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C^ℓ-contact and C^ℓ-right equivalences of real semi-quasihomogeneous C^ℓ function germs

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Abstract. In this paper we investigate the C^ℓ versions of contact and right equivalences of real semi-quasihomogeneous C^ℓ function germs, $1 \le \ell \le \infty$. The C^ℓ-right equivalence implies C^ℓ-contact equivalence for any $1 \le \ell \le \infty$ and in this work we show, up to certain conditions, that for semi-quasihomogeneous C^ℓ function germs the converse is also true (Theorem 1). As a consequence, concerning the particular case of quasihomogeneous C^ℓ function germs, we also have a similar result (Corollary 1) which recover a known result of M. Takahashi [14] for $\ell = \infty$. We note that we are considering semi-quasihomogeneous function germs with no additional hypothesis of isolated singularity at zero.

1. Introduction

For any ℓ with $1 \le \ell \le \infty$, two C^ℓ function germs $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ are:

- C^ℓ*-right equivalent* if there exists a C^ℓ-diffeomorphism germ $h : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ such that $g = f \circ h$.
- C^ℓ*-contact equivalent* if there exist a C^ℓ-diffeomorphism germ $h : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a non-zero C^ℓ function germ $M : (\mathbf{R}^n, 0) \to \mathbf{R}$, with $M(0) \neq 0$, such that $g = M \cdot f \circ h$.

These two equivalence relations are denoted by C^ℓ-$\mathscr{R}$ and C^ℓ-$\mathscr{K}$ equivalences, respectively. Also, when $l = \infty$ we just write $\mathscr{R}$ instead of C^∞-$\mathscr{R}$ and $\mathscr{K}$ instead of C^∞-$\mathscr{K}$, respectively.

It is easy to see that C^ℓ-$\mathscr{R}$-equivalence implies C^ℓ-$\mathscr{K}$-equivalence, but the converse does not hold in general. For instance, if $\ell = \infty$, the germs $f(x) = x^2$ and $g(x) = -x^2$ are $\mathscr{K}$-equivalent but they are not $\mathscr{R}$-equivalent. Hence, it seems an important problem to clarify the relationship between C^ℓ-$\mathscr{R}$ and C^ℓ-$\mathscr{K}$ equivalences. Recently, this subject was studied by some authors when $l = \infty$ and for the class of quasihomogeneous C^∞ function germs (cf.

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[14], [1], [2], [3]). However, there are few results investigating the C^ℓ-versions of these two equivalence relations for C^ℓ function germs, $1 \le \ell < \infty$. Kuiper in [6] studied the C^1-$\mathcal{R}$-equivalence of functions near isolated critical points. Bromberg and Medrano in [5] treated the C^ℓ-$\mathcal{R}$-sufficiency of quasihomogeneous functions. In [11], Ruas and Saia gave estimates for the degree of C^ℓ-$\mathcal{R}$ and C^ℓ-$\mathcal{K}$ determinacy of quasihomogeneous C^∞ map germs.

In this paper we consider the class of semi-quasihomogeneous C^ℓ function germs. The main result is Theorem 1, where we give a sufficient condition under which C^ℓ-$\mathcal{K}$-equivalence implies C^ℓ-$\mathcal{R}$-equivalence for C^ℓ function germs. This result is inspired in the Takahashi's paper [14]. As a consequence, concerning the particular case of quasihomogeneous C^ℓ function germs, $1 \le \ell \le \infty$, we also have a similar result (Corollary 1) which recover the Takahashi's result ([14, Theorem 1.1 p. 830]) when $\ell = \infty$.

2. Definitions and notations

For any ℓ with $1 \le \ell \le \infty$ denote by $\mathcal{E}_n^{[\ell]}$ the set of all germs of C^ℓ functions $(\mathbf{R}^n, 0) \to \mathbf{R}$. We shall not distinguish between germs and representative functions. Denote by $m_n^{[\ell]} = \{f \in \mathcal{E}_n^{[\ell]} \mid f(0) = 0\}$. Given a function germ $f \in \mathcal{E}_n^{[\ell]}$, Jf denotes the Jacobian ideal of f.

The C^ℓ-$\mathcal{R}$-equivalence between two C^ℓ function germs f and g will be denoted by $f \overset{C^\ell-\mathcal{R}}{\sim} g$, while the C^ℓ-$\mathcal{K}$-equivalence of them will be denoted by $f \overset{C^\ell-\mathcal{K}}{\sim} g$.

A C^ℓ function germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ is called *quasihomogeneous* of type $(r_1, \dots, r_n; d)$ if it satisfies the following equation

$$f(\lambda \cdot x) = \lambda^d f(x_1, \dots, x_n)$$

for all $\lambda > 0$ and $x = (x_1, \dots, x_n)$, where $\lambda \cdot x = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n)$.

With respect to the given weights $(r_1, \dots, r_n)$, for each monomial $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, where $\alpha = (\alpha_1, \dots, \alpha_n)$, call $\operatorname{fil}(x^\alpha) = \sum_{i=1}^n \alpha_i r_i$. A *filtration* in $\mathcal{E}_n^{[\ell]}$ is defined via the function

$$\operatorname{fil}(f) = \min\left\{ \operatorname{fil}(x^\alpha) \;\middle|\; \frac{\partial^\alpha f}{\partial x^\alpha}(0) \neq 0 \right\}$$

for each $f \in \mathcal{E}_n^{[\ell]}$.

Definition 1. A C^ℓ function germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ is called *semi-quasihomogeneous* of type $(r_1, \dots, r_n; d)$ if $f = q + \phi$ where $q(x_1, \dots, x_n)$ is a quasihomogeneous C^ℓ function germ of type $(r_1, \dots, r_n; d)$ and ϕ is a C^ℓ function germ with $\operatorname{fil}(\phi) > \operatorname{fil}(f)$.

We remark that a quasihomogeneous germ is clearly semi-quasihomogeneous, because the filtration of null polynomial is equal to infinity. Notice also that we are not considering any finite determinacy condition for both functions f and q.

From [11, Lemma 2] and [12, Lemma 2.1] it is possible to characterize a large class of C^ℓ functions of type $\frac{f_1}{f_2}$, where f_1 and f_2 are quasihomogeneous polynomial function germs.

EXAMPLE 1. *The germ* $f(x, y) = x^6 - y^{12} + \frac{x^{10}}{x^4+y^8}$ *is quasihomogeneous of type* $(2, 1; 12)$ *of class* C^5.

EXAMPLE 2. *The germ* $f(x, y) = \frac{x^{10}}{x^4+y^8} \cdot \frac{x^{10}}{x^4+y^8}$ *is quasihomogeneous of type* $(2, 1; 24)$ *of class* C^{11}.

EXAMPLE 3. *Let* $f(x, y) = \frac{x^{10}}{x^4+y^8} + \left(\frac{x^{12}}{x^4+y^8} + x^{14} + x^{10}y^8\right)$. *This germ is semi-quasihomogeneous of type* $(2, 1; 12)$ *of class* C^5. *Here* $f = q + \phi$ *with* $q(x, y) = \frac{x^{10}}{x^4+y^8}$, $\phi(x, y) = \frac{x^{12}}{x^4+y^8} + x^{14} + x^{10}y^8$ *and* $fil(\phi) = fil\left(\frac{x^{12}}{x^4+y^8} + x^{14} + x^{10}y^8\right) = 16$ *because for the fixed weights* $(2, 1)$, $fil(x^{14} + x^{10}y^8) = 28$.

3. Main results

The main result of this paper is the following:

THEOREM 1. *Let* $f, g \in \mathscr{E}_n^{[\ell]}$ *which are* C^ℓ*-contact equivalent. Suppose that* $f = q + \phi$ *is a semi-quasihomogeneous germ such that the following conditions are satisfied:*

(1) *For all* $i = 1, \dots, n$, *there exist* C^ℓ *function germs* $b_i^j \in m_n^{[\ell]}$ *such that*

$$\frac{\partial \phi}{\partial x_i}(x) = \sum_{j=1}^{n} b_i^j(x) \frac{\partial q}{\partial x_j}(x).$$

(2) *The germ* ϕ *can be written as* $\phi(x) = \sum_{j=1}^{n} a_j(x) \frac{\partial q}{\partial x_j}(x)$ *with* a_j *in* $m_n^{[\ell]}$, *for all* $j = 1, \dots, n$.

Under these conditions, f *is* C^ℓ*-*$\mathscr{R}$*-equivalent to* g *or* C^ℓ*-*$\mathscr{R}$*-equivalent to* $-g$.

When $f \overset{C^\ell - \mathscr{R}}{\sim} g$ or $f \overset{C^\ell - \mathscr{R}}{\sim} -g$ we will denote it by $f \overset{C^\ell - \mathscr{R}_\pm}{\sim} g$.

PROOF OF THEOREM 1. The proof of Theorem 1 follows the strategy given by Takahashi (cf. [14, Theorem 1.1]) in the case where $\ell = \infty$ and f is a quasihomogeneous C^∞ function germ.

STEP 1. Consider $g = M \cdot f \circ h$, by hypothesis.

STEP 2.

Lemma 1. *Suppose that f is a semi-quasihomogeneous C^ℓ function germ as in Theorem* 1. *Then, for any non-zero constant $c \in \mathbf{R}$, $c \cdot f \overset{C^\ell-\mathscr{R}_\pm}{\sim} f$.*

STEP 3. For any C^ℓ-diffeomorphism $h : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ we have that $c \cdot f$ and $(c \cdot f) \circ h = c \cdot (f \circ h)$ are C^ℓ-$\mathscr{R}$-equivalent.

STEP 4.

Lemma 2. *Suppose that $f : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ is a semi-quasihomogeneous C^ℓ function germ satisfying the hypotheses of Theorem* 1 *and let $M : (\mathbf{R}^n, 0) \to \mathbf{R}$ be a C^ℓ function germ with $M(0) \neq 0$. Then $M \cdot f$ and $M(0) \cdot f$ are C^ℓ-$\mathscr{R}$-equivalent.*

Now we conclude the proof of Theorem 1. As f is semi-quasihomogeneous, it follows from STEP 2 that for any non-zero real constant $c = M(0) \neq 0$,

$$f \overset{C^\ell-\mathscr{R}_\pm}{\sim} M(0) \cdot f.$$

From STEP 3, $M(0) \cdot f \overset{C^\ell-\mathscr{R}}{\sim} M(0) \cdot f \circ h$ and from STEP 4, $M(0) \cdot f \circ h \overset{C^\ell-\mathscr{R}}{\sim} M \cdot f \circ h$. Therefore

$$f \overset{C^\ell-\mathscr{R}_\pm}{\sim} M(0) \cdot f \overset{C^\ell-\mathscr{R}}{\sim} M(0) \cdot f \circ h \overset{C^\ell-\mathscr{R}}{\sim} M \cdot f \circ h,$$

that is,

$$f \overset{C^\ell-\mathscr{R}_\pm}{\sim} M \cdot f \circ h.$$

As $M \cdot f \circ h = g$ (STEP 1), Theorem 1 is proved. □

The Lemmas 1 and 2 will be proved in the next section.

If we consider f just a quasihomogeneous C^ℓ function germ then we have the following consequence:

Corollary 1. *Let $1 \leq \ell \leq \infty$. If $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ are C^ℓ function germs which are C^ℓ-$\mathscr{K}$-equivalent and one is quasihomogeneous then f is C^ℓ-$\mathscr{R}$-equivalent to g or C^ℓ-$\mathscr{R}$-equivalent to $-g$.*

Remark 1. *Notice if f is just a quasihomogeneous C^ℓ function germ, then the two hypotheses of Theorem* 1 *are trivially satisfied.*

We also recover the result of Takahashi [14] for the special case of C^∞ function germs:

Corollary 2 ([14, Theorem 1.1 p. 830]). *If $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ are C^∞ function germs which are $\mathscr{K}$-equivalent and one is quasihomogeneous then f is $\mathscr{R}$-equivalent to g or $\mathscr{R}$-equivalent to $-g$.*

Remark 2. *For C^∞ map germs, replacing C^ℓ-right equivalence by C^ℓ-right-left equivalence, Nishimura* [9] *proposed a systematic method for giving some affirmative answers to the C^ℓ recognition problem on C^ℓ-right-left equivalence of two given C^∞ map germs which are C^ℓ-contact equivalent,* $1 \leq \ell \leq \infty$.

4. Proof of the Lemmas

Proof of Lemma 1. Consider the family $F_t = q + t\phi$, where q and ϕ are as in Theorem 1. Without loss of generality, we assume the weights $r_1 \leq r_2 \leq \cdots \leq r_n$. Then, $F_0 = q$, $F_1 = q + \phi = f$ and $\frac{\partial F_t}{\partial t} = \phi$.

We construct a vector field $\xi : (\mathbf{R}^n \times [0,1], 0) \to (\mathbf{R}^n, 0)$ satisfying the equation

$$-\frac{\partial F_t}{\partial t} = \frac{\partial F_t}{\partial x} \circ \xi, \qquad \xi_i(0,t) = 0,\ i = 1, \dots, n. \tag{1}$$

This is equivalent to

$$\sum_{i=1}^{n} \frac{\partial F_t}{\partial x_i}(x,t)\xi_i(x,t) + \frac{\partial F_t}{\partial t}(x,t) = 0$$

which in turn is equivalent to

$$\sum_{i=1}^{n} \xi_i(x,t)\left[\frac{\partial q}{\partial x_i}(x) + t\frac{\partial \phi}{\partial x_i}(x)\right] + \phi(x) = 0. \tag{2}$$

Using the hypotheses (1) and (2) in Theorem 1 we can rewrite (2) as

$$\sum_{i=1}^{n} \xi_i(x,t)\left[\frac{\partial q}{\partial x_i}(x) + t\sum_{j=1}^{n} b_i^j(x)\frac{\partial q}{\partial x_j}(x)\right] + \sum_{j=1}^{n} a_j(x)\frac{\partial q}{\partial x_j}(x) = 0, \tag{3}$$

where $a_j, b_i^j \in m_n^{[\ell]}$, for all $i, j = 1, \dots, n$. Notice that the coefficient of $\dfrac{\partial q}{\partial x_i}$ is equal to

$$t\sum_{k=1}^{n} b_k^i \xi_k + \xi_i + a_i, \qquad \forall i = 1, \dots, n.$$

If ξ satisfies the following condition

$$A(x,t)\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{pmatrix}, \tag{4}$$

where the matrix A, omitting the variables, is given by

$$A = \begin{pmatrix} (1+tb_1^1) & tb_2^1 & \cdots & tb_n^1 \\ tb_1^2 & (1+tb_2^2) & \cdots & tb_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ tb_1^n & tb_2^n & \cdots & (1+tb_n^n) \end{pmatrix},$$

then there exists a vector field ξ such that (1) holds.

Since $a_j(0) = 0$ and also $b_i^j(0) = 0$ for all $i, j = 1, \dots, n$ (because $a_k, b_i^j \in m_n^{[\ell]}$), evaluating the matrix $A(x,t)$ in $(0,t)$ one has $\det A(0,t) \neq 0$ and Equation (4) can be solved with respect to $\xi_i(x,t)$.

Remark that $\xi_i(0,t) = 0$ and by Thom-Levine type Theorem [10], the family F_t is C^ℓ-$\mathcal{R}$-trivial and then f is C^ℓ-$\mathcal{R}$-equivalent to q. Consequently, $c \cdot f$ and $c \cdot q$ are C^ℓ-$\mathcal{R}$-equivalent for any non-zero constant $c \in \mathbf{R}$.

As q is quasihomogeneous, it is an immediate consequence of the Euler relation that $c \cdot q \overset{C^\ell-\mathcal{R}_\pm}{\sim} q$ for a non-zero constant $c \in \mathbf{R}$.

Then

$$f \overset{C^\ell-\mathcal{R}}{\sim} q \overset{C^\ell-\mathcal{R}_\pm}{\sim} c \cdot q \overset{C^\ell-\mathcal{R}}{\sim} c \cdot f.$$

That is, $c \cdot f \overset{C^\ell-\mathcal{R}_\pm}{\sim} f$ for any non-zero constant $c \in \mathbf{R}$, as required. □

Remark 3. *In the particular case where f is a quasihomogeneous C^ℓ function germ, $1 \leq \ell \leq \infty$, the Lemma 1 is trivial thanks to Euler relation and thus it is unnecessary.*

Proof of Lemma 2. Write $f = q + \phi$ where $q(x_1, \dots, x_n)$ is a quasihomogeneous C^ℓ function germ of type $(r_1, \dots, r_n; d)$ with $r_1 \leq r_2 \leq \cdots \leq r_n$ and ϕ is a C^ℓ function germ with $\mathrm{fil}(\phi) > \mathrm{fil}(f)$, then

$$f(\lambda^{r_1}x_1, \dots, \lambda^{r_n}x_n) = \lambda^d q(x_1, \dots, x_n) + \phi(\lambda^{r_1}x_1, \dots, \lambda^{r_n}x_n). \tag{5}$$

From the hypothesis $\dfrac{\partial \phi}{\partial x_i}(x) = \sum_{j=1}^n b_i^j(x) \dfrac{\partial q}{\partial x_j}(x)$ and differentiating the expression (5) with respect to λ one has

$$\begin{aligned} &\sum_{i=1}^n \frac{\partial f}{\partial x_i}(\lambda^{r_1}x_1, \dots, \lambda^{r_n}x_n) r_i \lambda^{r_i-1} x_i \\ &\quad = d\lambda^{d-1} q(x) + \sum_{i=1}^n \frac{\partial \phi}{\partial x_i}(\lambda^{r_1}x_1, \dots, \lambda^{r_n}x_n) r_i \lambda^{r_i-1} x_i \\ &\quad = d\lambda^{d-1} q(x) + \sum_{i=1}^n \left(\sum_{j=1}^n b_i^j(\lambda^{r_1}x_1, \dots, \lambda^{r_n}x_n) \frac{\partial q}{\partial x_j}(\lambda^{r_1}x_1, \dots, \lambda^{r_n}x_n) \right) r_i \lambda^{r_i-1} x_i. \end{aligned}$$

As $f = q + \phi$, omitting the variables for simplicity, it follows that

$$\frac{\partial f}{\partial x_i} = \frac{\partial q}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = \frac{\partial q}{\partial x_i} + \sum_{j=1}^{n} b_i^j \frac{\partial q}{\partial x_j}. \tag{6}$$

Since q is quasihomogeneous, we get the famous Euler relation:

$$q = \sum_{i=1}^{n} \frac{r_i}{d} x_i \frac{\partial q}{\partial x_i}. \tag{7}$$

To prove that $M \cdot f$ and $M(0) \cdot f$ are C^ℓ-$\mathscr{R}$-equivalent we construct an appropriate family $F(x,t)$ with $F_0 = M(0) \cdot f$, $F_1 = M \cdot f$ such that F_t is C^ℓ-$\mathscr{R}$-trivial.

Let $F : (\mathbf{R}^n \times [0,1], 0) \to (\mathbf{R}, 0)$ be given by

$$F(x,t) = M(tx) f(x), \qquad x \in \mathbf{R}^n,\ t \in [0,1].$$

Now we construct a vector field $\xi : (\mathbf{R}^n \times [0,1], 0) \to (\mathbf{R}^n, 0)$ satisfying the equation

$$-\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x} \circ \xi, \qquad \xi_i(0,t) = 0,\ i = 1, \ldots, n.$$

Observe that

$$\begin{aligned}
-\frac{\partial F}{dt} = \frac{\partial F}{dx} \circ \xi \Leftrightarrow & \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x,t)\xi_i(x,t) + \frac{\partial F}{\partial t}(x,t) = 0 \\
\Leftrightarrow & \sum_{i=1}^{n} \xi_i(x,t) \left[\frac{\partial M}{\partial x_i}(tx) t f(x) + M(tx) \frac{\partial f}{\partial x_i}(x) \right] \\
& + \sum_{i=1}^{n} \frac{\partial M}{\partial x_i}(tx) x_i f(x) = 0 \\
\Leftrightarrow & \sum_{i=1}^{n} \xi_i(x,t) M(tx) \frac{\partial f}{\partial x_i}(x) \\
& + \sum_{i=1}^{n} [\xi_i(x,t)t + x_i] \frac{\partial M}{\partial x_i}(tx)(q+\phi)(x) = 0.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
& \sum_{i=1}^{n} \xi_i(x,t) M(tx) \frac{\partial f}{\partial x_i}(x) \\
& \quad + \sum_{i=1}^{n} [\xi_i(x,t)t + x_i] \frac{\partial M}{\partial x_i}(tx) \left(q(x) + \sum_{j=1}^{n} a_j(x) \frac{\partial q}{\partial x_j} \right) = 0
\end{aligned} \tag{8}$$

where $a_j \in m_n^{[\ell]}$, for all $j = 1, \ldots, n$.

Using the expressions (6) and (7) we substitute $\dfrac{\partial f}{\partial x_i}$ and q in the expression (8) to obtain

$$\sum_{i=1}^{n} \xi_i(x,t) M(tx) \left[\frac{\partial q}{\partial x_i}(x) + \sum_{j=1}^{n} b_i^j(x) \frac{\partial q}{\partial x_j}(x) \right]$$
$$+ \sum_{i=1}^{n} [\xi_i(x,t)t + x_i] \frac{\partial M}{\partial x_i}(tx) \left[\sum_{j=1}^{n} \left(\frac{r_j}{d} x_j + a_j(x) \right) \frac{\partial q}{\partial x_j}(x) \right] = 0. \tag{9}$$

Reordering the expression (9)

$$\sum_{k=1}^{n} \Bigg\{ \xi_k(x,t) M(tx) + \sum_{j=1}^{n} \xi_j(x,t) M(tx) b_j^k(x)$$
$$+ \sum_{i=1}^{n} \left[(\xi_i(x,t)t + x_i) \frac{\partial M}{\partial x_i}(tx) \left(\frac{r_k}{d} x_k + a_k(x) \right) \right] \Bigg\} \frac{\partial q}{\partial x_k}(x) = 0.$$

Note that the coefficient of $\dfrac{\partial q}{\partial x_k}$ can be written as

$$\left(M(tx) b_1^k(x) + t \frac{\partial M}{\partial x_1}(tx) \left(\frac{r_k}{d} x_k + a_k(x) \right) \right) \xi_1(x,t)$$
$$+ \left(M(tx) b_2^k(x) + t \frac{\partial M}{\partial x_2}(tx) \left(\frac{r_k}{d} x_k + a_k(x) \right) \right) \xi_2(x,t) + \cdots$$
$$\vdots$$
$$+ \left(M(tx)(1 + b_k^k(x)) + t \frac{\partial M}{\partial x_k}(tx) \left(\frac{r_k}{d} x_k + a_k(x) \right) \right) \xi_k(x,t) + \cdots$$
$$\vdots$$
$$+ \left(M(tx) b_n^k(x) + t \frac{\partial M}{\partial x_n}(tx) \left(\frac{r_k}{d} x_k + a_k(x) \right) \right) \xi_n(x,t)$$
$$+ \sum_{i=1}^{n} x_i \frac{\partial M}{\partial x_i}(tx) \left(\frac{r_k}{d} x_k + a_k(x) \right).$$

That is,

$$\bar{A}(x,t) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix}, \tag{10}$$

where the matrix $\bar{A}$, omitting the variables, is given by

$$\bar{A} = \begin{pmatrix} M(1+b_1^1) + t\frac{\partial M}{\partial x_1}\Delta_1 & Mb_2^1 + t\frac{\partial M}{\partial x_2}\Delta_1 & \cdots & Mb_n^1 + t\frac{\partial M}{\partial x_n}\Delta_1 \\ Mb_1^2 + t\frac{\partial M}{\partial x_1}\Delta_2 & M(1+b_2^2) + t\frac{\partial M}{\partial x_2}\Delta_2 & \cdots & Mb_n^2 + t\frac{\partial M}{\partial x_n}\Delta_2 \\ \vdots & \vdots & \ddots & \vdots \\ Mb_1^n + t\frac{\partial M}{\partial x_1}\Delta_n & Mb_2^n + t\frac{\partial M}{\partial x_2}\Delta_n & \cdots & M(1+b_n^n) + t\frac{\partial M}{\partial x_n}\Delta_n \end{pmatrix}$$

where $\Delta_i = \left(\frac{r_i}{d}x_i + a_i(x)\right)$ and

$$B_k = -\sum_{i \neq k}^{n} x_i \frac{\partial M}{\partial x_i}(tx)\left(\frac{r_k}{d}x_k + a_k(x)\right), \qquad k = 1, \ldots, n.$$

Since $a_k(0) = 0$, evaluating the matrix $\bar{A}(x,t)$ in $(0,t)$ we obtain

$$\bar{A}(0,t) = \begin{pmatrix} M(0)(1+b_1^1(0)) & M(0)b_2^1(0) & \cdots & M(0)b_n^1(0) \\ M(0)b_1^2(0) & M(0)(1+b_2^2(0)) & \cdots & M(0)b_n^2(0) \\ \vdots & \vdots & \ddots & \vdots \\ M(0)b_1^n(0) & M(0)b_2^n(0) & \cdots & M(0)(1+b_n^n(0)) \end{pmatrix}.$$

Since also $b_i^j(0) = 0$ for all $i, j = 1, \ldots, n$, then $\det \bar{A}(0,t) = M(0)^n \neq 0$ and one can solve (10) with respect to $\xi_i(x,t)$.

Notice that $\xi_i(0,t) = 0$ and by Thom-Levine type Theorem [10] it follows that the flux given by integrating the C^ℓ vector field ξ realizes the C^ℓ-$\mathscr{R}$-triviality required of family F. Hence the proof of Lemma is complete. □

Remark 4. *In the particular case where f is a quasihomogeneous C^ℓ function germ, $1 \leq \ell \leq \infty$, the proof of the Lemma 2 is simpler and it becomes essentially the same of that done by Takahashi in* [14, Proposition 2.1 p. 830]. *In fact, we need just replacing $\mathscr{R}$-equivalence by C^ℓ-$\mathscr{R}$-equivalence; $\mathscr{K}$-equivalence by C^ℓ-$\mathscr{K}$-equivalence and $a_j \in m_n$ by $a_j \in m_n^{[\ell]}$ in the proof given by Takahashi.*

5. Examples

Example 4. *Consider $g(x,y) = q(x,y) = \frac{x^{10}}{x^4+y^8}$ the C^5 quasihomogeneous function germ of type $(2,1;12)$. Notice that we verify the C^ℓ-differentiability of f by applying* [12, Lemma 2.1]. *Let*

$$f(x,y) = \frac{x^{10}}{x^4+y^8} + \left(\frac{x^{12}}{x^4+y^8} + x^{14} + x^{10}y^8\right)$$

be semi-quasihomogeneous of type $(2,1;12)$ of class C^5.

Here $f = M \cdot q$, *with* $M(x,y) = 1 + x^2 + (x^4 + y^8)^2$ *and* $M(0) \neq 0$. *Since the hypotheses of Theorem* 1 *are satisfied,* $f \overset{C^\ell - \mathscr{R}_\pm}{\sim} q$.

Example 5. *Let* $f = \frac{x^{10}}{x^4+y^8} + x^2 \frac{x^{10}}{x^4+y^8} + \frac{x^4(x^4+y^8)^2}{x^2+y^4} \frac{x^{10}}{x^4+y^8}$, *here* q *is given in Example* 4 *and*

$$\phi(x,y) = x^2 q(x,y) + \frac{x^4(x^4+y^8)^2}{x^2+y^4} q(x,y).$$

Let $M(x,y) = 1 + x^2 + \dfrac{x^4(x^4+y^8)^2}{x^2+y^4}$. *Notice that* $M(0) \neq 0$ *and applying* [12, Lemma 2.1], M *is a function germ of class* C^9 *and* ϕ *is a function germ of class* C^7.

Note that $f = M \cdot q$ *and then* $f \overset{C^9 - \mathscr{K}}{\sim} q$. *From Theorem* 1, f *and* q *are* C^9-$\mathscr{R}_\pm$-*equivalent. Observe in this example that* f *is a semi-quasihomogeneous function germ while* q *is a quasihomogeneous function germ.*

Example 6. *Let* $M(x,y) = 1 + x^2 + \dfrac{x^4(x^4+y^8)^2}{x^2+y^4}$ *be the function germ given in Example* 5 *of class* C^9. *Let* $f = q + \phi$ *be a semi-quasihomogeneous function germ of class* C^5 *where* $q(x,y) = \frac{x^{10}}{x^4+y^8}$ *and* ϕ *is given in Example* 4.

Consider $\tilde{f} = M \cdot f$. *Then,* $\tilde{f}$ *is not a quasihomogeneous function germ and* $\tilde{f} \overset{C^9 - \mathscr{K}}{\sim} f$. *Then we also have* $\tilde{f} \overset{C^5 - \mathscr{K}}{\sim} f$. *Applying* [12, Lemma 2.1] *one obtains that* $\tilde{f}$ *is a* C^5 *function germ and also the hypotheses of Theorem* 1 *are satisfied. Then we can conclude that* $\tilde{f} \overset{C^5 - \mathscr{R}_\pm}{\sim} f$.

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