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The link surgery of $S^2 \times S^2$ and Scharlemann's manifolds

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Abstract. Fintushel-Stern's knot surgery has given many exotic 4-manifolds. We show that if an elliptic fibration has two, parallel, oppositely-oriented vanishing cycles (for example $S^2 \times S^2$ or Matsumoto's S^4), then the knot surgery does not change its differential structure. We also give a classification of link surgery of $S^2 \times S^2$ and a generalization of Akbulut's celebrated result that Scharlemann's manifold is standard.

1. Introduction

1.1. Knot surgery. We call a pair of manifolds an exotic pair, if they are homeomorphic but non-diffeomorphic. It has been an intriguing question to construct exotic pairs. In particular, 4-dimensional manifolds have given interesting examples. *Fintushel-Stern's knot surgery* in [7] is a powerful method to construct such 4-dimensional exotic pairs. Given a simply-connected 4-manifold X which contains a torus $T \subset X$ with the trivial normal bundle and a knot K in S^3, the knot surgery operation $X \rightsquigarrow X_K$ is defined by removing the neighborhood of T and regluing $(S^3 - \nu(K)) \times S^1$. The symbol ν represents the open neighborhood throughout the present article. Under favorable conditions (for example, the case that X contains the regular neighborhood C of the cusp singular fiber and T is a general fiber), the resulting 4-manifold X_K is simply-connected and has the same intersection form as X, hence it is homeomorphic to X by virtue of Freedman's celebrated theorem.

In [7], the following formula for the Seiberg-Witten invariant (SW-invariant) was establised.

$$SW_{X_K} = SW_X \cdot \Delta_K \tag{1}$$

Here Δ_K is the Alexander polynomial of K. This formula implies the knot surgery gives rise to many exotic pairs. If SW_X is non-trivial and $\Delta_K \neq 1$, then (X, X_K) is an exotic pair.

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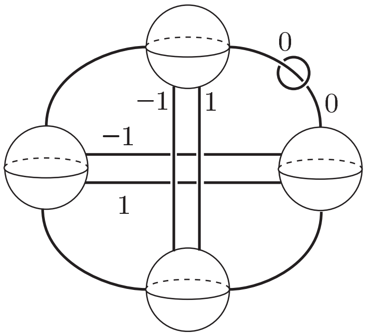

Fig. 1. Two parallel, oppositely-oriented cusp fibers in $S^2 \times S^2$.

One of the main purposes of this article is to show that there a lot of examples of Fintushel-Stern's knot surgery which do "not" produce exotic pairs. By the above argument, we need to focus on the case where $\Delta_K(t) = 1$ or $SW_X = 0$.

It is known that $X = S^2 \times S^2$ has trivial SW-invariant. The cusp neighborhood C can naturally be embedded inside X. In fact, X is diffeomorphic to the double $\bar{C} \cup C$ where $\bar{C}$ is C with the opposite orientation. Figure 1 describes the achiral elliptic fibration of X.

Definition 1. We denote the knot surgery $\bar{C} \cup C_K$ by A_K.

In [3] S. Akbulut showed that A_{3_1} is diffeomorphic to $S^2 \times S^2$. The proof essentially uses his other result [2]. Our first main theorem is:

Theorem 1. *A_K is diffeomorphic to $S^2 \times S^2$ for any knot K.*

We will prove this theorem in Section 3. The theorem shows the existence of infinitely many exotic embeddings of C into $S^2 \times S^2$.

1.2. Link surgery. Fintushel and Stern [7] defined *link surgery*, which is a link version of knot surgery. For an n-tuple $(X_1, X_2, \dots, X_n)$ of 4-manifolds, each of which contains a (specified) C, and an n-component (labeled) link L in S^3, we can define the *link surgery* $X(X_1, \dots, X_n; L)$. This is a variation of the fiber-sum operation connecting some manifolds rather than a surgery.

In the case of $X_i = S^2 \times S^2$ for any i, we denote the link surgery by A_L. Theorem 1 can be generalized to the link case as follows.

Theorem 2. *Let L be an n-component link. A_L is diffeomorphic to*

$$\begin{cases} \#^{2n-1} S^2 \times S^2, & \text{if } L \text{ is a proper link;} \\ \#^{2n-1} \mathbf{C}P^2 \#^{2n-1} \overline{\mathbf{C}P^2}, & \text{otherwise.} \end{cases}$$

In the proof, we give handle pictures of the link surgery $X(C, \dots, C; L)$ for a split link $L = K_1 \cup K_2$ or the Hopf link $L = H$.

1.3. Scharlemann's manifolds. Let $S^3_p(K)$ be the p-surgery along K in S^3, and $\gamma(\varepsilon)$ an embedded framed curve in $S^3_p(K)$. Here γ is a simple closed curve in $S^3 - \nu(K) \subset S^3_p(K)$ and ε is a framing of γ. The embedded curve induces a framed knot $\tilde{\gamma}$ in $S^3_p(K) \times S^1$ through $S^1 \xrightarrow{\gamma} S^3_p(K) \hookrightarrow S^3_p(K) \times S^1$. Here we obtain a manifold $B_{K,p}(\gamma(\varepsilon))$ (*Scharlemann's manifold*) by surgering out the neighborhood of $\tilde{\gamma}$ in $S^3_p(K) \times S^1$ and regluing $S^2 \times D^2$. Since the diffeomorphism type of $B_{K,p}(\gamma(\varepsilon))$ depends only on (K, p) and the free isotopy type of $\tilde{\gamma}$, we are concerned with the free homotopy class of $\gamma(\varepsilon)$. Thus the framings have two types in general.

If γ gives a normal generator in $\pi_1(S^3_p(K))$, then $B_{K,p}(\gamma(\varepsilon))$ is homeomorphic to $S^3 \times S^1 \# S^2 \times S^2$ or $S^3 \times S^1 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$ as can be seen from results presented in [8]. In the case of $p = -1$ we drop the suffix p of $B_{K,p}(\gamma(\varepsilon))$ as $B_K(\gamma(\varepsilon))$.

Scharlemann [15] studied the case where $(K, p) = (3_1, -1)$ and $\gamma = \gamma_0$ (the meridian of 3_1) and showed that $B_{3_1}(\gamma_0(1))$ has a fake self-homotopy structure on $S^3 \times S^1 \# S^2 \times S^2$. At that time the diffeomorphism type of $B_K(\gamma(\varepsilon))$ was not determined. After that, Akbulut [2] proved the following theorem using an amazingly difficult handle calculus.

Theorem 3 ([2]). *$B_{3_1}(\gamma_0(1))$ is diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$.*

It has been unknown whether Theorem 3 can be generalized to an arbitrary knot. We will prove the following as the third main theorem.

Theorem 4. *Let K be any knot in S^3 and $\gamma_0 \subset S^3_{-1}(K)$ the meridian of K in the diagram. $B_K(\gamma_0(1))$ is diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$.*

In the second half of Section 5.2, we will consider the diffeomorphism type of $B_{3_1}(\gamma(\varepsilon))$ for homotopy classes except $\gamma_0(\varepsilon)$.

Theorem 1 and 4 are proven by S. Akbulut in [5] independently. Our proofs are based on Lemma 5 regarding knot surgery in some achiral elliptic fibration.

Acknowledgement

The problem of whether A_K is an exotic $S^2 \times S^2$ or not, was asked by Professor Manabu Akaho ([1]). This paper gives a negative but complete answer to his question. I thank him for motivating me to study the attractive 4-dimensional world.

I thank Masaaki Ue, Kouichi Yasui, Shohei Yamada and Çağrı Karakurt for their many useful comments. I would also like to thank the anonymous referee for reading my manuscript intensively and patiently and giving helpful advice and comments in our correspondence.

2. Preliminaries

2.1. The neighborhoods of singular fibers and the knot surgery. First we recall the fishtail neighborhood F and cusp neighborhood C. The definition of such singular fibers can also be seen in [9]. We define two more neighborhoods of some singular fibers.

Definition 2 (Fishtail (or cusp) neighborhood). A *fishtail* (or *cusp*) *neighborhood* F (or C) is an elliptic fibration over D^2 with one fishtail (or cusp) singular fiber. The handle picture is the top-left (or top-right) in Figure 2. The neighborhood C (or F) includes self-intersection 0 torus as the general fiber.

Definition 3 (Symmetric fishtail (or cusp) neighborhood). We denote a fiber-sum of two parallel oppositely-oriented fishtail (or cusp) fibers over D^2 by SyF (or SyC). The handle picture is the bottom-left (or bottom-right) in Figure 2. The neighborhood SyF (or SyC) includes self-intersection 0

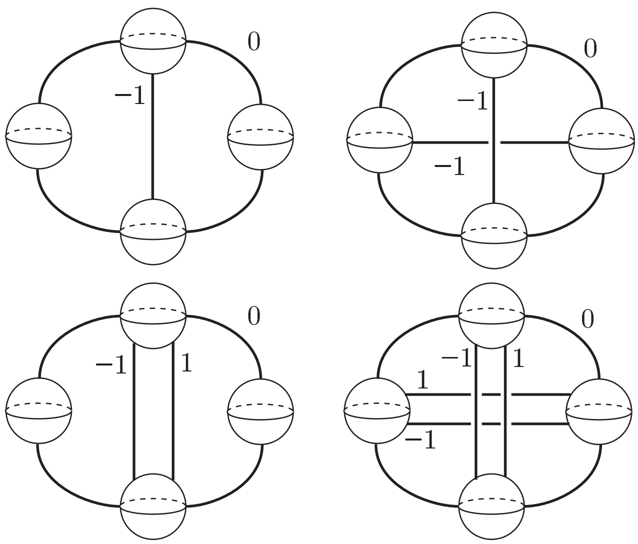

Fig. 2. F, C, SyF, and SyC.

torus as the general fiber. We call SyC (or SyF) *symmetric cusp* (or *fishtail*) *neighborhood*.

The diagrams in Figure 2 give the obvious embeddings $F \hookrightarrow SyF$ and $C \hookrightarrow SyC$.

Let X be a 4-manifold that contains C or F, and K a knot in S^3. The symbol $\bar{\nu}$ represents the closed neighborhood.

DEFINITION 4. We define *Fintushel-Stern's knot surgery* $X_{K,n}$ as

$$X_{K,n} := [X - \nu(T)] \cup_{\varphi_n} [(S^3 - \nu(K)) \times S^1].$$

Here the gluing map is the following:

$$\varphi_n : \partial\bar{\nu}(K) \times S^1 \to \partial\bar{\nu}(T) = T^2 \times \partial D^2$$

such that the map φ_n induces the following on the 1st homology:

$$[\{\text{the meridian of } K\} \times \{\text{pt}\}], \qquad [\{\text{pt}\} \times S^1] \mapsto \alpha, \beta$$

$$[\{\text{the longitude of } K\} \times \{\text{pt}\}] + n[\{\text{the meridian of } K\} \times \{\text{pt}\}]$$
$$\mapsto [\{\text{pt}\} \times \partial D^2] \tag{2}$$

where α, β are generators of $H_1(T^2)$. When X contains F, we assume that α is the class of the vanishing cycle. In the case of $n = 0$, we denote the result of the knot surgery simply by X_K.

2.2. The logarithmic transformation. The purpose of the present section is to define the logarithmic transformation. Let X be an oriented 4-manifold and $T \subset X$ an embedded torus with self-intersection 0.

DEFINITION 5. Let γ be an essential simple closed curve in T and φ a homeomorphism $\partial D^2 \times T^2 \to \partial\nu(T)$ satisfying $\varphi(\partial D^2 \times \{\text{pt}\}) = q(\{\text{pt}\} \times \gamma) + p(\partial D^2 \times \{\text{pt}\})$. Removing $\nu(T)$ from X and regluing $D^2 \times T^2$ via φ, we obtain the following manifold:

$$X(T, p, q, \gamma) := [X - \nu(T)] \cup_\varphi D^2 \times T^2.$$

We call this manifold the *logarithmic transformation* with the data (T, p, q, γ).

It is well-known that the diffeomorphism type of the logarithmic transformation depends only on the data (T, p, q, γ). The integer p is the *multiplicity* of the logarithmic transformation, γ the *direction* and q the *auxiliary multiplicity*.

If $p = 1$, then we call $X(T, 1, q, \gamma)$ a *q-fold Dehn twist* of $\partial\nu(T)$ along T parallel to γ.

Lemma 1 (Lemma 2.2 in [10]). *Suppose* $N = D^2 \times S^1 \times S^1$ *is embedded in a* 4*-manifold* X. *Suppose there is a disk* $D \subset X$ *intersecting* N *precisely in* $\partial D = \{q\} \times S^1$ *for some* $q \in \partial D^2 \times S^1$, *and that the normal framing of* D *in* X *differs from the product framing on* $\partial D \subset \partial N$ *by* ± 1 *twist. Then the diffeomorphism type of* X *does not change if we remove* N *and reglue it by a k-fold Dehn twist of* ∂N *along* $S^1 \times S^1$ *parallel to* $\gamma = \{q\} \times S^1$.

The submanifold $N \cup \nu(D)$ in Lemma 1 is diffeomorphic to the fishtail neighborhood F. Lemma 1 implies the following.

Lemma 2. *Let* X *be a* 4*-manifold containing* F. *Then a k-fold Dehn twist of a neighborhood of the general fiber parallel to the vanishing cycle of the fishtail fiber does not change the differential structure.*

3. Knot surgery case

3.1. 1-strand twist. Let X be a 4-manifold containing C, K_1 any knot in S^3, and K_2 the meridian of K_1. The torus $T_2 := K_2 \times S^1 \subset [S^3 - \nu(K_1)] \times S^1 \subset X_{K_1}$ has self-intersection 0. We denote the trivial normal bundle by $N_2 := \nu(K_2) \times S^1$.

Definition 6 (1-strand twist). We call the n-fold Dehn twist $X_{K_1}(T_2, 1, n, K_2)$ *(n-fold)* 1*-strand twist* of X_{K_1} along K_2.

Lemma 3. *The n-fold* 1*-strand twist of* X_{K_1} *along* K_2 *does not change the differential structure.*

Proof. Any parallel copy $K_2' \subset \partial N_2$ of K_2 moved through the use of obvious trivialization of N_2 is isotopic to one of vanishing cycles of C_{K_1}. Thus there exists a disk $D \subset C_{K_1}$ with $\partial D = K_2'$ whose framing of ∂D coming from the trivialization of $\nu(D)$ differs from the normal framing of the trivialization of N_2 by -1. Hence $N_2 \cup \nu(D)$ is diffeomorphic to the fishtail neighborhood.

Therefore Lemma 2 gives the following:

$$X_{K_1,n} \cong X_{K_1,0} = X_{K_1}. \qquad \square$$

This diffeomorphism can also be seen using handle calculus as in Figure 3, which was pointed out by S. Akbulut in [2]. The left in Figure 3 is the 4_1 surgery of the cusp neighborhood. The dashed circle in Figure 3 is the inverse image of $\{\mathrm{pt}\} \times \partial D^2$ via φ_0 (see (2)). Sliding the top -1-framed 2-handle over one of two 0-framed 2-handles below, we get the right-top one in Figure 3. Sliding the upper 0-framed 2-handle over the -1-framed 2-handle, we have the right-bottom picture. This diffeomorphism changes the gluing map φ_0 to φ_1. Iterating the process or the inverse one, we obtain Lemma 3.

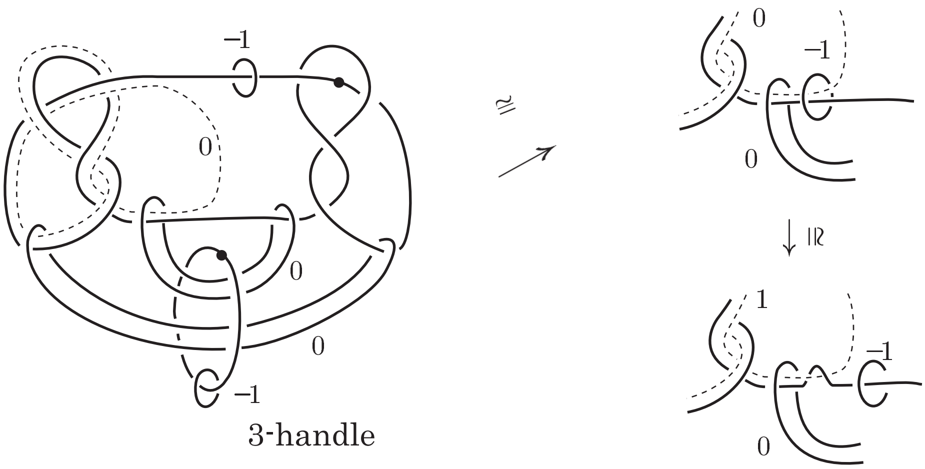

Fig. 3. A diagram C_{4_1} as an example with the attaching circle (the dashed circle) and the framing change.

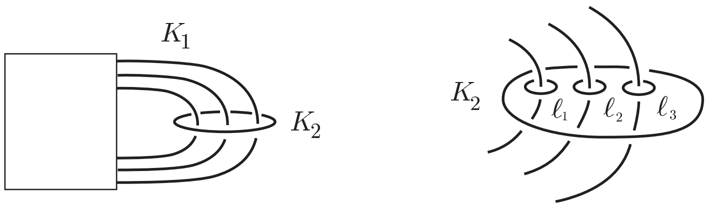

Fig. 4. $L = K_1 \cup K_2$ and ℓ_1, ℓ_2, ℓ_3.

3.2. 3-strand twist. Finding a hidden fishtail neighborhood in SyF_K or SyC_K, we give a diffeomorphism using 3-strand twist.

Let L be a 2-component link as in Figure 4. The left box is some tangle which presents K_1. Let X be a 4-manifold containing SyC or SyF. Along the general torus fiber in the fibration, we perform the knot surgery X_{K_1}. The torus $T_2 = K_2 \times S^1 \subset [S^3 - \nu(K_1)] \times S^1$ has the trivial neighborhood in X_{K_1}. We denote the neighborhood of the torus by N_2.

Definition 7 (3-strand twist). Let X be a 4-manifold containing C or F. We call the n-fold Dehn twist $X_{K_1}(T_2, 1, n, K_2)$ *(n-fold)* 3-*strand twist* along K_2.

Lemma 4. *For a manifold X containing SyC or SyF, the* 3-*strand twist of X_{K_1} along K_2 does not change the differential structure.*

Proof. Our main strategy here is to construct a fishtail neighborhood in which $K_2 \times S^1$ is a general fiber. Here we can find an obvious three-punctured

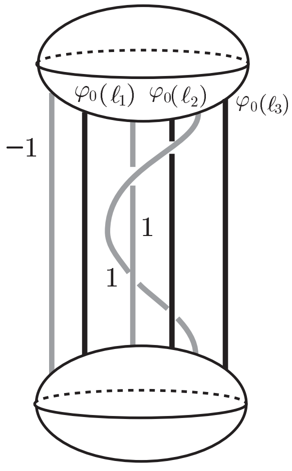

Fig. 5. An isotopy of $\varphi_0(\ell_i)$.

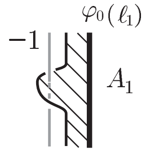

Fig. 6. A_1.

disk P whose boundaries are K_2, ℓ_1, ℓ_2, and ℓ_3 as indicated in Figure 4. Here each meridian ℓ_i lies in the boundary of N_1 which is the neighborhood of K_1. Figure 5 describes the submanifold of SyF and SyC in Figure 2 which is modified as follows. We take the middle 1-handle and two 2-handles running the 1-handle in Figure 2, and add a 1-framed 2-handle, which is cancelled with a 3-handle by one slide to another 1-framed 2-handle. Each image $\varphi_0(\ell_i)$ is parallel to two vanishing cycles of SyC or SyF in X_{K_1} as in Figure 5.

We construct mutually disjoint three annuli A_1, A_2 and A_3 such that one component of each ∂A_i is $\varphi_0(\ell_i)$. In addition, these annuli and P are also disjoint because P is embedded in the $[S^3 - \nu(K_1)] \times S^1$ part. A_1 is indicated in Figure 6 and the right side of ∂A_1 is $\varphi_0(\ell_1)$. A_2 and A_3 are indicated in the left and right in Figure 7 respectively. A_3 runs through the carved 2-handle (the dotted 1-handle) once. The right sides of ∂A_2 and ∂A_3 are $\varphi_0(\ell_2)$ and $\varphi_0(\ell_3)$. From the pictures obviously A_1, A_2 and A_3 are disjoint annuli in X_{K_1}.

The other sides of ∂A_i coincide with the boundaries of 2-disks parallel to the cores of the 2-handles in Figure 5. The three 2-disks are disjoint from $P \cup A_1 \cup A_2 \cup A_3$ since these 2-handles are disjoint from P and A_i. Capping

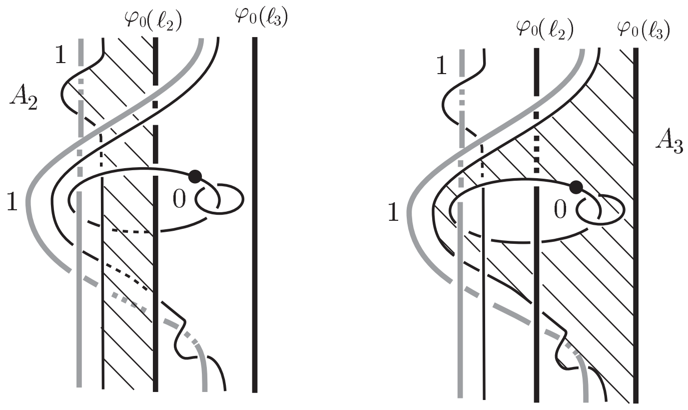

Fig. 7. Two embedded annuli A_2, A_3.

the 2-disks C_1, C_2 and C_3 to three components of $\partial(P \cup A_1 \cup A_2 \cup A_3) - K_2$, we obtain an embedded disk

$$D := P \cup A_1 \cup A_2 \cup A_3 \cup C_1 \cup C_2 \cup C_3$$

in X_{K_1} whose boundary is K_2.

The restriction on $\partial v(D)$ of the normal framing of $v(D)$ differs from the framing of K_2 induced by the normal bundle of N_2 by $-1+1+1=1$. Therefore $N_2 \cup v(D)$ is diffeomorphic to $\overline{F}$.

Alternatively, sliding the canceling 0-framed 2-handle to the -1-framed 2-handle, we can construct an embedding $F \hookrightarrow X_{K_1}$, in which the general fiber of F is T_2.

Applying Lemma 2 to this situation, we obtain the assertion of Lemma 4. □

For a 4-manifold X satisfying the assumption of Lemma 4, we can also prove that any odd-strand twist does not change the differential structure.

3.3. Proof of Theorem 1.

Proof. Since $\overline{C} \cup C$ includes SyC as in Figure 1, the 3-strand twist of A_{K_1} along K_2 does not change the differential structure, namely we have $A_{K_1} \cong \overline{C} \cup C_{K_3,n}$. The integer n is one of ∓ 1, ∓ 9. K_3 is the knot obtained by the ± 1-Dehn surgery along K_2 as in Figure 8. By using 1-strand twist in Section 3.1 we have $A_{K_3} \cong \overline{C} \cup C_{K_3,n} \cong A_{K_1}$.

Y. Ohyama in [14] proved that for any knot K there exists a finite sequence of local 3-strand twists: $K = k_0 \to k_1 \to \cdots \to k_n =$ unknot. The

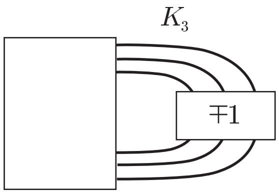

Fig. 8. K_3: The knot obtained by ± 1-Dehn surgery along K_2. The right box is the ∓ 1 full twist.

sequence implies a sequence of 4-dimensional diffeomorphisms:

$$A_K = A_{k_0} \cong A_{k_1} \cong \cdots \cong A_{k_n} = S^2 \times S^2. \qquad \square$$

The argument in the proof of Theorem 1.1 can be summarized as follows:

Lemma 5. *Any knot surgery of any achiral elliptic fibration containing SyF (or SyC) does not change the differential structure.*

Y. Matsumoto's achiral elliptic fibration on S^4 in [12] includes SyF. The handle picture can be seen in Figure 8.38 in [9].

Corollary 1. *Any knot surgery along a general fiber in Matsumoto's elliptic fibration on S^4 (such that the meridian of the knot is isotopic to the vanishing cycle) is diffeomorphic to the standard S^4.*

3.4. Infinitely many exotic embeddings. Using the diffeomorphism, we obtain infinitely many embeddings:

$$C \hookrightarrow C \cup \overline{C_K} = S^2 \times S^2. \tag{3}$$

We can obtain the following:

Corollary 2. *There exist infinitely many (mutually non-diffeomorphic) exotic embddings $C \hookrightarrow S^2 \times S^2$. Namely the embeddings give infinitely many exotic complements.*

Proof. We show that the complements $\overline{C_K}$ of the embeddings (3) give infinitely many mutually homeomorphic but non-diffeomorphic 4-manifolds. The cusp neighborhood C is embedded in K3 surface $E(2)$ as a neighborhood of a singular fiber of the elliptic surface. The group of self-diffeomorphisms up to isotopy on $\partial C \cong \Sigma(2,3,6)$ is $\mathbf{Z}/2\mathbf{Z}$ in the same way as the proofs of Lemma 8.3.10 in [9] and Lemma 3.7 in [11]. The nontrivial self-diffeomorphism is a 180° rotation of ∂C about the horizontal line in the top-right picture in Figure 2. Since the diffeomorphism is caused by a symmetry on 0-framed trefoil, this diffeomorphism extends to $E(2)$ (see also the proof of Theorem 0.1 in [3]).

Thus, if $E(2)_{K_1}$ and $E(2)_{K_2}$ are non-diffeomorphic for some knots K_1, K_2, then C_{K_1} and C_{K_2} are non-diffeomorphic. The formula (1) and $SW_{E(2)} = 1$ give infinitely many differential structures in $\{C_K | K : \text{knot}\}$. The homeomorphism $C \approx C_K$ for any knot K is due to the fact $C \cup \overline{C_K} \cong S^2 \times S^2$ (spin) and the result (0.8) Proposition-(iii) in [6]. Therefore $\{C_K | K : \text{knot}\}$ includes infinitely many differential structures. □

4. Link surgery case

In this section we draw a handle picture of the link surgery operation $X(C, \dots, C; L)$ in the cases where L is a split link and is the Hopf link. Finally we will prove A_L is the standard manifold (Theorem 2).

Let $L = K_1 \cup \cdots \cup K_n$ be an n-component link and X_i $(i = 1, \dots, n)$ oriented 4-manifolds each of which contains the cusp neighborhood C_i. Let T_i be a general fiber of C_i. Let φ_i be the maps

$$\varphi_i : \partial \bar{v}(K_i) \times S^1 \to \partial \bar{v}(T_i) = T_i \times \partial D^2$$

satisfying

$$\varphi_i(l_i \times \{\text{pt}\}) = \{\text{pt}\} \times \partial D^2$$

$$\varphi_i(m_i \times \{\text{pt}\}) = \alpha_i, \qquad \varphi_i(\{\text{pt}\} \times S^1) = \beta_i,$$

where l_i and m_i are the longitude and meridian of K_i and α_i, β_i are two circles in $\partial \bar{v}(T_i)$ corresponding to a basis in $H_1(T_i)$.

Definition 8. We define the *link surgery (operation)* as

$$\coprod_{i=1}^{n} X_i \mapsto [X_i - v(T_i)] \cup_{\varphi_i} [S^3 - v(L)] \times S^1.$$

Here the gluing maps are φ_i. We denote the link surgery operation of $(X_1, \dots, X_n)$ along a link L by $X(X_1, \dots, X_n; L)$.

Due to Fintushel and Stern's result [7], the SW-invariant of $X(X_1, \dots, X_n; L)$ is computed as follows:

$$SW_{X(X_1,\dots,X_n;L)} = \Delta_L(t_1, \dots, t_n) \cdot \prod_{i}^{n} SW_{E(1)\#_{T=T_i} X_i},$$

where $\Delta_L(t_1, \dots, t_n)$ is the multivariable Alexander polynomial of L and $E(1)\#_{T=T_i} X_i$ is the fiber-sum of the elliptic fibration $E(1)$ and X_i along general fibers T and T_i respectively. The definition of the fiber-sum can be seen in [7].

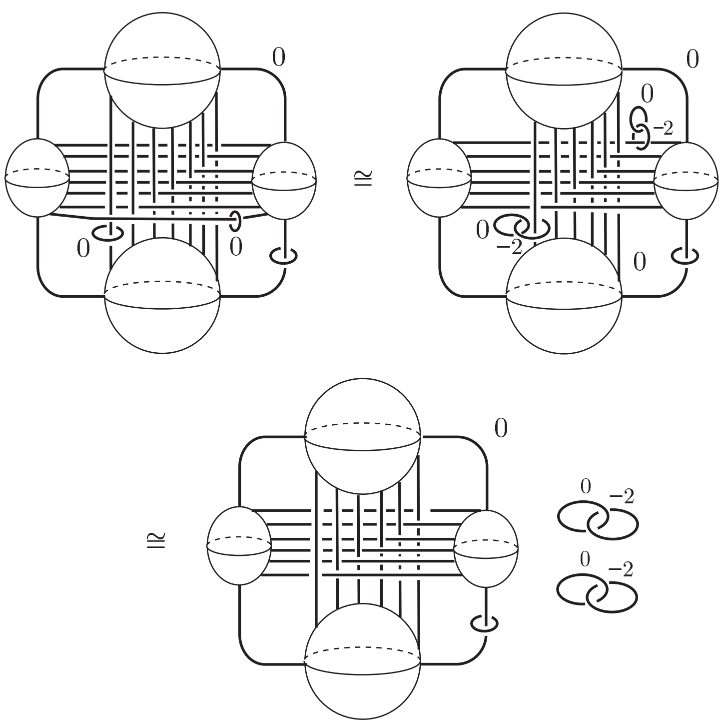

Fig. 9. $E(1)\#_{T=T_i}S^2 \times S^2 = E(1)\#^2 S^2 \times S^2$

Here we consider the link surgery operation of $\coprod_{i=1}^{n} S^2 \times S^2$ along any n-component link L. We denote the operation by A_L. The following diffeomorphism

$$E(1)\#_{T=T_i}S^2 \times S^2 \cong E(1)\#^2 S^2 \times S^2 = \#^3\mathbf{C}P^2\#^{11}\overline{\mathbf{C}P^2} \tag{4}$$

holds. The diagram of the fiber-sum $E(1)\#_{T=T_i}S^2 \times S^2$ is the leftmost figure in Figure 9 (where the diagram of $E(1)$ is Figure 8.10 in [9]). Several handle slides get two connected-sum components of $S^2 \times S^2$ (see Figure 9). The second equality in (4) is well-known. Thus the vanishing theorem of SW-invariant implies $SW_{A_L} = 0$.

We prepare several lemmas to prove Theorem 2.

Lemma 6. *Let $L = U_1 \cup U_2$ be a* 2*-component unlink. Then the handle picture of $X(C, C; L)$ is Figure* 11.

Suppose that $L = L_1 \cup L_2$ is any split link. Then the handle picture of $X(C, C; L)$ is obtained by replacing the two dotted 1*-handles in Figure* 11 *with the slice* 1*-handles corresponding to L_1 and L_2.*

In particular, in the case where $L = L' \cup U$ is an n-component link and U is a split unknot,

$$A_{L' \cup U} \cong A_{L'} \#^2 S^2 \times S^2.$$

Proof. Let $L = K_1 \cup K_2$ be a split link. First we consider the case where K_1, K_2 are both unknots U_1, U_2. Let D_1 and D_2 be the splitting 3-disks of U_1 and U_2 satisfying $D_1 \cup D_2 = S^3$, $D_1 \cap D_2 = S^2$, and $U_i \subset \mathrm{int}(D_i)$. Then we get a decomposition $[S^3 - \nu(L)] \times S^1 = [(D_1 - \nu(U_1)) \cup (D_2 - \nu(U_2))] \times S^1$. Each component $[D_i - \nu(U_i)] \times S^1$ is diffeomorphic to $D^2 \times S^1 \times S^1 - \nu(\beta_i)$ (see Figure 10), where β_i is $\{p_i\} \times S^1$ and p_i is a point in $D^2 \times S^1$.

The handle picture of $D^2 \times T^2 - \nu(\beta_1)$ is the left in Figure 13. The $S^2 \times S^1$ component $\partial \nu(\beta_1)$ of the boundary corresponds to the cylinder in the picture. The gluing of $D^2 \times T^2 - \nu(\beta_1)$ and $D^2 \times T^2 - \nu(\beta_2)$ along the $S^2 \times S^1$ component using the identity map has the handle picture of the right in Figure 13. With the dotted 1-handles description, the handle picture of $X(C, C; L)$ is Figure 11. Two boundary components of $X(C, C; L)$ are described as two spaces segmented by the attaching sphere of the 3-handle in Figure 11.

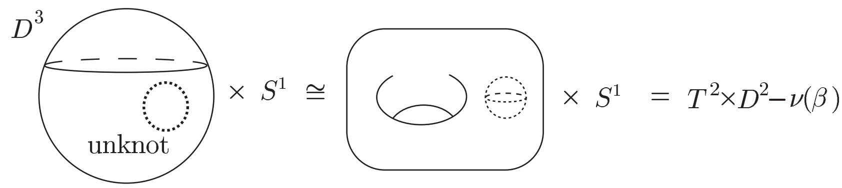

Fig. 10. $[D^3 - \nu(\text{unknot})] \times S^1 \cong D^2 \times T^2 - \nu(\beta)$

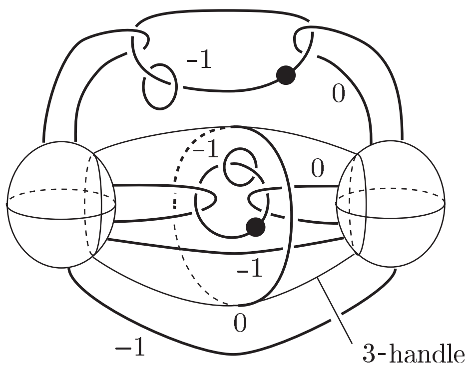

Fig. 11. The handle picture of $X(C, C; U_0 \cup U_1)$.

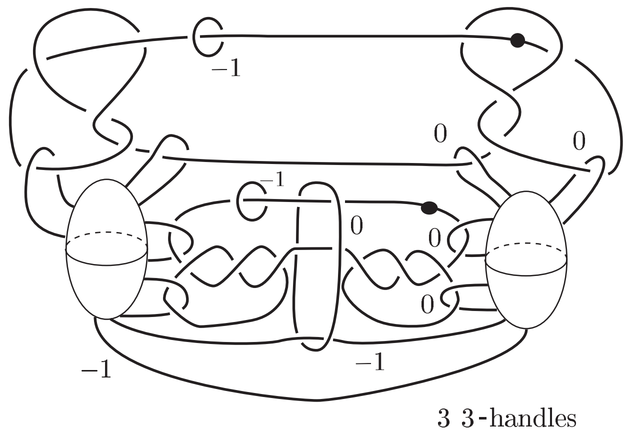

Fig. 12. $X(C, C; 3_1 \coprod 4_1)$

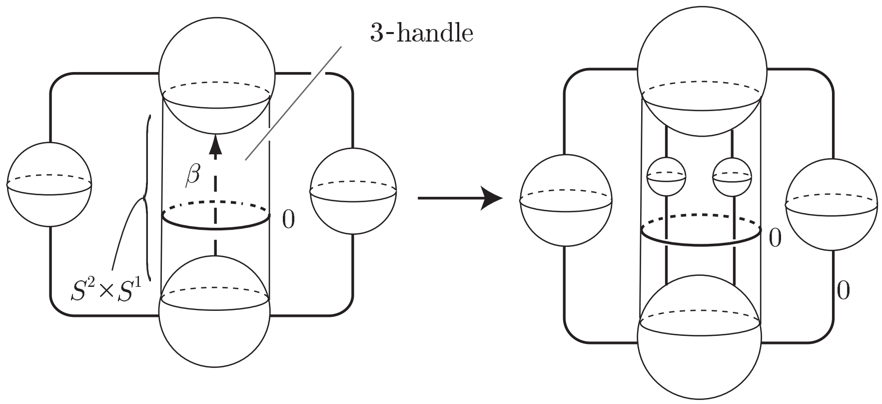

Fig. 13. $T^2 \times D^2 - \nu(\beta) \to (T^2 \times D^2 - \nu(\beta_1)) \cup (T^2 \times D^2 - \nu(\beta_2))$.

In the case where $L = K_1 \cup K_2$ is any split 2-component link, the handle picture of $X(C, C; L)$ can be drawn replacing the solid torus in Figure 10 with the knot complement $D^3 - \nu(K_i)$. The replacement of handle pictures can be viewed as in [3]. For example in the case of $K_1 = 3_1$ and $K_2 = 4_1$, the handle picture is Figure 12.

In particular if K_2 is the unknot, then A_L gives rise to two connected-sum components of $S^2 \times S^2$, as can be seen in Figures 14 and 15, therefore $A_{L' \cup U} \cong A_{L'} \#^2 S^2 \times S^2$ holds. The unlabeled links in the figures stand for 0-framed 2-handles. □

Next we draw a handle picture of $X(C, C; H)$ for the Hopf link and we compute A_H.

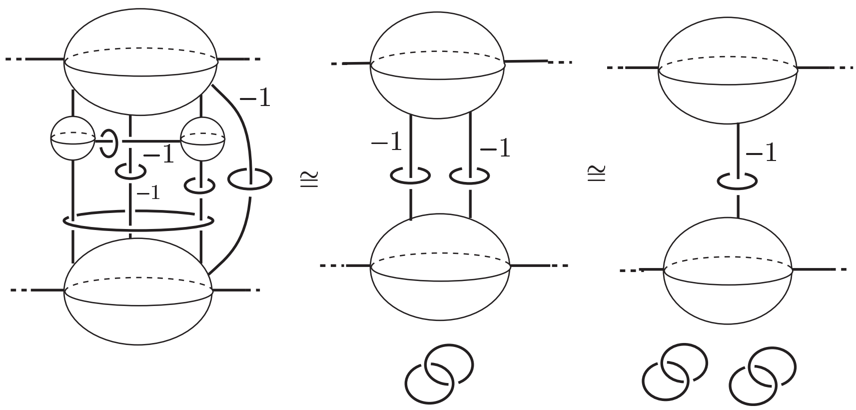

Fig. 14. The handle picture of $A_{L' \cup U} = A_{L'} \#^2 S^2 \times S^2$.

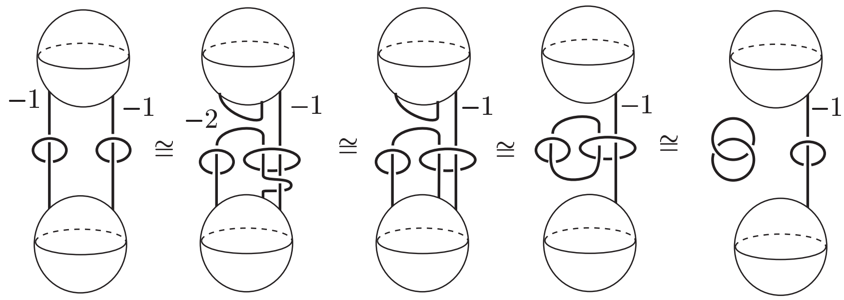

Fig. 15. To make an $S^2 \times S^2$-component from two parallel -1-framed 2-handles.

LEMMA 7. *Let H be the Hopf link. Then A_H is diffeomorphic to* $\#^3(\mathbf{C}P^2 \# \overline{\mathbf{C}P^2})$.

PROOF. The complement $[S^3 - \nu(H)] \times S^1$ is diffeomorphic to $T^3 \times I$ (the left in Figure 16), where I is the interval $[0,1]$ and the unlabeled links are 0-framed 2-handles.

Since the meridians and longitudes of the Hopf link exchange the roles each other, the locations of vanishing cycles are $\alpha = (S^1, \mathrm{pt}, \mathrm{pt}, 0)$, $\beta = (\mathrm{pt}, S^1, \mathrm{pt}, 0)$, $\beta' = (\mathrm{pt}, S^1, \mathrm{pt}, 1)$, and $\eta = (\mathrm{pt}, \mathrm{pt}, S^1, 1)$. Attaching four -1-framed 2-handles to $T^3 \times I$, we get the picture of $X(C, C; H)$ (the right in Figure 16). Next, attaching four vanishing cycles with opposite orientation (four meridional 0-framed 2-handles), and two sections (two 0-framed 2-handles) to two boundaries of $X(C, C; H)$, we get A_H (the top-left handle decomposition in Figure 17). The decomposition can be modified into the top-right picture in Figure 17 by two handle slides as indicated in the top-left picture. The

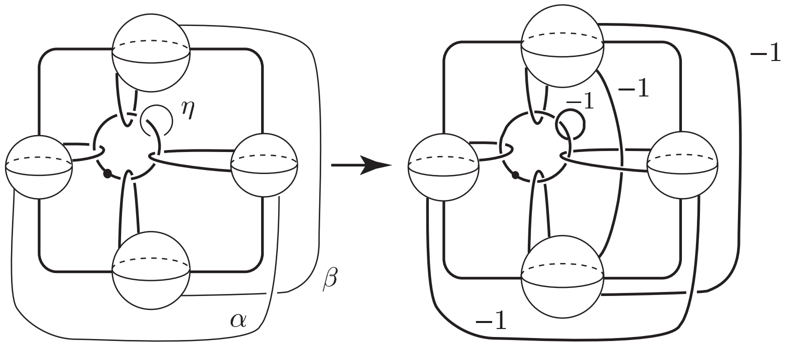

Fig. 16. $T^2 \times S^1 \times I \to X(C, C; H)$.

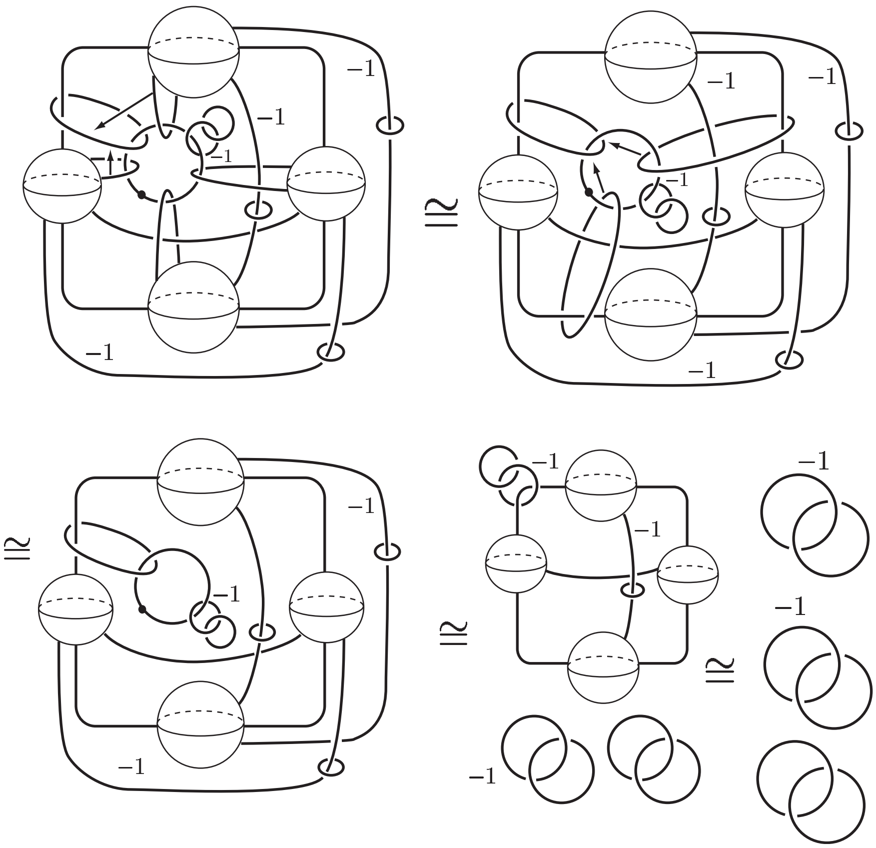

Fig. 17. The handle picture of $A_H = \#^3(\mathbf{C}P^2 \# \overline{\mathbf{C}P^2})$.

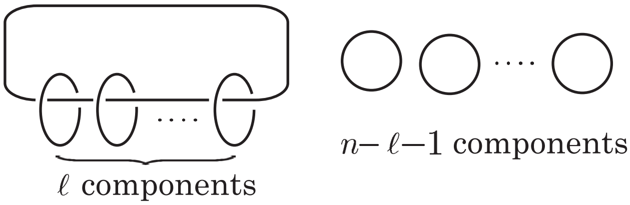

Fig. 18. The representatives $L_{n,\ell}$ ($\ell = 0, \dots, n-1$) of $\mathscr{L}_n$

resulting picture can be modified into the bottom-left picture by two handle slides indicated by the two arrows in the top-right picture. Two (unlinked) 0-framed 2-handles obtained by this modification are canceled with two 3-handles. By applying Figure 15 and easy handle calculus, the bottom-left picture can be modified into the bottom-middle picture in Figure 17. This picture is the diagram of $\#^3(\mathbf{C}P^2\#\overline{\mathbf{C}P^2})$ using handle calculus. □

At this point we can prove Theorem 2.

Proof. Let $L = K_1 \cup K_2 \cup \cdots \cup K_n$ be any n-component link. The set $\tilde{\mathscr{L}}_n$ of all n-component links up to local 3-strand twist consists of 2^{n-1} classes due to Nakanishi and Ohyama's results [14, 13]. Forgetting the ordering of the components of any link in $\tilde{\mathscr{L}}_n$, we get a set $\mathscr{L}_n$. The set $\mathscr{L}_n$ has n classes. A standard representative in each class is a link $L_{n,\ell}$ ($\ell = 0, 1, \dots, n-1$) as presented in Figure 18. Applying 3-strand twist to link surgery operation A_L, we have only to consider the diffeomorphism type of $A_{L_{n,\ell}}$ for some ℓ.

Notice that $L_{n,0}$ is the representative of all proper links ($\overset{\text{def}}{\Leftrightarrow} \sum_{i \neq j} lk(K_i, K_j) \equiv 0 \pmod 2\ {}^{\forall}i$) and $L_{n,\ell}$ ($\ell > 0$) are the representatives of improper link ($\overset{\text{def}}{\Leftrightarrow}$ not proper link).

Now suppose that $1 \leq \ell \leq n-2$. Applying Lemma 6 to the $(n-\ell-1)$-component unlink, we have

$$A_{L_{n,\ell}} = A_{L_{\ell+1,\ell}} \#^{2(n-\ell-1)} S^2 \times S^2.$$

Since ℓ parallel meridians in the remaining components construct a fiber-sum of ℓ copies of SyC, by using Figure 15 we have

$$A_{L_{\ell+1,\ell}} = A_H \#^{2(\ell-1)} S^2 \times S^2.$$

Using Lemma 7, we have

$$\begin{aligned} A_{L_{n,\ell}} &= \#^3(\mathbf{C}P^2\#\overline{\mathbf{C}P^2})\#^{2(\ell-1)} S^2 \times S^2 \#^{2(n-\ell-1)} S^2 \times S^2 \\ &= \#^{2n-1}(\mathbf{C}P^2\#\overline{\mathbf{C}P^2}). \end{aligned}$$

Suppose that $\ell = 0$. The link $L_{n,0}$ is the n-component unlink. Thus, using Lemma 6 we have

$$A_{L_{n,0}} = S^2 \times S^2 \#^{2(n-1)} S^2 \times S^2 \cong \#^{2n-1} S^2 \times S^2.$$

Suppose that $\ell = n-1$. Since the link $L_{n,n-1}$ does not have unlink component,

$$A_{L_{n,n-1}} = \#^3(\mathbf{C}P^2 \# \overline{\mathbf{C}P^2}) \#^{2(n-2)} S^2 \times S^2 \cong \#^{2n-1}(\mathbf{C}P^2 \# \overline{\mathbf{C}P^2}).$$

Therefore

$$A_L \cong \begin{cases} A_{L_{n,0}} \cong \#^{2n-1} S^2 \times S^2 & L \text{ is proper} \\ A_{L_{n,\ell}} \cong \#^{2n-1}(\mathbf{C}P^2 \# \overline{\mathbf{C}P^2}) & \text{otherwise.} \end{cases}$$ □

5. Scharlemann's manifolds

Let K be a knot in S^3 and $\gamma(\varepsilon)$ an embedded framed curve in $S^3_p(K)$, i.e., γ is a simple curve in $S^3_p(K)$ and ε is a framing of γ. The framed curve $\gamma(\varepsilon)$ gives a framed curve $\tilde{\gamma}$ in $S^3_p(K) \times S^1$, as mentioned in Section 1.3. To consider the isotopy type of $\tilde{\gamma}$, it is enough to consider ε as the (mod 2)-framing. Figure 19 is an example of framed curve presentations. We identify ε with an element of $\mathbf{Z}/2\mathbf{Z}$.

Definition 9. The 0-framing is defined as the Seifert framing of a curve embedded in the surgery presentation (p-surgery along K).

Definition 10. We fix a diagram of γ in the surgery presentation of $S^3_p(K)$. Let $\gamma(\varepsilon)$ be an embedded framed curve in $S^3_p(K)$. Namely the induced framing on $\tilde{\gamma}$ gives a trivialization $t_\varepsilon : \bar{\nu}(\tilde{\gamma}) \cong D^3 \times S^1$.

We define the (ε)-*surgery* along γ as

$$B_{K,p}(\gamma(\varepsilon)) := [S^3_p(K) \times S^1 - \nu(\tilde{\gamma})] \cup_{\psi_\varepsilon} S^2 \times D^2.$$

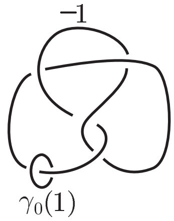

Fig. 19. A curve γ_0 with (mod 2)-framing.

The gluing map ψ_ε is the composition of the identity map $S^2 \times \partial D^2 \to \partial D^3 \times S^1$ and the restriction of t_ε^{-1} to the boundary. We call $B_{K,p}(\gamma(\varepsilon))$ *Scharlemann's manifold*. In the case of $p = -1$, we drop the suffix p.

The diffeomorphism type of $B_{K,p}(\gamma(\varepsilon))$ depends only on the homotopy type of $\gamma(\varepsilon)$ in $S^3_p(K)$. This operation coincides with taking the boundary after attaching a 5-dimensional 2-handle along $\tilde{\gamma}$ with the framing ε.

5.1. Scharlemann's manifolds along the meridian curves. In this section, we consider Scharlemann's manifolds with respect to the meridian γ_0 of K as in Figure 19. We remark the following.

Remark 1. Let γ_0 be the meridian circle in $S^3_{-1}(K)$. All Scharlemann's manifolds $B_K(\gamma_0(0))$ are diffeomorphic to $S^3 \times S^1 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$.

In the case of $\varepsilon = 1$, we note the relationship between $B_K(\gamma_0(1))$ and the knot surgery of the fishtail neighborhood.

Lemma 8. *$B_K(\gamma_0(1))$ is diffeomorphic to $\overline{F} \cup F_K$.*

Proof. Performing the knot surgery for $\overline{F} \cup F$, we have

$$\overline{F} \cup F_K = \overline{F} \cup [F - \nu(T)] \cup_{\varphi_0} [(S^3 - \nu(K)) \times S^1].$$

The handle picture is Figure 20 (the case of $K = 4_1$).

The surgery along $\tilde{\gamma}_0$ in $S^3_{-1}(K) \times S^1$ is the right in Figure 21. Hence we get the following diffeomorphisms.

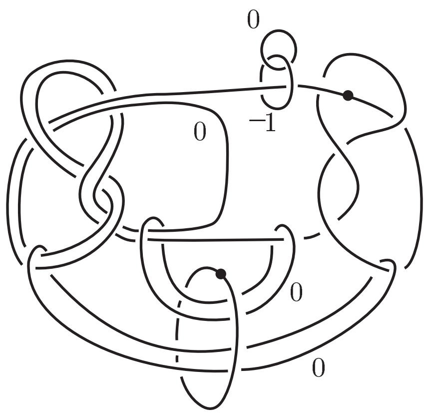

Fig. 20. $\overline{F} \cup [F - \nu(T)] \cup_{\varphi_0} [(S^3 - \nu(K)) \times S^1]$.

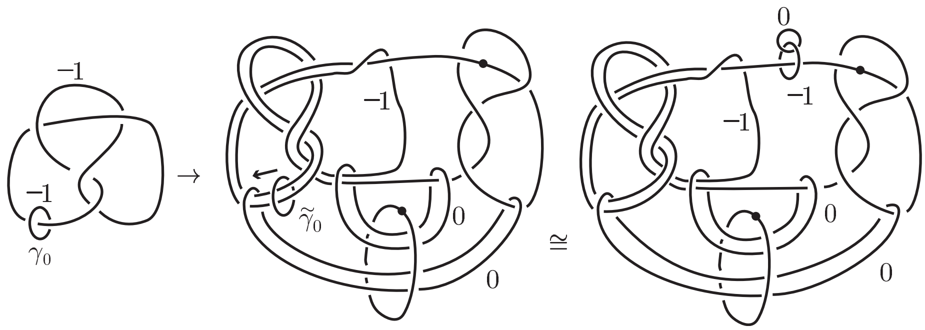

Fig. 21. The surgery along $\tilde{\gamma}_0$ with the framing 1.

$$\begin{aligned} B_K(\gamma_0(1)) &= [S^3_{-1}(K) \times S^1 - \nu(\tilde{\gamma}_0)] \cup_{\psi_1} S^2 \times D^2 \\ &\cong \bar{F} \cup (F - \nu(T)) \cup_{\varphi_{-1}} [S^3 - \nu(K)] \times S^1 \qquad \text{(See Figure 3 and 21.)} \\ &\cong \bar{F} \cup (F - \nu(T)) \cup_{\varphi_0} [S^3 - \nu(K)] \times S^1 \qquad \text{(Lemma 3)} \\ &= \bar{F} \cup F_K \qquad \square \end{aligned}$$

Here we prove Theorem 4.

PROOF. Since $\bar{F} \cup F$ contains SyF, the application of Lemma 5 to this situation gives the following:

$$\bar{F} \cup F_K \cong \bar{F} \cup F \cong S^3 \times S^1 \# S^2 \times S^2.$$

Here the last diffeomorphism is due to Figure 22. □

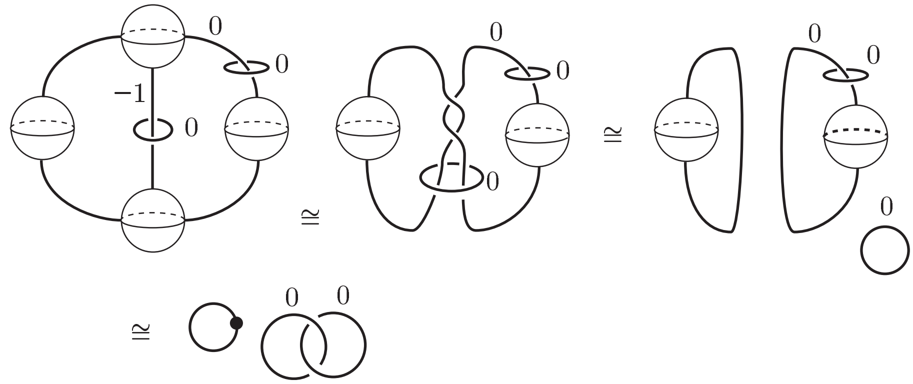

Fig. 22. $F \cup \bar{F} = S^3 \times S^1 \# S^2 \times S^2$.

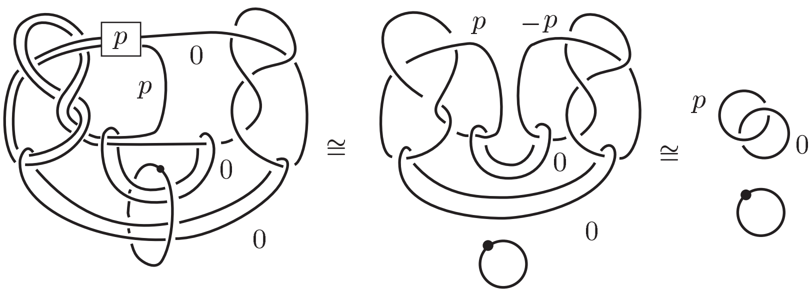

Fig. 23. $B_{K,p}(\gamma_0(0))$.

Corollary 3. *Let γ_0 be a meridian of K in the surgery presentation of $S^3_p(K)$. $B_{K,p}(\gamma_0(\varepsilon))$ is classified as follows:*

$$B_{K,p}(\gamma_0(\varepsilon)) = \begin{cases} S^3 \times S^1 \# S^2 \times S^2 & (\varepsilon - 1)p \equiv 0 \ (2) \\ S^3 \times S^1 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2} & (\varepsilon - 1)p \equiv 1 \ (2). \end{cases}$$

Proof. In the case of $\varepsilon = 1$, using the 1-strand twist, we have

$$B_{K,p}(\gamma_0(1)) \cong B_K(\gamma_0(1)) \cong S^3 \times S^1 \# S^2 \times S^2.$$

In the case of $\varepsilon = 0$, in the same way as Remark 1, we obtain

$$B_{K,p}(\gamma_0(0)) \cong \begin{cases} S^3 \times S^1 \# S^2 \times S^2 & p \equiv 0 \ (2) \\ S^3 \times S^1 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2} & p \equiv 1 \ (2) \end{cases}$$

(see Figure 23). □

Remark 2. $B_K(\gamma_0(1))$ is obtained from A_K as a surgery along an embedded S^2. The neighborhood of the sphere Σ is the union of the bottom 0-framed 2-handle and the 4-handle (the left of Figure 24). Attaching the 3-handle and 4-handle to the complement gets $B_K(\gamma_0(1))$ (the right of Figure 24). The circle δ in Figure 24 is the core circle of $S^1 \times D^3$ attached.

Remark 3. In [4] Akbulut got a plug twisting $(W_{1,2}, f)$ satisfying $E(1) = N \cup_{\mathrm{id}} W_{1,2}$ and $E(1)_{2,3} = N \cup_f W_{1,2}$. The definitions of plug, N and $W_{1,2}$ are written down in [4]. In the same way as [4] we can also show that there exist infinitely many plug twistings $(W_{1,2}, f_K)$ of $E(1)$ with the same plug $W_{1,2}$. As a result each of such plug twistings satisfies $E(1) = M \cup_{\mathrm{id}} W_{1,2}$ and $E(1)_K = M \cup_{f_K} W_{1,2}$. Infinite variations of Alexander polynomial imply the existence of infinite embeddings $W_{1,2} \hookrightarrow M \cup_{id} W_{1,2} = E(1)$.

5.2. Scharlemann's manifold along non-meridian curves. In this section we consider $B_{3_1}(\gamma(\varepsilon))$ in the case where γ is not homotopic to the meridian curve.

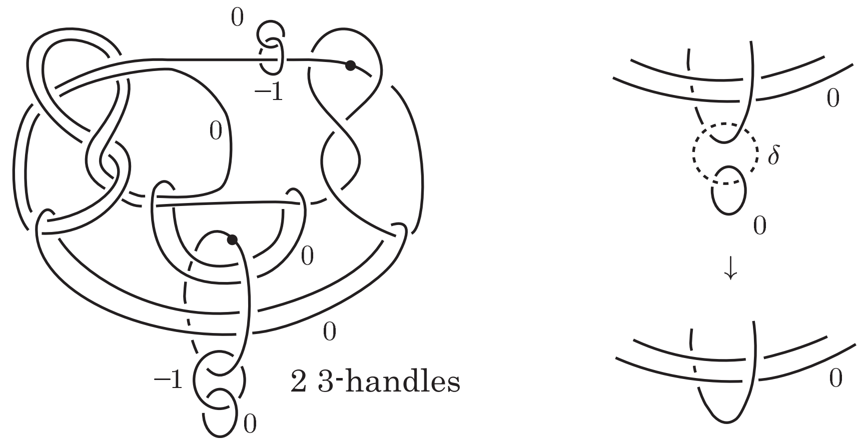

Fig. 24. The left: A_K. The right: surgery $B_K(\gamma_0(-1)) \cong [A_K - \nu(\Sigma)] \cup S^1 \times D^3$.

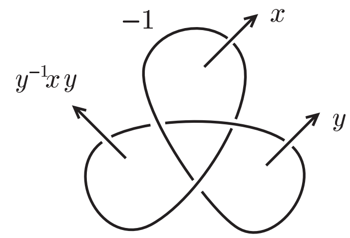

Fig. 25. The generators x, y of $\pi_1(S^3_{-1}(3_1))$.

The fundamental group of $S^3_{-1}(3_1)$ is isomorphic to

$$\pi = \pi_1(S^3_{-1}(3_1)) = \langle x, y \mid x^5 = (xy)^3 = (xyx)^2 \rangle \cong \tilde{A}_5.$$

These elements x, y are two generators as in Figure 25.

The set

$$[S^1, S^3_{-1}(3_1)] = \pi/\text{conj}. \tag{5}$$

of free homotopy classes of maps $S^1 \to S^3_{-1}(3_1)$ possesses 9 classes as follows:

Classes	$[e]$	$[x^5]$	$[xyx]$	$[x]$	$[x^2]$	$[x^3]$	$[x^4]$	$[xy]$	$[(xy)^2]$
Orders	1	2	4	10	5	10	5	6	3

Each of the classes is a normal generator of the fundamental group except for $[e]$, $[x^5]$. Since $[x]$ corresponds to the meridian curve, this case is already classified. We take a concrete presentation of $\gamma(\varepsilon)$ in $S^3_{-1}(3_1)$, and regard the

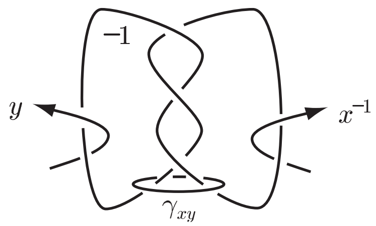

Fig. 26. γ_{xy}

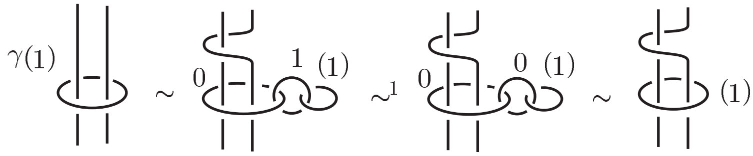

Fig. 27. A full-twist along $\gamma(1)$.

presentation as the diffeomorphism type of $B_{3_1}(\gamma(\varepsilon))$. We prove the case of $[xy]$.

Proposition 1. *Let γ_{xy} be a presentation in Figure 26, where $[\gamma_{xy}] = [xy]$. $B_{3_1}(\gamma_{xy}(1))$ is diffeomorphic to $S^3 \times S^1 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$.*

Here we define some notations in the diagrams for the proofs. The curves with (1) or (0) mean the (1) or (0)-surgery along the curves. The notations $\sim$ and $\sim^1$ throughout this section stand for some 4-dimensional diffeomorphism induced from a 3-manifold homeomorphism and a 1-strand twist, respectively.

By using 3-dimensional diffeomorphisms and 1-strand twists we get the diffeomorphism as in Figure 27. We can extend Figure 27 to any twist along $\gamma(1)$ as follows:

Lemma 9 (A full-twist along $\gamma(1)$). *A full-twist of any number of strands along $\gamma(1)$ does not change the diffeomorphism type of the 4-manifold: If a framed link $(K'; p')$ is obtained from $(K; p)$ by a full-twist along $\gamma(1)$, then $B_{K',p'}(\gamma(1))$ is diffeomorphic to $B_{K,p}(\gamma(1))$. We call such a deformation a full-twist along $\gamma(1)$.*

Proof. A Dehn twist (that is, 1-strand twist as in Lemma 3) along a curve parallel to γ does not change the differential structure because $\gamma(1)$ plays a role in the vanishing cycles in a fishtail neighborhood. □

Remark 4. To avoid reader's confusion, we must note on the difference between two kinds of twists (see Figure 28):

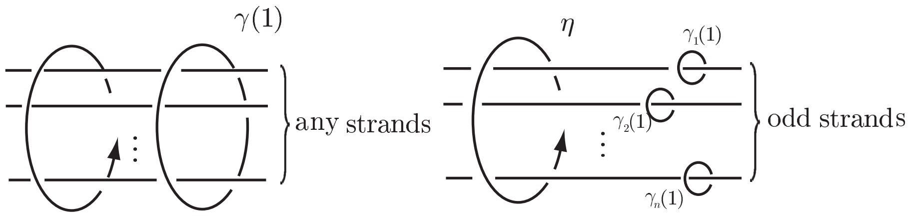

Fig. 28. A full-twist along $\gamma(1)$ and odd-strand twist ($n \equiv 1(2)$).

a full-twist along $\gamma(1)$ (Lemma 9);

an odd-strand twist (Definition 7).

The former (the left picture in Figure 28) is a full-twist along a curve isotopic to γ in Lemma 9. Even if any number of strands pierce a disk bounded by γ, we can get the diffeomorphism by the twist along γ. The latter (the right picture in Figure 28) is a full-twist along a curve η that satisfies the following: The odd strands of the former's type and the curve η are boundaries of an embedded punctured disk. Such a twist is explained in the last paragraph of Section 3.2. Even if there exists no 1-framed curve isotopic to the curve η, we can get the diffeomorphism by the twist along η. Hence a single 1-strand twist is in the intersection of two kinds of twists, and in other words, two kinds of twists above are interpreted as two types of generalizations of 1-strand twist.

Thus, Lemma 4 *cannot* be generalized to any even-strand twist case, because it is the latter's type twist. Any odd-strand twist is interpreted as 'a kind of 1-strand twist' given by a summation of odd 1-strand twists as in Figure 28 ((odd number) $\times 1 \equiv 1(2)$). This summation is due to the proof of Theorem 1. At any rate for a twist to give a 4-dimensional diffeomorphism we require an odd situation.

We use the same notation $\sim^1$ for any full-twist along $\gamma(1)$ in Lemma 9. Here we prove Proposition 1.

Proof. By using Figure 29 and Corollary 3 we have

$$B_{3_1}(\gamma_{xy}(1)) \cong B_{\text{unknot},3}(\gamma_0(0)) \cong S^3 \times S^1 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2}. \qquad \square$$

Here we will argue several other cases.

Proposition 2. *We fix presentations of* γ_{x^2}, γ_{x^3} *and* γ_{x^4} *as in the leftmost pictures in Figure* 30, 31, *and* 32 *respectively.* $B_{3_1}(\gamma_{x^2}(1))$, $B_{3_1}(\gamma_{x^3}(0))$ *and* $B_{3_1}(\gamma_{x^4}(1))$ *are diffeomorphic to* $S^3 \times S^1 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$.

Proof. In the case of $B_{3_1}(\gamma_{x^2}(1))$, by using Figure 30 and Corollary 3 we have $B_{3_1}(\gamma_{x^2}(1)) \cong S^3 \times S^1 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$.

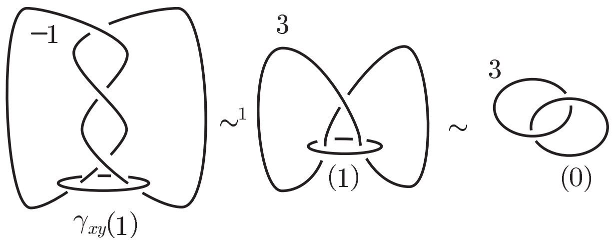

Fig. 29. $B_{3_1}(\gamma_{xy}(1)) \cong B_{\text{unknot},3}(\gamma_0(0))$.

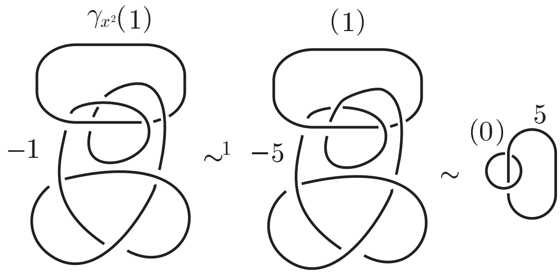

Fig. 30. $B_{3_1}(\gamma_{x^2}(1)) \cong B_{\text{unknot},5}(\gamma_0(0))$.

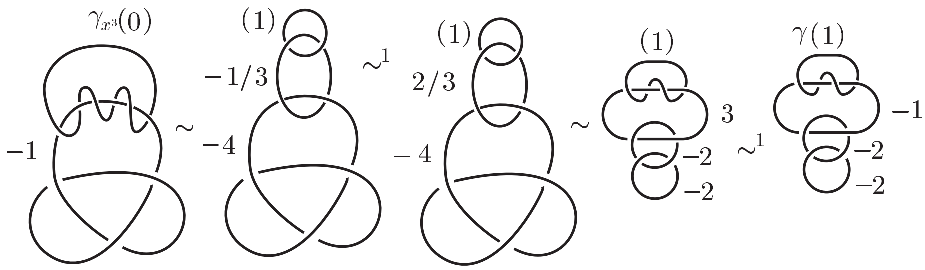

Fig. 31. The diffeomorphism for $B_{3_1}(\gamma_{x^3}(0))$.

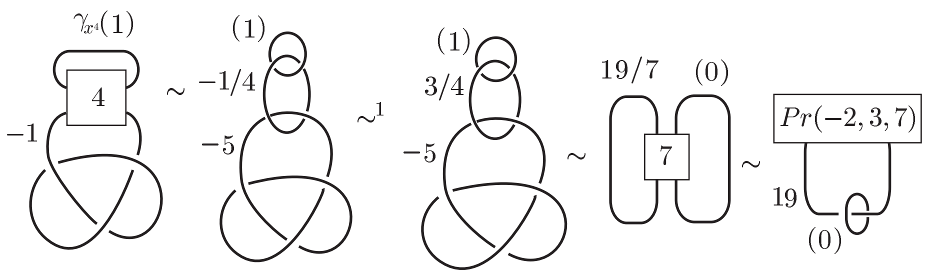

Fig. 32. The diffeomorphism for $B_{3_1}(\gamma_{x^4}(1))$.

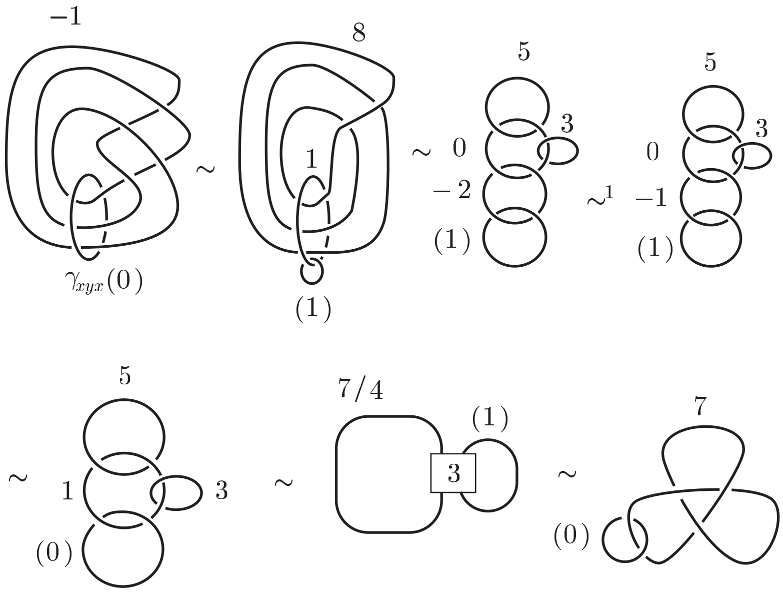

Fig. 33. The diffeomorphism for $B_{3_1}(\gamma_{xyx}(0))$.

In the case of $B_{3_1}(\gamma_{x^3}(0))$, γ in the last picture in Figure 31 presents the positive (2,7)-torus knot with the odd framing. It is obviously homotopic to the unknot. Namely the manifold $B_{3_1}(\gamma_{x^3}(1))$ is diffeomorphic to $S^3 \times S^1 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$.

In the case of $B_{3_1}(\gamma_{x^4}(1))$, the last picture in Figure 32 gives $S^3 \times S^1 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$ in the similar way. Here $Pr(-2,3,7)$ is the $(-2,3,7)$-pretzel knot. □

PROPOSITION 3. *We fix presentations γ_{xyx} and $\gamma_{(xy)^2}$ as in the leftmost pictures in Figure 33 and 34 respectively. $B_{3_1}(\gamma_{xyx}(0))$ and $B_{3_1}(\gamma_{(xy)^2}(1))$ are diffeomorphic to $S^3 \times S^1 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$.*

PROOF. In the case of $B_{3_1}(\gamma_{xyx}(0))$, the framed curve $\gamma_{xyx}(0)$ is homotopic to the curve in the first picture in Figure 33 due to $xyy^{-1}xy \sim x^2y \sim xyx$. Figure 33 implies $B_{3_1}(\gamma_{xyx}(0)) \cong S^3 \times S^1 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$.

In the case of $B_{3_1}(\gamma_{(xy)^2}(1))$ the deformation as in Figure 34 gets $S^3 \times S^1 \# \mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$. Here $T_{2,-7}$ is the negative $(2,7)$-torus knot. □

In the end of the paper we raise a question.

QUESTION 1. *In the following manifolds*

$$B_{3_1}(\gamma_{xy}(0)),\ B_{3_1}(\gamma_{x^2}(0)),\ B_{3_1}(\gamma_{x^3}(1)),\ B_{3_1}(\gamma_{x^4}(0)),\ B_{3_1}(\gamma_{xyx}(1)),\ B_{3_1}(\gamma_{(xy)^2}(0)),$$

does there exist any non-standard manifold?

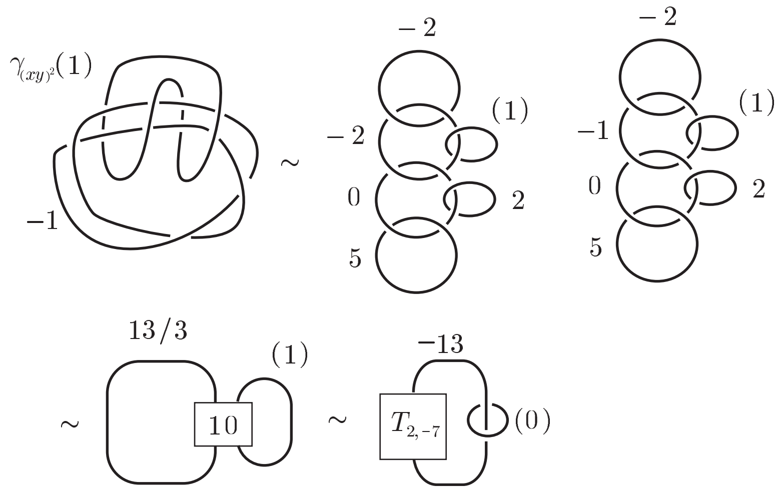

Fig. 34. The diffeomorphism for $B_{3_1}(\gamma_{(xy)^2}(1))$.

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