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7-colored 2-knot diagram with six colors

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ABSTRACT. It is known that any 7-colorable knot in 3-space is presented by a diagram assigned by four of the seven colors. In this paper, we prove the existence of a 7-colorable 2-knot in 4-space such that any non-trivial 7-coloring requires at least six of the seven colors.

1. Introduction

Harary and Kauffman [4] studied the number of colors on the arcs of a p-colored knot diagram. Let $\gamma : \{\text{the arcs of } D\} \to \mathbf{Z}/p\mathbf{Z}$ be a non-trivial p-coloring for a knot diagram D, and $N(D,\gamma) = \#\mathrm{Im}(\gamma) > 1$ the cardinality of the image. We denote by $C_p(K)$ the minimal number of $N(D,\gamma)$ for all p-colored diagrams (D,γ) of a knot K in $\mathbf{R}^3$. The notation $C_p(K)$ is originally used in [4], and also written as $\mathrm{mincol}_p(K)$ in some papers (cf. [6, 7]).

This number can be extended to a p-colorable 2-knot F in $\mathbf{R}^4$ naturally. It is not difficult to see that $C_3(F) = 3$ for any 3-colorable (2-)knot F. For the case $p = 5$, it is proved in [13] that

- $C_5(K) = 4$ for any 5-colorable knot K,
- $4 \le C_5(F) \le 5$ for any 5-colorable 2-knot F,
- $C_5(F) = 4$ for any 5-colorable ribbon 2-knot F, and
- there are infinitely many 5-colorable 2-knots F such that $C_5(F) = 5$.

On the other hand, for the case $p = 7$, it is proved in [10] that

- $C_7(K) = 4$ for any 7-colorable knot K,
- $4 \le C_7(F) \le 7$ for any 7-colorable 2-knot F, and
- $C_7(F) = 4$ for any 7-colorable ribbon 2-knot F.

Therefore, it is natural to ask whether there is a 7-colorable 2-knot F with $C_7(F) > 4$. The aim of this paper is to answer this question affirmatively.

THEOREM 1. *There are infinitely many 7-colorable 2-knots F such that $C_7(F) = 6$.*

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It is still an open question whether there is a 7-colorable 2-knot F with $C_7(F) = 5$ or 7. We remark that any p-colorable (2-)knot F satisfies $C_p(F) > \log_2 p + 1$ (cf. [9]).

This paper is organized as follows. In Section 2, we prove $C_7(F) \geq 6$ if F satisfies a certain condition on the quandle cocycle invariant (Theorem 2). In Section 3, we construct a 7-colored diagram with six colors of a 2-twist-spun 5_2-knot (Theorem 3) and prove Theorem 1.

2. Quandle cocycle invariants

A 2-*knot* is a 2-dimensional sphere embedded in $\mathbf{R}^4$ smoothly, and its *diagram* is the image under a projection of $\mathbf{R}^4$ onto $\mathbf{R}^3$ equipped with crossing information. Refer to [3] for more details. Throughout this section, we assume that all the 2-knots and their diagrams are oriented.

A 2-knot diagram consists of connected compact surfaces called *sheets*. Each sheet of a 2-knot diagram has an orientation arrow, say $\vec{v}$, such that the triple $(\vec{u}_1, \vec{u}_2, \vec{v})$ matches the orientation of $\mathbf{R}^3$, where $(\vec{v}_1, \vec{v}_2)$ defines the orientation of the surface.

Let t be a triple point of a diagram D of a 2-knot F. In a neighborhood of t, there are eight regions of $\mathbf{R}^3 \backslash D$. The *specified region* at t is the one of them such that the orientation arrows $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ on the top, middle, and bottom sheets, respectively, point away from the region. The *sign* of a triple point t is *positive* if $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ matches the orientation of $\mathbf{R}^3$. Otherwise, the sign is *negative*. We denote it by $\varepsilon(t)$.

For an odd prime p, a *p-coloring* for a diagram D is a map

$$\gamma : \{\text{the sheets of } D\} \to \mathbf{Z}/p\mathbf{Z}$$

such that $x_1 + x_2 \equiv y \pmod{p}$ holds at any double point, where x_1 and x_2 are the colors assigned to the lower sheets and y the one to the upper.

Fix a p-coloring γ for D. The *color* of t with respect to γ is an ordered triple

$$(a(t), b(t), c(t)) \in (\mathbf{Z}/p\mathbf{Z})^3$$

such that $a(t)$, $b(t)$, and $c(t)$ are the colors of the bottom, middle, and top sheets, respectively, adjacent to the specified region at t. See the left of Figure 1. Such a triple point is also illustrated by a crossing with four regions as in the right. We say that t is *degenerate* with respect to γ if $a(t) = b(t)$ or $b(t) = c(t)$, and otherwise *non-degenerate*.

For $n = 2$ or 3, let C_n' be the free abelian group generated by the n-tuples $(a_1, a_2, \ldots, a_n) \in (\mathbf{Z}/p\mathbf{Z})^n$, and C_n'' the subgroup of C_n' generated by the ele-

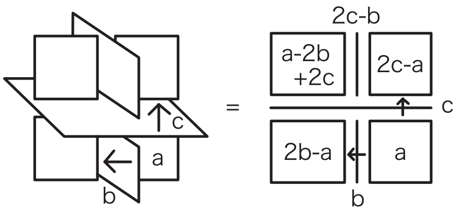

Fig. 1

ments such that $a_i = a_{i+1}$ for some $1 \le i \le n-1$. Take the quotient group $C_n = C'_n / C''_n$.

The 3-*chain* $\xi(D, \gamma)$ associated with (D, γ) is defined by

$$\xi(D, \gamma) = \sum_t \varepsilon(t)(a(t), b(t), c(t)) \in C_3.$$

By definition, any trivial p-coloring γ satisfies $\xi(D, \gamma) = 0$.

Let $\partial'_3 : C'_3 \to C'_2$ be the boundary map defined by

$$\partial'_3(a, b, c) = (a, c) - (a, b) - (2b - a, c) + (2c - a, 2c - b).$$

Since $\partial'_3(C''_3) \subset C''_2$, ∂'_3 induces the map $\partial_3 : C_3 \to C_2$ naturally. It is proved in [2] that any 3-chain $\xi(D, \gamma)$ is a 3-cycle; that is, $\partial_3(\xi(D, \gamma)) = 0$.

Let $\theta : C_3 \to \mathbf{Z}/p\mathbf{Z}$ be the homomorphism defined by

$$\theta(a, b, c) = (a - b)\frac{b^p + (2c - b)^p - 2c^p}{p} \in \mathbf{Z}/p\mathbf{Z}$$

for each generator (a, b, c) of C_3 (cf. [1, 8]). The *quandle cocycle invariant* $\Phi_p(F)$ [2] is the multi-set defined by

$$\Phi_p(F) = \{\theta(\xi(D, \gamma)) \in \mathbf{Z}/p\mathbf{Z} \mid \gamma : \text{a } p\text{-coloring for } D\}.$$

For a p-coloring γ for D and elements $k, l \in \mathbf{Z}/p\mathbf{Z}$, we define the p-coloring $k\gamma + l$ to be the composition of γ and the affine map $f : \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$ with $f(x) = kx + l$.

Lemma 1 ([12]). $\theta(\xi(D, k\gamma + l)) = k^2\theta(\xi(D, \gamma))$.

Let $\varphi_p(F)$ denote the number of 0's in the multi-set $\Phi_p(F)$. A p-coloring γ is *trivial* if it is a constant map. Since there are p trivial p-colorings for D and each trivial p-coloring contributes $0 \in \Phi_p(F)$, we have $\varphi_p(F) \ge p$.

Lemma 2 ([13]). *Let F be a p-colorable 2-knot. If $\varphi_p(F) = p$, then any non-trivially p-colored diagram (D, γ) of F has a non-degenerate triple point.*

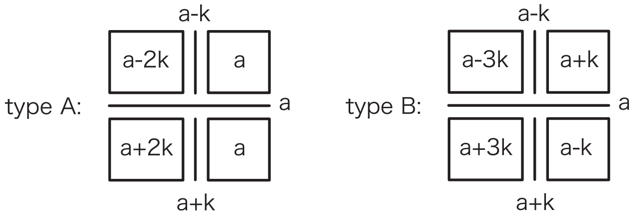

Fig. 2

Assume that $p \geq 7$. Let t be a non-degenerate triple point of (D, γ). We say that t is of *type A* or *type B* with respect to γ, respectively, according to whether the coloring γ around t is as in left or right of Figure 2, where $a, k \in \mathbf{Z}/p\mathbf{Z}$ with $k \neq 0$. Otherwise, t is of *type C*.

LEMMA 3. *Let γ be a non-trivial p-coloring for D with $p \geq 7$, and t a non-degenerate triple point of (D, γ).*

(i) *If t is of type A or B, then five colors in $\mathbf{Z}/p\mathbf{Z}$ appear on the sheets around t. If t is of type C, then the seven colors appear around t.*

(ii) *$N(D, \gamma) \geq 5$.*

PROOF. (i) Let $(a + ik, a + k, a)$ be the color of t for some $a, i, k \in \mathbf{Z}/p\mathbf{Z}$. Since t is non-degenerate, we have $i \neq 1$ and $k \neq 0$. Then the colors of the sheets are

$$a \text{ (top)}, \quad a \pm k \text{ (middle)}, \quad \text{and} \quad a \pm ik, \quad a \pm (2 - i)k \text{ (bottom)}.$$

By definition, t is of type A if $i = 0, 2$ and of type B if $i = -1, 3$. In both cases, the number of the distinct colors are five. Otherwise, the above seven colors are mutually distinct, and we have the conclusion.

(ii) This follows from (i) immediately.

LEMMA 4. *Let S and S' be subsets of $\mathbf{Z}/p\mathbf{Z}$.*

(i) *If $\#S = \#S' = 2$, then there is an affine map $f : \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$ such that $f(S) = S'$.*

(ii) *If $\#S = \#S' = n - 2$, then there is an affine map $f : \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$ such that $f(S) = S'$.*

PROOF. (i) For $S = \{a, b\}$ and $S' = \{c, d\}$, the affine map

$$f(x) = \frac{(c - d)x + (ad - bc)}{(a - b)}$$

satisfies $f(a) = c$ and $f(b) = d$.

(ii) It is sufficient to apply (i) to the sets $(\mathbf{Z}/p\mathbf{Z}) \backslash S$ and $(\mathbf{Z}/p\mathbf{Z}) \backslash S'$.

Now we consider the case $p = 7$.

Lemma 5. *Let* (D, γ) *be a 7-colored diagram with* $\mathrm{Im}(\gamma) = \{0, 1, 2, 5, 6\}$. *Then the color of a non-degenerate triple point with respect to* γ *is equal to one of the following triples;*

$$(0,1,0), \quad (2,1,0), \quad (0,6,0), \quad (5,6,0),$$
$$(5,2,0), \quad (6,2,0), \quad (2,5,0), \quad (1,5,0).$$

Proof. By Lemma 3(i), t is of type A or B. If t is of type A, we have

$$\{a, a \pm k, a \pm 2k\} = \{0, 1, 2, 5, 6\}.$$

This implies that $a = 0$ and $k = 1, 6$; in fact, by taking the sum of the elements in each set, we have $5a = 0$ and $\{\pm k, \pm 2k\} = \{\pm 1, \pm 2\}$. Similarly, if t is of type B, we have

$$\{a, a \pm k, a \pm 3k\} = \{0, 1, 2, 5, 6\}.$$

This implies that $a = 0$ and $k = 2, 5$. Therefore, the sheets around t are colored as shown in Figure 3. In each type, we have four kinds of colors of t according to the orientations of the top and middle sheets.

Proposition 1. *Let* (D, γ) *be a 7-colored diagram. If* $N(D, \gamma) = 5$, *then we have* $\theta(\xi(D, \gamma)) = 0$.

Proof. By Lemmas 1, 4(ii), and 5, we may assume that $\mathrm{Im}(\gamma) = \{0, 1, 2, 5, 6\}$ and

$$\xi(D, \gamma) = \alpha_1(0,1,0) + \alpha_2(2,1,0) + \alpha_3(0,6,0) + \alpha_4(5,6,0)$$
$$+ \beta_1(5,2,0) + \beta_2(6,2,0) + \beta_3(2,5,0) + \beta_4(1,5,0)$$

for some integers α_i and β_i $(i = 1, 2, 3, 4)$. It follows by the definition of ∂_3 that

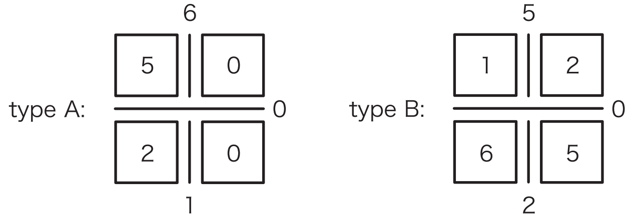

Fig. 3

$$\begin{aligned}\partial_3(\xi(D,\gamma)) &= (-\alpha_1+\alpha_3)(0,1)+(\alpha_1-\alpha_3)(0,6)\\ &\quad+(-\beta_3+\beta_4)(1,0)+(\beta_2-\beta_4)(1,5)\\ &\quad+(-\alpha_1+\alpha_2+\beta_3-\beta_4)(2,0)+(-\alpha_2+\alpha_4)(2,1)+(\beta_1-\beta_3)(2,5)\\ &\quad+(-\alpha_3+\alpha_4+\beta_1-\beta_2)(5,0)+(-\beta_1+\beta_3)(5,2)+(\alpha_2-\alpha_4)(5,6)\\ &\quad+(-\beta_1+\beta_2)(6,0)+(-\beta_2+\beta_4)(6,2).\end{aligned}$$

Since $\partial_3(\xi(D,\gamma))=0$, we have

$$\alpha_1=\alpha_2=\alpha_3=\alpha_4 \quad \text{and} \quad \beta_1=\beta_2=\beta_3=\beta_4.$$

Therefore, it holds that

$$\begin{aligned}\theta(\xi(D,\gamma)) &= \alpha_1\{\theta(0,1,0)+\theta(2,1,0)+\theta(0,6,0)+\theta(5,6,0)\}\\ &\quad+\beta_1\{\theta(5,2,0)+\theta(6,2,0)+\theta(2,5,0)+\theta(1,5,0)\}\\ &= \alpha_1(6+1+1+6)\frac{1^7+6^7-2\cdot 0^7}{7}\\ &\quad+\beta_1(3+4+4+3)\frac{2^7+5^7-2\cdot 0^7}{7}=0.\end{aligned}$$

Theorem 2. *Let F be a 7-colorable 2-knot. If $\varphi_7(F)=7$, then $C_7(F)\geq 6$.*

Proof. Assume that $C_7(F)\leq 5$. Let (D,γ) be a non-trivially 7-colored diagram with $N(D,\gamma)=C_7(F)$. Since $\varphi_7(F)=7$, we have $N(D,\gamma)=5$ by Lemmas 2 and 3(ii). It follows by Proposition 1 that $\theta(\xi(D,\gamma))=0$. This implies that $\varphi_7(F)>7$ and we have a contradiction.

3. Twist-spun 5_2-knots

Let T be a tangle diagram of a knot K. We consider a sequence of tangle diagrams as shown in Figure 4. We perform a Reidemeister move I to

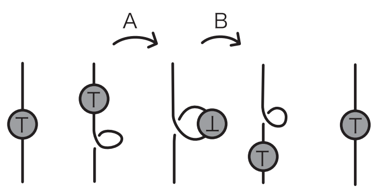

Fig. 4

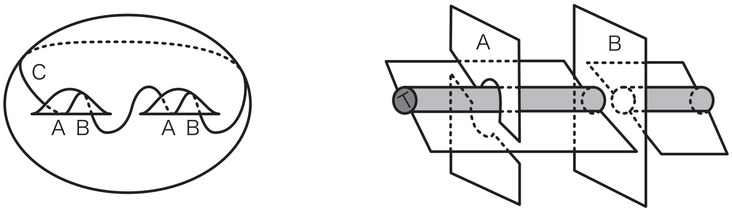

Fig. 5

produce a crossing below T, slide T over the crossing (the process A) and then under the crossing (the process B), and perform a Reidemeister move I to eliminate the crossing above T. This sequence presents a full twist of the tangle in the meridional direction.

For an integer $n > 0$, we construct a 2-knot diagram D_n by piling up the above sequence n times. Namely, we take a 2-sphere in $\mathbf{R}^3$ with n pairs of branch points connected by n double-point arcs. See the left of Figure 5. Let C be a closed curve which travels around the sphere intersecting each double-point arc twice. Then we replace a neighborhood of C by the product $T \times S^1$ to obtain D_n.

At the intersections between C and each double-point arc, crossing information is given in such a way that T goes over the transverse sheet (process A) and under the the transverse sheet (process B) as shown in the right of Figure 5. Then it is proved in [11] that D_n represents the *n-twist-spinning, $\tau^n K$, of K* [14].

Lemma 6 ([1]). *The n-twist-spun knot $\tau^n K$ is p-colorable if and only if K is p-colorable and n is even. Moreover, any p-coloring for T can be extended to that for D_n uniquely.*

Let K be the 5_2-knot. We consider a 7-coloring γ for D_2 of the 2-twist-spun 5_2-knot as shown in Figure 6. We first color the tangle diagram T at the upper left of the figure. The coloring for T does not change after passing over the transverse sheets in the process A. Since the end-arcs of T admit the color 0, so do the outermost transverse sheets. The coloring for the other sheets comes from the shadow coloring for the complementary regions of T.

The process B in the first twist changes the coloring for T in such a way that $x \mapsto 2 \cdot 0 - x = -x \pmod 7$. In fact, T passes under the transverse sheet with color 0. See the lower left of the figure.

We proceed the same argument on the coloring for T under the second twist. The coloring for T after two twists is coincident with the original one as shown in the lower right of the figure.

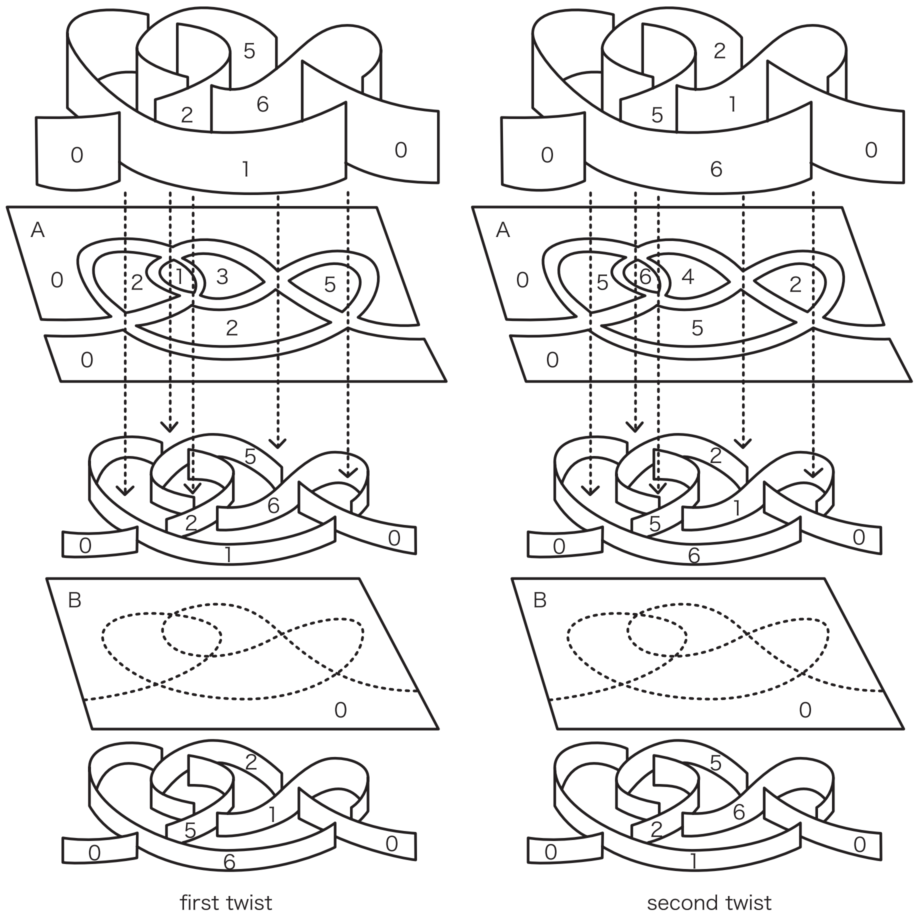

Fig. 6

REMARK 1. *The diagram D_2 has twenty sheets, and the numbers of the sheets colored by $0, 1, \ldots, 6$ are given by the following table.*

color	0	1	2	3	4	5	6	*total*
number of sheets	2	3	5	1	1	5	3	20

Therefore, we have $N(D_2, \gamma) = 7$.

THEOREM 3. *Let K be the 5_2-knot. Then $\tau^{2n}K$ $(n > 0)$ has a 7-colored diagram (D, γ) with $N(D, \gamma) = 6$; that is, $C_7(\tau^{2n}K) \leq 6$.*

PROOF. We consider the case $n = 1$. The other cases are similarly proved.

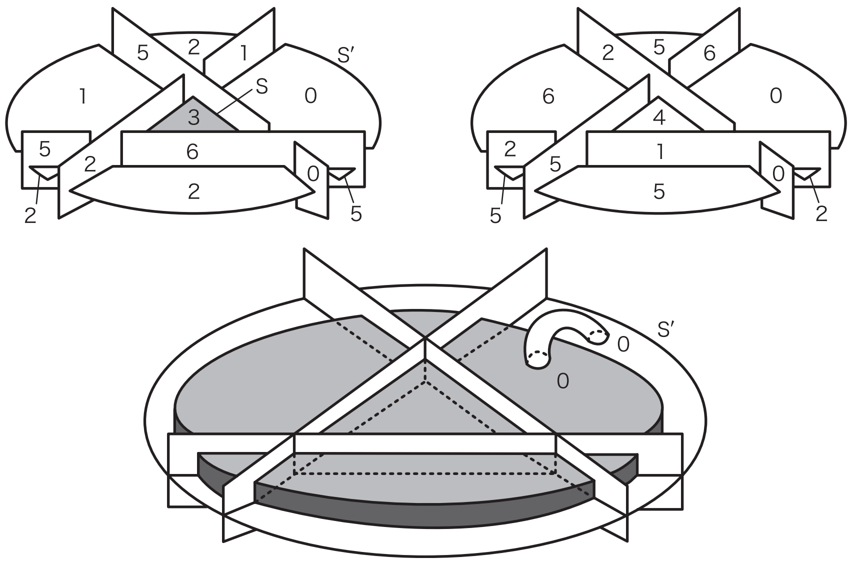

Fig. 7

Let (D_2, γ) be the 7-colored diagram as above. There is a unique sheet S with color 3. The neighborhood of S is illustrated in the upper left of Figure 7. Let S' be the sheet with 0 next to S as shown in the figure.

We deform the diagram D_2 as follows: We wrap S with S' by swelling S' like a balloon such that S' is higher than any other sheets in 4-space. See the bottom of the figure. This modification is similar to the ones used in [10, 13].

Then the colors of the sheets inside the balloon are obtained from those of the original sheets by mapping $x \mapsto -x$. See the upper right of Figure 7. Since there is no sheet with color 3 in the obtained diagram, we have the conclusion.

4. Quandle cocycle invariants of twist-spun 5_2-knot

Let K be an oriented p-colorable knot, and $\tau^{2n}K$ the $2n$-twist-spinning of K. It is known that the quandle cocycle invariant $\Phi_p(\tau^{2n}K)$ is calculated from a diagram of K instead of that of $\tau^{2n}K$ as follows (cf. [1, 12]).

We take a diagram D of K and fix a base point on it different from the crossings. Let $\bar{D}$ denote the plane curve obtained from D by ignoring crossing information of D. Each p-coloring $\gamma : \{\text{the arcs of } D\} \to \mathbf{Z}/p\mathbf{Z}$ for D defines a *shadow* p-coloring

$$\gamma' : \{\text{the connected regions of } \mathbf{R}^2 \backslash \bar{D}\} \to \mathbf{Z}/p\mathbf{Z}$$

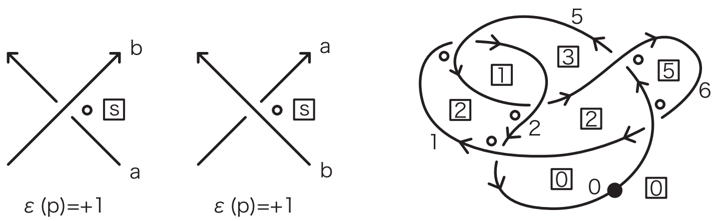

Fig. 8

uniquely such that (i) $s_1 + s_2 \equiv 2x \pmod{p}$ holds along any arc of D, where x is the color of the arc, and s_1 and s_2 are the shadow colors of the regions adjacent to the arc, and that (ii) the shadow colors of the regions adjacent to the base point are the same as the color of the arc containing the base point.

We denote by $\varepsilon(p)$ the sign of a crossing p of D. In a neighborhood of p, there are four regions of $\mathbf{R}^2 \backslash \bar{D}$. The *specified region* at p is the one of them which is on the right sides of the upper and lower arcs.

The *color* of p with respect to γ is an ordered triple

$$(s(p), a(p), b(p)) \in (\mathbf{Z}/p\mathbf{Z})^3$$

such that $s(p)$ is the shadow color of the specified region, and $a(p)$ and $b(p)$ are the colors of the lower and upper arcs adjacent to the specified region. See the left of Figure 8, where the specified region is marked with a small circle and the shadow color is surrounded by a square.

The 3-chain $\eta(D, \gamma)$ associated with (D, γ) is defined by

$$\eta(D, \gamma) = \sum_p \varepsilon(p)(s(p), a(p), b(p)) \in C_3.$$

Then we have the following.

THEOREM 4 ([1, 12]). *For any p-colorable knot K, it holds that*

$$\Phi_p(\tau^{2n}K) = \{2n\theta(\eta(D, \gamma)) \in \mathbf{Z}/p\mathbf{Z} \mid \gamma : a \ p\text{-coloring for } D\}.$$

In particular, if the number of p-colorings for K is exactly p^2, then it holds that

$$\Phi_p(\tau^{2n}K) = \{2n\theta(\eta(D, \gamma))k^2 \ \ (p \ \ \text{times}) \mid k \in \mathbf{Z}/p\mathbf{Z}\}$$

for any non-trivial p-coloring γ.

THEOREM 5. *Let K be the 5_2-knot. If $n \not\equiv 0 \pmod 7$, then $C_7(\tau^{2n}K) = 6$.*

PROOF. We have $C_7(\tau^{2n}K) \leq 6$ by Theorem 3. To prove $C_7(\tau^{2n}K) \geq 6$, we calculate the quandle cocycle invariant $\Phi_7(\tau^{2n}K)$ by using Theorem 4. We remark that the number of 7-colorings for K is equal to 7^2.

We consider the 7-coloring γ for the diagram D of K as shown in the right of Figure 8. Since the 3-chain associated with (D, γ) is given by

$$\eta(D, \gamma) = +(2,1,5) + (2,5,2) + (2,2,1) + (5,0,6) + (5,6,0),$$

we have

$$\begin{aligned}\theta(\eta(D,\gamma)) &= \frac{1+2^7-2\cdot 5^7}{7} - 3\frac{5^7+6^7-2\cdot 2^7}{7} + 5\frac{5^7-2\cdot 6^7}{7} - \frac{6^7+1}{7} \\ &= \frac{7\cdot 2^7 - 14\cdot 6^7}{7} = 2^7 - 2\cdot 6^7 \equiv 128 - 2\cdot(-1) \equiv 4 \pmod 7.\end{aligned}$$

Therefore, it follows by Theorem 4 that

$$\Phi_7(\tau^{2n}K) = \{\underbrace{0,\dots,0}_{7}, \underbrace{n,\dots,n}_{14}, \underbrace{2n,\dots,2n}_{14}, \underbrace{4n,\dots,4n}_{14}\}.$$

Since $\varphi_7(\tau^{2n}K) = 7$ for $n \not\equiv 0 \pmod 7$, we have $C_7(\tau^{2n}K) \geq 6$ by Theorem 2.

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