the answer to Question 2 is negative in general.

THEOREM 1.2. *There exists a four-dimensional simply-connected solvable Lie group G which satisfies the following:*

(1) *There exists a left-invariant Riemannian metric on G such that it is not a solvsoliton, and the corresponding submanifold is minimal.*
(2) *There exists a left-invariant Riemannian metric on G such that it is a solvsoliton, and the corresponding submanifold is not minimal.*

However we know many examples so that the corresponding submanifold to a solvsoliton is minimal. For example, in four-dimensional cases, if the nilradical is abelian, then the corresponding submanifold to a solvsoliton is minimal. We expect that Question 2 has a positive answer under certain additional conditions, which will be studied in the forthcoming papers.

The contents of this paper is as follows. In Section 2, we recall the notion of the corresponding submanifolds to left-invariant Riemannian metrics, and some necessary facts on reductive homogeneous spaces. In Sections 3 and 4, we prove Theorems 1.1 and 1.2 respectively.

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2. Preliminaries

We recall the notion of the corresponding submanifolds in Subsection 2.1. In Subsection 2.2, we also recall some necessary facts on reductive homogeneous spaces which we need to study the minimality of the corresponding submanifolds.

2.1. The corresponding submanifolds. In this subsection, we recall the notion of the corresponding submanifolds to left-invariant Riemannian metrics. For details we refer to [4, 8].

First of all, we recall the space of left-invariant Riemannian metrics, which will be the ambient space of the corresponding submanifolds. Let G be an n-dimensional simply-connected Lie group and $\mathfrak{g}$ be the Lie algebra of G. We consider the set of all left-invariant Riemannian metrics on G, which can naturally be identified with

$$\widetilde{\mathfrak{M}} := \{\langle\, ,\rangle \mid \text{an inner product on } \mathfrak{g}\}.$$

We identify $\mathfrak{g}$ with $\mathbf{R}^n$ from now on. Then, since $\mathrm{GL}_n(\mathbf{R})$ acts transitively on $\widetilde{\mathfrak{M}}$ by

$$g.\langle\cdot\,,\cdot\rangle := \langle g^{-1}\cdot, g^{-1}\cdot\rangle \qquad (\text{for } g \in \mathrm{GL}_n(\mathbf{R}), \langle\, ,\rangle \in \widetilde{\mathfrak{M}}),$$

we have an identification

$$\widetilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbf{R})/\mathrm{O}(n).$$

Note that $\tilde{\mathfrak{M}}$ equipped with the natural $\mathrm{GL}_n(\mathbf{R})$-invariant Riemannian metric is a noncompact Riemannian symmetric space. In order to describe this natural metric, we recall a general theory of reductive homogeneous spaces. Let U/K be a reductive homogeneous space, that is, there exists an Ad_K-invariant subspace $\mathfrak{m}$ of $\mathfrak{u}$ satisfying

$$\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{m}. \tag{2.1}$$

Note that $\mathfrak{u}$ and $\mathfrak{k}$ are the Lie algebras of U and K, respectively, and $\oplus$ denotes the direct sum as vector spaces. The decomposition (2.1) is called a *reductive decomposition*. Let us denote by $\pi : U \to U/K$ the natural projection, and by $o := \pi(e)$ the origin of U/K. We identify $\mathfrak{m}$ with the tangent space $\mathrm{T}_o(U/K)$ at o by the isomorphism

$$\mathrm{d}\pi_e|_{\mathfrak{m}} : \mathfrak{m} \to \mathrm{T}_o(U/K).$$

This identification induces a one-to-one correspondence between the set of U-invariant Riemannian metrics on U/K and the set of Ad_K-invariant inner products on $\mathfrak{m}$.

Now one can see that $\tilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbf{R})/\mathrm{O}(n)$ is a reductive homogeneous space, whose reductive decomposition is given by the subspace

$$\mathrm{sym}(n) := \{X \in \mathfrak{gl}_n(\mathbf{R}) \mid X = {}^tX\}.$$

Here $\mathfrak{gl}_n(\mathbf{R})$ is the Lie algebra of $\mathrm{GL}_n(\mathbf{R})$. We define the $\mathrm{Ad}_{\mathrm{O}(n)}$-invariant inner product on $\mathrm{sym}(n)$ by

$$\langle X, Y \rangle := \mathrm{tr}(XY) \qquad (\text{for } X, Y \in \mathrm{sym}(n)).$$

We call the $\mathrm{GL}_n(\mathbf{R})$-invariant Riemannian metric corresponding to the above $\mathrm{Ad}_{\mathrm{O}(n)}$-invariant inner product the *natural Riemannian metric*.

Next, we recall the notion of "isometric up to scaling" on $\tilde{\mathfrak{M}}$. This gives an equivalence relation on $\tilde{\mathfrak{M}}$.

DEFINITION 2.1. Two inner products $\langle\, , \rangle_1$ and $\langle\, , \rangle_2$ on $\mathfrak{g}$ are said to be *isometric up to scaling* if there exist $k > 0$ and an automorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ such that $\langle \cdot , \cdot \rangle_1 = k\langle \varphi(\cdot), \varphi(\cdot) \rangle_2$.

Note that above equivalence relation gives the equivalence relation of left-invariant Riemannian metrics on Lie groups. Assume that inner products $\langle\, , \rangle_1$ and $\langle\, , \rangle_2$ on $\mathfrak{g}$ are isometric up to scaling. Then, the corresponding left-invariant Riemannian metrics on G are isometric up to scaling as Riemannian metrics.

DEFINITION 2.2. For each inner product $\langle\, , \rangle$ on $\mathfrak{g}$, we call its equivalence class $[\langle\, , \rangle]$ the *corresponding submanifold to* $\langle\, , \rangle$.

Note that

$$[\langle\,,\rangle] := \{\langle\,,\rangle' \in \tilde{\mathfrak{M}} \mid \langle\,,\rangle' \sim \langle\,,\rangle\},$$

where $\langle\,,\rangle' \sim \langle\,,\rangle$ means that $\langle\,,\rangle'$ and $\langle\,,\rangle$ are isometric up to scaling. Let us denote by

$$\mathbf{R}^{\times} := \{c \cdot \mathrm{id} : \mathfrak{g} \to \mathfrak{g} \mid c \in \mathbf{R} \setminus \{0\}\},$$

$$\mathrm{Aut}(\mathfrak{g}) := \{\varphi : \mathfrak{g} \to \mathfrak{g} \mid \text{an automorphism}\}.$$

Then, the subgroup $\mathbf{R}^{\times}\,\mathrm{Aut}(\mathfrak{g})$ of $\mathrm{GL}_n(\mathbf{R})$ acts naturally on $\tilde{\mathfrak{M}}$. Let us denote by $\mathbf{R}^{\times}\,\mathrm{Aut}(\mathfrak{g}).\langle\,,\rangle$ the $\mathbf{R}^{\times}\,\mathrm{Aut}(\mathfrak{g})$-orbit through $\langle\,,\rangle$.

Proposition 2.3 ([8], Theorem 2.5). *Let $\langle\,,\rangle$ be an inner product on $\mathfrak{g}$. Then, the corresponding submanifold $[\langle\,,\rangle]$ is a homogeneous submanifold with respect to $\mathbf{R}^{\times}\,\mathrm{Aut}(\mathfrak{g})$, that is,*

$$[\langle\,,\rangle] = \mathbf{R}^{\times}\,\mathrm{Aut}(\mathfrak{g}).\langle\,,\rangle.$$

Next we recall the "moduli space" $\mathfrak{PM}$. We need $\mathfrak{PM}$ to examine the minimality of the corresponding submanifolds.

Definition 2.4. For a Lie algebra $\mathfrak{g}$, the quotient space of $\tilde{\mathfrak{M}}$ by the equivalence relation in Definition 2.1 is called the *moduli space of left-invariant Riemannian metrics on* $\mathfrak{g}$, and denoted by

$$\mathfrak{PM} := \{[\langle\,,\rangle] \mid \langle\,,\rangle \in \tilde{\mathfrak{M}}\}.$$

To determine $\mathfrak{PM}$ explicitly, we will use the following expression as a double coset space.

Proposition 2.5 ([8], Theorem 2.5). *If $\dim \mathfrak{g} = n$, then we have*

$$\mathfrak{PM} = \mathbf{R}^{\times}\,\mathrm{Aut}(\mathfrak{g}) \backslash \mathrm{GL}_n(\mathbf{R}) / \mathrm{O}(n).$$

Let $[[g]]$ denote the double coset of $g \in \mathrm{GL}_n(\mathbf{R})$, that is,

$$[[g]] := \mathbf{R}^{\times}\,\mathrm{Aut}(\mathfrak{g}) \cdot g \cdot \mathrm{O}(n).$$

Denote by $\langle\,,\rangle_0 \in \tilde{\mathfrak{M}} = \mathrm{GL}_n(\mathbf{R})/\mathrm{O}(n)$ the origin. Then, the map

$$\mathbf{R}^{\times}\,\mathrm{Aut}(\mathfrak{g}) \backslash \mathrm{GL}_n(\mathbf{R}) / \mathrm{O}(n) \to \mathfrak{PM} : [[g]] \mapsto [g.\langle\,,\rangle_0],$$

gives a bijection.

A subset $\mathfrak{U} \subset \mathrm{GL}_n(\mathbf{R})$ is called a *system of representatives* of $\mathfrak{PM}$ if

$$\mathfrak{PM} = \{[g.\langle\,,\rangle_0] \mid g \in \mathfrak{U}\}.$$

One can easily see the following.

LEMMA 2.6. *Let $\mathfrak{g}$ be an n-dimensional Lie algebra. Then $\mathfrak{U} \subset \mathrm{GL}_n(\mathbf{R})$ is a system of representatives of $\mathfrak{PM}$ if and only if for each $g \in \mathrm{GL}_n(\mathbf{R})$, there exists $g' \in \mathfrak{U}$ such that $g' \in [[g]]$.*

2.2. Standard facts on reductive homogeneous spaces. In this subsection, we review some of the standard facts on reductive homogeneous spaces and their homogeneous submanifolds. We refer to [1, 3].

Let U/K be a reductive homogeneous space with a reductive decomposition

$$\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{m}.$$

Recall that $\mathfrak{m}$ is identified with the tangent space $\mathrm{T}_o(U/K)$. In the following, we equip a U-invariant Riemannian metric g on U/K.

We here recall a formula for the Levi-Civita connection ∇ of g. For any $X \in \mathfrak{u}$, we define the fundamental vector field X^* on U/K by

$$X_p^* = \frac{d}{dt}(\exp tX).p|_{t=0} \qquad (\text{for } p \in U/K).$$

Let $X, Y, Z \in \mathfrak{u}$. Then one knows (see [3]):

$$\begin{aligned} X_o^* &= \mathrm{d}\pi_e(X), \\ [X^*, Y^*] &= -[X, Y]^*, \\ 2g(\nabla_{X^*} Y^*, Z^*) &= g([X^*, Y^*], Z^*) + g([X^*, Z^*], Y^*) + g(X^*, [Y^*, Z^*]). \end{aligned} \tag{2.2}$$

We now consider homogeneous submanifolds in $(U/K, g)$. Let U' be a Lie subgroup of U, and consider the orbit $U'.o$ through the origin o. Let $\mathfrak{u}'$ be the Lie algebra of U', and denote by $\langle\, , \rangle$ the inner product on $\mathfrak{m}$ corresponding to g. We define

$$\mathfrak{m}' := \mathrm{d}\pi_e(\mathfrak{u}') \cong \mathrm{T}_o(U'.o).$$

Denote by $\mathfrak{m} \ominus \mathfrak{m}'$ the orthogonal complement of $\mathfrak{m}'$ in $\mathfrak{m}$ with respect to $\langle\, , \rangle$. Then, the second fundamental form $h : \mathfrak{m}' \times \mathfrak{m}' \to \mathfrak{m} \ominus \mathfrak{m}'$ of $U'.o$ at o is defined by

$$h(X_o^*, Y_o^*) := (\nabla_{X^*} Y^* - \nabla'_{X^*} Y^*)_o \qquad (\text{for } X, Y \in \mathfrak{u}'),$$

where ∇' is the Levi-Civita connection of $U'.o$ with respect to the induced metric. Take $Z \in \mathfrak{u}$ satisfying $Z_o^* \in \mathfrak{m} \ominus \mathfrak{m}'$. From (2.2), one obtains

$$2\langle h(X_o^*, Y_o^*), Z_o^* \rangle = \langle [Z, X]_o^*, Y_o^* \rangle + \langle X_o^*, [Z, Y]_o^* \rangle. \tag{2.3}$$

The mean curvature vector of $U'.o$ at o is defined by

$$H := -(1/k)\,\mathrm{tr}(h) = -(1/k)\sum h(E_i', E_i'),$$

where $\{E_i'\}$ is an orthonormal basis of $\mathfrak{m}'$, and k is the dimension of $U'.o$. We call $U'.o$ *minimal* if its mean curvature vector H is equal to zero. Note that we also call $U'.o$ minimal when the codimension of $U'.o$ is equal to zero. Let $l(\neq 0)$ be the codimension of $U'.o$, and $\{\xi_1', \dots, \xi_l'\}$ be the basis of $\mathfrak{m} \ominus \mathfrak{m}'$. Then $U'.o$ is minimal if and only if

$$\sum \langle h(E_i', E_i'), \xi_j' \rangle = 0 \tag{2.4}$$

for each $j = 1, \dots, l$.

3. Four-dimensional nilsolitons

Our goal of this section is to prove Theorem 1.1. We first recall that all four-dimensional simply-connected nilpotent Lie groups admit nilsolitons ([10]). After that we examine the minimality of the corresponding submanifold to each left-invariant Riemannian metric.

We discuss solvsolitons in the Lie algebra. First of all, let us recall the definition of a solvsoliton.

Definition 3.1. An inner product $\langle , \rangle$ on a solvable Lie algebra $\mathfrak{g}$ is called a *solvsoliton* if the Ricci operator satisfies

$$\mathrm{Ric}_{\langle , \rangle} = cI + D \qquad (\text{for some } c \in \mathbf{R} \text{ and } D \in \mathrm{Der}(\mathfrak{g})).$$

If $\mathfrak{g}$ is nilpotent, then a solvsoliton on $\mathfrak{g}$ is called a *nilsoliton*. We also recall a classification of four-dimensional nilpotent Lie algebras.

Proposition 3.2 ([16]). *Let $\mathfrak{g}$ be a four-dimensional nilpotent real Lie algebra. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:*

- $\mathbf{R}^4$, *an abelian Lie algebra,*
- $\mathfrak{h}_3 \oplus \mathbf{R} := \mathrm{span}\{e_1, \dots, e_4 \mid [e_1, e_2] = e_3\}$,
- $\mathfrak{n}_4 := \mathrm{span}\{e_1, \dots, e_4 \mid [e_1, e_2] = e_3, [e_1, e_3] = e_4\}$.

Note that $\mathfrak{h}_3 = \mathrm{span}\{e_1, e_2, e_3 \mid [e_1, e_2] = e_3\}$ is the Heisenberg Lie algebra.

In the abelian case, it is well known that there exists only one left-invariant Riemannian metric up to isometry and scaling, which is flat. Furthermore the corresponding submanifold coincides with the ambient space $\tilde{\mathfrak{M}}$, which is minimal.

For $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathbf{R}$ or $\mathfrak{n}_4$, let us denote by $\langle\, , \rangle_0$ the inner product on $\mathfrak{g}$ so that the above basis $\{e_1, \ldots, e_4\}$ is orthonormal. By Lauret, nilsolitons on four-dimensional Lie algebras have been classified.

Proposition 3.3 ([10]). *Let $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathbf{R}$ or $\mathfrak{n}_4$, and $\langle\, , \rangle$ be an inner product on $\mathfrak{g}$. Then the inner product $\langle\, , \rangle$ is a nilsoliton if and only if $\langle\, , \rangle \in [\langle\, , \rangle_0]$.*

Proposition 3.3 follows from the arguments about $\mathcal{N}_4$ in [10, Section 5]. Note that $\mathcal{N}_4$ is the set of all nilpotent Lie brackets on a four-dimensional real vector space. We also refer to [19, Table 2], a classification table of nilsolitons in four-dimensional cases.

Next we study the minimality of the corresponding submanifolds to nilsolitons on $\mathfrak{h}_3 \oplus \mathbf{R}$ and $\mathfrak{n}_4$. In the case of $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathbf{R}$, it is known that $\mathfrak{PM} = \{\mathrm{pt}\}$ ([8, 11]). Then the corresponding submanifold $[\langle\, , \rangle_0]$ coincides with the ambient space $\tilde{\mathfrak{M}}$, which is minimal.

Therefore we only need to consider the case of $\mathfrak{g} = \mathfrak{n}_4$. We first calculate $\mathrm{Der}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{g})$. Recall that they are defined by

$$\mathrm{Der}(\mathfrak{g}) = \{D \in \mathfrak{gl}(\mathfrak{g}) \mid D[\cdot\, , \cdot] = [D(\cdot), \cdot] + [\cdot, D(\cdot)]\},$$
$$\mathrm{Aut}(\mathfrak{g}) = \{\varphi \in \mathrm{GL}(\mathfrak{g}) \mid \varphi[\cdot\, , \cdot] = [\varphi(\cdot), \varphi(\cdot)]\}.$$

By direct calculations, one can obtain matrix expressions of $\mathrm{Der}(\mathfrak{n}_4)$ and $\mathrm{Aut}(\mathfrak{n}_4)$ with respect to the basis $\{e_1, \ldots, e_4\}$ as follows:

$$\mathrm{Der}(\mathfrak{n}_4) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ x_{31} & x_{43} & x_{11} + x_{22} & 0 \\ x_{41} & x_{42} & x_{43} & 2x_{11} + x_{22} \end{pmatrix} \right\},$$

$$\mathrm{Aut}(\mathfrak{n}_4) = \left\{ \begin{pmatrix} x_{11} & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ x_{31} & x_{32} & x_{11}x_{22} & 0 \\ x_{41} & x_{42} & x_{11}x_{32} & x_{11}^2 x_{22} \end{pmatrix} \middle|\, x_{11}, x_{22} \neq 0 \right\}. \tag{3.1}$$

Lemma 3.4. *Let $\mathfrak{g} = \mathfrak{n}_4$, Then the following $\mathfrak{U}$ is a system of representatives of $\mathfrak{PM}$:*

$$\mathfrak{U} = \left\{ g_{(\lambda_1, \lambda_2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda_2 & 1 \end{pmatrix} \middle|\, \lambda_1 > 0, \lambda_2 \in \mathbf{R} \right\}.$$

Proof. Take any $g \in GL_4(\mathbf{R})$. By Lemma 2.6, we only need to show that

$$\exists g_{(\lambda_1, \lambda_2)} \in \mathfrak{U} : g_{(\lambda_1, \lambda_2)} \in [[g]].$$

First of all, there exists $k \in O(4)$ such that

$$gk = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & 0 \\ a_4 & a_5 & a_6 & 0 \\ a_7 & a_8 & a_9 & a_{10} \end{pmatrix}, \qquad a_1, a_3, a_6, a_{10} > 0.$$

By (3.1), one has

$$\varphi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a_2/a_1 & 1 & 0 & 0 \\ -a_4/a_1 & 0 & 1 & 0 \\ (a_2a_8 - a_3a_7)/(a_1a_3) & -a_8/a_3 & 0 & 1 \end{pmatrix} \in \mathrm{Aut}(\mathfrak{n}_4).$$

This yields that

$$[[g]] \ni \varphi_1 gk = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_3 & 0 & 0 \\ 0 & a_5 & a_6 & 0 \\ 0 & 0 & a_9 & a_{10} \end{pmatrix}.$$

By (3.1), one can take

$$\varphi_2 = a_{10}/(a_1a_6) \begin{pmatrix} a_6/a_{10} & 0 & 0 & 0 \\ 0 & a_1/a_6 & 0 & 0 \\ 0 & 0 & a_1/a_{10} & 0 \\ 0 & 0 & 0 & a_1a_6/a_{10}^2 \end{pmatrix} \in \mathbf{R}^\times \mathrm{Aut}(\mathfrak{n}_4).$$

This yields that

$$[[g]] \ni \varphi_2\varphi_1 gk = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (a_3a_{10})/a_6^2 & 0 & 0 \\ 0 & a_5/a_6 & 1 & 0 \\ 0 & 0 & a_9/a_{10} & 1 \end{pmatrix}.$$

By (3.1), one has

$$\varphi_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -(a_5a_6)/(a_3a_{10}) & 1 & 0 \\ 0 & (a_5^2a_6^2)/(a_3^2a_{10}^2) & -(a_5a_6)/(a_3a_{10}) & 1 \end{pmatrix} \in \mathrm{Aut}(\mathfrak{n}_4).$$

This gives

$$[[g]] \ni \varphi_3\varphi_2\varphi_1 g k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (a_3a_{10})/a_6^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_9' & 1 \end{pmatrix}.$$

By putting $\lambda_1 := (a_3a_{10})/a_6^2 > 0$, and $\lambda_2 := a_9'$, we complete the proof. □

PROPOSITION 3.5. *Let* $\mathfrak{g} = \mathfrak{n}_4$. *Then* $\mathbf{R}^\times \operatorname{Aut}(\mathfrak{n}_4).\langle\,,\rangle_0$ *is the unique minimal orbit.*

PROOF. Take any $\langle\,,\rangle$. By Lemma 3.4, there exist $\lambda_1 > 0$, and $\lambda_2 \in \mathbf{R}$ such that

$$\mathbf{R}^\times \operatorname{Aut}(\mathfrak{n}_4).\langle\,,\rangle = \mathbf{R}^\times \operatorname{Aut}(\mathfrak{n}_4).(g_{(\lambda_1,\lambda_2)}.\langle\,,\rangle_0).$$

Let us define

$$U' := g_{(\lambda_1,\lambda_2)}^{-1}(\mathbf{R}^\times \operatorname{Aut}(\mathfrak{n}_4))g_{(\lambda_1,\lambda_2)}.$$

Then, since $g_{(\lambda_1,\lambda_2)}^{-1}$ gives an isometry of the space $\widetilde{\mathfrak{M}}$, one has an isometric congruence

$$\mathbf{R}^\times \operatorname{Aut}(\mathfrak{n}_4).(g_{(\lambda_1,\lambda_2)}.\langle\,,\rangle_0) \cong U'.\langle\,,\rangle_0.$$

Hence we have only to study $U'.\langle\,,\rangle_0$. Let $\mathfrak{u}'$ be the Lie algebra of U'. By the expression of $\mathbf{R} \oplus \operatorname{Der}(\mathfrak{n}_4)$, one can directly calculate

$$\begin{aligned}
\mathfrak{u}' &= g_{(\lambda_1,\lambda_2)}^{-1}(\mathbf{R} \oplus \operatorname{Der}(\mathfrak{n}_4))g_{(\lambda_1,\lambda_2)} \\
&= \left\{ \begin{pmatrix} r + x_{11} & 0 & 0 & 0 \\ (1/\lambda_1)x_{21} & r + x_{22} & 0 & 0 \\ x_{31} & \lambda_1 x_{43} & r + x_{11} + x_{22} & 0 \\ -\lambda_2 x_{31} + x_{41} & \lambda_1(-\lambda_2 x_{43} + x_{42}) & \lambda_2 x_{11} + x_{43} & r + 2x_{11} + x_{22} \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} r' & 0 & 0 & 0 \\ x_{21}' & -x_{11} + x_{22}' & 0 & 0 \\ x_{31} & \lambda_1 x_{43} & x_{22}' & 0 \\ x_{41}' & x_{42}' & \lambda_2 x_{11} + x_{43} & x_{11} + x_{22}' \end{pmatrix} \right\}.
\end{aligned}$$

Let us denote by E_{ij} the matrix whose (i, j)-entry is 1 and others are 0. It is easy to see that $\{E_1, \ldots, E_8\}$ given by the following is a basis of $\mathfrak{u}'$:

$$\begin{aligned}
&E_1 := E_{11}, \qquad E_2 := E_{22} + E_{33} + E_{44}, \qquad E_3 := E_{21}, \\
&E_4 := E_{31}, \qquad E_5 := E_{41}, \qquad E_6 := E_{42}, \qquad\qquad (3.2) \\
&E_7 := \lambda_1 E_{32} + E_{43}, \qquad E_8 := -E_{22} + \lambda_2 E_{43} + E_{44}.
\end{aligned}$$

Let us put

$$E_i' := (E_i)_o^* = (1/2)(E_i + {}^tE_i) \qquad (i = 1, \dots, 8).$$

Then $\{E_1', \dots, E_8'\}$ is a basis of $\mathfrak{m}' := d\pi_e(\mathfrak{u}') = \{(1/2)(X + {}^tX) \mid X \in \mathfrak{u}'\}$. We also define

$$\xi_1 := -E_{22} + 2E_{33} - E_{44},$$

$$\xi_2 := \lambda_1\lambda_2(E_{22} - E_{44}) - 4E_{32} + 4\lambda_1 E_{43},$$

and put

$$\xi_j' := (\xi_j)_0^* = (1/2)(\xi_j + {}^t\xi_i) \qquad (j = 1, 2).$$

As mentioned in Section 2, the inner product on $\mathfrak{m} := \mathrm{sym}(n)$ is given by

$$\langle X, Y \rangle := \mathrm{tr}(XY) \qquad (\text{for } X, Y \in \mathfrak{m}).$$

Hence, one can see that $\{\xi_1', \xi_2'\}$ is a basis of $\mathfrak{m} \ominus \mathfrak{m}'$. Here we take an orthonormal basis of $\mathfrak{m}'$. Let us put

$$X_1 := E_1, \qquad X_2 := (1/\sqrt{3})E_2, \qquad X_3 := \sqrt{2}E_3, \qquad X_4 := \sqrt{2}E_4,$$

$$X_5 := \sqrt{2}E_5, \qquad X_6 := \sqrt{2}E_6, \qquad X_7 := \sqrt{2/(1+\lambda_1^2)}E_7,$$

$$X_8 := T(-(\lambda_2/(1+\lambda_1^2))E_7 + E_8),$$

where $T = \sqrt{2(1+\lambda_1^2)/(\lambda_1^2\lambda_2^2 + 4(1+\lambda_1^2))}$. Furthermore we also put

$$X_i' := (X_i)_o^* = (1/2)(X_i + {}^tX_i) \qquad (i = 1, \dots, 8).$$

Since

$$\langle E_7', E_7' \rangle = (1+\lambda_1^2)/2, \qquad \langle E_7', E_8' \rangle = \lambda_2/2, \qquad \langle E_8', E_8' \rangle = (4+\lambda_2^2)/2,$$

one can see that $\{X_1', \dots, X_8'\}$ is an orthonormal basis of $\mathfrak{m}'$.

We show that $U'.\langle\ ,\ \rangle_0$ is minimal if and only if $(\lambda_1, \lambda_2) = (1, 0)$. By (2.4), $U'.\langle\ ,\ \rangle_0$ is minimal if and only if

$$\begin{cases} \sum \langle h(X_i', X_i'), \xi_1' \rangle = 0, \\ \sum \langle h(X_i', X_i'), \xi_2' \rangle = 0. \end{cases}$$

Our first claim is that

$$\sum \langle h(X_i', X_i'), \xi_1' \rangle = 0 \qquad \text{if and only if } \lambda_1 = 1. \tag{3.3}$$

We calculate $\sum \langle h(X_i', X_i'), \xi_1' \rangle$. By direct calculations, one has

$$[\xi_1, E_7] = 3(\lambda_1 E_{32} - E_{43}), \qquad [\xi_1, E_8] = -3\lambda_2 E_{43}.$$

Then, we obtain the bracket products $[\xi_1, X_i]$ as follows:

$$[\xi_1, X_3] = -X_3, \qquad [\xi_1, X_4] = 2X_4, \qquad [\xi_1, X_5] = -X_5,$$

$$[\xi_1, X_7] = 3\sqrt{2/(1+\lambda_1^2)}(\lambda_1 E_{32} - E_{43}),$$

$$[\xi_1, X_8] = (-3\lambda_1\lambda_2 T/(1+\lambda_1^2))(E_{32} + \lambda_1 E_{43}),$$

and others are equal to zero. These yield that

$$\langle [\xi_1, X_3]_o^*, (X_3)_o^* \rangle = \langle -X_3', X_3' \rangle = -1,$$

$$\langle [\xi_1, X_4]_o^*, (X_4)_o^* \rangle = \langle 2X_4', X_4' \rangle = 2,$$

$$\langle [\xi_1, X_5]_o^*, (X_5)_o^* \rangle = \langle -X_5', X_5' \rangle = -1,$$

$$\begin{aligned}\langle [\xi_1, X_7]_o^*, (X_7)_o^* \rangle &= 3\sqrt{2/(1+\lambda_1^2)}\langle \lambda_1 (E_{32})_o^* - (E_{43})_o^*, X_7' \rangle \\ &= 3(\lambda_1^2 - 1)/(1+\lambda_1^2), \\ \langle [\xi_1, X_8]_o^*, (X_8)_o^* \rangle &= (-3\lambda_1\lambda_2 T/(1+\lambda_1^2))\langle (E_{32})_o^* + \lambda_1 (E_{43})_o^*, X_8' \rangle \\ &= -3\lambda_1^2\lambda_2^2(\lambda_1^2 - 1)/((1+\lambda_1^2)(\lambda_1^2\lambda_2^2 + 4(1+\lambda_1^2))).\end{aligned}$$

Therefore, by (2.3), we obtain

$$\sum \langle h(X_i', X_i'), \xi_1' \rangle = \frac{12(\lambda_1^2 - 1)}{\lambda_1^2\lambda_2^2 + 4(1+\lambda_1^2)}.$$

Since $\lambda_1 > 0$, this yields our first claim (3.3).

Our second claim is, under the assumption $\lambda_1 = 1$, that

$$\sum \langle h(X_i', X_i'), \xi_2' \rangle = 0 \qquad \text{if and only if } \lambda_2 = 0. \tag{3.4}$$

From now on we assume $\lambda_1 = 1$. Then, note that

$$X_7 = E_7 = E_{32} + E_{43},$$

$$X_8 = 2/\sqrt{\lambda_2^2 + 8}((-\lambda_2/2)E_7 + E_8),$$

$$\xi_2 = \lambda_2(E_{22} - E_{44}) - 4E_{32} + 4E_{43}.$$

We calculate $\sum \langle h(X_i', X_i'), \xi_2' \rangle$. By direct calculations, we have

$$[\xi_2, X_3] = \lambda_2 X_3 - 4X_4, \qquad [\xi_2, X_4] = 4X_5,$$
$$[\xi_2, X_5] = -\lambda_2 X_5, \qquad [\xi_2, X_6] = -2\lambda_2 X_6,$$
$$[\xi_2, X_7] = 8E_6 - \lambda_2 E_7 = 4\sqrt{2}X_6 - \lambda_2 X_7,$$
$$[\xi_2, X_8] = \sqrt{\lambda_2^2 + 8}(E_{32} - E_{43}),$$

and others are equal to zero. These yield that

$$\langle [\xi_2, X_3]_o^*, (X_3)_o^* \rangle = \langle \lambda_2 X_3' - 4X_4', X_3' \rangle = \lambda_2,$$
$$\langle [\xi_2, X_4]_o^*, (X_4)_o^* \rangle = \langle 4X_5', X_4' \rangle = 0,$$
$$\langle [\xi_2, X_5]_o^*, (X_5)_o^* \rangle = \langle -\lambda_2 X_5', X_5' \rangle = -\lambda_2,$$
$$\langle [\xi_2, X_6]_o^*, (X_6)_o^* \rangle = \langle -2\lambda_2 X_6', X_6' \rangle = -2\lambda_2,$$
$$\langle [\xi_2, X_7]_o^*, (X_7)_o^* \rangle = \langle 4\sqrt{2}X_6' - \lambda_2 X_7', X_7' \rangle = -\lambda_2,$$
$$\begin{aligned}\langle [\xi_2, X_8]_o^*, (X_8)_o^* \rangle &= \sqrt{\lambda_2^2 + 8}\langle (E_{32})_o^* - (E_{43})_o^*, X_8' \rangle \\ &= 2\langle (E_{32})_o^* - (E_{43})_o^*, (-\lambda_2/2)E_7' + E_8' \rangle = -\lambda_2.\end{aligned}$$

We thus obtain $\sum \langle h(X_i', X_i'), \xi_2' \rangle = -4\lambda_2$. This yields our second claim (3.4). This completes the proof of (2). □

By (3.2), all orbits of the action of $\mathbf{R}^\times \operatorname{Aut}(\mathfrak{n}_4)$ have dimension eight. Hence this action is of cohomogeneity two, and has no singular orbits.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Let us consider the four-dimensional solvable Lie algebra

$$\mathfrak{s}_4 := \operatorname{span}\{e_1, \ldots, e_4 \mid [e_1, e_2] = e_2, [e_1, e_3] = -e_3, [e_2, e_3] = e_4\}.$$

We will write $\langle , \rangle_0$ the inner product on $\mathfrak{s}_4$ so that the above basis is orthonormal.

We first study $\operatorname{Der}(\mathfrak{s}_4)$ and $\operatorname{Aut}(\mathfrak{s}_4)$ with respect to the above basis $\{e_1, \ldots, e_4\}$. By direct calculations, we have

$$\operatorname{Der}(\mathfrak{s}_4) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ -x_{43} & x_{22} & 0 & 0 \\ -x_{42} & 0 & x_{33} & 0 \\ x_{41} & x_{42} & x_{43} & x_{22} + x_{33} \end{pmatrix} \right\}, \qquad (4.1)$$

$$\mathrm{Aut}(\mathfrak{s}_4) \supset \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ x_{31} & 0 & x_{33} & 0 \\ x_{41} & -x_{31}x_{22} & -x_{21}x_{33} & x_{22}x_{33} \end{pmatrix} \middle| x_{22}, x_{33} \neq 0 \right\}. \quad (4.2)$$

PROPOSITION 4.1. *Let* $\mathfrak{g} = \mathfrak{s}_4$, *Then the following* $\mathfrak{U}$ *is a system of representatives of* $\mathfrak{PM}$*:*

$$\mathfrak{U} = \left\{ g_{(\lambda_1, \lambda_2, \lambda_3, \lambda_4)} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & \lambda_3 & \lambda_4 & 1 \end{pmatrix} \middle| \lambda_1 > 0,\ \lambda_2, \lambda_3, \lambda_4 \in \mathbf{R} \right\}.$$

PROOF. Take any $g \in \mathrm{GL}_4(\mathbf{R})$. By Lemma 2.6, we only need to show that

$$\exists g_{(\lambda_1, \lambda_2, \lambda_3, \lambda_4)} \in \mathfrak{U} : g_{(\lambda_1, \lambda_2, \lambda_3, \lambda_4)} \in [[g]].$$

One knows there exists $k \in \mathrm{O}(4)$ such that

$$gk = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & 0 \\ a_4 & a_5 & a_6 & 0 \\ a_7 & a_8 & a_9 & a_{10} \end{pmatrix}, \qquad a_1, a_3, a_6, a_{10} > 0.$$

By (4.2), we can take

$$\varphi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a_2/a_1 & 1 & 0 & 0 \\ -a_4/a_1 & 0 & 1 & 0 \\ A & a_4/a_1 & a_2/a_1 & 1 \end{pmatrix} \in \mathrm{Aut}(\mathfrak{s}_4),$$

where $A = (-a_1 a_7 - 2a_2 a_4)/a_1^2$. This yields that

$$[[g]] \ni \varphi_1 g k = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_3 & 0 & 0 \\ 0 & a_5 & a_6 & 0 \\ 0 & a_8' & a_9' & a_{10} \end{pmatrix}.$$

Furthermore, from (4.2), one can take

$$\varphi_2 = a_{10}/(a_3 a_6) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_6/a_{10} & 0 & 0 \\ 0 & 0 & a_3/a_{10} & 0 \\ 0 & 0 & 0 & (a_3 a_6)/a_{10}^2 \end{pmatrix} \in \mathbf{R}^\times \mathrm{Aut}(\mathfrak{s}_4).$$

This gives us that

$$[[g]] \ni \varphi_2\varphi_1 g k = \begin{pmatrix} (a_1a_{10})/(a_3a_6) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a_5/a_6 & 1 & 0 \\ 0 & a_8'/a_{10} & a_9'/a_{10} & 1 \end{pmatrix}.$$

By putting $\lambda_1 := (a_1a_{10})/(a_3a_6) > 0$, $\lambda_2 = a_5/a_6$, $\lambda_3 = a_8'/a_{10}$, and $\lambda_4 = a_9'/a_{10}$, we complete the proof. □

By Lauret [14], solvsolitons on four-dimensional simply-connected solvable Lie groups have been classified, and it is known that $\mathfrak{s}_4$ admits a solvsoliton.

Proposition 4.2. *An inner product* $\langle\,,\rangle$ *on* $\mathfrak{s}_4$ *is a solvsoliton if and only if* $[\langle\,,\rangle] = [g_{(\sqrt{3}/2,0,0,0)}.\langle\,,\rangle_0]$.

Proof. It has been proved by Lauret that a given solvable Lie algebra can admit at most one solvsoliton up to isometry and scaling ([14, Theorem 5.1]). Hence it is sufficient to show that $g_{(\sqrt{3}/2,0,0,0)}.\langle\,,\rangle_0$ is a solvsoliton. Here, recall that $g_{(\sqrt{3}/2,0,0,0)}.\langle\,,\rangle_0$ is an inner product so that $\{(\sqrt{3}/2)e_1, e_2, e_3, e_4\}$ is orthonormal. Note that the nilradical of $\mathfrak{s}_4$ coincides with the Heisenberg Lie algebra $\mathfrak{h}_3 = \mathrm{span}\{e_2, e_3, e_4\}$, and $\mathfrak{s}_4 = \mathbf{R}e_1 \oplus \mathfrak{h}_3$ is the orthogonal decomposition with respect to $g_{(\sqrt{3}/2,0,0,0)}.\langle\,,\rangle_0$. Then, by Lauret's theorem ([14, Theorem 4.8]), we only need to show that $g_{(\sqrt{3}/2,0,0,0)}.\langle\,,\rangle_0$ satisfies the following conditions:

(1) $(\mathfrak{h}_3, g_{(\sqrt{3}/2,0,0,0)}.\langle\,,\rangle_0|_{\mathfrak{h}_3\times\mathfrak{h}_3})$ is a nilsoliton with Ricci operator $\mathrm{Ric} = cI + D$, for some $c < 0$ and $D \in \mathrm{Der}(\mathfrak{h}_3)$.
(2) $[e_1, e_1] = 0$.
(3) ${}^t(\mathrm{ad}\, e_1) \in \mathrm{Der}(\mathfrak{s}_4)$.
(4) $g_{(\sqrt{3}/2,0,0,0)}.\langle e_1, e_1\rangle_0 = -(1/c)\,\mathrm{tr}\{(1/2)(\mathrm{ad}\, e_1 + {}^t(\mathrm{ad}\, e_1))\}^2$.

By direct calculations, we obtain that $g_{(\sqrt{3}/2,0,0,0)}.\langle\,,\rangle_0|_{\mathfrak{h}_3\times\mathfrak{h}_3}$ is a nilsoliton on $\mathfrak{h}_3$ with $c = -3/2$, namely Condition (1) holds. It is obvious that Conditions (2) and (3) hold. By direct calculations, we obtain Condition (4). □

To prove Theorem 1.2, we consider $g_{(t,0,0,0)}.\langle\,,\rangle_0$ for $t > 0$, which is a curve through $\langle\,,\rangle_0$ and $g_{(\sqrt{3}/2,0,0,0)}.\langle\,,\rangle_0$.

Proposition 4.3. *Let* $\mathfrak{g} = \mathfrak{s}_4$, *and* $t > 0$. *Then* $[g_{(t,0,0,0)}.\langle\,,\rangle_0]$ *is minimal if and only if* $t = 1$.

Proof. Let us define

$$U' := g_{(t,0,0,0)}^{-1}(\mathbf{R}^\times\,\mathrm{Aut}(\mathfrak{s}_4))g_{(t,0,0,0)}.$$

Then, since $g_{(t,0,0,0)}^{-1}$ gives an isometry, we have an isometric congruence

$$[g_{(t,0,0,0)}.\langle\,,\,\rangle_0] = \mathbf{R}^\times \operatorname{Aut}(\mathfrak{s}_4).(g_{(t,0,0,0)}.\langle\,,\,\rangle_0) \cong U'.\langle\,,\,\rangle_0.$$

Hence we have only to study $U'.\langle\,,\,\rangle_0$. Let $\mathfrak{u}'$ be the Lie algebra of U'. By (4.1), we have

$$\mathfrak{u}' := \left\{ \begin{pmatrix} r & 0 & 0 & 0 \\ -tx_{43} & r + x_{22} & 0 & 0 \\ -tx_{42} & 0 & r + x_{33} & 0 \\ tx_{41} & x_{42} & x_{43} & r + x_{22} + x_{33} \end{pmatrix} \right\}.$$

We take a basis $\{X_1, \dots, X_6\}$ of $\mathfrak{u}'$ as follows:

$$X_1 := (1/2)(E_{11} + E_{22} + E_{33} + E_{44}), \qquad X_2 := (1/2)(E_{11} + E_{22} - E_{33} - E_{44}),$$

$$X_3 := (1/2)(E_{11} - E_{22} + E_{33} - E_{44}), \qquad X_4 := (\sqrt{2/(t^2+1)})(-tE_{21} + E_{43}),$$

$$X_5 := (\sqrt{2/(t^2+1)})(-tE_{31} + E_{42}), \qquad X_6 := \sqrt{2}E_{41}.$$

Let us put

$$X_i' := (X_i)_o^* = (1/2)(X_i + {}^tX_i) \qquad (i = 1, \dots, 6).$$

Then $\{X_1', \dots, X_6'\}$ is an orthonormal basis of $\mathfrak{m}' = d\pi_e(\mathfrak{u}')$. Furthermore we take

$$\xi_1 := E_{11} - E_{22} - E_{33} + E_{44}, \qquad \xi_2 := E_{21} + tE_{43},$$

$$\xi_3 := E_{31} + tE_{42}, \qquad \xi_4 := E_{32},$$

and put

$$\xi_j' := (\xi_j)_o^* = (1/2)(\xi_j + {}^t\xi_j) \qquad (j = 1, \dots, 4).$$

Then $\{\xi_1', \dots, \xi_4'\}$ is a basis of $\mathfrak{m} \ominus \mathfrak{m}'$.

We prove that $[g_{(t,0,0,0)}.\langle\,,\,\rangle_0]$ is minimal if and only if $t = 1$. By (2.4), recall that $[g_{(t,0,0,0)}.\langle\,,\,\rangle_0]$ is minimal if and only if

$$\sum \langle h(X_i', X_i'), \xi_j' \rangle = 0$$

for each $j = 1, \dots, 4$.

Our claim is

$$\sum \langle h(X_i', X_i'), \xi_1' \rangle = 0 \qquad \text{if and only if } t = 1. \tag{4.3}$$

We calculate $\sum \langle h(X_i', X_i'), \xi_1' \rangle$. By direct calculations, we have

$$[\xi_1, X_4] = (2\sqrt{2/(t^2+1)})(tE_{21} + E_{43}),$$
$$[\xi_1, X_5] = (2\sqrt{2/(t^2+1)})(tE_{31} + E_{42}),$$

and others are equal to zero. Therefore, we have

$$\begin{aligned}\langle [\xi_1, X_4]_o^*, (X_4)_o^* \rangle &= (2\sqrt{2/(t^2+1)})\langle t(E_{21})_o^* + (E_{43})_o^*, X_4' \rangle \\ &= 2(1-t^2)/(1+t^2), \\ \langle [\xi_1, X_5]_o^*, (X_5)_o^* \rangle &= (2\sqrt{2/(t^2+1)})\langle t(E_{31})_o^* + (E_{42})_o^*, X_5' \rangle \\ &= 2(1-t^2)/(1+t^2).\end{aligned}$$

We thus obtain

$$\sum \langle h(X_i', X_i'), \xi_1' \rangle = 4(1-t^2)/(1+t^2).$$

Since $t > 0$, this yields (4.3).

We assume $t = 1$ from now on. Then, it is sufficient to show that

$$\sum \langle h(X_i', X_i'), \xi_j' \rangle = 0 \tag{4.4}$$

for each $j = 2, 3, 4$. Note that, when $t = 1$,

$$X_4 = -E_{21} + E_{43}, \qquad X_5 = -E_{31} + E_{42},$$
$$\xi_2 = E_{21} + E_{43}, \qquad \xi_3 = E_{31} + E_{42}.$$

We calculate $\sum \langle h(X_i', X_i'), \xi_j' \rangle$ for $j = 2, 3, 4$. The bracket products are given by

$$[\xi_2, X_3] = \xi_2, \qquad [\xi_2, X_5] = -\sqrt{2}X_6,$$
$$[\xi_3, X_2] = \xi_3, \qquad [\xi_3, X_4] = -\sqrt{2}X_6,$$
$$[\xi_4, X_2] = \xi_4, \qquad [\xi_4, X_3] = -\xi_4, \qquad [\xi_4, X_4] = -\xi_3,$$

and others are equal to zero. We thus obtain that

$$\langle [\xi_j, X_i]_o^*, (X_i)_o^* \rangle = 0,$$

for any $i = 1, \dots, 6$ and $j = 2, 3, 4$. These yield (4.4), and we complete the proof. $\square$

The next theorem follows from Propositions 4.2 and 4.3, immediately.

Theorem 4.4. *We have the following:*

(1) *Let* $\langle\,,\rangle = \langle\,,\rangle_0$. *Then* $\langle\,,\rangle$ *is not a solvsoliton, and the corresponding submanifold* $[\langle\,,\rangle]$ *is minimal.*

(2) *Let* $\langle\,,\rangle = g_{((\sqrt{3}/2,0,0,0))}\langle\,,\rangle_0$. *Then* $\langle\,,\rangle$ *is a solvsoliton, and the corresponding submanifold* $[\langle\,,\rangle]$ *is not minimal.*

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