HIROSHIMA MATH. J.
43 (2013), 239–264

Epimorphisms from 2-bridge link groups onto Heckoid groups (I)

In honour of J. Hyam Rubinstein and his contribution to mathematics

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(Received June 4, 2012)
(Revised August 2, 2012)

ABSTRACT. Riley "defined" the Heckoid groups for 2-bridge links as Kleinian groups, with nontrivial torsion, generated by two parabolic transformations, and he constructed an infinite family of epimorphisms from 2-bridge link groups onto Heckoid groups. In this paper, we make Riley's definition explicit, and give a systematic construction of epimorphisms from 2-bridge link groups onto Heckoid groups, generalizing Riley's construction.

1. Introduction

In [17], Riley introduced an infinite collection of Laurent polynomials, called the Heckoid polynomials, associated with a 2-bridge link K, and observed, through extensive computer experiments, that these Heckoid polynomials define the affine representation variety of certain groups, the Heckoid groups for K. To be more precise, he "defines" the Heckoid group of index $q \geq 3$ for K to be a Kleinian group generated by two parabolic transformations which are obtained by choosing a "right" root of the Heckoid polynomials (see [17, the paragraph following Theorem A in p. 390]). The classical Hecke groups, introduced in [6], are essentially the simplest Heckoid groups. Riley discussed relations of the Heckoid polynomials with the polynomials defining the nonabelian $SL(2, \mathbf{C})$-representations of 2-bridge link groups introduced in [16], and proved that each Heckoid polynomial divides the nonabelian representation polynomials of 2-bridge links $\tilde{K}$, where $\tilde{K}$ belongs to an infinite collection of 2-bridge links determined by K and the index q. This suggests that there are epimorphisms from the link group of $\tilde{K}$ onto the Heckoid group

The first author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2012R1A1A3009996). The second author was supported by JSPS Grants-in-Aid 22340013.
2010 *Mathematics Subject Classification*. Primary 57M25, 57M50.
Key words and phrases. Heckoid groups, 2-bridge links.

of index q for K, as observed in [17, the paragraph following Theorem B in p. 391].

The purpose of this paper is (i) to give an explicit combinatorial definition of the Heckoid groups for 2-bridge links (Definition 3.2), (ii) to prove that the Heckoid groups are identified with Kleinian groups generated by two parabolic transformations (Theorem 2.2), and (iii) to give a systematic construction of epimorphisms from 2-bridge link groups onto Heckoid groups, generalizing Riley's construction (Theorem 2.3 and Remark 4.4).

We note that the results (i) and (ii) are essentially contained in the work of Agol [1], in which he announces a complete classification of the non-free Kleinian groups generated by two-parabolic transformations. Moreover, this classification theorem gives a nice characterization of the Heckoid groups, by showing that they are exactly the Kleinian groups, with nontrivial torsion, generated by two-parabolic transformations.

The result (iii) is an analogy of the systematic construction of epimorphisms between 2-bridge link groups given in [14, Theorem 1.1]. In the sequel [10] of this paper, we prove, by using small cancellation theory, that the epimorphisms in Theorem 2.3 are the only upper-meridian-pair-preserving epimorphisms from 2-bridge link groups onto *even* Heckoid groups. This in turn forms an analogy of [9, Main Theorem 2.4], which gives a complete characterization of upper-meridian-pair-preserving epimorphisms between 2-bridge link groups.

This paper is organized as follows. In Section 2, we describe the main results. In Section 3, we give an explicit combinatorial definition of Heckoid groups. Sections 4, 5 and 6, respectively, are devoted to the proof of Theorem 2.3, the topological description of Heckoid orbifolds, and the proof of Theorem 2.2.

Throughout this paper, we denote the orbifold fundamental group of an orbifold X by $\pi_1(X)$.

2. Main results

Consider the discrete group, H, of isometries of the Euclidean plane $\mathbf{R}^2$ generated by the π-rotations around the points in the lattice $\mathbf{Z}^2$. Set $(\boldsymbol{S}^2, \boldsymbol{P}) := (\mathbf{R}^2, \mathbf{Z}^2)/H$ and call it the *Conway sphere*. Then $\boldsymbol{S}^2$ is homeomorphic to the 2-sphere, and $\boldsymbol{P}$ consists of four points in $\boldsymbol{S}^2$. We also call $\boldsymbol{S}^2$ the Conway sphere. Let $\boldsymbol{S} := \boldsymbol{S}^2 - \boldsymbol{P}$ be the complementary 4-times punctured sphere. For each $s \in \hat{\mathbf{Q}} := \mathbf{Q} \cup \{\infty\}$, let α_s be the simple loop in $\boldsymbol{S}$ obtained as the projection of a line in $\mathbf{R}^2 - \mathbf{Z}^2$ of slope s. Then α_s is *essential* in $\boldsymbol{S}$, i.e., it does not bound a disk in $\boldsymbol{S}$ and is not homotopic to a loop around a puncture. Conversely, any essential simple loop in $\boldsymbol{S}$ is isotopic to α_s for a unique

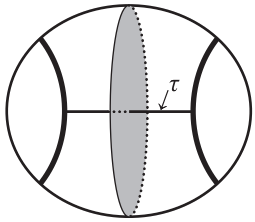

Fig. 1. A trivial tangle

$s \in \hat{\mathbf{Q}}$. Then s is called the *slope* of the simple loop. We abuse notation to denote by α_s the pair of conjugacy classes in $\pi_1(\boldsymbol{S})$ represented by the loop α_s with two possible orientations.

A *trivial tangle* is a pair (B^3, t), where B^3 is a 3-ball and t is a union of two arcs properly embedded in B^3 which is simultaneously parallel to a union of two mutually disjoint arcs in ∂B^3. Let τ be the simple unknotted arc in B^3 joining the two components of t as illustrated in Figure 1. We call it the *core tunnel* of the trivial tangle. Pick a base point x_0 in int τ, and let (μ_1, μ_2) be the generating pair of the fundamental group $\pi_1(B^3 - t, x_0)$ each of which is represented by a based loop consisting of a small peripheral simple loop around a component of t and a subarc of τ joining the circle to x_0. For any base point $x \in B^3 - t$, the generating pair of $\pi_1(B^3 - t, x)$ corresponding to the generating pair (μ_1, μ_2) of $\pi_1(B^3 - t, x_0)$ via a path joining x to x_0 is denoted by the same symbol. The pair (μ_1, μ_2) is unique up to (i) reversal of the order, (ii) replacement of one of the members with its inverse, and (iii) simultaneous conjugation. We call the equivalence class of (μ_1, μ_2) the *meridian pair* of $\pi_1(B^3 - t)$.

By a *rational tangle*, we mean a trivial tangle (B^3, t) which is endowed with a homeomorphism from $\partial(B^3, t)$ to $(\boldsymbol{S}^2, \boldsymbol{P})$. Through the homeomorphism we identify the boundary of a rational tangle with the Conway sphere. Thus the slope of an essential simple loop in $\partial B^3 - t$ is defined. We define the *slope* of a rational tangle to be the slope of an essential loop on $\partial B^3 - t$ which bounds a disk in B^3 separating the components of t. (Such a loop is unique up to isotopy on $\partial B^3 - t$ and is called a *meridian* of the rational tangle.) We denote a rational tangle of slope r by $(B^3, t(r))$. By van Kampen's theorem, the fundamental group $\pi_1(B^3 - t(r))$ is identified with the quotient $\pi_1(\boldsymbol{S})/\langle\langle \alpha_r \rangle\rangle$, where $\langle\langle \alpha_r \rangle\rangle$ denotes the normal closure.

For each $r \in \hat{\mathbf{Q}}$, the 2-*bridge link* $K(r)$ *of slope* r is defined to be the sum of the rational tangles of slopes ∞ and r, namely, $(S^3, K(r))$ is obtained from $(B^3, t(\infty))$ and $(B^3, t(r))$ by identifying their boundaries through the identity map on the Conway sphere $(\boldsymbol{S}^2, \boldsymbol{P})$. (Recall that the boundaries of rational tangles are identified with the Conway sphere.) $K(r)$ has one or two components according as the denominator of r is odd or even. We call $(B^3, t(\infty))$ and $(B^3, t(r))$, respectively, the *upper tangle* and *lower tangle* of the 2-bridge link. By van Kampen's theorem, the link group $G(K(r)) = \pi_1(S^3 - K(r))$ is obtained as follows:

$$G(K(r)) = \pi_1(S^3 - K(r)) \cong \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r \rangle\rangle.$$

We call the image in the link group of the meridian pair of $\pi_1(B^3 - t(\infty))$ (resp. $\pi_1(B^3 - t(r))$) the *upper meridian pair* (resp. *lower meridian pair*).

For a rational number r ($\neq \infty$) and an integer $n \geq 2$, the (*even*) *Heckoid orbifold, $\boldsymbol{S}(r;n)$, of index n for the* 2-*bridge link* $K(r)$ is the 3-orbifold as shown in Figure 2. Namely, the underlying space $|\boldsymbol{S}(r;n)|$ is $E(K(r))$ and the singular set is the *lower* tunnel, where the index of singularity is n. Here, the lower tunnel means the core tunnel of the lower tangle. the core tunnel The (*even*) *Hekoid group* $G(r;n)$ is defined to be the orbifold fundamental group $\pi_1(\boldsymbol{S}(r;n))$. By van Kampen's theorem for orbifold fundamental groups (cf. [4, Corollary 2.3]), we have

$$G(r;n) \cong \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r^n \rangle\rangle \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r^n \rangle\rangle.$$

In particular, the even Heckoid group $G(r;n)$ is a two-generator and one-relator group. We call the image in $G(r;n)$ of the meridian pair of $\pi_1(B^3 - t(\infty))$ the *upper meridian pair*.

The announcement by Agol [1] and the announcement made in the second author's joint work with Akiyoshi, Wada and Yamashita [2, Section 3 of Preface] (cf. Remark 6.1) suggest that the group $G(r;n)$ makes sense even when n is a half-integer greater than 1. The precise definition of $G(r;n)$ with $n > 1$ a half-integer is given in Definition 3.2, and a topological description of the corresponding orbifold, $\boldsymbol{S}(r;n)$, is given by Proposition 5.3 (see Figures 5 and

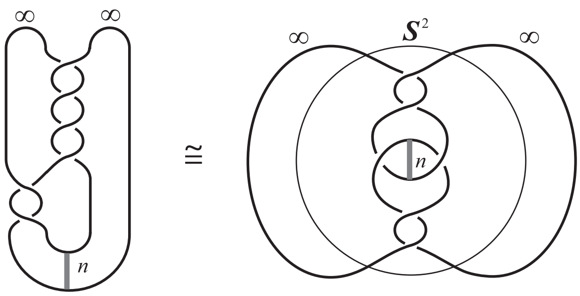

Fig. 2. The even Heckoid orbifold $\boldsymbol{S}(r;n)$ of index n for the 2-bridge link $K(r)$, where we employ Convention 5.1. Here $(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$ is the 2-bridge link with $r = 2/9 = [4, 2]$ (with a single component). The rational tangles $(B^3, t(\infty))$ and $(B^3, t(r))$, respectively, are the outside and the inside of the bridge sphere $\boldsymbol{S}^2$.

6). When $n > 1$ is a non-integral half-integer, $G(r;n)$ and $\boldsymbol{S}(r;n)$, respectively, are called the (*odd*) *Heckoid orbifold* and the (*odd*) *Heckoid group* of index n for $K(r)$. There is a natural epimorphism from $\pi_1(B^3 - t(\infty))$ onto the odd Heckoid group $G(r;n)$, and the image of the meridian pair of $\pi_1(B^3 - t(\infty))$ is called the *upper meridian pair* of $G(r;n)$. Thus the odd Heckoid groups are also two-generator groups. However, we show that they are not one-relator groups (Proposition 6.8).

Remark 2.1. Our terminology is slightly different from that of [17], where $G(r;n)$ is called the Heckoid group of index "$2n$" for $K(r)$. The Heckoid orbifold $\boldsymbol{S}(r;n)$ and the Heckoid group $G(r;n)$ are *even* or *odd* according to whether Riley's index $2n$ is even or odd.

We prove the following theorem, which was anticipated in [17] and is contained in [1] without proof.

Theorem 2.2. *For r a rational number and $n > 1$ an integer or a half-integer, the Heckoid group $G(r;n)$ is isomorphic to a geometrically finite Kleinian group generated by two parabolic transformations.*

In order to explain a systematic construction of epimorphisms from 2-bridge link groups onto Heckoid groups, we prepare a few notation. Let $\mathscr{D}$ be the *Farey tessellation*, that is, the tessellation of the upper half space $\mathbf{H}^2$ by ideal triangles which are obtained from the ideal triangle with the ideal vertices $0, 1, \infty \in \hat{\mathbf{Q}}$ by repeated reflection in the edges. Then $\hat{\mathbf{Q}}$ is identified with the set of the ideal vertices of $\mathscr{D}$. For each $r \in \hat{\mathbf{Q}}$, let Γ_r be the group of automorphisms of $\mathscr{D}$ generated by reflections in the edges of $\mathscr{D}$ with an endpoint r. Let $n > 1$ be an integer or a half-integer, and let $C_r(2n)$ be the group of automorphisms of $\mathscr{D}$ generated by the parabolic transformation, centered on the vertex r, by $2n$ units in the clockwise direction.

For r a rational number and $n > 1$ an integer or a half-integer, let $\Gamma(r;n)$ be the group generated by Γ_∞ and $C_r(2n)$. Then we have the following systematic construction of epimorphisms from 2-bridge link groups onto Heckoid groups.

Theorem 2.3. *Suppose that r is a rational number and that $n > 1$ is an integer or a half-integer. For $s \in \hat{\mathbf{Q}}$, if s or $s+1$ belongs to the $\Gamma(r;n)$-orbit of ∞, then there is an upper-meridian-pair-preserving epimorphism from $G(K(s))$ to $G(r;n)$.*

This theorem may be regarded as a generalization of Theorem B and Theorem 3 of Riley [17]. In fact, they correspond to the case when s belongs to the orbit of ∞ by the infinite cyclic subgroup $C_r(2n)$ of $\Gamma(r;n)$ (see Remark 4.4).

The above theorem is actually obtained from the following theorem.

THEOREM 2.4. *Suppose that r is a rational number and that $n > 1$ is an integer or a half-integer. Let s and s' be elements of $\hat{\mathbf{Q}}$ which belong to the same $\Gamma(r;n)$-orbit. Then the conjugacy classes α_s and $\alpha_{s'}$ in $G(r;n)$ are equal. In particular, if s belongs to the $\Gamma(r;n)$-orbit of ∞, then α_s is the trivial conjugacy class in $G(r;n)$.*

3. Combinatorial definition of Heckoid groups

In this section, we give an explicit combinatorial definition of even/odd Heckoid groups. Consider the $(2,2,2,\infty)$-orbifold, $\boldsymbol{O} := (\mathbf{R}^2 - \mathbf{Z}^2)/\hat{H}$, where $\hat{H}$ is the group generated by π-rotations around the points in $\left(\frac{1}{2}\mathbf{Z}\right)^2$. Note that $\boldsymbol{O}$ has a once-punctured sphere as the underlying space, and has three cone points of cone angle π. The orbifold fundamental group of $\boldsymbol{O}$ has the presentation

$$\pi_1(\boldsymbol{O}) = \langle P, Q, R \mid P^2 = Q^2 = R^2 = 1 \rangle,$$

where $D := (PQR)^{-1}$ is represented by the puncture of $\boldsymbol{O}$ (see Figure 3). For each $s \in \hat{\mathbf{Q}}$, the image of a straight line of slope s in $\mathbf{R}^2 - \mathbf{Z}^2$ disjoint from the singular set of $\hat{H}$ projects to a simple loop, β_s, in $\boldsymbol{O}$ disjoint from the cone points. Thus the loop β_s (with an orientation) represents a conjugacy class in $\pi_1(\boldsymbol{O})$. We abuse notation to denote by β_s the pair of the conjugacy classes in $\pi_1(\boldsymbol{O})$ represented by β_s with two possible orientations. Throughout this paper, we choose the generating set $\{P, Q, R\}$ of $\pi_1(\boldsymbol{O})$ so that the conjugacy classes β_0 and β_∞ are represented by RQ and PQ, respectively (see Figure 3 and [2, Section 2.1]).

The Conway sphere $\boldsymbol{S} = (\mathbf{R}^2 - \mathbf{Z}^2)/H$ is the $(\mathbf{Z}/2\mathbf{Z})^2$-covering of $\boldsymbol{O}$, and hence $\pi_1(\boldsymbol{S})$ is a normal subgroup of $\pi_1(\boldsymbol{O})$ such that $\pi_1(\boldsymbol{O})/\pi_1(\boldsymbol{S}) \cong (\mathbf{Z}/2\mathbf{Z})^2$. Each simple loop α_s in $\boldsymbol{S}$ doubly covers the simple loop β_s, and so we have $\alpha_s = \beta_s^2$ as conjugacy classes in $\pi_1(\boldsymbol{O})$.

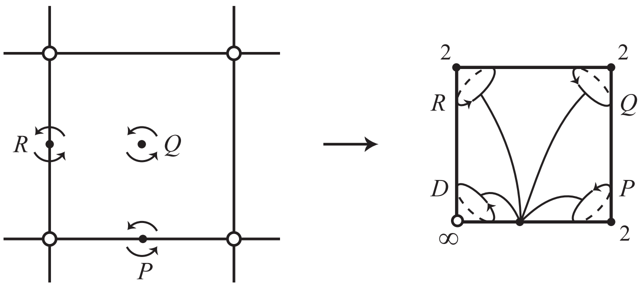

Fig. 3. The orbifold $\boldsymbol{O}$

For each $r \in \hat{\mathbf{Q}}$ and integer $m \geq 2$, consider the orbifold, $\boldsymbol{B}(r;m)$, as illustrated in Figure 4. In order to give its explicit description, we prepare notation following [13]. For an integer $m \geq 2$, let $D^2(m)$ be the *discal 2-orbifold* obtained from the unit disk D^2 in the complex plane by taking the quotient of the action generated by the $2\pi/m$-rotation $z \mapsto e^{2\pi i/m}z$. We call the product 3-orbifold $D^2(m) \times I$ with $I = [0,1]$ a 2-*handle orbifold*. The quotient orbifold of the unit 3-ball B^3 in $\mathbf{R}^3$ by the dihedral subgroup, D_{2m}, of $SO(3)$ of order $2m$ is denoted by $B^3(2,2,m)$ and is called a 3-*handle orbifold*. By using this notation, the orbifold $\boldsymbol{B}(r;m)$ has the following description (see Figure 4). Let $\check{\boldsymbol{O}}$ be the compact 2-orbifold obtained from $\boldsymbol{O}$ by removing an open regular neighborhood of the puncture. Then $\boldsymbol{B}(r;m)$ is obtained from the product orbifold $\check{\boldsymbol{O}} \times [0,1]$ by attaching 2- and 3-handle orbifolds as follows.

(1) Attach a 2-handle orbifold $D^2(m) \times I$ along the simple loop $\beta_r \times \{0\}$, i.e., identify $(\partial D^2(m)) \times I$ with an annular neighborhood of $\beta_r \times \{0\}$ in the boundary of $\check{\boldsymbol{O}} \times I$.

(2) Cap off the spherical orbifold boundary of the resulting orbifold by a 3-handle orbifold $B^3(2,2,m)$.

Note that the 2-dimensional orbifold $\check{\boldsymbol{O}}$ sits in the boundary of $\boldsymbol{B}(r;m)$; we call it the *outer boundary* of $\boldsymbol{B}(r;m)$, and denote it by $\partial_{out}\boldsymbol{B}(r;m)$. To be precise, as in the definition of rational tangles, $\boldsymbol{B}(r;m)$ is defined to be the orbifold as in Figure 4 which is endowed with a homeomorphism from $\partial_{out}\boldsymbol{B}(r;m)$ to $\check{\boldsymbol{O}}$. Thus, by van Kampen's theorem for orbifold fundamental groups [3, Corollary 2.3], we can identify the orbifold fundamental group $\pi_1(\boldsymbol{B}(r;m))$ with $\pi_1(\check{\boldsymbol{O}})/\langle\langle \beta_r^m \rangle\rangle = \pi_1(\boldsymbol{O})/\langle\langle \beta_r^m \rangle\rangle$. (Here we use the fact that the inclusion map $\partial B^3(2,2,m) \to B^3(2,2,m)$ induces an isomorphism between the orbifold fundamental groups.)

For a rational number r and an integer $m \geq 2$, let $\boldsymbol{O}(r;m)$ be the orbifold obtained by identifying $\boldsymbol{B}(\infty;2)$ and $\boldsymbol{B}(r;m)$, along their outer boundaries, via

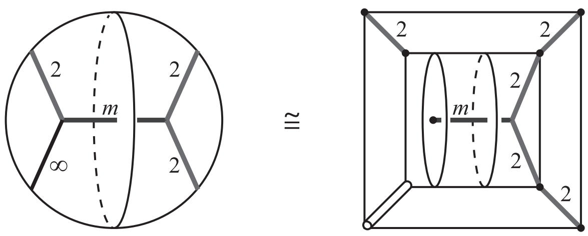

Fig. 4. The orbifold $\boldsymbol{B}(r;m) = (\check{\boldsymbol{O}} \times I) \cup (D^2(m) \times I) \cup B^3(2,2,m)$ with $r = \infty$, where we employ Convention 5.1.

their identification with $\check{\boldsymbol{O}}$. By van Kampen's theorem, the orbifold fundamental group of $\boldsymbol{O}(r;m)$ is given by the following formula:

$$\pi_1(\boldsymbol{O}(r;m)) \cong \pi_1(\boldsymbol{O})/\langle\langle \beta_\infty^2, \beta_r^m \rangle\rangle.$$

PROPOSITION 3.1. *For a rational number r and an integer $n > 1$, the even Heckoid orbifold $\boldsymbol{S}(r;n)$ is a $(\mathbf{Z}/2\mathbf{Z})^2$-covering of $\boldsymbol{O}(r;m)$, where $m = 2n$. In particular, the even Heckoid group $G(r;n)$ is identified with the image of the homomorphism, ψ, which is the following composition of two natural homomorphisms*

$$\pi_1(\boldsymbol{S}) \to \pi_1(\boldsymbol{O}) \to \pi_1(\boldsymbol{O}(r;m)).$$

PROOF. Let $\check{\boldsymbol{S}}$ be the compact 2-manifold obtained from $\boldsymbol{S}$ by removing open regular neighborhoods of the punctures. Then we see that the even Heckoid orbifold $\boldsymbol{S}(r;n)$ is obtained from $\check{\boldsymbol{S}} \times [-1,1]$ by attaching a 2-handle $D^2 \times I$ along $\alpha_\infty \times \{1\}$ and by attaching a 2-handle orbifold $D^2(n) \times I$ along $\alpha_r \times \{-1\}$. Note that the group $\hat{H}/H \cong (\mathbf{Z}/2\mathbf{Z})^2$ acts on $\check{\boldsymbol{S}}$ and the quotient is identified with $\check{\boldsymbol{O}}$. Since the loops α_∞ and α_r on $\check{\boldsymbol{S}}$ can be chosen so that they are invariant by the action, it extends to an action on $\boldsymbol{S}(r;n)$. Moreover the quotient of $\check{\boldsymbol{S}} \times [0,1] \cup D^2 \times I$ and that of $\check{\boldsymbol{S}} \times [-1,0] \cup D^2(n) \times I$ are identified with $\boldsymbol{B}(\infty;2)$ and $\boldsymbol{B}(r;m)$, respectively. Hence $\boldsymbol{S}(r;n)$ is a $(\mathbf{Z}/2\mathbf{Z})^2$-covering of $\boldsymbol{O}(r;m)$. Since the covering $\boldsymbol{S}(r;n) \to \boldsymbol{O}(r;m)$ is "induced" by the covering $\boldsymbol{S} \to \boldsymbol{O}$, and since the natural homomorphism $\pi_1(\boldsymbol{S}) \to \pi_1(\boldsymbol{S}(r;n))$ is surjective, we see that $G(r;n)$ is identified with $\mathrm{Im}(\psi)$. $\square$

This motivates us to introduce the following definition.

DEFINITION 3.2. For a rational number r and a non-integral half-integer n greater than 1, the (*odd*) *Heckoid group $G(r;n)$ of index n for $K(r)$* is defined to be the image, $\mathrm{Im}(\psi)$, of the natural map

$$\psi : \pi_1(\boldsymbol{S}) \to \pi_1(\boldsymbol{O}) \to \pi_1(\boldsymbol{O}(r;m)),$$

where $m = 2n$. The covering orbifold of $\boldsymbol{O}(r;m)$ corresponding to the subgroup $G(r;n)$ of $\pi_1(\boldsymbol{O}(r;m))$ is denoted by $\boldsymbol{S}(r;n)$ and is called the (*odd*) *Heckoid orbifold for the 2-bridge link $K(r)$ of index n*. (See Section 5 for a topological description of this orbifold.)

Note that ψ is equal to the composition

$$\pi_1(\boldsymbol{S}) \to \pi_1(B^3 - t(\infty)) = \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty \rangle\rangle \to \pi_1(\boldsymbol{O})/\langle\langle \beta_\infty^2 \rangle\rangle \to \pi_1(\boldsymbol{O}(r;m)).$$

Since $\pi_1(\boldsymbol{S}) \to \pi_1(B^3 - t(\infty))$ is surjective and since $\pi_1(B^3 - t(\infty))$ is a free group of rank 2, the Heckoid group $G(r;n)$ is generated by two elements. However, no odd Heckoid group is a one-relator group (see Proposition 6.8).

4. Proofs of Theorems 2.3 and 2.4

The following lemma, on the existence of certain self-homeomorphisms of the orbifold $\boldsymbol{B}(r;m)$, is the heart of Theorem 2.4. For the definition of a homeomorphism (diffeomorphism) between orbifolds, see [3, Section 2.1.3] or [8, p. 138].

Lemma 4.1. (1) *For $r \in \hat{\mathbf{Q}}$ and an integer $m \geq 2$, let F be a discal 2-suborbifold properly embedded in $\boldsymbol{B}(r;m)$ bounded by β_r, and let φ be the m-th power of the Dehn twist of the underlying space $|\boldsymbol{B}(r;m)|$, preserving the singular set, along the disk $|F|$. Then φ is a self-homeomorphism of the orbifold $\boldsymbol{B}(r;m)$, which induces the identity (outer) automorphism of $\pi_1(\boldsymbol{B}(r;m))$.*

(2) *For an integer $m \geq 2$, let γ be the reflection of $|\boldsymbol{B}(\infty;m)|$ of Figure 4 in the sheet of the figure. Then γ is a self-homeomorphism of the orbifold $\boldsymbol{B}(\infty;m)$. Moreover, if $m = 2$, then γ induces the identity (outer) automorphism of $\pi_1(\boldsymbol{B}(\infty;m))$.*

Proof. (1) To show the first assertion, we have only to check that each singular point x of $\boldsymbol{B}(r;m)$ has a neighborhood, U_x, such that the restriction of φ to U_x lifts to an equivariant homeomorphism from a manifold covering of U_x to that of $\varphi(U_x)$. But, this follows from the following observation. Let $p : D^2 \times [0,1] \to D^2(m) \times [0,1]$ be the universal covering of the 2-handle orbifold, given by $p(z,t) = (z^m, t)$, where we identify both D^2 and $|D^2(m)|$ with the unit disk in the complex plane. Let φ be the m-th power of the Dehn twist of $|D^2(m) \times [0,1]|$ given by $\varphi(z,t) = (e^{2\pi m t i}z, t)$. Then it is covered by the Dehn twist, $\tilde{\varphi}$, of $D^2 \times [0,1]$, defined by $\tilde{\varphi}(z,t) = (e^{2\pi t i}z, t)$, namely we have $\varphi \circ p = p \circ \tilde{\varphi}$. (The corresponding automorphism of the local group, $\mathbf{Z}/m\mathbf{Z}$, is the identity map.) Thus we have shown the first assertion that φ is a self-homeomorphism of the orbifold $\boldsymbol{B}(r;m)$.

To show the second assertion, we may assume $r = \infty$ without loss of generality. Then we can see by using Figure 3 that

$$(\varphi_*(P), \varphi_*(Q), \varphi_*(R)) = (\beta_\infty^m P \beta_\infty^{-m}, \beta_\infty^m Q \beta_\infty^{-m}, R),$$

where $\beta_\infty = PQ \in \pi_1(\boldsymbol{B}(\infty;m))$. Since $\beta_\infty^m = 1$ in $\pi_1(\boldsymbol{B}(\infty;m))$, we see that φ_* is the identity map.

(2) We show that γ satisfies the local condition (in the definition of a homeomorphism between orbifolds) at every singular point x. Suppose first that x is contained in the interior of an edge of the singular set. Then x has a neighborhood homeomorphic to the 2-handle orbifold $D^2(m) \times [0,1]$ such that the restriction of γ to it is given by $\gamma(z,t) = (\bar{z}, t)$. This is covered by the self-homeomorphism $\tilde{\gamma}$ of the universal cover $D^2 \times [0,1]$, defined by $\tilde{\gamma}(z,t) = (\bar{z}, t)$. (The corresponding automorphism of the local group, $\mathbf{Z}/m\mathbf{Z}$, of x is

given by $[k] \mapsto [-k]$ for every $[k] \in \mathbf{Z}/m\mathbf{Z}$.) Suppose next that x is the vertex, on which the edges of indices 2, 2, and m are incident. Then x has a neighborhood homeomorphic to the 3-handle orbifold $B^3(2,2,m) = B^3/D_{2m}$. Then the restriction of the map γ is covered by the reflection in the disk in B^3 containing the axes of the pair of order 2 generators of D_{2m}. (The corresponding automorphism of the local group, D_{2m}, of x is the identity map.) Thus we have shown that γ is a self-homeomorphism of the orbifold $\boldsymbol{B}(\infty; m)$.

To show the second assertion, observe by using Figure 3 that

$$(\gamma_*(P), \gamma_*(Q), \gamma_*(R)) = (P, \beta_\infty Q \beta_\infty^{-1}, \beta_\infty R \beta_\infty^{-1}).$$

By composing the inner automorphism $\iota : x \mapsto \beta_\infty^{-1} x \beta_\infty$, we have

$$(\iota \circ \gamma_*(P), \iota \circ \gamma_*(Q), \iota \circ \gamma_*(R)) = (\beta_\infty^{-1} P \beta_\infty, Q, R) = (QPQ, Q, R).$$

If $m = 2$, then $(PQ)^2 = \beta_\infty^2 = 1$ and so $\iota \circ \gamma_*(P) = Q(PQ) = Q(QP) = P$. Hence γ_* is the identity outer automorphism when $m = 2$. $\square$

We can easily observe that the restrictions of the homeomorphisms, φ and γ, to the outer boundary $\check{\boldsymbol{O}}$ act on the set of essential simple loops in $\check{\boldsymbol{O}}$ by the following rule.

(1) $\varphi(\beta_s) = \beta_{A_{(r;m)}(s)}$, where $A_{(r;m)}$ is the automorphism of the Farey tessellation $\mathscr{D}$ which is the parabolic transformation, centered at the vertex r, by m units in the clockwise direction (i.e., a generator of the infinite cyclic group $C_r(m)$).

(2) $\gamma(\beta_s) = \beta_{-s}$.

Hence we obtain the following corollary.

Corollary 4.2. *For any $r \in \hat{\mathbf{Q}}$ and an integer $m \geq 3$, the following hold.*

(1) *The conjugacy classes of $\pi_1(\boldsymbol{B}(r;m))$ determined by the simple loops β_s and $\beta_{A_{(r;m)}(s)}$ are identical. So, the same conclusion also holds for the conjugacy classes of the quotient group $\pi_1(\boldsymbol{O}(r;m)) \cong \pi_1(\boldsymbol{B}(r;m))/\langle\langle \beta_\infty^2 \rangle\rangle$.*

(2) *The conjugacy classes of $\pi_1(\boldsymbol{B}(\infty;2))$ determined by the simple loops β_s and β_{-s} are identical. So, the same conclusion also holds for the conjugacy classes of the quotient group $\pi_1(\boldsymbol{O}(r;m)) \cong \pi_1(\boldsymbol{B}(\infty;2))/\langle\langle \beta_r^m \rangle\rangle$.*

Proof of Theorem 2.4. Suppose that s and s' belong to the same $\Gamma(r;n)$-orbit. Since $\Gamma(r;n)$ of automorphisms of $\mathscr{D}$ is generated by the three transformations $s \mapsto -s$, $A_{(\infty;2)}$ and $A_{(r;m)}$ with $m = 2n$, we see by Corollary 4.2 that the conjugacy classes β_s and $\beta_{s'}$ in $\pi_1(\boldsymbol{O}(r;m))$ are equal. On the other hand, we can easily see that the natural action of $\pi_1(\boldsymbol{O})/\pi_1(\boldsymbol{S}) \cong (\mathbf{Z}/2\mathbf{Z})^2$ on the conjugacy classes in $\pi_1(\boldsymbol{S})$ preserves α_s, the pair of conjugacy classes represented by the loop α_s with two possible orientations. So, the same conclusion holds for the natural action of $\pi_1(\boldsymbol{O}(r:m))/\pi_1(\boldsymbol{S}(r;n))$ on the conjugacy classes in $\pi_1(\boldsymbol{S}(r;n))$. Hence the precending result implies that the conjugacy classes $\alpha_s = \beta_s^2$ and $\alpha_{s'} = \beta_{s'}^2$ in $G(r;n) = \pi_1(\boldsymbol{S}(r;n))$ are equal. □

Proof of Theorem 2.3. Suppose first that s belongs to the $\Gamma(r;n)$-orbit of ∞. Then the conjugacy class of α_s in $G(r;n) \subset \pi_1(\boldsymbol{O}(r;m))$ is trivial by Theorem 2.4. Since the conjugacy class of $\alpha_\infty = \beta_\infty^2$ in $\pi_1(\boldsymbol{O}(r;m))$ is also trivial by definition, the homomorphism $\pi_1(\boldsymbol{S}) \mapsto \pi_1(\boldsymbol{O}(r;m))$ descends to a homomorphism

$$G(K(s)) \cong \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_s \rangle\rangle \to \pi_1(\boldsymbol{O})/\langle\langle \beta_\infty^2, \beta_r^m \rangle\rangle \cong \pi_1(\boldsymbol{O}(r;m)).$$

Since the image of this homomorphism is equal to the Heckoid group $G(r;n)$ by Proposition 3.1 and Definition 3.2, we obtain an epimorphism $G(K(s)) \to G(r;n)$, which is apparently upper-meridian-pair-preserving.

Suppose next that $s+1$ belongs to the $\Gamma(r;n)$-orbit of ∞. Then there is an epimorphism $G(K(s+1)) \to G(r;n)$ by the above argument. Since there is an upper-meridian-pair-preserving isomorphism $G(K(s)) \cong G(K(s+1))$, we obtain the desired epimorphism. □

At the end of this section, we give a characterization of those rational numbers which belong to the $\Gamma(r;n)$-orbit of ∞. Since $G(r;n)$ is isomorphic to $G(r+1;n)$, we may assume in the remainder of this paper that $0 < r < 1$. For the continued fraction expansion

$$r = [a_1, a_2, \ldots, a_k] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_k}}}$$

where $k \geq 1$, $(a_1, \ldots, a_k) \in (\mathbf{Z}_+)^k$, and $a_k \geq 2$, let $\boldsymbol{a}$, $\boldsymbol{a}^{-1}$, $\varepsilon\boldsymbol{a}$ and $\varepsilon\boldsymbol{a}^{-1}$, with $\varepsilon \in \{-,+\}$, be the finite sequences defined as follows:

$$\boldsymbol{a} = (a_1, a_2, \ldots, a_k), \qquad \boldsymbol{a}^{-1} = (a_k, a_{k-1}, \ldots, a_1),$$
$$\varepsilon\boldsymbol{a} = (\varepsilon a_1, \varepsilon a_2, \ldots, \varepsilon a_k), \quad \varepsilon\boldsymbol{a}^{-1} = (\varepsilon a_k, \varepsilon a_{k-1}, \ldots, \varepsilon a_1).$$

Then we have the following proposition, which can be proved by the argument in [14, Section 5.1].

Proposition 4.3. *Let r be as above and $n > 1$ an integer or a half-integer. Set $m = 2n$. Then a rational number s belongs to the $\Gamma(r;n)$-orbit of ∞ if and only if s has the following continued fraction expansion:*

$$s = 2c + [\varepsilon_1\boldsymbol{a}, mc_1, -\varepsilon_1\boldsymbol{a}^{-1}, 2c_2, \varepsilon_2\boldsymbol{a}, mc_3, -\varepsilon_2\boldsymbol{a}^{-1}, \ldots, 2c_{2t-2}, \varepsilon_t\boldsymbol{a}, mc_{2t-1}, -\varepsilon_t\boldsymbol{a}^{(-1)}]$$

for some positive integer t, $c \in \mathbf{Z}$, $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t) \in \{-,+\}^t$ *and* $(c_1, c_2, \dots, c_{2t-1}) \in \mathbf{Z}^{2t-1}$.

Remark 4.4. Riley's Theorem B and Theorem 3 in [17] imply the following. Let α and β be relatively prime integers with $1 \le \beta < \alpha$. For integers $d \ge 2$, $m \ge 3$, and $e \ge 1$, consider the 2-bridge link $K(\beta^*/\alpha^*)$, where $(\alpha^*, \beta^*) = (\alpha^d m, \alpha^{d-1} m(\alpha - \beta) + e)$. Then there is an epimorphism from the link group $G(K(\beta^*/\alpha^*))$ onto the Heckoid group $G(\beta/\alpha; n)$, where $n = m/2$. This result corresponds to the case when $r = (\alpha - \beta)/\alpha$ and $s = [\boldsymbol{a}, mc, -\boldsymbol{a}^{-1}]$, where $c = \varepsilon\alpha^d$ with $\varepsilon = \pm 1$ in Proposition 4.3. In fact, a simple calculation shows

$$s = (\alpha^{d-1} m(\alpha - \beta) + (-1)^k \varepsilon)/(\alpha^d m) = \beta^*/\alpha^*,$$

where k is the length of $\boldsymbol{a}$ and ε is chosen so that $(-1)^k \varepsilon = e$. Thus Theorem 2.3 and Proposition 4.3 imply that there is an epimorphism from the link group $G(K(\beta^*/\alpha^*)) = G(K(s))$ onto the Heckoid group $G(r; n) \cong G(1 - r; n) = G(\beta/\alpha; n)$, recovering Riley's result.

5. Topological description of odd Heckoid orbifolds

In this section, we show, following the sketch of Agol [1], that the orbifold $\boldsymbol{O}(r; m)$ and the odd Heckoid orbifold $\boldsymbol{S}(r; n)$ are depicted as in Figures 5 and 6. Here, we employ the following convention.

Convention 5.1. Let Σ be a trivalent graph properly embedded in a compact 3-manifold M such that each edge e of Σ is given a *weight* $w(e) \in \mathbf{N}_{\ge 2} \cup \{\infty\}$. Here, a loop component of Σ is regarded as an edge. Assume that if v is a (trivalent) vertex and e_1, e_2, e_3 are the edges incident on v, then either some $w(e_i)$ is ∞ or the following inequality holds:

$$\frac{1}{w(e_1)} + \frac{1}{w(e_2)} + \frac{1}{w(e_3)} > 1.$$

Then the weighted graph (M, Σ, w) determines the following 3-orbifold.

(a) Let Σ_∞ be the subgraph consisting of those edges with weight ∞. Then the underlying space of the orbifold is the complement of an open regular neighborhood of Σ_∞.

(b) The singular set of the orbifold is the intersection of $\Sigma - \Sigma_\infty$ with the underlying space, where the index is given by the weight. (We identify an edge of the singular set with the corresponding edge of Σ.)

We denote the orbifold by the same symbol (M, Σ, w). The part of the boundary of the orbifold (M, Σ, w) contained in ∂M is called the *outer-boundary* of (M, Σ, w) and is denoted by $\partial_{out}(M, \Sigma, w)$.

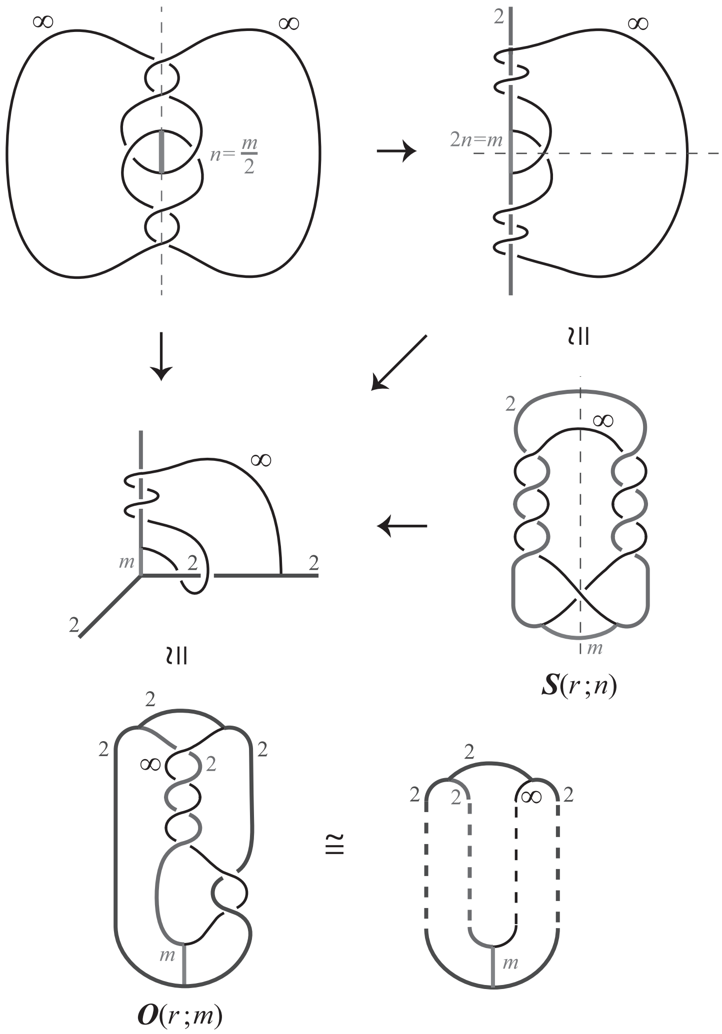

Fig. 5. The case when $K(r)$ is a knot and $m = 2n > 1$ is an odd integer. Here $r = 2/9 = [4, 2]$. The odd Heckoid orbifold $\boldsymbol{S}(r; n)$ (middle right) is a $\mathbf{Z}/2\mathbf{Z}$-covering of $\boldsymbol{O}(r; m)$ (lower left). The upper left figure is not an orbifold, but is a hyperbolic cone manifold. The odd Heckoid orbifold $\boldsymbol{S}(r; n)$ is the quotient of the cone manifold by the π-rotation around the axis containing the singular set.

In this section, we prove the following propositions.

Proposition 5.2. *For a rational number r and an integer $m \geq 2$, the orbifold $\boldsymbol{O}(r; m)$ is homeomorphic to the orbifold $(S^3, K(r) \cup \tau_+ \cup \tau_-, w)$, where τ_+ and τ_- are the upper and lower tunnels of $K(r)$ and the weight function w is defined by the following rule.*

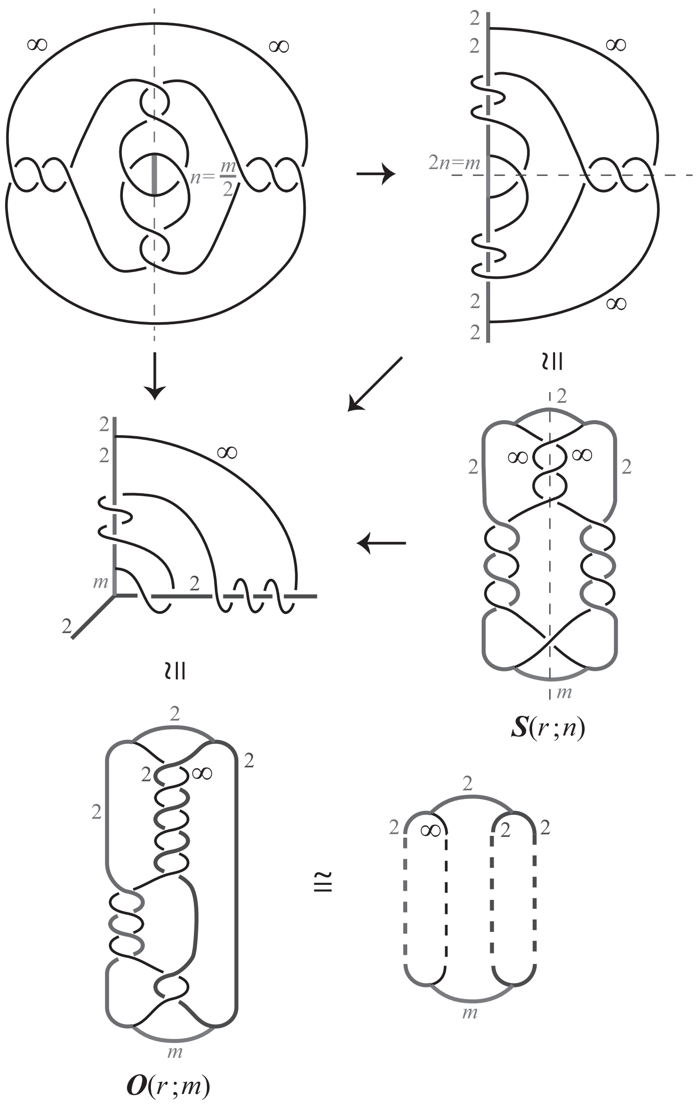

Fig. 6. The case when $K(r)$ is a 2-component link and $m = 2n > 1$ is an odd integer. Here $r = 9/56 = [6, 4, 2]$. The odd Heckoid orbifold $\boldsymbol{S}(r;n)$ (middle right) is a $\mathbf{Z}/2\mathbf{Z}$-covering of $\boldsymbol{O}(r;m)$ (lower left). The upper left figure is not an orbifold, but is a hyperbolic cone manifold. The odd Heckoid orbifold $\boldsymbol{S}(r;n)$ is the quotient of the cone manifold by the π-rotation around the axis containing the singular set.

(a) $w(\tau_+) = 2$ *and* $w(\tau_-) = m$.

(b) *One of the four edges, say* J, *of* $K(r) \cup \tau_+ \cup \tau_-$ *contained in* $K(r)$ *has weight* ∞ *and the remaining three edges have weight* 2.

PROPOSITION 5.3. *For a rational number $r = q/p$, where p and q are relatively prime integers such that $0 \le q < p$, and a non-integral half-integer n greater than* 1, *the odd Heckoid orbifold $\boldsymbol{S}(r;n)$ is described as follows.*

(1) *Suppose that $K(r)$ is a knot, i.e., p is odd* (*see Figure* 5). *Consider the* 2-*bridge knot $K(\hat{r})$, where $\hat{r} = (q/2)/p$ or $((p+q)/2)/p$ according to whether q is even or odd. Let τ_- be the lower tunnel of $K(\hat{r})$, and let J_1 and J_2 be the edges of $K(\hat{r}) \cup \tau_-$ such that $K(\hat{r}) = J_1 \cup J_2$. Then $\boldsymbol{S}(r;n)$ is homeomorphic to the orbifold $(S^3, K(\hat{r}) \cup \tau_-, \hat{w})$, where the weight function $\hat{w}$ is defined as follows.*
 (a) *$\hat{w}(\tau_-) = m$ with $m = 2n$.*
 (b) *$\hat{w}(J_1) = \infty$ and $\hat{w}(J_2) = 2$.*

(2) *Suppose that $K(r)$ has two components, i.e., p is even* (*see Figure* 6). *Consider the* 2-*bridge link $K(\hat{r})$, where $\hat{r} = q/(p/2)$. Let τ_+ and τ_- be the upper and lower tunnels of $K(\hat{r})$, and let J_1 and J_2 be the union of mutually disjoint arcs of $K(\hat{r}) = t(\infty) \cup t(\hat{r})$ bounded by $\partial(\tau_+ \cup \tau_-)$ such that $K(\hat{r}) = J_1 \cup J_2$ and such that $J_i \cap t(\infty)$ $(i = 1, 2)$ is equal to the closure of the intersection of $t(\infty)$ with a component of $B^3 - D_0$, where D_0 is a "horizontal" disk embedded in $(B^3, t(\infty))$ bounded by the slope* 0 *simple loop α_0, which intersects $t(\infty)$ transversely in two points and contains the core tunnel τ_+ of $(B^3, t(\infty))$* (*see Figure* 7(b)). *Then $\boldsymbol{S}(r;n)$ is homeomorphic to the orbifold $(S^3, K(\hat{r}) \cup \tau_+ \cup \tau_-, \hat{w})$, where the weight function $\hat{w}$ is defined as follows.*
 (a) *$\hat{w}(\tau_+) = 2$ and $\hat{w}(\tau_-) = m$.*
 (b) *The* (*two*) *components of J_1 have weight ∞, and the* (*two*) *components of J_2 have weight* 2.

REMARK 5.4. (1) Because of the $(\mathbf{Z}/2\mathbf{Z})^2$-symmetry of 2-bridge links, the choice of the edge J in $K(r)$ in Proposition 5.2 and that of the edges J_1 and J_2 in $K(\hat{r})$ in Proposition 5.3 do not affect the homeomorphism class of the resulting orbifolds.

(2) By the announcement in [2, Section 3 of Preface], there exist hyperbolic cone manifolds as illustrated in the upper left figures in Figures 5 and 6. The odd Heckoid orbifolds are $\mathbf{Z}/2\mathbf{Z}$-quotients of the cone manifolds.

PROOF OF PROPOSITION 5.2. Recall that $\boldsymbol{O}(r;m) = \boldsymbol{B}(\infty;2) \cup \boldsymbol{B}(r;m)$ and note that $(S^3, K(r) \cup \tau_+ \cup \tau_-, w) = (B^3, t(\infty) \cup \tau_+, w_+) \cup (B^3, t(r) \cup \tau_-, w_-)$, where $w_\pm$ are "restrictions" of w. We can observe that there are homeomorphisms $f_+ : \boldsymbol{B}(\infty;2) \to (B^3, t(\infty) \cup \tau_+, w_+)$ and $f_- : \boldsymbol{B}(r;m) \to (B^3, t(r) \cup \tau_-, w_-)$ such that the restriction of each of $f_\pm$ to the outer-boundary determines a homeomorphism from $\check{\boldsymbol{O}}$ to the 2-orbifold, $\check{\boldsymbol{S}}$, obtained from the Conway sphere $\boldsymbol{S}$ by removing an open regular neighborhood of a puncture and filling in order 2 cone points to the remaining punctures. Moreover, each of the homeomorphisms maps the (isotopy class of the) simple loop β_s in $\boldsymbol{O}$ to the the (isotopy class of the) simple loop α_s in $\boldsymbol{S}$ for every $s \in \hat{\mathbf{Q}}$. Thus we can choose $f_\pm$ so that they are consistent with the gluing maps in the constructions of $\boldsymbol{O}(r;m)$ and $(S^3, K(r) \cup \tau_+ \cup \tau_-, w)$; so, f_+ and f_- determine the desired homeomorphism from $\boldsymbol{O}(r;m)$ to $(S^3, K(r) \cup \tau_+ \cup \tau_-, w)$. $\square$

PROOF OF PROPOSITION 5.3. By Definition 3.2, the odd Heckoid group $G(r;n)$ is equal to the kernel of the natural projection

$$\pi_1(\boldsymbol{O}(r;m)) \to \pi_1(\boldsymbol{O}(r;m))/\psi(\pi_1(\boldsymbol{S})),$$

where $m = 2n$ is an odd integer. Thus the Heckoid orbifold $\boldsymbol{S}(r;n)$ is the regular covering of $\boldsymbol{O}(r;m)$ with the covering transformation group

$$\pi_1(\boldsymbol{O}(r;m))/\psi(\pi_1(\boldsymbol{S})) \cong \pi_1(\boldsymbol{O})/\langle\langle \pi_1(\boldsymbol{S}), \beta_\infty^2, \beta_r^m \rangle\rangle.$$

Note that $\pi_1(\boldsymbol{O})/\pi_1(\boldsymbol{S}) \cong (\mathbf{Z}/2\mathbf{Z})^2$ is generated by the homology classes $[\beta_0]$ and $[\beta_\infty]$. Since $[\beta_r] = p[\beta_0] + q[\beta_\infty]$ and since m is odd, the covering transformation group is isomorphic to

$$\langle [\beta_0], [\beta_\infty] \mid p[\beta_0] + q[\beta_\infty] \rangle_{(2)} \cong \mathbf{Z}/2\mathbf{Z},$$

where the suffix (2) represents that this is a presentation as a $\mathbf{Z}/2\mathbf{Z}$-module. Let $\tilde{\boldsymbol{B}}(\infty;2)$ and $\tilde{\boldsymbol{B}}(r;m)$, respectively, be the inverse images of the suborbifolds $\boldsymbol{B}(\infty;2)$ and $\boldsymbol{B}(r;m)$ under the 2-fold covering $\boldsymbol{S}(r;n) \to \boldsymbol{O}(r;m)$. Then, by the above description of the covering transformation group, the covering orbifold $\tilde{\boldsymbol{B}}(\infty;2)$ and its covering involution, h_+, are described as follows.

(a) If $(p,q) \equiv (1,0) \pmod 2$, then $\tilde{\boldsymbol{B}}(\infty;2)$ is identified with the orbifold $(B^3, t(\infty), \hat{w}_+)$, where the weight function $\hat{w}_+$ takes the value ∞ on one of the components of $t(\infty)$ and the value 2 on the other component. Under this identification, the covering involution h_+ is the π-rotation whose axis contains the core tunnel (see Figure 7(a)).

(b) If $(p,q) \equiv (0,1) \pmod 2$, then $\tilde{\boldsymbol{B}}(\infty;2)$ is identified with the orbifold $(B^3, t(\infty) \cup \tau_+, \hat{w}_+)$, where τ_+ is the core tunnel, and the weight function $\hat{w}_+$ is given by the following rule: $\hat{w}_+(\tau_+) = 2$, and $\hat{w}_+$ takes the value ∞ on a pair of edges whose interiors are contained in one of the components the complement of the horizontal disk D_0 in B^3, and the value 2 on the remaining pair of edges. Under this identification, h_+ is the π-rotation whose axis bisects τ_+ (see Figure 7(b)).

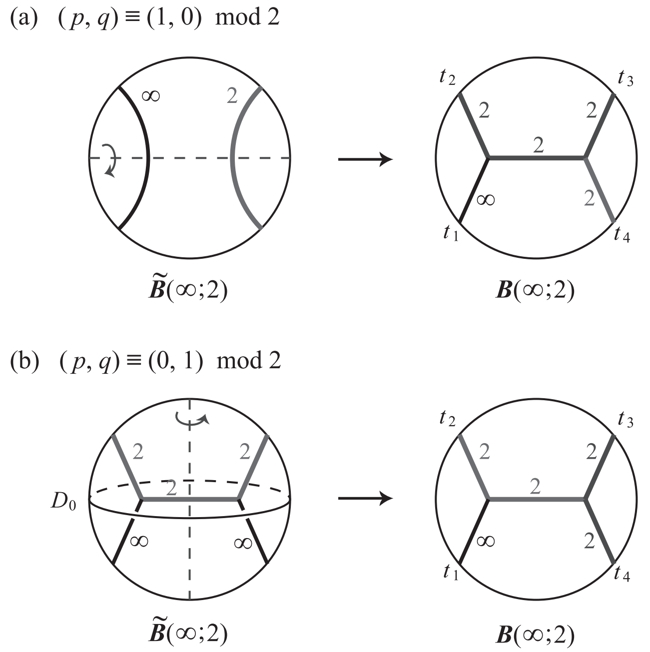

Fig. 7. The covering orbifold $\tilde{\boldsymbol{B}}(\infty;2)$ of $\boldsymbol{B}(\infty;2)$

We can easily observe the following:

Claim. *Under the identifications of the outer-boundaries* $\partial_{out}\tilde{\boldsymbol{B}}(\infty;2)$ *and* $\partial_{out}\boldsymbol{B}(\infty;2)$ *with* (*an orbifold obtained from*) $\boldsymbol{S}$, *as in the above and in the proof of Proposition* 5.2, *the covering projection* $\partial_{out}\tilde{\boldsymbol{B}}(\infty;2) \to \partial_{out}\boldsymbol{B}(\infty;2)$ *maps the pair of simple loops* $(\alpha_0, \alpha_\infty)$ *to* $(\alpha_0, \alpha_\infty^2)$ *or* $(\alpha_0^2, \alpha_\infty)$ *according to whether* $(p,q) \equiv (1,0)$ *or* $(0,1)$ $(\mathrm{mod}\ 2)$.

On the other hand, $\tilde{\boldsymbol{B}}(r;m)$ is identified with the orbifold $(B^3, t_- \cup \tau_-, \hat{w}_-)$, where (B^3, t_-) is a 2-string trivial tangle, τ_- is the core tunnel of (B^3, t_-), and where the weight function $\hat{w}_-$ is given by the following rule: $\hat{w}_-(\tau_-) = m$, and $\hat{w}_-$ takes the value ∞ on one of the four edges of $t_- \cup \tau_-$ whose union is equal to t_-, and the value 2 on the remaining three edges. The covering involution, h_-, of $\tilde{\boldsymbol{B}}(r;m) \cong (B^3, t_- \cup \tau_-, \hat{w}_-)$ is the π-rotation whose axis bisects τ_- (cf. Figure 7(b)).

By these observations concerning the suborbifolds $\tilde{\boldsymbol{B}}(\infty;2)$ and $\tilde{\boldsymbol{B}}(r;m)$, the odd Heckoid orbifold $\boldsymbol{S}(r;n) = \tilde{\boldsymbol{B}}(\infty;2) \cup \tilde{\boldsymbol{B}}(r;m)$ is regarded as the union of the orbifold $(B^3, t_- \cup \tau_-, \hat{w}_-)$ and the orbifold $(B^3, t(\infty), \hat{w}_+)$ or $(B^3, t(\infty) \cup \tau_+, \hat{w}_+)$ according to whether $(p,q) \equiv (1,0)$ or $(0,1)$ $(\mathrm{mod}\ 2)$. This implies that

$\boldsymbol{O}(r;n)$ is constructed from some 2-bridge link as in the proposition. The remaining task is to identify the slope, $\hat{r}$, of the 2-bridge link. To this end, pick a disk $\tilde{D}$ properly embedded in $(B^3, t_- \cup \tau_-, \hat{w}_-) \cong \tilde{\boldsymbol{B}}(r;m)$ which intersects the singular set transversely in a single point in the interior of τ_-, such that $\tilde{D}$ is mapped homeomorphically by the covering projection to a disk in $\boldsymbol{B}(r;m)$ bounded by the loop α_r. Then the slope $\hat{r}$ of the 2-bridge link is equal to the slope of the simple loop $\partial\tilde{D}$ in $\partial_{out}\tilde{\boldsymbol{B}}(\infty;2)$. (Here $\partial_{out}\tilde{\boldsymbol{B}}(\infty;2)$ is identified with the outer boundary of $(B^3, t(\infty), \hat{w}_+)$ or $(B^3, t(\infty) \cup \tau_+, \hat{w}_+)$; so the slope of $\partial\tilde{D}$ in it is defined.) By using the Claim in the above, we can see that $\hat{r} = (q/2)/p$ or $q/(p/2)$ according as $(p,q) \equiv (1,0)$ or $(0,1) \pmod 2$. This completes the proof of the proposition except when $(p,q) \equiv (1,1) \pmod 2$. This remaining case can be settled by using the fact that there is a homeomorphism from $(S^3, K(q/p))$ to $(S^3, K((p+q)/p))$ sending the upper/lower tunnels of $K(q/p)$ to those of $K((p+q)/p)$. $\square$

6. Heckoid groups as two-parabolic Kleinian groups

In this section, we prove Theorem 2.2, which is contained in the announcement by Agol [1]. As noted in [1], the proof relies on the orbifold theorem and is analogous to the arguments in [7, Proof of Theorem 9].

REMARK 6.1. This theorem also follows from the announcement made in the second author's joint work with Akiyoshi, Wada and Yamashita [2, Section 3 of Preface]. Note, however, that there is an error in the assertion 5 in Page IX in Preface, though a special case is treated correctly in [2, Proposition 5.3.9]. In fact, the first sentence of the assertion should be read as follows: The holonomy group of $M(\theta^-, \theta^+)$ is discrete if and only if $\theta^{\pm} \in \{2\pi/n \mid n \in \frac{1}{2}\mathbf{N}_{\geq 2}\} \cup \{0\}$. The second author would also like to note that this assertion can be proved by using the argument of Parkkonen in [15, Lemma 7.5]; this was forgotten to mention in [2], though the paper is included in the bibliography.

In order to prove Theorem 2.2, we prove that $\boldsymbol{O}(r;m)$ with $m = 2n \geq 3$ admits a hyperbolic structure. Throughout this section, we identify $\boldsymbol{O}(r;m)$ with the orbifold $(S^3, K(r) \cup \tau_+ \cup \tau_-, w)$ in Proposition 5.2. We denote by B^3_+ and B^3_- the 3-balls of S^3 bounded by the bridge sphere of $K(r)$ such that

$$(S^3, K(r) \cup \tau_+ \cup \tau_-, w) = (B^3_+, t(\infty) \cup \tau_+, w_+) \cup (B^3_-, t(r) \cup \tau_-, w_-).$$

We refer to [4, Introduction and Section 8] (cf. [3, Chapter 2], [5, Chapter 2] and [8, Chapter 6]) for standard terminologies for orbifolds.

Lemma 6.2. *For a rational number r and an integer $m \geq 3$, the following hold.*

(1) *$\boldsymbol{O}(r;m)$ does not contain a bad 2-suborbifold.*

(2) *Any football $S^2(p,p)$ in $\boldsymbol{O}(r;m)$ bounds a discal 3-suborbifold.*

(3) *$\boldsymbol{O}(r;m)$ does not contain an essential turnover.*

(4) *$\boldsymbol{O}(r;m)$ is topologically atoroidal, i.e., it does not contain an essential orientable toric 2-suborbifold.*

Proof. (1) Suppose that $\boldsymbol{O}(r;m)$ contains a bad 2-suborbifold, F. Then F is either a *teardrop* $S^2(p)$ or a *spindle* $S^2(p,q)$ with $1 < p < q$. Since the indices of the singular set of $\boldsymbol{O}(r;m)$ are 2 and $m(\geq 3)$, and since the underlying 2-sphere $|F|$ intersects $K(r)$ in an even number of points, we see that $|F|$ is disjoint from $K(r)$ and intersects (at least) one of the unknotting tunnels $\tau_{\pm}$ transversely in a single point, where $F \cong S^2(2)$, $S^2(m)$ or $S^2(2,m)$. Since the endpoints of each of the unknotting tunnels are contained in $K(r)$, this implies that $K(r)$ is a split link, a contradiction. Hence $\boldsymbol{O}(r;m)$ cannot contain a bad 2-suborbifold.

(2) Let F be a suborbifold of $\boldsymbol{O}(r;m)$ which is a football. As in (1), we see that one of the following holds.

(i) $|F|$ intersects $K(r)$ in two points, where $F \cong S^2(2,2)$.

(ii) $|F|$ is disjoint from $K(r)$ and intersects one of the unknotting tunnels $\tau_{\pm}$ in two points and does not intersect the other unknotting tunnel, where $F \cong S^2(2,2)$ or $S^2(m,m)$.

Suppose that condition (i) holds. Then $|F|$ is disjoint from $\tau_+ \cup \tau_-$, and so either τ_+ and τ_- are separated by $|F|$, or $\tau_+ \cup \tau_-$ is contained in a single component of $S^3 - |F|$. If τ_+ and τ_- are separated by $|F|$, then $|F|$ must intersect $K(r)$ in at least four points, a contradiction. Hence $\tau_+ \cup \tau_-$ is contained in a single component of $S^3 - |F|$. Let B_1^3 and B_2^3 be the 3-balls in S^3 bounded by $|F|$, such that $\tau_+ \cup \tau_- \subset B_2^3$. Set $K_i = B_i^3 \cap K(r)$ $(i = 1,2)$. Then the genus 3 open handle body $S^3 - (K(r) \cup \tau^+ \cup \tau^-)$ is the union of $B_1^3 - K_1$ and $B_2^3 - (K_2 \cup \tau^+ \cup \tau^-)$ along the open annuls $|F| - K(r)$, and hence the rank 3 free group, $\pi_1(S^3 - (K(r) \cup \tau^+ \cup \tau^-))$, is the free product of $\pi_1(B_1^3 - K_1)$ and $\pi_1(B_2^3 - (K_2 \cup \tau^+ \cup \tau^-))$ with the infinite cyclic amalgamated subgroup $\pi_1(|F| - K(r))$. Since $H_1(B_1^3 - K_1) \cong \mathbf{Z}$, this implies $\pi_1(B_1^3 - K_1) \cong \mathbf{Z}$. Hence (B_1^3, K_1) is a trivial 1-string tangle. Thus $(B_1^3, B_1^3 \cap (K(r) \cup \tau_+ \cup \tau_-)) = (B_F^3, B_1^3 \cap K(r))$ determines a discal 3-suborbifold of $\boldsymbol{O}(r;m)$ bounded by F, and therefore F is inessential.

Suppose that condition (ii) holds. For simplicity, we assume that $|F|$ intersects τ_+ in two points and does not intersect τ_-. (The other case is treated similarly.) Let B_F^3 be the 3-ball bounded by $|F|$ such that $B_F^3 \cap \tau_+$ is a subarc of τ_+. Then $(B_F^3, B_F^3 \cap (K(r) \cup \tau_+ \cup \tau_-)) = (B_F^3, B_F^3 \cap \tau_+)$ is a trivial 1-string tangle, because τ_+ is contained in a trivial constituent knot in the spatial graph $K(r) \cup \tau_+ \cup \tau_-$. Hence it determines a discal 3-suborbifold of $\boldsymbol{O}(r;m)$ bounded by F.

(3) Suppose that $\boldsymbol{O}(r;m)$ contains an essential turnover $F \cong S^2(p,q,r)$. Then either $|F|$ is disjoint from $K(r)$, or $|F|$ intersects $K(r)$ in two points. In the first case, $|F|$ intersects $\tau_+ \cup \tau_-$ in three points and hence $|F|$ intersects τ_+ or τ_- in an odd number of points. As in (1), it follows that $K(r)$ is a split link, a contradiction. Hence we may assume that $|F|$ intersects $K(r)$ in two points, and therefore $|F|$ is disjoint from τ_+ or τ_-. For simplicity, we assume that $|F|$ is disjoint from τ_-. (The other case is treated similarly.) By using the fact that $|F|$ is also disjoint from $\partial \boldsymbol{O}(r;m)$ and the fact that $(B^3_-, t(r) \cup \tau_-)$ is a relative regular neighborhood of τ_- in $(S^3, K(r) \cup \tau_+ \cup \tau_-)$, we can see that F is isotopic to a 2-suborbifold which is disjoint from the suborbifold $(B^3_-, t(r) \cup \tau_-, w_-)$. Hence we may assume that F is contained in the interior of the suborbifold $(B^3_+, t(\infty) \cup \tau_+, w_+)$. Let t_i $(1 \le i \le 4)$ be the edges of $t(\infty) \cup \tau_+$ as illustrated in the right figures in Figure 7. Note that $t(\infty) = \bigcup_{i=1}^4 t_i$, $w_+(t_1) = \infty$ and $w_+(t_i) = 2$ $(2 \le i \le 4)$. Thus $|F|$ is disjoint from t_1 and $|F|$ intersects $(t(\infty) - t_1) \cup \tau_+$ transversely in three points. Let D_h be the disk properly embedded in B^3_+ determined by the plane in which Figure 7 is drawn. Then D_h contains the graph $t(\infty) \cup \tau_+$. We may assume that $|F|$ is transversal to D_h and hence $|F| \cap D_h$ consists of mutually disjoint circles. By using the irreducibility of $B^3_+ - (t(\infty) \cup \tau_+)$, we may assume, by a standard argument, that no component of $|F| \cap D_h$ bounds a disk disjoint from $t(\infty) \cup \tau_+$. Then it follows that $|F| \cap D_h$ must consist of a single circle which intersects τ_+, t_3 and t_4 in a single point. Let D_F be the disk in D_h bounded by the circle $D_h \cap |F|$, and let B^3_F be the 3-ball in B^3_+ bounded by $|F|$. Then D_F is properly embedded in B^3_F, and $B^3_F \cap (t(\infty) \cup \tau_+) = D_F \cap (t(\infty) \cup \tau_+)$. Hence $(B^3_F, B^3_F \cap (t(\infty) \cup \tau_+))$ determines a discal 3-orbifold bounded by the turnover F, a contradiction.

(4) Suppose that $\boldsymbol{O}(r;m)$ contains an essential pillow $F \cong S^2(2,2,2,2)$. Then $|F|$ is disjoint from τ_-, which has index $m \ge 3$, and hence we may assume, as in (3), that $|F|$ is contained in the suborbifold $(B^3_+, t(\infty) \cup \tau_+, w_+)$. Under the notation in (3), $|F|$ is disjoint from t_1 and $|F|$ intersects $t(\infty) \cup \tau_+$ transversely in four points. We may also assume that $|F|$ is transversal to the disk D_h introduced in (3) and hence $|F| \cap D_h$ consists of mutually disjoint circles. By using the irreducibility of $B^3_+ - (t(\infty) \cup \tau_+)$ and the assumption that F is essential, we may assume that no component of $|F| \cap D_h$ bounds a disk disjoint from $t(\infty) \cup \tau_+$. Hence we see that either (i) $|F| \cap D_h$ consists of a single loop which intersects $(t(\infty) - t_1) \cup \tau_+$ in four points, or (ii) $|F| \cap D_h$ consists of two loops each of which intersects $(t(\infty) - t_1) \cup \tau_+$ in two points. In either case, we can find an "outermost disk" δ in D_h satisfying the following conditions.

(a) $\delta \cap |F|$ is an arc, c, in $\partial\delta$.
(b) $\delta \cap (t(\infty) \cup \tau_+)$ is an arc, c', in $\partial\delta$ which is contained in the interior of an edge of the graph $t(\infty) \cup \tau_+$.
(c) $\partial\delta = c \cup c'$.
(d) δ is contained in the 3-ball, B^3_F, in B^3_+ which is bounded by $|F|$.

Then the frontier of a regular neighborhood of δ in B^3_F is a disk properly embedded in B^3_F disjoint from the singular set, whose boundary is an essential loop in the pillow F. This contradicts the assumption that F is essential. Hence $\boldsymbol{O}(r;m)$ does not contain an essential pillow.

Assume that $\boldsymbol{O}(r;m)$ contains an essential torus, F. Then F is a torus contained in $S^3 - (K(r) \cup \tau_+ \cup \tau_-)$, which is a genus 3 open handlebody. Hence F must be compressible in $S^3 - (K(r) \cup \tau_+ \cup \tau_-)$, a contradiction.

By the classification of toric 2-orbifolds, an orientable toric 2-orbifold is a torus, a turnover or a pillow. Hence by (3) and the above arguments, $\boldsymbol{O}(r;m)$ does not contain an essential orientable toric 2-orbifold. □

LEMMA 6.3. *For a rational number r and an integer $m \geq 3$, the orbifold $\boldsymbol{O}(r;m)$ is Haken, i.e., it is irreducible and does not contain an essential turnover, but contains an essential 2-suborbifold.*

PROOF. By Lemma 6.2(1), $\boldsymbol{O}(r;m)$ does not contain a bad 2-suborbifold. By Lemma 6.2(2) and (3), every orientable spherical 2-suborbifold of $\boldsymbol{O}(r;m)$ with nonempty singular set bounds a discal 3-suborbifold. Moreover any 2-sphere (i.e., spherical 2-suborbifold with empty singular set) of $\boldsymbol{O}(r;m)$ bounds a 3-ball in $\boldsymbol{O}(r;m)$, because Proposition 5.2 implies that the complement of an open regular neighborhood of the singular set of $\boldsymbol{O}(r;m)$ is homeomorphic to a genus 3 handlebody. Hence the orbifold $\boldsymbol{O}(r;m)$ is irreducible. Moreover, it does not contain an essential turnover by Lemma 6.2(3). Since $\partial\boldsymbol{O}(r;m) \cong S^2(2,2,2,m)$ is not a turnover, we see by [3, Proposition 4.6] that $\boldsymbol{O}(r;m)$ is Haken (see [4, Definition 8.0.1]). □

LEMMA 6.4. *For a rational number r and an integer $m \geq 3$, the orbifold $\boldsymbol{O}(r;m)$ is homotopically atoroidal, i.e., $\pi_1(\boldsymbol{O}(r;m))$ is not virtually abelian and every rank 2 free abelian subgroup of $\pi_1(\boldsymbol{O}(r;m))$ is peripheral.*

PROOF. Since $\boldsymbol{O}(r;m)$ is Haken by Lemma 6.3, we see by [19, Theorem A] (cf. [4, Proposition 8.2.2]) that $\boldsymbol{O}(r;m)$ is good, i.e., it has a manifold cover. Suppose on the contrary that $\boldsymbol{O}(r;m)$ is not homotopically atoroidal (see [4, Definition 8.2.13]). Then, since $\boldsymbol{O}(r;m)$ is topologically atoroidal by Lemma 6.2(4), we see by [4, Proposition 8.2.11] that $\boldsymbol{O}(r;m)$ is either Euclidean or Seifert fibered. (Here, we use the fact that $\boldsymbol{O}(r;m)$ is good.) This contradicts the fact that $\partial\boldsymbol{O}(r;m) \cong S^2(2,2,2,m)$ is not Euclidian. Hence the orbifold $\boldsymbol{O}(r;m)$ is homotopically atoroidal. □

COROLLARY 6.5. *For a rational number r and an integer $m \geq 3$, the interior of $\boldsymbol{O}(r;m)$ has a geometrically finite hyperbolic structure. In particular, $\boldsymbol{O}(r;m)$ is very good, i.e., it has a finite cover which is a manifold.*

PROOF. By Lemmas 6.3 and 6.4, $\boldsymbol{O}(r;m)$ is a homotopically atoroidal Haken 3-orbifold. Hence, by the orbifold theorem for Haken orbifolds [4, Theorem 8.2.14], $\boldsymbol{O}(r;m)$ is hyperbolic. Moreover, it follows from the proof of the theorem that the hyperbolic structure can be chosen to be geometrically finite. The last assertion follows from Selberg's Lemma [18] (cf. [12, Theorem 2.29]). □

Let $P = \mathrm{cl}(\partial \boldsymbol{B}(\infty;2) - \partial_{out}\boldsymbol{B}(\infty;2))$. Then $P \cong D^2(2,2)$ is an annular 2-suborbifold in $\partial \boldsymbol{O}(r;m)$, and the following lemma shows that $(\boldsymbol{O}(r;m), P)$ is a pared 3-orbifold (see [4, Definition 8.3.7]).

LEMMA 6.6. *For a rational number r and an integer $m \geq 3$, the pair $(\boldsymbol{O}(r;m), P)$ satisfies the following conditions, and hence it is a pared 3-orbifold.*

(1) *$\boldsymbol{O}(r;m)$ is irreducible and very good.*

(2) *P is incompressible.*

(3) *Every rank 2 free abelian subgroup of $\pi_1(\boldsymbol{O}(r;m))$ is conjugate to a subgroup of $\pi_1(P)$. (In fact, $\pi_1(\boldsymbol{O}(r;m))$ does not contain a rank 2 free abelian subgroup.)*

(4) *Any properly embedded annular 2-suborbifold $(A, \partial A) \subset (\boldsymbol{O}(r;m), P)$ whose boundary rests on essential loops in P is parallel to P.*

PROOF. (1) This follows from Lemma 6.3 and Corollary 6.5.

(2) Suppose that P is compressible. Then there is a discal orbifold $(F, \partial F)$ properly embedded in $(\boldsymbol{O}(r;m), P)$ such that ∂F is a loop in P parallel to ∂P. Since F has at most one cone point, $|F|$ is disjoint from τ_+ or τ_-. For simplicity, we assume that $|F|$ is disjoint from τ_-. (The other case is treated similarly.) By using the fact that ∂F is parallel to ∂P in $\partial \boldsymbol{O}(r;n)$, and the fact that $(B^3_-, t(r) \cup \tau_-)$ is a relative regular neighborhood of τ_- in $(S^3, K(r) \cup \tau_+ \cup \tau_-)$, we can see that F is isotopic to a 2-suborbifold which is disjoint from the suborbifold $(B^3_-, t(r) \cup \tau_-, w_-)$. Hence we may assume that F is contained in the interior of the suborbifold $(B^3_+, t(\infty) \cup \tau_+, w_+)$. Then, by looking at the intersection of $|F|$ with the disk D_h as in the proof of Lemma 6.2(3), we see that this cannot happen.

(3) Suppose that $\pi_1(\boldsymbol{O}(r;m))$ contains a rank 2 free abelian subgroup, H. Then H is conjugate to a subgroup of $j_*(\pi_1(\partial \boldsymbol{O}(r;m)))$ by Lemma 6.4, where j is the inclusion. If j_* is injective, then $j_*(\pi_1(\partial \boldsymbol{O}(r;m))) \cong \pi_1(S^2(2,2,2,m))$ is isomorphic to a Fuchsian group and hence it cannot contain a rank 2 free abelian subgroup, a contradiction. So, j_* is not injective.

By the loop theorem for good orbifolds [4, p. 133], $\partial\boldsymbol{O}(r;m)$ is compressible. Let $F \cong D^2(d)$ be a compressing disk for $\partial\boldsymbol{O}(r;m)$. Then $d = 1, 2$ or m, and ∂F is a loop in $\partial\boldsymbol{O}(r;m)$ separating the 4 singular points into two pairs of singular points. Thus the result of compression of $\partial\boldsymbol{O}(r;m)$ by F is a union of two 2-suborbifolds, $F_1 \cong S^2(2,2,d)$ and $F_2 \cong S^2(2,m,d)$. If $d = 1$, $F_2 \cong S^2(2,m)$ is a bad 2-suborbifold, a contradiction to Lemma 6.2(1). Hence $d = 2$ or m, and therefore F_1 and F_2 are turnovers. By Lemma 6.2(3), they must be inessential. Since none of them is boundary parallel, each F_i is a spherical turnover bounding a discal 3-orbifold. Note that the singular set of $\boldsymbol{O}(r;m)$ has exactly two vertices and the boundaries of regular neighborhoods of the vertices are $S^2(2,2,2)$ and $S^2(2,2,m)$ (see Proposition 5.2). Hence we see $d = 2$ and F_1 and F_2 are the boundaries of regular neighborhoods of the two vertices. Thus $\partial\boldsymbol{O}(r;m)$ is parallel to the boundary of the 3-orbifold obtained from the regular neighborhoods of the two vertices of the singular set by joining them by a tube around the unique edge of the singular set (of index 2) joining the two vertices. Thus $\pi_1(\boldsymbol{O}(r;m))$ is a free product of the dihedral groups of orders 4 and $2m$ with amalgamated subgroup isomorphic to $\mathbf{Z}/2\mathbf{Z}$. It is easy to see that such a group cannot contain a rank 2 free abelian subgroup. Hence, $\pi_1(\boldsymbol{O}(r;m))$ does not contain a rank 2 free abelian subgroup.

(4) Let $(A, \partial A)$ be an annular 2-suborbifold properly embedded in $(\boldsymbol{O}(r;m), P)$ whose boundary rests on essential loops in P.

Suppose first that A is an annulus. Then A is disjoint from $\tau_+ \cup \tau_-$. Since each component of ∂A is parallel to ∂P in $P \subset \partial\boldsymbol{O}(r;m)$, we may assume as in (2) that A is embedded in a regular neighborhood of the 2-suborbifold of $(S^3, K(r) \cup \tau_+ \cup \tau_-, w)$ determined by the 2-bridge sphere. Thus $(A, \partial A)$ is regarded as a suborbifold of $(\check{\boldsymbol{S}} \times [-1,1], P')$, where $\check{\boldsymbol{S}}$ is obtained from the Conway sphere $\boldsymbol{S}$ by removing an open regular neighborhood of a puncture and filling in order 2 cone points to the remaining punctures (cf. Proof of Proposition 5.2), and P' is the product annulus $\partial\check{\boldsymbol{S}} \times [-1,1]$. Consider a disk properly embedded in $|\check{\boldsymbol{S}}| \times [-1,1]$ which contains the singular set. By looking at the intersection of A with the disk, we can find a boundary compressing disk for A. By the irreducibility of the complement of the singular set of $\check{\boldsymbol{S}} \times [-1,1]$, this implies that A is parallel to an annulus in $P' \subset P$.

Suppose next that A is homeomorphic to $D^2(2,2)$. Then A is disjoint from τ_-, which has index $m \geq 3$. Since ∂A is parallel to ∂P in $P \subset \partial\boldsymbol{O}(r;m)$, we see as in the above that A is contained in the suborbifold $(B^3_+, t(\infty) \cup \tau_+, w_+)$. By looking at the intersection of $|A|$ with the disk D_h as in the proof of Lemma 6.2(3), we can see that A is parallel to the suborbifold of P bounded by ∂A. $\square$

Since $\pi_1(\boldsymbol{O}(r;m))$ is not virtually abelian by Lemma 6.4, we obtain the following proposition by Lemma 6.6 and by the orbifold theorem for Haken pared orbifolds [4, Theorem 8.3.9].

Proposition 6.7. *For a rational number r and an integer $m \geq 3$, the pared orbifold $(\boldsymbol{O}(r;m), P)$ is hyperbolic, i.e., there is a geometrically finite hyperbolic 3-orbifold M such that for some $\delta > 0$ and μ, $(\boldsymbol{O}(r;m), P)$ is homeomorphic to*

$$(\mathrm{thick}_\mu(C_\delta(M)), \partial\, \mathrm{thick}_\mu(C_\delta(M)) \cap \mathrm{thin}_\mu(C_\delta(M)),$$

where $C_\delta(M)$ is the closed δ-neighborhood of the convex core $C(M)$ of M, and $\mathrm{thick}_\mu(C_\delta(M))$ and $\mathrm{thin}_\mu(C_\delta(M))$ are μ-thick part and μ-thin part. Here μ is chosen so that $\mathrm{thin}_\mu(C_\delta(M))$ consists of only cuspidal part.

Proof of Theorem 2.2. By the above proposition, there is a faithful discrete representation $\rho : \pi_1(\boldsymbol{O}(r;m)) \to PSL(2, \mathbf{C})$ which maps the conjugacy class represented by the loop ∂P to a parabolic transformation. Recall that the Heckoid group $G(r;n) = \pi_1(\boldsymbol{S}(r;n))$ is a subgroup of $\pi_1(\boldsymbol{O}(r;m))$ of index 2 or 4 by Proposition 3.1 and Definition 3.2 and that it is generated by two elements in the conjugacy class of ∂P. Hence, the restriction of ρ to the subgroup $G(r;n)$ gives the desired isomorphism from $G(r;n)$ to a geometrically finite Kleinian group generated by two parabolic transformations. □

At the end of this section, we prove the following proposition, which illustrates a significant difference between odd Heckoid groups and even Heckoid groups.

Proposition 6.8. *No odd Heckoid group is a one-relator group.*

Proof. Consider an odd Heckoid orbifold $\boldsymbol{S}(r;n)$. By Proposition 5.3, the singular set of $\boldsymbol{S}(r;n)$ has two or four 1-dimensional strata. Note that the above proof of Theorem 2.2 shows that $\boldsymbol{S}(r;n)$ is hyperbolic, and so the interior of $\boldsymbol{S}(r;n)$ is homeomorphic to a hyperbolic orbifold $\mathbf{H}^3/\Gamma$, where Γ is a Kleinian group isomorphic to $\pi_1(\boldsymbol{S}(r;n))$. Hence $\Gamma \cong \pi_1(\boldsymbol{S}(r;n))$ has two or four conjugacy classes of maximal finite cyclic subgroups, accordingly. On the other hand, any one-relator group has a unique maximal finite cyclic subgroup up to conjugacy (see [11, Theorem IV.5.2]). Hence the odd Heckoid group $G(r;n) = \pi_1(\boldsymbol{S}(r;n))$ cannot be a one-relator group. □

Acknowledgements

The second author would like to thank Gerhard Burde for drawing his attention to the work of Riley [17] when he was visiting Frankfurt in 1997. The second author would also like to thank Ian Agol for sending him a copy of the slide of his talk [1]. Both authors would like to thank Yeonhee Jang, Makoto Ozawa and Toshio Saito for helpful discussions and information concerning Section 6. They would also like to thank the referee for very careful reading and helpful comments.

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