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An analogue of the spectral projection for homogeneous trees

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Abstract. We shall define the spectral projection on the homogeneous tree $\mathfrak{X}$, which is an analogue of the one given by Bray for semisimple Lie groups. We shall prove the Paley–Wiener theorem for the spectral projection on $\mathfrak{X}$. As an application, we present an elementary proof of the Paley–Wiener theorem for the Helgason–Fourier transform on $\mathfrak{X}$, which was obtained by Cowling and Setti.

1. Introduction

One of the main concerns in the harmonic analysis has been the characterization of the images of the Fourier transforms of various function spaces, such as a space of compactly supported smooth functions, Schwartz space and L^p Schwartz space. Even now, a number of authors consider these problems for the case of Lie groups or homogeneous spaces. In [2], Bray studied the spectral projection P_λ on the Riemannian symmetric space G/K of rank 1 and gave the characterization of the range of P_λ acting on $C_c^\infty(G/K)$. Here the spectral projection $P_\lambda f$ of $f \in C_c^\infty(G/K)$ is defined by

$$P_\lambda f(g) = (f * \phi_\lambda)(g) = \int_G f(g_1)\phi_\lambda(g_1^{-1}g)dg_1,$$

ϕ_λ denoting the zonal spherical function on G. Ionescu characterized the image of $L^2(G/K)$ under the spectral projection in [7], and Jana determined the image of the L^p Schwartz space $\mathscr{C}^p(G/K)$ in [8].

Many authors have pointed out the analogy between the harmonic analysis on homogeneous trees $\mathfrak{X}$ and that on Lie groups (see [4, 3, 5]). In particular, Cowling, Meda and Setti studied the Helgason–Fourier transform and its inverse transform in [4]. In the subsequent paper [5], they gave characterizations of the images of the space of compactly supported functions $C_c(\mathfrak{X})$ and the Schwartz space $\mathscr{C}(\mathfrak{X})$. We study here an analogue of the spectral projection for $\mathfrak{X}$. In this line of research, it is natural to study the characterization

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of these spaces under the spectral projection. In this paper, we shall give a characterization of the range of $C_c(\mathfrak{X})$ under the spectral projection P_s on $\mathfrak{X}$.

A brief outline of this note is as follows: Section 2 is devoted to the overview of the spherical representations on the homogeneous trees and the definition of the Helgason–Fourier transform. In Section 3, we define the generalized spherical functions relative to the n-th martingale difference on $\mathfrak{X}$. We write down the Helgason–Fourier transform in terms of the generalized spherical functions. In Section 5, we shall give a characterization of $C_c(\mathfrak{X})$ under the spectral projection. Our proof is made in parallel with the discussion of [2] for semisimple Lie groups. As an application of our result, we shall give an elementary proof of the Paley–Wiener theorem for the Helgason–Fourier transform due to Cowling and Setti [5]. Our proof depends only on the Paley–Wiener theorem for the Fourier cosine transform on torus $\mathbf{T}$.

2. Notation and preliminaries

To begin with, let us fix some notation and terminology. For more information, the reader is referred to the book [6] or the survey [4].

Let $q \geq 2$ and $\mathfrak{X}$ be a homogeneous tree of degree $q+1$. It carries a natural distance d, $d(x, y)$ being the number of edges between the vertices x and y. We fix a reference point o in $\mathfrak{X}$ and write $|x| = d(x, o)$. Let $x, y \in \mathfrak{X}$. When $x, y \in \mathfrak{X}$ belong to the same edge, they are said to be adjacent and we write $x \sim y$. The geodesic path starting at x and ending at y means the sequence $\{x_0, x_1, \dots, x_n\}$ in $\mathfrak{X}$ satisfying $d(x, y) = n$, $x_0 = x$, $x_n = y$ and $d(x_i, x_j) = |i - j|$. For any $x, y \in \mathfrak{X}$, there exists the unique geodesic path joining x and y and will be denoted by $[x, y]$. For $x \in \mathfrak{X}$ and $n \leq |x|$, we write $x^{(n)}$ for the element in $[o, x]$ such that $|x^{(n)}| = n$.

A geodesic ray ω in $\mathfrak{X}$ is an infinite sequence $\{\omega_n : n \in \mathbf{Z}_{\geq 0}\}$ satisfying $d(\omega_i, \omega_j) = |i - j|$. Let ω and ω' be geodesic rays. We say that ω and ω' are equivalent if there exist $i \in \mathbf{Z}_{\geq 0}$ and $j \in \mathbf{Z}_{\geq 0}$ such that $\omega_n = \omega'_{n+i}$ for all $n \geq j$. The Poisson boundary is the set of equivalence classes of all geodesic rays and will be denoted by Ω. For $\omega \in \Omega$, we choose the representative of ω starting at o and denote it by ω again. In this paper, the geodesic rays are always interpreted as the representative starting at o.

Let $x, y \in \mathfrak{X}$ and $\omega \in \Omega$. We use the notation $c(x, y)$ to denote the confluence point of the geodesic paths $[o, x]$ and $[o, y]$. Similarly, $c(x, \omega)$ denotes the confluence point of the geodesic path $[o, x]$ and the geodesic ray ω. We write $\mathfrak{B}_n$ for the closed ball centered at o of radius n and $\mathfrak{S}_n$ for the sphere centered at o of radius n, respectively. For convenience, we set $\mathfrak{B}_{-1} = \emptyset$. Let $w_n = \text{Card } \mathfrak{S}_n$, Card S indicating the cardinality of the set S. Then it is known that $w_n = (q+1)q^{n-1}$ for $n \geq 1$ and $w_0 = 1$.

We denote by G the group of isometries of $\mathfrak{X}$ and by K the stabilizer of o in G. Then G/K can be identified with $\mathfrak{X}$ via the correspondence $g \mapsto g \cdot o$. We endow the group G with the Haar measure dg such that the mass of K is equal to 1. Let $C(G/K)$ denote the space of continuous functions on G/K and $C_c(G/K)$ the subspace of $C(G/K)$ with compact support. Then, under the above identification, we have for $f \in C_c(G/K)$ that

$$\int_G f(g)dg = \sum_{x \in \mathfrak{X}} f(x).$$

For $g \in G$, we put

$$\sigma(g) = |g \cdot o|,$$

$$\Omega(g) = (q+1)q^{\sigma(g)-1} \qquad (\text{for } g \neq o), \quad \Omega(o) = 1.$$

We set

$$E(x) = \{\omega \in \Omega : x = \omega_{|x|}\}. \tag{1}$$

We define the K-invariant, G-quasi-invariant probability measure ν on Ω by

$$\nu(E(o)) = \nu(\Omega) = 1,$$

$$\nu(E(x)) = \frac{1}{(q+1)q^{|x|-1}} \qquad (x \in \mathfrak{X} \backslash \{o\}).$$

Let $\mathscr{M}$ denote the σ-algebra generated by $E(x)$. Then $(\Omega, \mathscr{M}, \nu)$ is a measure space. For $E \in \mathscr{M}$, χ_E indicates the characteristic function of E. Let $\mathscr{M}_n$ denote the σ-subalgebra of $\mathscr{M}$ generated by $E(x)$ with $|x| \leq n$. For a $\mathscr{M}$-measurable function η, we indicate by $\mathbf{E}_n\eta$ the conditional expectation of η relative to $\mathscr{M}_n$, that is,

$$\mathbf{E}_n\eta(\omega) = \frac{1}{\nu(E(\omega_n))} \int_{E(\omega_n)} \eta(\omega')d\nu(\omega'). \tag{2}$$

Here we set $\mathbf{E}_{-1}\eta = 0$. With these conventions, the set $\{\mathbf{E}_n\eta : n \in \mathbf{Z}_{\geq 0}\}$ is a martingale associated to $\eta \in L^1(\Omega)$. Let us set $\mathbf{D}_n\eta = \mathbf{E}_n\eta - \mathbf{E}_{n-1}\eta$. Then $\mathbf{D}_n\eta$ is called the n-th martingale difference of $\eta \in L^1(\Omega)$. $\mathbf{D}_n\eta$ is written as

$$\mathbf{D}_n\eta(\omega) = \int_\Omega \delta_n(\omega, \omega')\eta(\omega')d\nu(\omega'),$$

where

$$\delta_n(\omega, \omega') = \nu(E(\omega_n))^{-1}\chi_{E(\omega_n)}(\omega') - \nu(E(\omega_{n-1}))^{-1}\chi_{E(\omega_{n-1})}(\omega').$$

For the explicit expression of $\delta_n(\omega, \omega')$, see [9, Proposition 4.3]. The height function $h_\omega(x)$ of $x \in \mathfrak{X}$ with respect to $\omega \in \Omega$ is defined by

$$h_\omega(x) = \lim_{m \to \infty} d(x, \omega_m). \tag{3}$$

By definition, the Poisson kernel $p(g, \omega)$ is the Radon–Nikodym derivative $d\nu(g^{-1}\omega)/d\nu(\omega)$. As shown in [6, p. 37], it holds that

$$p(x, \omega) = q^{h_\omega(x)}.$$

In analogy with the terminology for semisimple Lie groups, we define the Poisson transform of $\eta \in L^2(\Omega)$ by

$$P^s\eta(x) = \int_\Omega p(x, \omega)^{1/2+\sqrt{-1}s}\eta(\omega)d\nu(\omega) \qquad (s \in \mathbf{C}). \tag{4}$$

We set, for $n \in \mathbf{Z}_{\geq 0}$,

$$S(n, x) = \begin{cases} \{x\} & (|x| \leq n), \\ \{y \in \mathfrak{X} : |y| = |x|, y^{(n)} = x^{(n)}\} & (|x| > n). \end{cases}$$

For a function f on $\mathfrak{X}$, we define its average $\varepsilon_n f$ by

$$\varepsilon_n f(x) = \frac{1}{\text{Card } S(n, x)} \sum_{y \in S(n, x)} f(y). \tag{5}$$

We write $f^\# = \varepsilon_0 f$ and call $f^\#$ the spherical mean of f. For a function f on $\mathfrak{X}$ and $n \in \mathbf{Z}_{\geq 0}$, we define

$$\Delta_n f(x) = \varepsilon_n f(x) - \varepsilon_{n-1} f(x).$$

Here we set $\varepsilon_{-1} f = 0$. The Laplace operator $\mathscr{L}$ on $\mathfrak{X}$ is defined by

$$\mathscr{L}f(x) = \frac{1}{q+1} \sum_{y \sim x} f(y). \tag{6}$$

As described in [6, p. 35], it is satisfied that

$$\mathscr{L}P^s\eta(x) = \lambda(s)P^s\eta(x), \tag{7}$$

where $\lambda(s) = \{\sqrt{q}/(q+1)\}\cos(s \log q)$.

We say that a function f on $\mathfrak{X}$ is radial if $f(x)$ depends only on $|x|$. For a function space $E(\mathfrak{X})$, we denote the subspace of radial functions in $E(\mathfrak{X})$ by $E(\mathfrak{X})^\#$. We naturally identify $E(\mathfrak{X})$ with $E(G/K)$ and $E(\mathfrak{X})^\#$ with $E(K\backslash G/K)$, respectively. The convolution $f * \varphi$ of $f \in \mathscr{D}(\mathfrak{X})$ and $\varphi \in \mathscr{D}(\mathfrak{X})^\#$ is given by

$$(f * \varphi)(g) = \int_G f(g_1)\varphi(g_1^{-1}g)dg_1. \tag{8}$$

3. The Helgason–Fourier transform on $\mathfrak{X}$

Retain the notation in §2. We shall first review the spherical representations of G and the Helgason–Fourier transform on $\mathfrak{X}$ to explain the notation and parametrization. We use the notation $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ to denote the canonical inner product and the corresponding norm on $L^2(\Omega)$, respectively. We set $\tau = 2\pi/\log q$ and $\mathbf{T} = \mathbf{R}/\tau\mathbf{Z}$. We say that a function $F(s)$ on $\mathbf{R}$ is Weyl-invariant if it satisfies $F(s) = F(-s)$ and $F(s+\tau) = F(s)$.

Let $s \in \mathbf{C}$. Define the action π_s of G on $L^2(\Omega)$ by the formula

$$(\pi_s(g)\eta)(\omega) = p(g \cdot o, \omega)^{(1/2)+\sqrt{-1}s}\eta(g^{-1}\omega). \tag{9}$$

Let $s \in \mathbf{C}$ be such that $s \notin \pm\frac{1}{2}\sqrt{-1} + \frac{\tau}{2}\mathbf{Z}$. According to [6, p. 44], the intertwining operator I_s between π_s and π_{-s} is defined by

$$I_s = (P^{-s})^{-1}P^s. \tag{10}$$

For $f \in C_c(\mathfrak{X})$, we define its Helgason–Fourier transform by

$$\tilde{f}(s,\omega) = (\pi_s(f)1)(\omega) = \sum_{x \in \mathfrak{X}} f(x)p(x,\omega)^{(1/2)+\sqrt{-1}s}. \tag{11}$$

Then as indicated in [4, Proposition 2.6], the following inversion formula holds:

$$f(x) = c_G \int_{\Omega}\int_{\mathbf{T}} \tilde{f}(s,\omega)p(x,\omega)^{(1/2)-\sqrt{-1}s}|c(s)|^{-2}ds\,dv(\omega), \tag{12}$$

where $c_G = q/\{2\tau(q+1)\}$ and

$$c(s) = \frac{\sqrt{q}}{q+1} \cdot \frac{q^{(1/2)+\sqrt{-1}s} - q^{-(1/2)-\sqrt{-1}s}}{q^{\sqrt{-1}s} - q^{-\sqrt{-1}s}} \tag{13}$$

is a c-function. Further, the Helgason–Fourier transform extends to an isometric mapping from $L^2(\mathfrak{X})$ into $L^2(\mathbf{T} \times \Omega, c_G|c(s)|^{-2}ds\,dv(\omega))$ and its range coincides with the subspace of $L^2(\mathbf{T} \times \Omega, c_G|c(s)|^{-2}ds\,dv(\omega))$ consisting of functions which satisfy the following symmetry condition:

$$\int_{\Omega} F(s,\omega)p(x,\omega)^{(1/2)-\sqrt{-1}s}dv(\omega) = \int_{\Omega} F(-s,\omega)p(x,\omega)^{(1/2)+\sqrt{-1}s}dv(\omega). \tag{14}$$

Let $\mathscr{H}_n$ denote the subspace of $L^2(\Omega)$ comprised of all functions F such that $\mathbf{D}_n F = F$. Let $a \in \mathfrak{X}\backslash\{o\}$ and for ease of notation write a' for $a^{(|a|-1)}$. We define the function ξ_a on Ω by $\xi_o(\omega) = 1$ and for $a \neq o$

$$\xi_a(\omega) = v(E(a))^{-1}\chi_{E(a)}(\omega) - v(E(a'))^{-1}\chi_{E(a')}(\omega). \tag{15}$$

Then it is easy to check that $\mathbf{D}_{|a|}\xi_a = \xi_a$.

Let $a \in \mathfrak{X}$ and $s \in \mathbf{C}$. We define the function $\Phi_{a,s}$ on $\mathfrak{X}$ by

$$\Phi_{a,s}(x) = \int_{\Omega} p(x,\omega)^{1/2+\sqrt{-1}s} \xi_a(\omega) d\nu(\omega). \tag{16}$$

We call $\Phi_{a,s}$ the generalized spherical function on $\mathfrak{X}$. When $a = o$, $\Phi_{o,s}$ coincides with the spherical function ϕ_s on $\mathfrak{X}$, which is defined in the preceding papers [4, 3, 5]. By the definition of the generalized spherical function, it holds that

$$\Phi_{a,s}(g \cdot o) = \langle \pi_s(g)1, \xi_a \rangle, \qquad \Phi_{a,s}(x) = P^s \xi_a(x).$$

Define the function $Q_n(s)$ on $\mathbf{C}$ by

$$Q_0(s) = 1, \tag{17}$$

$$Q_n(s) = \frac{\sqrt{q}}{q+1} q^{-n/2} q^{\sqrt{-1}(n-1)s} (q^{1/2+\sqrt{-1}s} - q^{-1/2-\sqrt{-1}s}) \qquad (n \geq 1). \tag{18}$$

We note that the function $Q_n(s)$ is an analogue of Kostant's polynomial for semisimple Lie groups. In the following, we use the notation ψ to denote

$$\psi(n,x) = \frac{\sin(ns \log q)}{\sin(s \log q)}, \qquad (n \in \mathbf{Z}_{\geq 0}, s \in \mathbf{R}).$$

Applying Theorem 2.1 in [9], we can immediately obtain the explicit expression of the generalized spherical function $\Phi_{a,s}$.

Proposition 1. *We have the following expressions:*
(1) *(The case $a \neq o$) Let $\omega \in E(x)$. Then we have*

$$\Phi_{a,s}(x) = \begin{cases} 0 & (|x| < |a|), \\ q^{-(|x|-|a|)/2} \psi(|x| - |a| + 1, s) Q_{|a|}(s) \xi_a(\omega) & (|x| \geq |a|). \end{cases}$$

(2) *(The case $a = o$) We have*

$$\phi_s(x) = q^{-(1/2+\sqrt{-1}s)|x|} \left\{ 1 + \frac{q}{q+1} (1 - q^{-1-\sqrt{-1}2s}) \sum_{j=1}^{|x|} q^{\sqrt{-1}2js} \right\}.$$

Remark 1. *Taking into account $c(s) + c(-s) = 1$, we can easily check that the expressions of $\phi_s(x)$ in Proposition 1 coincide with the ones described in* [5, p. 138].

Finally in this section, we remark that the Paley–Wiener theorem for the spherical transform was already proved by Betori and Pagliacci [1]. Let $\mathscr{A}$ denote the Abel transform on $\mathfrak{X}$. For unexplained notation and discussion, see [1] or [4]. In [1, Theorem 2.7], they proved that $\mathscr{A}$ is a bicontinuous isomorphism of $C_c(\mathfrak{X})^{\#}$ onto $C_{\mathrm{ev}}(\mathbf{Z})$. They also showed that $\operatorname{supp} f \subseteq \mathscr{B}_N$ if and only if supp $\mathscr{A}f \subseteq [-N, N]$. Let $\mathscr{F}$ denote the Fourier transform on $\mathbf{Z}$. Then the spherical transform factors as $\tilde{f} = \mathscr{F}(\mathscr{A}f)$. Therefore, using the result of Betori and Pagliacci and applying the Paley–Wiener theorem on $\mathbf{Z}$, we have the following proposition.

Proposition 2. *Let $f \in C_c(\mathfrak{X})^{\#}$ be such that* supp $f \subseteq \mathscr{B}_N$. *Then the spherical transform $\tilde{f}$ satisfies the following conditions:*

(1) $\tilde{f}(s)$ *is smooth on* $\mathbf{T}$,

(2) $\tilde{f}(s) = \tilde{f}(-s)$ *and* $\tilde{f}(s+\tau) = \tilde{f}(s)$,

(3) $\tilde{f}(s)$ *extends to a holomorphic function on* $\mathbf{C}$ *and there exists a constant* $C > 0$ *such that*

$$|\tilde{f}(s)| \leq Cq^{N|\Im s|}.$$

Conversely, if $F(s)$ satisfies the above conditions (1)–(3), *then there exists $f \in C_c(\mathfrak{X})^{\#}$ with* supp $f \subseteq \mathscr{B}_N$ *such that* $\tilde{f} = F$.

The above proposition can be obtained independently by the method of Cowling and Setti in [5], and so we use this proposition to prove Proposition 5 in §5.

4. Spectral projection on $\mathfrak{X}$

In this section, following the analogy with the case of semisimple Lie groups, we shall give the definition of the spectral projection on $\mathfrak{X}$.

For $f \in C_c(\mathfrak{X})$, we define the spectral projection $P_s f$ by

$$P_s f(x) = (f * \phi_s)(x) = \int_G f(g_1)\phi_s(g_1^{-1}g)dg_1, \tag{19}$$

where $x = g \cdot o$. Applying the functional equation of the spherical function in [6, p. 55] to the right-hand side of (19) and using Fubini's theorem, we obtain

$$\begin{aligned} P_s f(x) &= \int_G f(g_1 \cdot o) \int_\Omega p(g_1 \cdot o, \omega)^{1/2+\sqrt{-1}s} p(g \cdot o, \omega)^{1/2-\sqrt{-1}s} d\nu(\omega) dg_1 \\ &= \int_\Omega \tilde{f}(s, \omega) p(x, \omega)^{1/2-\sqrt{-1}s} d\nu(\omega). \end{aligned} \tag{20}$$

By using (20), the inversion formula (12) is expressed as

$$f(x) = c_G \int_{\mathbf{T}} P_s f(x) |c(s)|^{-2} ds. \tag{21}$$

To investigate more properties of the spectral projection, we shall compute $\Delta_n P_s f(x)$ below.

Proposition 3. *Let* $f \in C_c(\mathfrak{X})$. *Then*

$$\Delta_n P_s f(x) = \int_\Omega \Phi_{\omega_n, -s}(x) \tilde{f}(s, \omega) d\nu(\omega).$$

Proof. If $s \in -\frac{1}{2}\sqrt{-1} + \frac{\tau}{2}\mathbf{Z}$, then $\Phi_{\omega_n, -s}(x) = 0$ and $P_s f(x) = 0$, so that the assertion is trivial. Hence we can assume $s \notin -\frac{1}{2}\sqrt{-1} + \frac{\tau}{2}\mathbf{Z}$. Under this assumption, using [9, Lemma 3.3], we have

$$\begin{aligned} \Delta_n P_s f(x) &= (\Delta_n P^{-s} \tilde{f}(s, \cdot))(x) \\ &= (P^{-s} \mathbf{D}_n \tilde{f}(s, \cdot))(x) \\ &= P^{-s}\left(\sum_{y \in \mathfrak{X}} f(y) \Phi_{(\cdot)_n, s}(y)\right)(x) \\ &= \sum_{y \in \mathfrak{X}} f(y) \left\{ \int_\Omega p(x, \omega)^{1/2 - \sqrt{-1}s} \Phi_{\omega_n, s}(y) d\nu(\omega) \right\}. \end{aligned} \tag{22}$$

On the other hand,

$$\begin{aligned} &\int_\Omega p(x, \omega)^{1/2 - \sqrt{-1}s} \Phi_{\omega_n, s}(y) d\nu(\omega) \\ &\quad = \int_\Omega p(x, \omega)^{1/2 - \sqrt{-1}s} \int_\Omega p(y, \omega')^{1/2 + \sqrt{-1}s} \delta_n(\omega, \omega') d\nu(\omega') d\nu(\omega) \\ &\quad = \int_\Omega p(y, \omega')^{1/2 + \sqrt{-1}s} \int_\Omega p(y, \omega)^{1/2 - \sqrt{-1}s} \delta_n(\omega, \omega') d\nu(\omega) d\nu(\omega') \\ &\quad = \int_\Omega p(y, \omega')^{1/2 + \sqrt{-1}s} \Phi_{\omega'_n, -s}(x) d\nu(\omega'). \end{aligned} \tag{23}$$

Substituting (23) into (22), we obtain that

$$\begin{aligned} \Delta_n P_s f(x) &= \sum_{y \in \mathfrak{X}} f(y) \int_\Omega p(y, \omega')^{1/2 + \sqrt{-1}s} \Phi_{\omega'_n, -s}(x) d\nu(\omega') \\ &= \int_\Omega \left(\sum_{y \in \mathfrak{X}} f(y) p(y, \omega')^{1/2 + \sqrt{-1}s}\right) \Phi_{\omega'_n, -s}(x) d\nu(\omega') \\ &= \int_\Omega \Phi_{\omega'_n, -s}(x) \tilde{f}(s, \omega') d\nu(\omega'). \end{aligned}$$

This concludes the proof. □

Finally in this section, we list the essential properties of the spectral projection.

Corollary 1. *The spectral projection* P_s *has the following properties:*
(1) $s \mapsto P_s f(x)$ *is a Weyl-invariant holomorphic function on* $\mathbf{C}$,
(2) $\mathscr{L} P_s f(x) = \lambda(s) P_s f(x)$,
(3) $Q_n(-s)^{-1} \Delta_n P_s f(x)$ *is holomorphic on* $\mathbf{C}$.

Remark 2. *Since* $\Delta_n P_s f(x)$ *is an even function with respect to the variable* s, *we see that* $Q_n(s)^{-1} Q_n(-s)^{-1} \Delta_n P_s f(x)$ *is also holomorphic on* $\mathbf{C}$.

5. The Paley–Wiener theorem for the spectral projection

In this section, we shall characterize the image of $C_c(\mathfrak{X})$ under the spectral projection on $\mathfrak{X}$. As an application of this, we shall give an elementary proof of the Paley–Wiener theorem for the Helgason–Fourier transform, which is proved by Cowling and Setti in [5].

Let $N \in \mathbf{Z}_{\geq 0}$. Let $C_N(\mathfrak{X})$ denote the subset of $C_c(\mathfrak{X})$ consisting of all $f \in C_c(\mathfrak{X})$ such that $\operatorname{supp} f \subseteq \mathfrak{B}_N$. $\mathscr{T}_N(\mathbf{T} \times \mathfrak{X})$ denotes the set comprised of all functions F on $\mathbf{T} \times \mathfrak{X}$ satisfying the following conditions:

(N1) $F(s, x)$ is a Weyl-invariant smooth function on $\mathbf{R}$ with respect to the variable s,
(N2) for each $s \in \mathbf{R}$, $\mathscr{L} F(s, x) = \lambda(s) F(s, x)$,
(N3) for each $x \in \mathfrak{X}$, $F(s, x)$ extends to a Weyl-invariant holomorphic function on $\mathbf{C}$,
(N4) for each $n \in \mathbf{Z}_{\geq 0}$, $Q_n(-s)^{-1} \Delta_n F(s, x)$ is holomorphic on $\mathbf{C}$ and there exists a constant $C_N > 0$ which does not depend on the choice of n such that

$$|Q_n(-s)^{-1} \Delta_n F(s, x)| \leq C_N q^{(|x| - n + N)|\Im s|}.$$

We set

$$\mathscr{T}(\mathbf{T} \times \mathfrak{X}) = \bigcup_{N=0}^{\infty} \mathscr{T}_N(\mathbf{T} \times \mathfrak{X}).$$

We shall first show the following proposition, which is the assertion about the necessary condition in the Paley–Wiener theorem for the spectral projection.

Proposition 4. *Let* $f \in C_N(\mathfrak{X})$. *Then* $F(s, x) = P_s f(x)$ *belongs to* $\mathscr{T}_N(\mathbf{T} \times \mathfrak{X})$.

Proof. The conditions (N1)–(N3) are already proved in Corollary 1. We show here that the condition (N4) is fulfilled. By the definition of the Helgason–Fourier transform (11), we can easily see that

$$|\tilde{f}(s,\omega)| \leq \sum_{x \in \mathfrak{B}_N} |f(x)| q^{h_\omega(x)/2} q^{|x| \cdot |\Im s|} \leq C_N' q^{N|\Im s|} \tag{24}$$

for some constant $C_N' > 0$. From Proposition 3, $Q_n(-s)^{-1} \Delta_n F(s,x)$ is holomorphic on $\mathbf{C}$ and it is satisfied that

$$Q_n(-s)^{-1} \Delta_n F(s,x) = q^{-(|x|-n)/2} \psi(|x| - n + 1, s) \int_\Omega \xi_{\omega_n}(\omega') \tilde{f}(s,\omega) d\nu(\omega)$$

for $\omega' \in E(\omega_n)$. Noting

$$|\xi_a(\omega)| \leq \frac{q^{|a|}(q^2-1)}{q^2}, \qquad |q^{-n/2}\psi(n+1,s)| \leq \frac{q+1}{q-1} q^{n|\Im s|} \qquad (s \in \mathbf{C}),$$

we can find a constant $C_N > 0$ so that

$$|Q_n(-s)^{-1} \Delta_n F(s,x)| \leq C_N q^{(|x|-n+N)|\Im s|},$$

concluding the proof. □

The difficult part of the proof of the Paley–Wiener theorem is to prove that it is also the sufficient condition.

Let $F \in \mathscr{T}_N(\mathbf{T} \times \mathfrak{X})$ and set

$$f(x) = c_G \int_{\mathbf{T}} F(s,x) |c(s)|^{-2} ds. \tag{25}$$

Then from the condition (N4) with $n = 0$, we see that

$$\begin{aligned} |f(x)| &\leq C_N c_G \int_{\mathbf{T}} |c(s)|^{-2} ds \\ &\leq C_N c_G \int_{\mathbf{T}} \frac{4(q+1)^2}{(q-1)^2} \sin^2(s \log q) ds = \frac{q(q+1)}{(q-1)^2} C_N, \end{aligned}$$

and hence $f(x)$ is bounded on $\mathfrak{X}$.

We put $f_n(x) = \Delta_n f(x)$. Then

$$f_n(x) = c_G \int_{\mathbf{T}} F_n(s,x) |c(s)|^{-2} ds,$$

where $F_n(s,x) = \Delta_n F(s,x)$. By the definition of Δ_n, we observe that $F_n(s,x)$ satisfies the conditions (N3) and (N4) again. The following lemma is obtained in the same way as in [5].

LEMMA 1. *Let* $N \in \mathbf{Z}_{>0}$, $F \in \mathscr{T}_N(\mathbf{T} \times \mathfrak{X})$ *and* $a \in \mathfrak{S}_n$. *If* $n > N$ *then* $F_n(s,a) = 0$ *for all* $s \in \mathbf{T}$.

PROOF. We set $\phi(s) = Q_n(-s)^{-1} F_n(s,a)$. Then the condition (N4) yields that $\phi(s)$ is an entire function of exponential type N. We use the Paley–Wiener theorem on $\mathbf{Z}$ to write

$$\phi(s) = \sum_{k \in \mathbf{Z}} \phi(k) q^{\sqrt{-1}ks},$$

where $\phi(k) = 0$ unless $-N \le k \le N$. It follows from the condition (N3) that

$$\phi(-s) = Q_n(s)^{-1} F_n(-s,a) = \frac{Q_n(-s)}{Q_n(s)} Q_n(-s)^{-1} F_n(s,a) = \frac{Q_n(-s)}{Q_n(s)} \phi(s).$$

As shown in [5, pp. 241–242], it is satisfied that

$$\sum_{k \in \mathbf{Z}} \phi(k) q^{-\sqrt{-1}ks} = \sum_{k \in \mathbf{Z}} \left[-q^{-2\sqrt{-1}s(n-1)-1} + (1-q^{-2}) \sum_{\ell=0}^{\infty} q^{-2\sqrt{-1}s(\ell+n)-\ell} \right] \times \phi(k) q^{\sqrt{-1}ks},$$

and hence

$$\phi(k) = -q^{-1}\phi(-k+2n-2) + (1-q^{-2}) \sum_{\ell=0}^{\infty} q^{-\ell} \phi(-k+2n+2\ell). \tag{26}$$

From this, when $n > N+1$, it is easily verified that $\phi(k) = 0$ for all $k \in \mathbf{Z}$. In case $n = N+1$, (26) yields

$$\phi(k) = -q^{-1}\phi(-k+2N),$$

and so $\phi(N) = 0$. Therefore, in this case, $\phi(k) = 0$ for all $k \in \mathbf{Z}$. This concludes the proof. □

Using these facts, we shall prove the following proposition.

PROPOSITION 5. *Let* $F \in \mathscr{T}_N(\mathbf{T} \times \mathfrak{X})$. *Then we have for each* $n \in \mathbf{Z}_{\ge 0}$ *that* $f_n \in C_N(\mathfrak{X})$. *Moreover, if* $n > N$, *then* $f_n(x) = 0$ *for all* $x \in \mathfrak{X}$.

PROOF. We first consider the case when $n = 0$. Since f_0 and F_0 are the spherical means of f and F, respectively, they are radial functions on $\mathfrak{X}$. In addition, the condition (N4) is written as

$$|F_0(s,x)| \le C_N q^{N|\Im s|}.$$

Consequently, from Proposition 2, we have $f_0 \in C_N(\mathfrak{X})$.

Let us next assume $n \in \mathbf{Z}_{>0}$. It is to be noted that $f_n(x) = 0$ when $|x| < n$. From this, we may assume $|x| \geq n$. We put $a = x^{(n)}$ and choose an $\omega \in E(x)$. Because $F_n = \Delta_n F_n$ and $\mathscr{L} F_n = \lambda(s) F_n$, it follows from [9, Lemma 3.2] that

$$F_n(s, x) = q^{-(|x|-|a|)/2} \psi(|x| - |a| + 1, s) F_n(s, a). \tag{27}$$

In the case when $n > N$, Lemma 1 yields that $F_n(s, a) = 0$ and therefore $f_n(x) = 0$ for all $x \in \mathfrak{X}$.

Suppose that $n \leq N$. We set

$$g_a(s) = Q_n(s)^{-1} Q_n(-s)^{-1} F_n(s, a).$$

Then (27) is written as

$$F_n(s, x) = q^{-(|x|-|a|)/2} \psi(|x| - |a| + 1, s) g_a(s) Q_n(s) Q_n(-s).$$

On the other hand, we have

$$\begin{aligned} f_n(x) &= c_G \int_{\mathbf{T}} F_n(s, x) |c(s)|^{-2} ds \\ &= q^{-(|x|-|a|)/2} c_G \int_{\mathbf{T}} g_a(s) \psi(|x| - |a| + 1, s) Q_n(s) Q_n(-s) |c(s)|^{-2} ds. \end{aligned} \tag{28}$$

We here compute $Q_n(s) Q_n(-s) |c(s)|^{-2}$. Since

$$\begin{aligned} Q_n(s) &= q^{-n/2} q^{\sqrt{-1}(n-1)s} (q^{\sqrt{-1}s} - q^{-\sqrt{-1}s}) c(s), \\ Q_n(-s) &= q^{-n/2} q^{-\sqrt{-1}(n-1)s} (q^{-\sqrt{-1}s} - q^{\sqrt{-1}s}) c(-s), \end{aligned}$$

we see that

$$\begin{aligned} Q_n(s) Q_n(-s) &= q^{-n} |c(s)|^2 (q^{\sqrt{-1}s} - q^{-\sqrt{-1}s})^2 (-1) \\ &= 4 q^{-n} |c(s)|^2 \sin^2(s \log q). \end{aligned}$$

Accordingly we have

$$Q_n(s) Q_n(-s) |c(s)|^{-2} = 4 q^{-n} \sin^2(s \log q). \tag{29}$$

Substituting (29) into (28), we obtain

$$f_n(x) = 4 q^{-n} q^{-(|x|-|a|)/2} c_G \int_{\mathbf{T}} g_a(s) \psi(|x| - |a| + 1, s) \sin^2(s \log q) ds. \tag{30}$$

By the condition (N4), we observe that

$$|Q_n(-s)^{-1} F_n(s, a)| \leq C q^{N|\Im s|}.$$

We pick $\Lambda \in \mathbf{R}$ so that $\Lambda < 1/2$. Since the zeros of $Q_n(s)$ lie in the set $\frac{1}{2}\sqrt{-1} + \frac{\tau}{2}\mathbf{Z}$, we can find a constant $d > 0$ such that

$$Q_n(s) \geq dq^{n|\Im s|}$$

for $\Im s < \Lambda$. Then, by an argument similar to that in [2, Theorem 3.2(J)], we can see

$$|g_a(s)| \leq Cq^{(N-n)|\Im s|}$$

for $\Im s < \Lambda$. As $g_a(s)$ is a holomorphic function on $\mathbf{C}$, we have

$$|g_a(s)| \leq Cq^{(N-n)|\Im s|}.$$

We here apply the Paley–Wiener theorem for the Fourier transform on $\mathbf{Z}$ to the expression (30). We consequently obtain that $f_n(x) = 0$ for $|x| - |a| + 1 > N - n + 1$. This concludes the proof. □

Using Proposition 5, we can obtain the following proposition.

Proposition 6. *Let $F \in \mathscr{T}_N(\mathbf{T} \times \mathfrak{X})$. We set*

$$f(x) = c_G \int_{\mathbf{T}} F(s, x)|c(s)|^{-2} ds.$$

Then $f \in C_N(\mathfrak{X})$.

Proof. Let $n \in \mathbf{Z}_{\geq 0}$ and set $f_n(x) = \Delta_n f(x)$. Then Proposition 5 yields that $f_n \in C_N(\mathfrak{X})$ and $f_n = 0$ when $n > N$. Let $x \in \mathfrak{X}$ be such that $|x| > N$. We choose an integer M so that $|x| \leq M$. Then $f(x)$ can be written as the following finite sum:

$$f(x) = \varepsilon_M f(x) = f_0(x) + f_1(x) + \cdots + f_N(x).$$

Since $f_n \in C_N(\mathfrak{X})$, we have $f \in C_N(\mathfrak{X})$. □

Summarizing the arguments in this section, we arrive at the following theorem.

Theorem 1. *The spectral projection P_s gives a linear isomorphism from $C_c(\mathfrak{X})$ onto $\mathscr{T}(\mathbf{T} \times \mathfrak{X})$. Moreover, the image of $C_N(\mathfrak{X})$ under P_s coincides with $\mathscr{T}_N(\mathbf{T} \times \mathfrak{X})$ for all $N \in \mathbf{Z}_{\geq 0}$.*

In the remainder of this section, we shall give an elementary proof of the Paley–Wiener theorem for the Helgason–Fourier transform due to Cowling and Setti. Our proof is a direct consequence of Theorem 1.

Let $\mathscr{Z}_N(\mathbf{T} \times \Omega)$ denote the set of all functions F on $\mathbf{T} \times \Omega$ satisfying the following conditions:

(H1) $F(s,\omega)$ is a smooth function on $\mathbf{T}$ with respect to the variable s,
(H2) $F(s+\tau,\omega) = F(s,\omega)$,
(H3) $F(s,\omega)$ extends to a τ-periodic holomorphic function on $\mathbf{C}$ and there exists a constant $C_N > 0$ such that

$$|F(s,\omega)| \le C_N q^{N|\Im s|},$$

(H4) F satisfies the symmetry condition (14).

With the notation above, Cowling and Setti have proved the following theorem.

Theorem 2 ([5, Theorem 1]). *The Helgason–Fourier transform gives a linear isomorphism of* $C_N(\mathfrak{X})$ *onto* $\mathscr{Z}_N(\mathbf{T}\times\Omega)$.

In order to prove the above theorem, Cowling and Setti investigated $\dim \mathscr{Z}_N(\mathbf{T}\times\Omega)$ and showed that $\dim \mathscr{Z}_N(\mathbf{T}\times\Omega) = \mathrm{Card}\,\mathscr{B}_N$. Our proof is a consequence of Theorem 1 and simpler than the one of Cowling and Setti.

Proof. Let $F \in \mathscr{Z}_N(\mathbf{T}\times\Omega)$. We first show that the Poisson transform $P^{-s}F(s,\cdot)$ of $F(s,\omega)$ satisfies the conditions (N1)–(N4). The condition (N2) is already shown in Corollary 1. The symmetry condition (14) and the condition (H2) imply that $P^{-s}F(s,\cdot)$ is Weyl-invariant. Thus Corollary 1 yields that the conditions (N1) and (N3) are fulfilled. By the definition of the Poisson transform, we have

$$\begin{aligned} |P^{-s}F(s,x)| &\le \int_\Omega |p(x,\omega)^{1/2-\sqrt{-1}s}|\,|F(s,\omega)|d\nu(\omega) \\ &\le \int_\Omega q^{h_\omega(x)/2} q^{|\Im s|\cdot|x|} C_N q^{N|\Im s|}\,d\nu(\omega) \\ &\le C_N q^{(|x|+N)|\Im s|}. \end{aligned}$$

Thus the condition (N4) is an immediate corollary of Proposition 3. Therefore we see that $P^{-s}F \in \mathscr{T}_N(\mathbf{T}\times\mathfrak{X})$. We set

$$f(x) = c_G \int_{\mathbf{T}} P^{-s}F(s,x)|c(s)|^{-2}ds. \tag{31}$$

Applying here Proposition 6, we have $f \in C_N(\mathfrak{X})$. This concludes the proof. □

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