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Measures with maximum total exponent and generic properties of C^1 expanding maps

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Abstract. We show that a generic C^1 expanding map on a compact Riemannian manifold has a unique measure of maximum total exponent which is fully supported and of zero entropy. We also show that for $r \geq 2$ a generic C^r expanding map does not have fully supported measures of maximum total exponent.

1. Introduction

Let M be an N-dimensional, compact, connected, smooth Riemannian manifold without boundary and let T be a C^1 expanding map of M. Recall that a C^1 map $T : M \to M$ is called expanding if there exist $c > 0$ and $\lambda > 1$ such that $\|DT^n(x)v\|_{T^n x} \geq c\lambda^n \|v\|_x$ holds for any $x \in M$, $v \in T_x M$ and non-negative integer n, where $DT : TM \to TM$; $(x, v) \mapsto DT(x)v$ is the tangent map of T and $\|v\|_x$ is the norm of v induced by the Riemannian metric of M. Clearly such a map T is surjective since it is an open map and its image is compact. We denote by $\mathscr{M}(T)$ the totality of invariant Borel probability measures of T. Since the absolute value of the determinant of the matrix representation of $DT(x) : T_x M \to T_{Tx} M$ is independent of the choice of orthonormal bases of $T_x M$ and $T_{Tx} M$, we can define the determinant $|\det D(T)(\cdot)| : M \to \mathbf{R}$ of the tangent map DT. Let $J(T)(x) = |\det D(T)(x)|$ and consider the quantity

$$\lambda(T, \mu) = \int_M \log J(T)(x) d\mu.$$

We may call $\lambda(T, \mu)$ total exponent of T with respect to $\mu \in \mathscr{M}(T)$ since the following formula holds

$$\int_M \log J(T)(x) d\mu = \int_M \sum_{j=1}^{s(x)} k(j, x)\chi(j, x) d\mu,$$

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where $\{\chi(1,x) < \cdots < \chi(s(x),x)\}$ is the totality of distinct Lyapunov exponents of T at x and $k(j,x)$ is the multiplicity of $\chi(j,x)$ for each j.

Put

$$\lambda(T) = \sup\{\lambda(T,\mu) : \mu \in \mathcal{M}(T)\}.$$

An element μ in $\mathcal{M}(T)$ is called a measure with maximum total exponent if $\lambda(T,\mu) = \lambda(T)$. We denote by $\mathcal{L}(T)$ the set of measures with maximum total exponent. It is easy to show that $\mathcal{L}(T)$ is not empty since $\mathcal{M}(T)$ is compact in the weak $*$ topology and the map $\mu \mapsto \lambda(T,\mu)$ is continuous.

For any nonnegative integer r, $C^r(M,M)$ denotes the space of C^r maps on M endowed with the C^r topology. Note that a sequence T_n converges in $C^r(M,M)$ if and only if all the derivatives of T_n of order less than or equal to r converge uniformly on M. Let $\mathcal{E}^r(M,M)$ be the space of C^r expanding maps of M. Then it is easy to see that it is an open subspace in $C^r(M,M)$. Recall that a subset of a topological space X is called residual if it contains a set expressed as a countable intersection of open dense subsets in X. A topological space X is called a Baire space if any residual subset is dense in X. For a Baire space, consider a property P with respect to elements in X. We say that the property P is generic or a generic element satisfies P if there exists a residual subset each member of which satisfies P. It is well known that the topological space $C^r(M,M)$ is a Baire space, consequently, so is $\mathcal{E}^r(M.M)$.

The purpose of this paper is to prove the following theorems.

THEOREM 1. *Each of the following properties for element T in $\mathcal{E}^1(M,M)$ is generic.*

(1) *T has a unique measure with maximum total exponent.*
(2) *Any measure with maximum total exponent for T has zero entropy.*
(3) *Any measure with maximum total exponent for T is fully supported.*

On the contrary, we have the following when $r \geq 2$.

THEOREM 2. *For $r \geq 2$, a generic element in $\mathcal{E}^r(M,M)$ has no fully supported measures with maximum total exponent.*

The same kind of theorems are first proved by Jenkinson and Morris in [8] for expanding maps on the circle. Afterward, inspired by their results, the second author of this paper extended those to expanding maps of the n-torus in his dissertation [14]. We should note that these results are similar in spirit to some theorems in ergodic optimization. So we explain about some preceding results briefly (see the survey by Jenkinson in [7] for the further discussion and references). Consider a continuous map T on a compact metric space

X and a continuous function $f : X \to \mathbf{R}$. The main interests in ergodic optimization are invariant probability measures μ which maximize the integral $\int_X f \, d\mu$. Such measures are referred to maximizing measures for f. Bousch and Jenkinson showed that for a fixed expanding map of the circle, a generic continuous function has a unique maximising measure with full support in Theorem C of [2] (cf. Proposition 9 in [1]). On the other hand Brémont proved in [4] that for any fixed continuous map on a compact metric space, any maximizing measure for a generic continuous function has zero entropy. Therefore we see that for any expanding map on the circle, a generic continuous function has a unique maximizing measure with full support and zero entropy. One of the main results in Jenkinson and Morris [8] asserts that for a generic but not fixed C^1 expanding map T on the circle, $\log J(T)$ has a unique maximizing measure with full support and zero entropy. In other words we generalize the results in [8] on the unit circle to those on a compact manifold admitting C^1 expanding maps.

The ideas of our proofs are essentially the same as those in [8]. But the technique in [8] seems to have some difficulties to be applied to the general expanding maps. So we need to make modifications of lemmas in [8] so that they can work in our general case. In particular, the crucial step of constructing an auxiliary perturbation of C^1 map along a periodic orbit is given with full generality.

In Section 2, we summarize some fundamental results on expanding maps. Section 3 is devoted to the construction of an appropriate perturbation. Finally proofs of Theorem 1 and Theorem 2 are given in Section 4.

2. Preliminaries

In this section we summarize the results which are needed in the proof of those theorems in Introduction. Most of them are well know facts for expanding maps, so we just give references or sketch the proof.

First we need the following lemma. For the proof consult Lemma 20 of Section 3 in [15] (see also Section 7.26–Section 7.30 in [10] and Section 3 in Bowen [3]).

Lemma 1. *Let T be an element in $\mathscr{E}^1(M, M)$. Then for any $\beta > 0$ there exists a Markov partition for T with diameter less than β, i.e. we can construct a finite cover $\{R_1, \ldots, R_q\}$ of M by closed sets satisfying the following conditions.*

(1) $\overline{\operatorname{int} R_i} = R_i$ *for each i.*

(2) $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ *for $i \neq j$.*

(3) $\bigcup_{i=1}^{q} \operatorname{int} R_i$ *is dense in M.*

(4) $T(\bigcup_{i=1}^{q} \partial R_i) \subset \bigcup_{i=1}^{q} \partial R_i$.

(5) *If* $T(\mathrm{int}\, R_i) \cap \mathrm{int}\, R_j \neq \emptyset$, *then* $R_j \subset TR_i$.
(6) $\max_{1 \le i \le q} \mathrm{diam}_d(R_i) < \beta$, *where* $\mathrm{diam}_d(R)$ *is the diameter of* $R \subset M$ *with respect to the distance* d *induced by the Riemannian metric on* M.

For $T \in \mathscr{E}^1(M, M)$, choose $\beta > 0$ so small that T maps any ball of radius less than β homeomorphically onto its image. Let $\mathscr{R} = \{R_1, \dots, R_q\}$ be a Markov partition of T with diameter less than β. Now we define a subshift of finite type in the usual way as follows. Put

$$\Sigma = \{\xi = (\xi_i)_{i \ge 0} : A(\xi_i \xi_{i+1}) = 1 \text{ for each } i \ge 0\},$$

where $A = (A(ij))$ is a $q \times q$ matrix given by

$$A(ij) = \begin{cases} 1 & \text{if } T(\mathrm{int}\, R_i) \cap \mathrm{int}\, R_j \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Now we consider the shift transformation $\sigma : \Sigma \to \Sigma$ satisfying $(\sigma\xi)_i = \xi_{i+1}$ for any $i \ge 0$ and $\xi \in \Sigma$. We choose $c > 0$ and $\lambda > 1$ such that $\|DT^n(x)v\|_{T^n x} \ge c\lambda^n \|v\|_x$ for any $x \in M$, $v \in T_x M$ and nonnegative integer n. Putting $\theta = 1/\lambda$, we define $d_\theta : \Sigma \times \Sigma \to \mathbf{R}$ by $d_\theta(\xi, \eta) = \theta^{n(\xi,\eta)}$, where $n(\xi, \eta) = \inf\{i \ge 0 : \xi_i \neq \eta_i\}$. Here we regard $\inf \emptyset$ and θ^∞ as $+\infty$ and 0, respectively. For $\xi \in \Sigma$, we see that $\bigcap_{i=0}^{n-1} T^{-i} R_{\xi_i}$ is nonempty and $\mathrm{diam}_d(\bigcap_{i=0}^{n-1} T^{-i} R_{\xi_i}) \le (1/c)\theta^n \mathrm{diam}_d(M)$ by the choice of β. Therefore $\bigcap_{i=0}^{\infty} T^{-i} R_{\xi_i}$ consists of a single point and we can define a map $\pi : \Sigma \to M$ so that $\pi(\xi)$ is the single point in $\bigcap_{i=0}^{\infty} T^{-i} R_{\xi_i}$. Moreover we have the following.

LEMMA 2. (1) π *is a Lipschitz continuous map from* (Σ, d_θ) *to* (M, d).
(2) $\bigcap_{j=0}^{\infty} T^{-j}(\bigcup_{i=1}^{q} \mathrm{int}\, R_i)$ *is dense in* M.
(3) *For any* $x \in \bigcap_{j=0}^{\infty} T^{-j}(\bigcup_{i=1}^{q} \mathrm{int}\, R_i)$, $\pi^{-1}\{x\}$ *is a single point set.*
(4) π *is surjective.*
(5) $\pi \circ \sigma = T \circ \pi$ *holds.*
(6) *The subshift* (Σ, σ) *of finite type is topological mixing. i.e.* $A^{n_0} > 0$ *for some positive integer* n_0.

PROOF. The assertion (1) is obvious from the fact that $d(\pi\xi, \pi\eta) \le \mathrm{diam}_d(\bigcap_{i=0}^{n(\xi,\eta)-1} T^{-i} R_{\xi_i}) \le (1/c)\theta^{n(\xi,\eta)} \mathrm{diam}_d(M)$ holds. The assertions (2)–(5) are a sort of exercises of general topology. To prove the assertion (6), we have only to show that T is topological mixing. But this is a well known fact. For example, by using the lifting of T to the universal covering space of M, we can easily see that for any nonempty open set $U \subset M$, there exists an integer $n > 0$ such that $T^n U = M$.

As a corollary we obtain the following lemma which is a modification of Lemma 1 in [8] (see also [9] and [12]).

Lemma 3. *If T is an element in $\mathscr{E}^1(M,M)$, then we obtain the following.*

(1) *Let $\mathscr{M}_p(T)$ be the set of invariant probability measures each of which is supported on a periodic point. Then $\mathscr{M}_p(T)$ is dense in $\mathscr{M}(T)$ in the weak $*$ topology.*

(2) *Let Y be a proper closed subset of M. Then for any $\mu \in \mathscr{M}(T)$ with $\operatorname{supp}\mu \subset Y$, there exists a sequence $\mu_n \in \mathscr{M}(T)$ converging to μ in the weak $*$ topology such that μ_n is supported on a periodic orbit and $\operatorname{supp}\mu_n \cap (\bigcap_{i=0}^{\infty} T^{-i}Y) = \varnothing$ for each n.*

Proof. Since the subshift (Σ,σ) of finite type constructed in Lemma 2 is topological mixing, it satisfies the specification property (see [6]). Consequently so does T. Therefore the proofs of Lemma 1 and Lemma 2 for the closed set $A = \bigcap_{i=0}^{\infty} T^{-i}Y$ in [12] do work. Thus we obtain (1) and (2).

Shub proved the conjugacy theorem of expanding maps in [13] via Contraction Principle. His proof leads us to the following statement which corresponds to Lemma 2 in [8].

Lemma 4. *Let T_k be a sequence of elements in $\mathscr{E}^1(M,M)$ satisfying the conditions:* (i) *there exist $c>0$ and $\lambda>1$ independent of k such that $\|DT_k^n(x)v\|_{T_k^n x} \geq c\lambda^n\|v\|_x$ for any $x \in M$, $v \in T_xM$ and nonnegative integer n;* (ii) *T_k converges to $T \in \mathscr{E}^1(M,M)$ in the C^0 topology. Then for sufficiently large k, there exists a homeomorphism $h_k : M \to M$ such that $h_k \circ T = T_k \circ h_k$, and both h_k and h_k^{-1} converge to the identity id_M in the C^0 topology.*

Proof. By virtue of Theorem (α) in [13] and its proof, we see that any two homotopic expanding endomorphisms T and S of a compact manifold, there exists a unique homeomorphism $h : M \to M$ such that $h \circ T = S \circ h$. On the other hand the condition (ii) of the lemma implies that there exists an integer k_0 such that T_k is homotopic to T for any $k \geq k_0$. Therefore we conclude that for any $k \geq k_0$, there exists a unique homeomorphism $h_k : M \to M$ such that $h_k \circ T = T_k \circ h_k$. Thus it remains to verify that both h_k and h_k^{-1} converge to the identity id_M in the C^0 topology. To this end we recall the strategy in [13].

We denote by $p : \tilde{M} \to M$ the universal covering space of M endowed with the Riemannian metric which is the pull-back of the Riemannian metric on M by the natural projection p. Let V be the set of continuous maps $F : \tilde{M} \to \tilde{M}$ which are lifting of continuous maps of M into itself such that $\gamma \circ F = F \circ \gamma$ holds for any covering transformation γ. We define a function $D : V \times V \to \mathbf{R}$ defined by $D(F,G) = \sup_{x \in \tilde{M}} d(pFx, pGx)$, where d is the distance function on M induced by the Riemannian metric. We easily see that (V,D) is a complete metric space. Consider homotopic elements $T, S \in \mathscr{E}^1(M,M)$. We assume that S satisfies the condition (i) in the statement of the lemma. Let $\tilde{T}$ and $\tilde{S}$ be the liftings of T and S, respectively. We note that the lifting of an expanding map on the universal covering space becomes a diffeomorphism. Since T and S are homotopic, we see that $\tilde{S}^{-1}F\tilde{T} \in V$ holds for any $F \in V$. Therefore we can consider a mapping $\Phi : V \to V$ by $F \mapsto \tilde{S}^{-1}F\tilde{T}$. Then it is not hard to see that $D(\Phi^n F, \Phi^n G) \le c^{-1}\lambda^{-n}D(F,G)$ for any $F, G \in V$ and $n \ge 0$. Therefore Contraction Principle yields that there exists a unique $H \in V$ with $\Phi H = H$. In fact, we can see that H is the lifting of the desired homeomorphism h satisfying $h \circ T = S \circ h$ in the same way as in the proof of Theorem 3 in [13]. Now we have

$$\begin{aligned}
D(H, \mathrm{id}_{\tilde{M}}) &\le D(H, \Phi^n(\mathrm{id}_{\tilde{M}})) + D(\Phi^n(\mathrm{id}_{\tilde{M}}), \mathrm{id}_{\tilde{M}}) \\
&\le c^{-1}\lambda^{-n}D(H, \mathrm{id}_{\tilde{M}}) + D(\tilde{S}^{-n}\tilde{T}^n, \mathrm{id}_{\tilde{M}}) \\
&\le c^{-1}\lambda^{-n}D(H, \mathrm{id}_{\tilde{M}}) + \sum_{k=1}^{n} D(\tilde{S}^{-k}\tilde{T}^k, \tilde{S}^{-(k-1)}\tilde{T}^{k-1}) \\
&= c^{-1}\lambda^{-n}D(H, Id_{\tilde{M}}) + \sum_{k=0}^{n-1} D(\Phi^k(\tilde{S}^{-1}\tilde{T}), \Phi^k\, \mathrm{id}_{\tilde{M}}) \\
&\le c^{-1}\lambda^{-n}D(H, Id_{\tilde{M}}) + \frac{\lambda}{c(\lambda-1)} D(\tilde{S}^{-1}\tilde{T}, \mathrm{id}_{\tilde{M}}).
\end{aligned}$$

Thus if we choose n satisfying $c^{-1}\lambda^{-n} < 1$, we have $D(H, \mathrm{id}_{\tilde{M}}) \le (\lambda/(\lambda-1))(\lambda^n/(c\lambda^n - 1))D(\tilde{S}^{-1}\tilde{T}, \mathrm{id}_{\tilde{M}})$. Consequently, we have $d_0(h, \mathrm{id}_M) \le (\lambda/(\lambda-1))(\lambda^n/(c\lambda^n - 1))d_0(S,T)$, where d_0 is the C^0 metric defined by $d_0(f,g) = \sup_{x \in M} d(f(x), g(x))$ for $f, g \in C^0(M,M)$.

By virtue of the above argument, we see that $d_0(h_k, \mathrm{id}_M) \to 0$ if $d_0(T_k, T) \to 0$. By the definition the metric d_0, we also have $d_0(h_k^{-1}, \mathrm{id}_M) \to 0$.

Next we summarize the results corresponding to Lemma 3 and Lemma 4 in [8].

LEMMA 5. *Let T_k be a sequence of elements in $\mathscr{E}^1(M,M)$ satisfying the conditions* (i) *and* (ii) *in* Lemma 4. *Then we have the following.*

(1) *Every $\mu \in \mathscr{M}(T)$ is the weak $*$ limit of a sequence of Borel probability measures $\{\mu_k\}$ with $\mu_k \in \mathscr{M}(T_k)$ for each k.*

(2) *Let $\{\mu_k\}$ be a sequence of Borel probability measures with $\mu_k \in \mathscr{M}(T_k)$ for each k. Then any weak $*$ accumulation point of the sequence belongs to $\mathscr{M}(T)$.*

(3) *Let $\{\mu_k\}$ be a sequence of Borel probability measures with $\mu_k \in \mathscr{M}(T_k)$ for each k. If μ_k converges to μ in the weak $*$ topology, we have*

$$\limsup_{k\to\infty} h(T_k, \mu_k) \le h(T, \mu).$$

Proof. By virtue of Lemma 4, we can prove the assertions (1) and (2) in the same way as the assertions (a) and (b) in Lemma 3 in [8].

If we verify that the entropy map $\mathcal{M}(T) \ni \mu \mapsto h(T,\mu) \in \mathbf{R}$ is upper semi-continuous for $T \in \mathcal{E}^1(M,M)$, we obtain the assertion (3) in the same way as Lemma 4 in [8] by using Lemma 4. It remains to show the upper semi-continuity of entropy map for T. Note that T is forward expansive, i.e. there exists $\beta > 0$ such that if $d(T^n x, T^n y) < \beta$ for any $n \geq 0$, then $x = y$. Thus it is easy to establish the upper semi-continuity of the entropy map in the same way as the proof of Theorem 8.2 in [16].

Finally we need the following.

Lemma 6. *If T_k is a sequence of elements in $\mathcal{E}^1(M,M)$ and converges to $T \in \mathcal{E}^1(M,M)$ in the C^1 topology, then for sufficiently large k, T_k satisfies the conditions* (i) *and* (ii) *in* Lemma 4. *Moreover, any weak $*$ accumulation point of a sequence μ_k with $\mu_k \in \mathcal{L}(T_k)$ belongs to $\mathcal{L}(T)$.*

Proof. We can easily see that the first assertion is true. Since $J(T_k)$ converges $J(T)$ uniformly on M, the second assertion follows from the assertions (1) and (2) in Lemma 5 in the same way as Lemma 5 in [8].

3. Construction of an auxiliary perturbation along a periodic orbit

In this section we construct an appropriate perturbation of an given C^1 map along its periodic point. Before giving them we need some definitions and notation.

Let U and V be neighborhoods of the origin in N-dimensional Euclidean space $\mathbf{R}^N$. Consider a C^1 map $F : U \to V$ which is locally diffeomorphic around each point of U and $F(0) = 0$. We denote by (e_i) the standard orthonormal basis of $\mathbf{R}^N$ with respect to the Euclidean metric. For our convenience we write $\partial F(x)$ for the matrix representation of the tangent map $DF(x)$ with respect to the standard orthonormal basis (e_i), i.e. the usual Jacobian matrix of F. We assume that U and V are endowed with Riemannian metrics g_U and g_V, respectively. Applying the Gram-Schmidt orthonormalization to the standard basis (e_i), we obtain matrix valued smooth functions $P(\cdot) : U \to GL(N,\mathbf{R})$ and $Q(\cdot) : V \to GL(N,\mathbf{R})$ such that $\alpha_i(x) = \sum_{j=1}^N P(ji)(x)e_j$ and $\beta_i(y) = \sum_{j=1}^N Q(ji)(y)e_j$ form orthonormal frames $\alpha(x) = (\alpha_i(x))$ and $\beta(y) = (\beta_i(y))$ of $\mathbf{R}^N = T_x U$ and $\mathbf{R}^N = T_y V$ with respect to the metrics g_U and g_V, respectively. The matrix representation of $DF(x) : T_x U \to T_{F(x)} V$ is given by $Q(F(x))^{-1}\partial F(x)P(x)$. Now we define $J(F) = |\det DF| : U \to \mathbf{R}$ by $J(F)(x) = |\det Q(F(x))^{-1}\partial F(x)P(x)|$ for each $x \in U$. Then it is easy to see that $J(F)$ is independent of the choice of orthonormal frames α and β. We can prove the following.

LEMMA 7. *There exists a positive number C depending only on F such that for any ε $(0 < \varepsilon < 1)$ and $\gamma > 0$, there exists $\delta_0 > 0$ such that for any δ with $0 < \delta < \delta_0$, we can find a C^1 map $F_\delta : U \to V$ satisfying the following properties with $G_\delta(x) = \log(J(F_\delta)(x)/J(F)(x))$.*

(1) $F_\delta(0) = 0$.

(2) $\sup_{x \in U} \|F_\delta(x) - F(x)\| < C\delta$, *where $\|\cdot\|$ denotes the Euclidean norm on $\mathbf{R}^N$.*

(3) $\sup_{x \in U} \|\partial F_\delta(x) - \partial F(x)\|_{GL} < C\varepsilon$, *where $\|\cdot\|_{GL}$ denotes the operator norm on $GL(N, \mathbf{R})$ induced by the Euclidean norm on $\mathbf{R}^N$.*

(4) $F_\delta(x) = F(x)$ *and* $G_\delta(x) = 0$ *if* $\|x\| \geq \delta$.

(5) $G_\delta(0) = \varepsilon$

(6) $\sup_{x \in U} G_\delta(x) < \gamma + \varepsilon$.

PROOF. Consider the functions

$$u(t) = \begin{cases} \exp(-1/t) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0 \end{cases} \qquad \text{and} \qquad v(t) = \tanh t.$$

For $a, \delta > 0$ small, we define a C^∞ map $\varDelta : \mathbf{R}^N \to \mathbf{R}^N$ vanishing for x with $\|x\| > \delta$ by

$$\varDelta(x) = \begin{pmatrix} \varDelta_1(x) \\ \vdots \\ \varDelta_N(x) \end{pmatrix} = u(\delta^2 - \|x\|^2) \begin{pmatrix} v(ax_1/u(\delta^2)) \\ \vdots \\ v(ax_N/u(\delta^2)) \end{pmatrix}, \tag{3.1}$$

where the choice of a will be specified later. Observe that the (i, j)-th element of the Jacobian matrix $\partial\varDelta(x)$ is given as

$$\begin{aligned} \frac{\partial \varDelta_i}{\partial x_j}(x) &= -2x_j u'(\delta^2 - \|x\|^2) v(ax_i/u(\delta^2)) \\ &\quad + a\frac{u(\delta^2 - \|x\|^2)}{u(\delta^2)} v'(ax_j/u(\delta^2))\delta(ij), \end{aligned} \tag{3.2}$$

where $\delta(ij)$ is the Kronecker delta. Therefore, we easily see that

$$|\varDelta_i(x)| \leq \exp(-1/\delta^2), \qquad \text{and} \qquad \left|\frac{\partial \varDelta_i}{\partial x_j}(x)\right| \leq \frac{2}{\delta^3}\exp(-1/\delta^2) + a\delta(ij) \tag{3.3}$$

if $\delta^2 \leq 1/2$. Define F_δ by

$$F_\delta(x) = F(x) + \partial F(0)\varDelta(x). \tag{3.4}$$

Then the Jacobian matrix ∂F_δ is given by

$$\partial F_\delta(x) = \partial F(x) + \partial F(0)\partial\Delta(x). \tag{3.5}$$

From the first inequality in (3.3), there exists δ_1 such that $F_\delta(U) \subset V$ holds if $\delta < \delta_1$. Since $\Delta(0) = 0$ by the definition (3.1), this yields the assertion (1).

Using the first inequality in (3.3) again, we can find $C_1 > 0$ and $\delta_2 > 0$ with $\delta_2 < \delta_1$ depending only on F, $\sup_{x \in U}\|F_\delta(x) - F(x)\| < C_1\delta$ holds for $\delta < \delta_2$. This yields the assertion (2). The assertion (4) is obvious since $\Delta(x) = 0$ for x with $\|x\| \geq \delta$.

In order to verify the other assertions, we consider the matrix representation of $DF_\delta(x)$ with respect to the orthonormal frames α and β. We have

$$J(F_\delta)(x) = |\det(Q(F_\delta(x))^{-1}\partial F(x)P(x) + Q(F_\delta(x))^{-1}\partial F(0)\partial\Delta(x)P(x))|. \tag{3.6}$$

Therefore we obtain

$$\begin{aligned} J(F_\delta)(0) &= |\det(Q(0)^{-1}\partial F(0)P(0) + Q(0)^{-1}\partial F(0)\partial\Delta(0)P(0))| \\ &= |\det(Q(0)^{-1}\partial F(0)P(0))|\,|\det(I_N + \partial\Delta(0)))| \\ &= J(F)(0)(1+a)^N. \end{aligned} \tag{3.7}$$

Note that we have used the fact $\partial\Delta(0) = aI_N$, where I_N is the identity matrix. Now we are in a position to specify the choice of a. If we put $a = \exp(\varepsilon/N) - 1$, we have $G_\delta(0) = \varepsilon$. Then we can choose a number $C_2 \geq C_1$ depending only on F and a positive number δ_3 depending only on F and ε such that $\sup_{x \in U}\|\partial F_\delta(x) - \partial F(x)\|_{GL} < C_2\varepsilon$ holds for any $\delta < \delta_3$. Thus the assertion (3) is valid.

It remains to show the last assertion (6). By virtue of the assertion (4), we have only to evaluate $G_\delta(x)$ for x with $\|x\| \leq \delta$. We will use an elementary inequality

$$\left|\frac{\det L_2}{\det L_1}\right| \leq \exp(N\|L_1^{-1}\|_{GL}\|L_2 - L_1\|_{GL}) \tag{3.8}$$

for $L_1, L_2 \in GL(N, \mathbf{R})$. This is verified as follows. From $L_1^{-1}L_2 = I_N + L_1^{-1}(L_2 - L_1)$, we have $\|L_1^{-1}L_2\|_{GL} \leq 1 + \|L_1^{-1}(L_2 - L_1)\|_{GL}$. Since it is easy to see that $|\alpha| \leq \|A\|_{GL}$ holds for any matrix $A \in GL(N, \mathbf{R})$ and for any eigenvalue α of A, we obtain

$$|\det L_2/\det L_1| = |\det L_1^{-1}L_2| \leq \|L_1^{-1}L_2\|_{GL}^N \leq (1 + \|L_1^{-1}(L_2 - L_1)\|_{GL})^N.$$

Now the inequality (3.8) is an easy consequence of the inequality $1 + \lambda \leq e^\lambda$ for $\lambda \geq 0$.

First using the equation (3.6) we have

$$J(F_\delta)(x) = J(F)(x)\left|\frac{\det Q(F(x))}{\det Q(F_\delta(x))}\right|$$
$$\times\left|\frac{\det(I_N + \partial F(x)^{-1}\partial F(0)\partial\Delta(x))}{\det(I_N + \partial\Delta(x))}\right| |\det(I_N + \partial\Delta(x))|. \tag{3.9}$$

Since $Q : V \to GL(N, \mathbf{R})$ is C^∞ and $\|x\| \le \delta$, we see that

$$\|Q(F_\delta(x))^{-1}\|_{GL}\|Q(F_\delta(x)) - Q(F(x))\|_{GL} < C_3\delta \tag{3.10}$$

for a positive constant C_3 depending only on F. In addition, it is not hard to see that

$$\|(I_N + \partial\Delta(x))^{-1}\|_{GL}\|(\partial F(x)^{-1}\partial F(0) - I_N)\partial\Delta(x)\|_{GL}$$
$$< C_4 \sup_{x:\|x\|\le\delta} \|\partial F(x) - \partial F(0)\|_{GL} \tag{3.11}$$

for a positive constant C_4 depending only on F. Next we evaluate $|\det(I_N + \partial\Delta(x))|$ as follows. By virtue of the second inequality in (3.3), its (i, j)-th element satisfies $|(I_N + \partial\Delta(x))(ij)| < \delta(ij)\exp(\varepsilon/N) + C_5\delta$, where C_5 is a large constant depending only on F. Thus by using the definition of the determinant, we have

$$|\det(I_N + \partial\Delta(x))|$$
$$< (e^{\varepsilon/N} + C_5\delta)^N + \sum_{j=0}^{N-1}\frac{N!}{(N-j)!j!}(e^{\varepsilon/N} + C_5\delta)^j(C_5\delta)^{N-j}(N-j)!$$
$$< e^{\varepsilon}(1 + C_6\delta) \tag{3.12}$$

for large C_6 depending only on F. Combining (3.8), (3.9), (3.10), (3.11) and (3.12), we obtain

$$\frac{J(F_\delta)(x)}{J(F)(x)} < (1 + C_6\delta)\exp\left(C_3\delta + C_4 \sup_{x:\|x\|\le\delta}\|\partial F(x) - \partial F(0)\|_{GL} + \varepsilon\right).$$

This yields

$$G_\delta(x) < (C_3 + C_6)\delta + C_4 \sup_{x:\|x\|\le\delta}\|\partial F(x) - \partial F(0)\|_{GL} + \varepsilon.$$

Hence, putting $C = C_2$, we can find a positive number $\delta_0 < \delta_3$ such that all the assertions in the lemma are valid.

The following theorem is the main result in this section.

Theorem 3. *Let T be an element in $C^1(M,M)$ such that $J(T)(x) \neq 0$ holds for every point $x \in M$. Assume that T has a periodic point x_0 with least period p. Then for any ε with $0 < \varepsilon < 1$ and $\gamma > 0$, there exists a positive number $\delta_0 > 0$ such that for each δ with $0 < \delta < \delta_0$, we can find an open neighborhood U_δ^i of $T^i x_0$ for each $i = 0, 1, \ldots, p-1$. and an element T_δ of $C^1(M,M)$ satisfying the following.*

(1) *$T^i x_0 = T_\delta^i x_0$ for each $i = 0, 1, \ldots, p-1$.*
(2) *$U_\delta^i \cap U_\delta^j = \varnothing$ if $i \neq j$.*
(3) *$Tx = T_\delta x$ for any $x \in M \backslash \bigcup_{i=0}^{p-1} U_\delta^i$.*
(4) *If $0 < \delta' < \delta$, then we have $\overline{U_{\delta'}^i} \subset U_\delta^i$.*
(5) *For any charts (U, φ), (V, ψ) with $TU \subset V$ and any compact set $K \subset U$, $T_\delta U \subset V$ we have $\sup_{x \in K} \|\psi \circ T_\delta(x) - \psi \circ T(x)\| < C_{\varphi,\psi,K}\delta$ and $\sup_{x \in K} \|\partial(\psi \circ T_\delta \circ \varphi^{-1})(\varphi(x)) - \partial(\psi \circ T \circ \varphi^{-1})(\varphi(x))\|_{GL} < C_{\varphi,\psi,K}\varepsilon$, where $C_{\varphi,\psi,K}$ is a positive constant depending only on T, K, φ, and ψ.*
(6) *Define $G_\delta : M \to \mathbf{R}$ by $G_\delta(x) = \log(J(T_\delta)(x)/J(T)(x))$. Then we have $G_\delta(T_\delta^i x_0) = \varepsilon$ for each $i = 0, 1, \ldots, p-1$.*
(7) *$\sup_{x \in M} G_\delta(x) < \gamma + \varepsilon$ holds.*

In particular, we can choose $\{U_\delta^i\}_{0<\delta<\delta_0}$ so that $\operatorname{diam}_d U_\delta^i < \delta$ and $\bigcap_{\delta:0<\delta<\delta_0} U_\delta^i = \{T^i x_0\}$ for each $i = 0, 1, \ldots, p-1$, where $\operatorname{diam}_d(A)$ is the diameter of $A \subset M$ with respect to the distance d of M as before.

Proof. For each $i = 0, 1, \ldots, p-1$, choose a chart (V_i, φ_i) around $T^i x_0$ and an open set U_i such that $T^i x_0 \in U_i \subset \overline{U_i} \subset V_i$, $\overline{TU_i} \subset V_{i+1}$, and $\varphi_i(T^i x_0) = 0$, where we regard U_p and (V_p, φ_p) as U_0 and (V_0, φ_0) respectively. Moreover we assume that V_i's are mutually disjoint. Consider $T_{i,i+1} = \varphi_{i+1} \circ T \circ \varphi_i^{-1} : \varphi_i(U_i) \to \varphi_{i+1}(V_{i+1})$. Each $\varphi_i(V_i)$ is endowed with the Riemannian metric g_i which is the push-forward of the Riemannian metric of the manifold M by φ_i. Obviously we can apply Lemma 7 to the case when $U = \varphi_i(U_i)$, $V = \varphi_{i+1}(V_{i+1})$, $F = T_{i.i+1}$, $g_U = g_i$ and $g_V = g_{i+1}$. Note that there exist a positive constant c_M depending only on M and $\delta_0 > 0$ such that $0 < \delta < \delta_0$ yields that $B_{c_M\delta}(0) \subset \varphi_i U_i$ and $\operatorname{diam}_d(\varphi_i^{-1}(B_{c_M\delta}(0)) < \delta$ for each $i = 0, 1, \ldots, p-1$, where $B_r(0) = \{x \in \mathbf{R}^N : \|x\| < r\}$. We denote by $T_{i,i+1,\delta}$ the map corresponding to F_δ. Define a map $T_{i,\delta} : U_i \to V_{i+1}$ by $T_{i,\delta} = \varphi_{i+1}^{-1} \circ T_{i,i+1,\delta} \circ \varphi_i$.

Now put $U_\delta^i = \varphi_i^{-1}(B_{c_M\delta}(0))$ and define a map T_δ by

$$T_\delta x = \begin{cases} T_{i,\delta} x & \text{if } x \in U_i \text{ for } 0 \leq i \leq p-1, \\ Tx & \text{if } x \in M \backslash \bigcup_{i=0}^{p-1} U_i. \end{cases}$$

We just verify that U_δ^i's and T_δ satisfy the assertions in the theorem. The assertions (1)–(4) are obvious from the definition of U_δ^i's and T_δ. T_δ is of class

C^1 on U_i and coincides T on $U_i \backslash \overline{U_\delta^i}$ by definition. Therefore T_δ is of C^1 on M. The assertion (5) of the theorem immediately follows from the assertions (2), (3), and (4) in Lemma 7. By virtue of the assertions (5) and (6) in Lemma 7, we easily see the validity of the other assertions concerned with G_δ if we notice that $J(T)(x) = J(T_{i,i+1})(\varphi_i(x))$ and $J(T_\delta)(x) = J(T_{i,i+1,\delta})(\varphi_i(x))$ by definition.

From the definition of U_δ^i above, the last assertion of the theorem is clearly valid.

As a corollary we obtain a modification of Lemma 7 in [8] which plays a crucial role in our argument.

COROLLARY 1. *Let T be an element in $\mathscr{E}^1(M,M)$ and let x_0 be a periodic point of T with least period p. Then there exists $\varepsilon_0 > 0$ such that for any ε with $0 < \varepsilon < \varepsilon_0$ and $\gamma > 0$, there exists a positive number $\delta_0 > 0$ such that for each δ with $0 < \delta < \delta_0$, we can find an open neighborhood U_δ^i of $T^i x_0$ for each $i = 0, 1, \dots, p-1$ and an element T_δ of $\mathscr{E}^1(M,M)$ satisfying* (1)–(7) *in the statement of* Theorem 3. *In particular, we can choose $\{U_\delta^i\}_{0<\delta<\delta_0}$ so that* $\operatorname{diam}_d U_\delta^i < \delta$ *and* $\bigcap_{\delta:0<\delta<\delta_0} U_\delta^i = \{T^i x_0\}$ *for each* $i = 0, 1, \dots, p-1$.

PROOF. Let us consider the case when the map T in Theorem 3 is an element in $\mathscr{E}^1(M,M)$. Note that if the map T_δ obtained in Theorem 3 could be an element in $\mathscr{E}^1(M,M)$ for any $\varepsilon < 1$, there would be nothing to be proved. On the other hand the second inequality in the assertion (5) of Theorem 3 guarantees that there exists $\varepsilon_0 > 0$ depending only on T such that if $\varepsilon < \varepsilon_0$, then T_δ is expanding.

4. Proof of theorems

As we have constructed the perturbation in the previous section, the arguments in this section are almost the same as those in Section 3 and Section 4 in [8].

It is well known that one can define a distance function ρ on the set $\mathscr{M}(M)$ of Borel probability measures on M such that it induces the weak $*$ topology on $\mathscr{M}(M)$ and satisfies the condition

$$\rho((1-\lambda)\mu + \lambda\nu, \mu) \le \lambda \tag{4.1}$$

for every $\mu, \nu \in \mathscr{M}(M)$ and $\lambda \in (0,1)$. For each $\kappa > 0$ we consider the following sets.

$$\mathscr{R}_\kappa = \{T \in \mathscr{E}^1(M,M) : \operatorname{diam}_\rho(\mathscr{L}(T)) < \kappa\}$$

$$\mathscr{S}_\kappa = \left\{ T \in \mathscr{E}^1(M,M) : \sup_{\mu \in \mathscr{L}(T)} h(T,\mu) < \kappa h_{\mathrm{top}}(T) \right\},$$

where $h_{\mathrm{top}}(T)$ denotes the topological entropy of T. We need the following to show that the properties (1) and (2) in Theorem 1 is generic.

Proposition 1. *For each $\kappa > 0$, both $\mathscr{R}_\kappa$ and $\mathscr{S}_\kappa$ are open and dense subsets of $\mathscr{E}^1(M, M)$ in the C^1 topology.*

Proof. First we show that $\mathscr{E}^1(M,M)\backslash\mathscr{R}_\kappa$ and $\mathscr{E}^1(M,M)\backslash\mathscr{S}_\kappa$ are closed in $\mathscr{E}^1(M,M)$ in the C^1 topology. Assume that $T_k \in \mathscr{E}^1(M,M)\backslash\mathscr{R}_\kappa$ converges to $T \in \mathscr{E}^1(M,M)$ in the C^1 topology. Since $\mathscr{L}(T_k)$ is compact, we can find $\mu_k, \nu_k \in \mathscr{L}(T_k)$ satisfying $\rho(\mu_k, \nu_k) \geq \kappa$. By Lemma 6, their weak $*$ accumulation points are in $\mathscr{L}(T)$. Choosing subsequences if necessary, we may assume that there exist $\mu, \nu \in \mathscr{M}(M)$ such that $\rho(\mu_k, \mu)$, $\rho(\nu_k, \nu)$ converge to 0. Thus we have $\mathrm{diam}_\rho(\mathscr{L}(T)) \geq \kappa$. Hence we have $T \in \mathscr{E}^1(M,M)\backslash\mathscr{R}_\kappa$.

Next assume that $T_k \in \mathscr{E}^1(M,M)\backslash\mathscr{S}_\kappa$ converges to $T \in \mathscr{E}^1(M,M)$ in the C^1 topology. By Shub's theorem in [13], we may assume each T_k is topologically conjugate to T. Consequently $h_{\mathrm{top}}(T_k) = h_{\mathrm{top}}(T)$. Since $\mathscr{L}(T_k)$ is compact and the entropy map for an expanding map is upper semi-continuous (see Proof of Lemma 5 (3)), we can find ν_k such that $h(T_k, \nu_k) = \sup_{\mu \in \mathscr{L}(T_k)} h(T_k, \mu)$. Again choosing a subsequence if necessary, we may assume that $\rho(\nu_k, \nu)$ converges to 0. Lemma 6 yields $\nu \in \mathscr{L}(T)$. Moreover, by Lemma 5 (3), we have

$$h(T, \nu) \geq \limsup_{k\to\infty} h(T_k, \nu_k) \geq \kappa h_{\mathrm{top}}(T).$$

Hence we have $T \in \mathscr{E}^1(M,M)\backslash\mathscr{S}_\kappa$.

Now we have only to prove $\mathscr{R}_\kappa \cap \mathscr{S}_\kappa$ is dense in $\mathscr{E}^1(M,M)$. Choose any $T \in \mathscr{E}^1(M,M)$ and $\varepsilon > 0$ with $0 < \varepsilon < \varepsilon_0$, where, $\varepsilon_0 > 0$ is the positive number appearing in Corollary 1 depending only on T. By virtue of Lemma 3, there exists a periodic point x_0 with least period p such that the measure $\mu_0 = (1/p)\sum_{i=0}^{p-1} \delta_{T^i x_0}$ satisfies

$$\int_M \log J(T) d\mu_0 > \lambda(T) - \frac{\kappa\varepsilon}{8}. \tag{4.2}$$

Let T_δ be the perturbation obtained by applying Corollary 1 to T and x_0 with $\gamma = \kappa\varepsilon/8$. Note that since $\varepsilon < \varepsilon_0$, $T_\delta \in \mathscr{E}^1(M,M)$. Then (6) in Theorem 3 and the inequality (4.2) yields that

$$\begin{aligned} \lambda(T_\delta) \geq \int_M \log J(T_\delta) d\mu_0 &= \int_M \log J(T) d\mu_0 + \int_M G_\delta \, d\mu_0 \\ &> \lambda(T) + \left(1 - \frac{\kappa}{8}\right)\varepsilon. \end{aligned} \tag{4.3}$$

Take any strictly decreasing sequence $\delta_k \le \delta_0$ of positive numbers converging to 0, where δ_0 is as in the statement in Theorem 3. For the sake of simplicity, we write T_{δ_k} and G_{δ_k} as T_k and G_k, respectively. Choose $\nu_k \in \mathscr{L}(T_k)$. The first inequality in Theorem 3 (5), T_k converges to T in the C^0 topology. Therefore, by taking a subsequence if necessary, we may assume that ν_k converges to $\nu \in \mathscr{M}(T)$. Thus we have

$$\begin{aligned}\int_M G_k\, d\nu_k &= \int_M \log J(T_k) d\nu_k - \int_M \log J(T) d\nu_k \\ &= \lambda(T_k) - \int_M \log J(T) d\nu_k > \lambda(T_k) - \lambda(T) - \frac{\kappa\varepsilon}{8} \end{aligned} \tag{4.4}$$

for any k sufficiently large. Combining (4.3) with (4.4) we have

$$\int_M G_k\, d\nu_k > \left(1 - \frac{\kappa}{4}\right)\varepsilon. \tag{4.5}$$

Recalling the open sets U_δ^i in Theorem 3, we see that $U_k = \bigcup_{i=0}^{p-1} U_{\delta_k}^i$ satisfies that $\overline{U_{k+1}} \subset U_k$ and $\bigcap_{k=1}^{\infty} \overline{U_k} = O_T(x_0)$, where $O_T(x_0) = \{x_0, Tx_0, \dots, T^{p-1}x_0\}$. Now we evaluate $\nu_k(U_k)$ as follows.

$$\begin{aligned}\nu_k(U_k) &\ge \left(1 + \frac{\kappa}{8}\right)^{-1} \varepsilon^{-1} \int_{U_k} G_k\, d\nu_k = \left(1 + \frac{\kappa}{8}\right)^{-1} \varepsilon^{-1} \int_M G_k\, d\nu_k \\ &> \left(1 + \frac{\kappa}{8}\right)^{-1} \left(1 - \frac{\kappa}{4}\right) > 1 - \frac{3\kappa}{8}. \end{aligned}\tag{4.6}$$

In the above, the first inequality follows from (7) in Theorem 3 with $\gamma = \kappa\varepsilon/8$, the equality in the first line follows from (3) in Theorem 3, and the first inequality in the second line follows from (4.5). Using the well known fact that ν_k converges to ν in the weak $*$ topology if and only if $\limsup_{k\to\infty} \nu_k(F) \le \nu(F)$ holds for any closed set F, we can easily see from (4.6) that $\nu(\overline{U_k}) \ge 1 - (3/8)\kappa$ for each k. Thus we obtain $\nu(O_T(x_0)) \ge 1 - (3/8)\kappa$. Hence we can write ν as $\nu = (1 - 3\kappa/8)\mu_0 + (3\kappa/8)\hat{\nu}$ for some $\hat{\nu} \in \mathscr{M}(T)$. From the condition (4.1), this yields

$$\lim_{k\to\infty} \rho(\nu_k, \mu_0) = \rho(\nu, \mu_0) \le \frac{3\kappa}{8}. \tag{4.7}$$

In addition, by Lemma 5 (3), we have

$$\begin{aligned}\limsup_{k\to\infty} h(T_k, \nu_k) \le h(T, \nu) &= \left(1 - \frac{3\kappa}{8}\right) h(T, \mu_0) + \frac{3\kappa}{8} h(T, \hat{\nu}) \\ &\le \frac{3\kappa}{8} h_{\mathrm{top}}(T). \end{aligned}\tag{4.8}$$

Hence we have shown that the inequalities (4.7) and (4.8) hold for any accumulation point as $\delta \to 0$ of the sets $\mathscr{L}(T_\delta)$. This implies that for any ε with $(0 < \varepsilon < \varepsilon_0)$, there exists δ_1 such that $T_\delta \in \mathscr{R}_\kappa \cap \mathscr{S}_\kappa$ whenever $\delta < \delta_1$. Since the second inequality in (5) in Theorem 3 holds, we can choose ε so that T_δ belongs to a given neighborhood of T in the C^1 topology.

To prove that the property (3) in Theorem 1 is generic, we show the following.

Proposition 2. *For a nonempty closed proper subset Y of M, consider the set*

$$M^1(Y) = \{T \in \mathscr{E}^1(M, M) : \operatorname{supp} \mu \subset Y \text{ holds for some } \mu \in \mathscr{L}(T)\}.$$

Then $M^1(Y)$ is a closed and nowhere dense subset of $\mathscr{E}^1(M, M)$ in the C^1 topology.

Proof. First we show that $M^1(Y)$ is closed in $\mathscr{E}^1(M, M)$. Assume that $T_k \in M^1(Y)$ converges to $T \in \mathscr{E}^1(M, M)$ in the C^1 topology. Note that the sequence T_k satisfies the conditions (i) and (ii) in Lemma 4. Let $\mu_k \in \mathscr{L}(T_k)$ satisfy $\operatorname{supp} \mu_k \subset Y$ and let μ be an accumulation point of them. We may assume that μ_k converges to μ in the weak $*$ topology. From Lemma 6 μ turns out to be an element in $\mathscr{L}(T)$. Moreover, since Y is closed, we have $\mu(Y) \geq \limsup_{k\to\infty} \mu_k(Y) = 1$. Consequently we see $\mu(Y) = 1$ and $M^1(Y)$ is closed.

Next we show that $M^1(Y)$ is nowhere dense in $\mathscr{E}^1(M, M)$. If $M^1(Y)$ is not empty, take any $T \in M^1(Y)$ and $Y_\infty = \bigcap_{j=0}^{\infty} T^{-j} Y$. Choose any ε with $0 < \varepsilon < \varepsilon_0$, where ε_0 is the same as in Corollary 1 as before. Note that if $\mu \in \mathscr{M}(T)$ satisfies $\mu(Y) = 1$, we also have $\mu(Y_\infty) = 1$. By Lemma 3, we can find a periodic point x_0 of T with least period p satisfying $Y_\infty \cap O_T(x_0) = \varnothing$ and

$$\int_M \log J(T) d\mu_0 > \lambda(T) - \varepsilon, \tag{4.9}$$

where μ_0 denotes the T-invariant probability measure supported on $O_T(x_0)$ as before.

We can find a positive integer N_0 such that $Y_{N_0} = \bigcap_{j=0}^{N_0} T^{-j} Y$ satisfies $Y_{N_0} \cap O_T(x_0) = \varnothing$. For each $\kappa > 0$ consider the set

$$Y_{N_0,\kappa} = \bigcup_{S \in \mathscr{E}^1(M,M): d_0(S,T) \leq \kappa} \bigcap_{j=0}^{N_0} S^{-j} Y,$$

where $d_0(S, T)$ is the usual C^0-metric defined by $d_0(S, T) = \sup_{x \in M} d(Sx, Tx)$.

It is not hard to see that $\bigcap_{\kappa>0} Y_{N_0,\kappa} = Y_{N_0}$. Now applying Theorem 3 to T and x_0 with $\gamma = 1$, we construct C^1 map T_δ. As stated in Corollary 1, there exists $\varepsilon_0 > 0$ depending only on T such that T_δ is expanding for any δ sufficiently small. Note that by virtue of the first inequality in (5) in Theorem 3, there exists $C_T > 0$ depending only on T such that $d_0(T_\delta, T) < C_T\delta$. Choose $\delta > 0$ so that

$$\inf_{y \in Y_{N_0, C_T\delta}} \inf_{0 \le i \le p-1} d(T^i x_0, y) > \delta.$$

Since the diameter of each U_δ^i in Theorem 3 is less than δ, the assertion (3) in Theorem 3 yields that $T = T_\delta$ on the set $Y_{N_0, C_T\delta}$. Consequently $T = T_\delta$ on the set $\overline{Y_{N_0, C_T\delta}}$ and $O_T(x_0)$ is also a periodic orbit of T_δ by Theorem 3 (1). Note that if $\mu \in \mathcal{M}(T_\delta)$ satisfies $\mu(Y) = 1$, we have $\mu(\overline{Y_{N_0, C_T\delta}}) = 1$. In particular, for a Borel probability measure with $\mu(\overline{Y_{N_0, C_T\delta}}) = 1$, μ is T-invariant if and only if it is T_δ-invariant.

We show that $\mu \in \mathcal{M}(T_\delta)$ with $\mu(Y) = 1$ cannot be an element of $\mathcal{L}(T_\delta)$. Assume that $\mu \in \mathcal{M}(T_\delta)$ satisfies $\mu(Y) = 1$. Then we have

$$\int_M \log J(T_\delta) d\mu = \int_M \log J(T) d\mu \le \lambda(T) \tag{4.10}$$

since $T = T_\delta$ on $\overline{Y_{N_0, C_T\delta}}$. On the other hand, the inequality (4.9) and Theorem 3 (6) yields

$$\lambda(T) < \int_M \log J(T) d\mu_0 + \varepsilon = \int_M \log J(T_\delta) d\mu_0.$$

Combining this with (4.10), we arrive at

$$\int_M \log J(T_\delta) d\mu = \int_M \log J(T) d\mu \le \lambda(T) < \int_M \log J(T_\delta) d\mu_0 \le \lambda(T_\delta).$$

Choose any T in $M^1(Y)$ and consider any neighborhood of T in the C^1 topology. If $\varepsilon > 0$ and $\delta > 0$ are small enough, the map T_δ constructed in Corollary 1 can be found in the neighborhood since the assertion (5) in Theorem 3 holds. The argument above implies that if $\delta > 0$ is sufficiently small, we see that $T_\delta \notin M^1(Y)$. Hence $M^1(Y)$ has no interior points.

PROOF OF THEOREM 1. We can easily verify that the set of $T \in \mathcal{E}^1(M, M)$ satisfying the property (1) and the set of $T \in \mathcal{E}^1(M, M)$ satisfying the property (2) are given by $\bigcap_{n=1}^{\infty} \mathcal{R}_{1/n}$ and $\bigcap_{n=1}^{\infty} \mathcal{S}_{1/n}$, respectively. Thus, properties (1) and (2) are generic by Proposition 1.

Since M is a compact metric space, we can find a countable family $\{Y_n\}$ of closed proper subsets of M such that any closed proper subset Y of M turns out to be a subset of Y_n for some n. Indeed, let $\{B_n\}$ be a countable open base of M consisting of open balls. We may assume that the radius of each B_n is so small that B_n is a proper subset of M. Putting $Y_n = M \backslash B_n$, we obtain the desired family of closed proper subsets of M. We easily see that the set $T \in \mathscr{E}^1(M, M)$ satisfying the property (3) is given by $\bigcap_{n=1}^{\infty}(\mathscr{E}^1(M, M) \backslash M^1(Y_n))$. Since each $M^1(Y_n)$ is a closed and nowhere dense subset of $\mathscr{E}^1(M, M)$ by Proposition 2, we arrive at the desired result. $\square$

Finally we prove Theorem 2. To this end we consider the symbolic dynamics (Σ, σ) in Lemma 2 and a function $V : \Sigma \to \mathbf{R}$ which is d_θ-Lipschitz continuous. Denote by $\mathscr{M}(\sigma)$ the set of σ-invariant Borel probability measures on Σ. Put

$$\lambda(\sigma, V, \nu) = \int_{\Sigma} V \, d\nu,$$

$$\lambda(\sigma, V) = \sup\left\{\int_{\Sigma} V \, d\nu : \nu \in \mathscr{M}(\sigma)\right\}$$

and denote by $\mathscr{L}(\sigma, V)$ the set of measures in $\mathscr{M}(\sigma)$ satisfying $\lambda(\sigma, V) = \lambda(\sigma, V, \nu)$. Since $\mathscr{M}(\sigma)$ is compact with respect to the weak $*$ topology, the continuity of V yields that $\mathscr{L}(\sigma, V)$ is nonempty. We need the following fact that can be find in Savchenko [11]. We state it with proof for the reader's convenience.

Lemma 8. *There exists a unique nonnegative d_θ-Lipschitz continuous function $\varphi : \Sigma \to \mathbf{R}$ satisfying the following properties.*

(1) $V \le \varphi \circ \sigma - \varphi + \lambda(\sigma, V)$ *on* Σ.

(2) *For any nonnegative function* $\psi : \Sigma \to \mathbf{R}$ *satisfying* $V \le \psi \circ \sigma - \psi + \lambda(\sigma, V)$ *on* Σ, *we have* $\varphi \le \psi$ *on* Σ.

(3) *For* $\nu \in \mathscr{L}(\sigma, V)$, *we have* $V = \varphi \circ T - \varphi + \lambda(\sigma, V)$ *on* $\operatorname{supp} \nu$.

Proof. We just follow the same lines as the proof of Proposition 11 in [5]. We may assume that $\lambda(\sigma, V) = 0$. Define φ by

$$\varphi(\xi) = \sup\{S_n V(\eta) : n \ge 0 \text{ and } \sigma^n \eta = \xi\},$$

where $S_n V(\eta) = \sum_{j=0}^{n-1} V(\sigma^j \eta)$ for $n \ge 1$ and $S_0 = 0$. First we show that $0 \le \varphi(\xi) < +\infty$ for each $\xi \in \Sigma$. Since $S_0 V = 0$, we have only to show that the set $\{S_n V(\eta) : n \ge 0 \text{ and } \sigma^n \eta = \xi\}$ is bounded from above. From Lemma 2 (6), There exists an integer $n_0 > 0$ such that for any $\eta \in \Sigma$ and $n \ge 0$ we have $\sigma^{n+n_0}(Z(\eta[0, n-1])) = \Sigma$, where $\eta[0, n-1]$ is the word $\eta_0 \eta_1 \ldots \eta_{n-1}$ and $Z(\eta[0, n-1])$ is the cylinder set $\{\zeta : \zeta_j = \eta_j \text{ for } 0 \le j \le n-1\}$. Thus for each

η with $\sigma^{n+n_0}\eta = \xi$, we can find a fixed point η^0 of σ^{n+n_0} in $Z(\eta[0, n-1])$. Note that $\lambda(\sigma, V) = 0$ yields $S_{n+n_0}V(\eta^0) \le 0$. On the other hand we have

$$|S_{n+n_0}V(\eta) - S_{n+n_0}V(\eta^0)| \le [V]_\theta \sum_{j=0}^{n+n_0-1} d_\theta(\sigma^j\eta, \sigma^j\eta^0)$$
$$\le \left(\frac{1}{1-\theta} + n_0\right)[V]_\theta,$$

where $[V]_\theta$ is the Lipschitz constant of V. Thus we easily see that $\varphi(\xi) \le (1/(1-\theta) + n_0)[V]_\theta$.

Next we show that φ is d_θ-Lipschitz continuous. Note that if ξ and η in Σ satisfy $\xi_0 = \eta_0$, then $\zeta_0\zeta_1 \dots \zeta_{n-1} \cdot \xi \in \Sigma$ yields $\zeta_0\zeta_1 \dots \zeta_{n-1} \cdot \eta \in \Sigma$ for a word $\zeta_0\zeta_1 \dots \zeta_{k-1}$, where $w \cdot w'$ denotes the concatenation of words w and w'. Therefore, we have $|S_nV(\zeta_0\zeta_1 \dots \zeta_{n-1} \cdot \xi) - S_nV(\zeta_0\zeta_1 \dots \zeta_{n-1} \cdot \eta)| \le (1/(1-\theta))[V]_\theta d_\theta(\xi, \eta)$. Thus φ is continuous. Moreover, we see easily that

$$|\varphi(\xi) - \varphi(\eta)| \le \max\left(\max_{\zeta\in\Sigma} |\varphi(\zeta)|, \frac{[V]_\theta}{1-\theta}\right) d_\theta(\xi, \eta)$$

holds for any $\xi, \eta \in \Sigma$. By definition the inequality $V \le \varphi \circ \sigma - \varphi$ is valid on Σ. Now proof of (1) is complete.

Next, let ψ be a nonnegative function satisfying $V \le \psi \circ \sigma - \psi$ on Σ. Then for any pair $(\xi, \eta) \in \Sigma \times \Sigma$ with $\sigma^n\eta = \xi$, we have $\psi(\xi) \ge S_nV(\eta) + \psi(\eta)$. This clearly implies that the assertion (2) is valid.

Finally, let $\nu \in \mathscr{L}(\sigma, V)$. Combining the fact that $\int_\Sigma (\varphi \circ \sigma - \varphi - V) d\nu = 0$ with the assertion (1), we obtain $\varphi \circ \sigma - \varphi - V = 0$ ν-a.e. The continuity of V and φ implies that this equality holds everywhere on $\operatorname{supp} \nu$.

Proof of Theorem 2. Put $V = \log J(T) \circ \pi$. Note that since T is of class C^2, it is easy to see that V is a d_θ-Lipschitz continuous function on Σ.

First we verify that $\pi_*(\mathscr{L}(\sigma, V)) = \mathscr{L}(T)$ as follows, where π_* is the push-forward of π defined by

$$\int_M f \, d\pi_*\nu = \int_\Sigma f \circ \pi \, d\nu$$

for $f \in C(M)$. Note that π_* is surjective since so is π. Moreover, we have $\pi_*(\mathscr{M}(\sigma)) = \mathscr{M}(T)$ by Lemma 2 (5) (see, for example, Proposition 3.2 and Proposition 3.11 in [6]). Thus for any $\mu \in \mathscr{L}(T)$, there exists $\nu \in \mathscr{M}(\sigma)$ such that $\pi_*\nu = \mu$ and $\lambda(T) = \lambda(\sigma, V, \nu)$. Clearly we have

$$\lambda(T) = \int_M \log J(T) d\mu = \int_\Sigma V \, d\nu \le \lambda(\sigma, V). \tag{4.11}$$

On the other hand, for any $\nu' \in \mathscr{M}(\sigma)$, we have

$$\int_{\Sigma} V\, d\nu' = \int_{M} \log J(T) d\pi_* \nu' \le \lambda(T).$$

Thus we obtain $\lambda(\sigma, V) \le \lambda(T)$. Combining this with (4.11), we conclude that

$$\int_{\Sigma} V\, d\nu = \lambda(\sigma, V) = \lambda(T).$$

Next we show that if there exists an element in $\mathscr{L}(T)$ with support M, then $\mathscr{M}(T)$ coincides $\mathscr{L}(T)$, namely, $\lambda(T) = \int_M \log J(T) d\mu$ holds for any $\mu \in \mathscr{M}(T)$. Let μ be an element in $\mathscr{L}(T)$ with support M. Then from the argument above, we find $\nu \in \mathscr{L}(\sigma, V)$ such that $\pi_* \nu = \mu$. We can show that $\operatorname{supp} \nu = \Sigma$. Indeed, choose any $\xi \in \Sigma$, then we see that $\bigcap_{k=0}^{n-1} T^{-j} \operatorname{int} R_{\xi_j} \neq \emptyset$ for any positive integer n since the Markov partition satisfies (3), (4), and (5) in Lemma 1. Therefore $Z(\xi[0, n-1]) \supset \pi^{-1}(\bigcap_{j=0}^{n-1} T^{-j} \operatorname{int} R_{\xi_j})$ holds. Thus we have $\nu(Z(\xi[0, n-1])) \ge \nu(\pi^{-1}(\bigcap_{j=0}^{n-1} T^{-j} \operatorname{int} R_{\xi_j})) > 0$ for all $n \ge 0$. This yields $\operatorname{supp} \nu = \Sigma$. Now by virtue of Lemma 8, there exists a d_θ-Lipschitz continuous function $\varphi : \Sigma \to \mathbf{R}$ satisfying $V = \varphi \circ \sigma - \varphi + \lambda(\sigma, V)$ on Σ. This implies that $\lambda(\sigma, V) = \int_\Sigma V\, d\nu$ holds for any $\nu \in \mathscr{M}(\sigma)$. Consequently, we have $\lambda(T) = \int_M \log J(T) d\mu$ for any $\mu \in \mathscr{M}(T)$.

For any neighborhood of T in the C^r topology, it is not hard to construct an element $S \in \mathscr{E}^r(M, M)$ such that there exists a fixed point x_0 and a periodic point y_0 with least period $p \ge 2$ such that $\log J(S)(x_0) \neq (1/p) \log J(S^p)(y_0)$. Thus the set $\mathscr{F}^r(M, M)$ of the maps T such that $\mathscr{L}(T) = \mathscr{M}(T)$ is nowhere dense. In addition clearly it is closed. Therefore we arrive at the desired result. $\square$

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