

Geodesic orbit manifolds and Killing fields of constant length

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ABSTRACT. The goal of this paper is to clarify connections between Killing fields of constant length on a Riemannian geodesic orbit manifold (M, g) and the structure of its full isometry group. The Lie algebra of the full isometry group of (M, g) is identified with the Lie algebra of Killing fields \mathfrak{g} on (M, g) . We prove the following result: If \mathfrak{a} is an abelian ideal of \mathfrak{g} , then every Killing field $X \in \mathfrak{a}$ has constant length. On the ground of this assertion we give a new proof of one result of C. Gordon: Every Riemannian geodesic orbit manifold of nonpositive Ricci curvature is a symmetric space.

1. Introduction

All manifolds in this paper are supposed to be connected. At first, we recall and discuss important definitions.

DEFINITION 1. A Riemannian manifold (M, g) is called a manifold with homogeneous geodesics or a geodesic orbit manifold (shortly, a *GO-manifold*) if any geodesic γ of M is an orbit of a 1-parameter subgroup of the full isometry group of (M, g) .

DEFINITION 2. A Riemannian manifold $(M = G/H, g)$, where H is a compact subgroup of a Lie group G and g is a G -invariant Riemannian metric, is called a space with homogeneous geodesics or a geodesic orbit space (shortly, a *GO-space*) if any geodesic γ of M is an orbit of a 1-parameter subgroup of the group G .

This terminology was introduced in [14] by O. Kowalski and L. Vanhecke, who initiated a systematic study of such spaces. In the same paper, O. Kowalski and L. Vanhecke classified all GO-spaces of dimension ≤ 6 . Many

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interesting results about GO-manifolds and its subclasses one can find in [1, 2, 3, 4, 5, 7, 9, 15, 16, 18], and in the references therein. In [10], C. Gordon obtained some structure results on GO-spaces, in particular, the following one: Every Riemannian GO-manifold of nonpositive Ricci curvature is a symmetric space.

The goal of this paper is to clarify connections between Killing fields of constant length on a Riemannian GO-manifold (M, g) and the structure of its full isometry group. The Lie algebra of the full (connected) isometry group of (M, g) is identified naturally with the Lie algebra of Killing fields \mathfrak{g} on (M, g) . We prove the following result: If \mathfrak{a} is an abelian ideal of \mathfrak{g} , then every Killing field $X \in \mathfrak{a}$ has constant length (Theorem 1). On the ground of this theorem we give a new proof of the above mentioned result of C. Gordon on GO-manifolds with nonpositive Ricci curvature (Theorem 2).

2. Notation and useful facts

Let (M, g) be a GO-manifold and G be its connected full isometry group. Obviously, (M, g) is homogeneous and $M = G/H$, where H is the isotropy subgroup at a point $o \in M$. Since H is compact, there is an $\text{Ad}(H)$ -invariant decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad (1)$$

where $\mathfrak{g} = \text{Lie}(G)$ and $\mathfrak{h} = \text{Lie}(H)$. The Riemannian metric g is G -invariant and is determined by an $\text{Ad}(H)$ -invariant Euclidean metric $g = (\cdot, \cdot)$ on the space \mathfrak{m} which is identified with the tangent space at the initial point $o = eH$.

In what follows we identify elements of \mathfrak{g} with corresponded Killing vector fields on (M, g) .

Now, we recall some well known formulas for a homogeneous Riemannian manifold $(M, g = (\cdot, \cdot))$ [8]. Let us choose some g -orthonormal basis (X_i) in \mathfrak{m} . Consider also a vector $Z \in \mathfrak{m}$ defined by the condition $(Z, X) = \text{trace}(\text{ad}_X)$ for every $X \in \mathfrak{m}$. Therefore, $Z = 0$ iff the Lie algebra \mathfrak{g} (and the Lie group G) is unimodular. The following formula (that is more simple for the unimodular case) is useful for the Ricci curvature calculation:

$$\begin{aligned} \text{Ric}(X, X) = & -\frac{1}{2}B_{\mathfrak{g}}(X, X) - \frac{1}{2}\sum_i |[X, X_i]_{\mathfrak{m}}|^2 \\ & + \frac{1}{4}\sum_{i,j} ([X_i, X_j]_{\mathfrak{m}}, X)^2 - ([Z, X]_{\mathfrak{m}}, X), \end{aligned} \quad (2)$$

where $B_{\mathfrak{g}}$ is the Killing form of the Lie algebra \mathfrak{g} , $X \in \mathfrak{m}$, and $V_{\mathfrak{m}}$ means the \mathfrak{m} -part of a vector $V \in \mathfrak{g}$.

LEMMA 1 ([14]). *A homogeneous Riemannian manifold $(M = G/H, g)$ with the reductive decomposition (1) is a GO-space if and only if for any $X \in \mathfrak{m}$ there is $H_X \in \mathfrak{h}$ such that*

$$([H_X + X, Y]_{\mathfrak{m}}, X) = 0 \quad \text{for all } Y \in \mathfrak{m}.$$

This lemma shows that the property to be a GO-space depends only on the reductive decomposition (1) and the Euclidean metric g on \mathfrak{m} . In other words, if $(M = G/H, g)$ is a GO-space, then any locally isomorphic homogeneous Riemannian space $(M' = G'/H', g')$, where G' is locally isomorphic to G , is a GO-space. Also a direct product of Riemannian manifolds is a manifold with homogeneous geodesics if and only if each factor is a manifold with homogeneous geodesics.

For any subspace $l \subset \mathfrak{g}$ and any $U \in \mathfrak{g}$ we use a symbol ad_U^l for the restriction of the operator ad_U to l , i.e. $\text{ad}_U^l : l \rightarrow l$, $\text{ad}_U^l(X) = [U, X]_l$.

LEMMA 2. *Suppose that $(M = G/H, g)$ is a GO-space. Let \mathfrak{m}_1 and \mathfrak{m}_2 be $\text{Ad}(H)$ -invariant subspaces of \mathfrak{m} such that $(\mathfrak{m}_1, \mathfrak{m}_2) = 0$ and $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Then for any $U \in \mathfrak{m}_1$ the operator $\text{ad}_U^{\mathfrak{m}_2}$ is skew-symmetric. If, in addition, $[\mathfrak{h}, \mathfrak{m}_1] = 0$, then the operator $\text{ad}_U^{\mathfrak{m}}$ is skew-symmetric.*

PROOF. For any $X \in \mathfrak{m}_2$ there is $H_X \in \mathfrak{h}$ such that $([H_X + X, Y]_{\mathfrak{m}}, X) = 0$ for all $Y \in \mathfrak{m}$ (see Lemma 1). Therefore,

$$0 = ([H_X + X, U]_{\mathfrak{m}}, X) = ([H_X, U]_{\mathfrak{m}}, X) + ([X, U]_{\mathfrak{m}}, X) = ([X, U]_{\mathfrak{m}}, X),$$

because $[H_X, U] \in \mathfrak{m}_1$ and $X \in \mathfrak{m}_2$. If $[\mathfrak{h}, \mathfrak{m}_1] = 0$, then the same is true for any $X \in \mathfrak{m}$. This proves the lemma. \square

LEMMA 3 ([10]). *Let $(M = G/H, g)$ be a GO-space, then the group G is unimodular.*

PROOF. Here we give a more direct proof, than the original one in [10]. Suppose that the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ is not unimodular and consider its proper subspace

$$\mathfrak{u} = \{X \in \mathfrak{g} \mid \text{trace}(\text{ad}_X) = 0\}.$$

Obviously, $\mathfrak{h} \subset \mathfrak{u}$. Since

$$\text{ad}_{[X, Y]} = [\text{ad}_X, \text{ad}_Y] \quad \text{and} \quad \text{trace}(\text{ad}_{[X, Y]}) = \text{trace}([\text{ad}_X, \text{ad}_Y]) = 0,$$

then $[\mathfrak{u}, \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{u}$, hence, \mathfrak{u} is an ideal of \mathfrak{g} . Consider $\mathfrak{m}_1 = \mathfrak{m} \cap \mathfrak{u}$ and let \mathfrak{m}_2 be a (non-trivial) g -orthogonal complement to \mathfrak{m}_1 in \mathfrak{m} . Since \mathfrak{u} is an ideal of \mathfrak{g} and g is $\text{ad}(\mathfrak{h})$ -invariant, then \mathfrak{m}_1 and \mathfrak{m}_2 are $\text{ad}(\mathfrak{h})$ -invariant. On the other hand, $[\mathfrak{h}, \mathfrak{m}_2] \subset [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{u}$. Therefore, $[\mathfrak{h}, \mathfrak{m}_2] = 0$. Now, consider any non-trivial $Y \in \mathfrak{m}_2$. By our construction, $\text{trace}(\text{ad}_Y) \neq 0$. On the other hand,

by Lemma 1 for every $X \in \mathfrak{m}$ there is $H_X \in \mathfrak{h}$ such that $([H_X + X, Y]_{\mathfrak{m}}, X) = 0$. Since $[\mathfrak{h}, \mathfrak{m}_2] = 0$, we get $([X, Y]_{\mathfrak{m}}, X) = 0$ (see also Lemma 2), that implies $\text{trace}(\text{ad}_Y) = 0$, a contradiction. \square

LEMMA 4. *Let (M, g) be a Riemannian manifold and X be a Killing field on (M, g) . Consider any point $x \in M$ such that $X(x) \neq 0$. Then the integral curve of X through the point x is a geodesic if and only if x is a critical point of the function $y \in M \mapsto g_y(X, X)$.*

PROOF. In fact, this is proved in Proposition 5.7 of Chapter VI in [13]. \square

We will use a symbol M_x for the tangent space of a manifold M at a point $x \in M$.

LEMMA 5. *Let (M, g) be a Riemannian manifold and \mathfrak{g} be its Lie algebra of Killing fields. Then (M, g) is a GO-manifold if and only if for any $x \in M$ and any $v \in M_x$ there is $X \in \mathfrak{g}$ such that $X(x) = v$ and x is a critical point of the function $y \in M \mapsto g_y(X, X)$. If (M, g) is homogeneous, then the latter condition is equivalent to the following one: for any $Y \in \mathfrak{g}$ the equality $g_x([Y, X], X) = 0$ holds.*

PROOF. By Lemma 4 an integral curve of X through the point $x \in M$ is geodesic if and only if x is a critical point of the function $y \in M \mapsto g_y(X, X)$. If (M, g) is homogeneous, then it is equivalent to the condition

$$Y \cdot g(X, X)|_x = 2g_x([Y, X], X) = 0$$

for every $Y \in \mathfrak{g}$. \square

In what follows we need the following

PROPOSITION 1 (*Theorem 2.10 in [19]*). *If a Killing vector field X on a compact Riemannian manifold M satisfies the condition $\text{Ric}(X, X) \leq 0$, then X is parallel on M and $\text{Ric}(X, X) \equiv 0$.*

COROLLARY 1. *If a compact homogeneous Riemannian manifold (M, g) has nonpositive Ricci curvature, then it is a Euclidean torus. In particular, its full connected isometry group is abelian.*

3. Main results

At first, we get the following remarkable result.

THEOREM 1. *Let (M, g) be a GO-manifold, \mathfrak{g} is its Lie algebra of Killing fields. Suppose that \mathfrak{a} is an abelian ideal of \mathfrak{g} . Then any $X \in \mathfrak{a}$ has constant length on (M, g) .*

PROOF. Let x be any point in M . We will prove that x is a critical point of the function $y \in M \mapsto g_y(X, X)$. Since (M, g) is homogeneous, then (by Lemma 5) it suffices to prove that $g_x([Y, X], X) = 0$ for every $Y \in \mathfrak{g}$.

Consider any $Y \in \mathfrak{a}$, then $Y \cdot g(X, X) = 2g([Y, X], X) = 0$ on M , since \mathfrak{a} is abelian.

Now, consider $Y \in \mathfrak{g}$ such that $g_x(Y, U) = 0$ for every $U \in \mathfrak{a}$. We will prove that $g_x([Y, X], X) = 0$. By Lemma 5, for the vector $X(x) \in M_x$ there is a Killing field $Z \in \mathfrak{g}$ such that $Z(x) = X(x)$ and $g_x([V, Z], Z) = 0$ for any $V \in \mathfrak{g}$. In particular, $g_x([Y, Z], Z) = 0$. Now, $W = X - Z$ vanishes at x and we get

$$\begin{aligned} g_x([Y, X], X) &= g_x([Y, Z + W], Z + W) = g_x([Y, Z + W], Z) \\ &= g_x([Y, Z], Z) + g_x([Y, W], Z) = g_x([Y, W], Z). \end{aligned}$$

Note that $g_x([Y, W], Z) = -g_x([W, Y], Z) = g_x(Y, [W, Z]) = 0$ because $W(x) = 0$ ($0 = W \cdot g(Y, Z)|_x = g_x([W, Y], Z) + g_x(Y, [W, Z])$) and $[W, Z] = [X, Z] \in \mathfrak{a}$. Therefore, $g_x([Y, X], X) = 0$. Hence, x is a critical point of the function $y \in M \mapsto g_y(X, X)$.

Since every $x \in M$ is a critical point of the function $y \in M \mapsto g_y(X, X)$, then X has constant length on (M, g) . \square

REMARK 1. *This result can be easily generalized to some cases when \mathfrak{g} is a subalgebra of the Lie algebra of the full connected isometry group of (M, g) . It suffices to assume that a connected subgroup G (with the Lie algebra \mathfrak{g}) of the full isometry group of (M, g) is such that $(M = G/H, g)$ is a GO-space.*

In the rest of the paper we reprove the following

THEOREM 2 (C. Gordon [10]). *Every Riemannian GO-manifold of non-positive Ricci curvature is symmetric.*

REMARK 2. *It should be noted that the original proof of this theorem (Theorem 5.1 in [10]) has an error in the claim “Since U^*/L^* is a compact homogeneous space, its Ricci curvature Ric^* is nonnegative”. Nevertheless, this error could be corrected, and the proof in [10] requires only a little modification. But here we present a more simple proof, in which some constructions from [10] are essentially used.*

It suffices to prove Theorem 2 for simply connected manifolds. Indeed, if a Riemannian homogeneous manifold M has a Riemannian symmetric space of nonpositive Ricci curvature (equivalently, nonpositive sectional curvature) as a universal covering space, then it is symmetric too [17, 12].

Let (M, g) be a simply connected GO-manifold with nonpositive Ricci curvature, and let G be its full connected isometry group. We know that G is

unimodular (Lemma 3), and the isotropy subgroup H must be connected. At first, we reduce the problem to the case when G is semisimple.

PROPOSITION 2. *Let (M, g) be a simply connected GO-manifold with nonpositive Ricci curvature. Then it is a direct metric product of a Euclidean space \mathbf{E}^m and a simply connected GO-manifold (M_1, g_1) (with nonpositive Ricci curvature) with semisimple full isometry groups.*

PROOF. Recall that a Lie algebra \mathfrak{g} is semisimple if and only if it has no nontrivial abelian ideal (see, e.g. Section 1.4.4 in [11]). Let \mathfrak{g} be the Lie algebra of Killing fields on (M, g) . Now, suppose that \mathfrak{a} is a nontrivial abelian ideal of \mathfrak{g} . By Theorem 1, any nontrivial Killing field $X \in \mathfrak{a}$ has constant length on (M, g) . Since the Ricci curvature of (M, g) is nonpositive, then by Theorem 4 in [6] we get that $\text{Ric}(X, X) = 0$, moreover, the Killing field X is parallel on (M, g) , and the Riemannian manifold (M, g) is a direct metric product of two Riemannian manifolds, one of which is a one-dimensional manifold \mathbf{E}^1 tangent to Killing field X , and another one, say (\tilde{M}, \tilde{g}) , is a simply connected GO-manifold with nonpositive Ricci curvature.

This procedure could be repeated with (\tilde{M}, \tilde{g}) etc., unless the last obtained Riemannian manifold has a semisimple full isometry group (i. e. the Lie algebra of Killing fields on the latter manifold has no nontrivial abelian ideal). This proves the proposition. \square

In what follows we suppose that the group G is semisimple. Now we consider a reductive decomposition (see (1))

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

where $\mathfrak{g} = \text{Lie}(G)$, $\mathfrak{h} = \text{Lie}(H)$, and \mathfrak{m} is an orthogonal complement to \mathfrak{h} in \mathfrak{g} with respect to the Killing form $B_{\mathfrak{g}}$ of the (semisimple) Lie algebra \mathfrak{g} . The Riemannian metric g is G -invariant and is determined by an $\text{Ad}(H)$ -invariant Euclidean metric $g = (\cdot, \cdot)$ on the space \mathfrak{m} . Now we consider a *maximal compactly embedded* subalgebra $\mathfrak{k} \subset \mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{k}$.

If $\mathfrak{h} = \mathfrak{k}$, then the manifold under consideration is a symmetric space [17, 12]. Suppose now, that $\mathfrak{h} \neq \mathfrak{k}$. Then there are $\text{Ad}(H)$ -invariant subspaces $\mathfrak{m}_1, \mathfrak{m}_2 \subset \mathfrak{m}$ such that $(\mathfrak{m}_1, \mathfrak{m}_2) = 0$, $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, and $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_1$.

Let K^* be a compact connected Lie group with the Lie algebra \mathfrak{k} and H^* be its subgroup corresponded to the subalgebra $\mathfrak{h} \subset \mathfrak{k}$. We have $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_1$, therefore \mathfrak{m}_1 could be identified with the tangent space at the point eH^* of a compact homogeneous manifold $M^* = K^*/H^*$. We consider a K^* -invariant Riemannian metric g^* on M^* that is generated with the inner product $(\cdot, \cdot)|_{\mathfrak{m}_1}$. Note that K^* may not act effectively on $M^* = K^*/H^*$, but this is not important for calculation of the Ricci curvature of (M^*, g^*) .

We choose a (\cdot, \cdot) -orthonormal basis X_1, X_2, \dots, X_r , $r = \dim(\mathfrak{m}_1)$, in \mathfrak{m}_1 , and (\cdot, \cdot) -orthonormal basis Y_1, Y_2, \dots, Y_s , $s = \dim(\mathfrak{m}_2)$, in \mathfrak{m}_2 . Denote the Ricci curvature of (M^*, g^*) by Ric^* and the Killing forms of \mathfrak{k} and \mathfrak{g} by $B_{\mathfrak{k}}$ and $B_{\mathfrak{g}}$ respectively. Using (2) and the fact that \mathfrak{g} is unimodular (Lemma 3 is not necessary, because every semisimple Lie algebra is unimodular), we get

$$\begin{aligned} \text{Ric}(X, X) &= -\frac{1}{2}B_{\mathfrak{g}}(X, X) - \frac{1}{2}\sum_i |[X, X_i]_{\mathfrak{m}}|^2 - \frac{1}{2}\sum_i |[X, Y_i]_{\mathfrak{m}}|^2 \\ &\quad + \frac{1}{4}\sum_{i,j} ([X_i, X_j]_{\mathfrak{m}}, X)^2 + \frac{1}{4}\sum_{i,j} ([Y_i, Y_j]_{\mathfrak{m}}, X)^2 + \frac{1}{2}\sum_{i,j} ([X_i, Y_j]_{\mathfrak{m}}, X)^2 \end{aligned}$$

and

$$\text{Ric}^*(X, X) = -\frac{1}{2}B_{\mathfrak{k}}(X, X) - \frac{1}{2}\sum_i |[X, X_i]_{\mathfrak{m}_1}|^2 + \frac{1}{4}\sum_{i,j} ([X_i, X_j]_{\mathfrak{m}_1}, X)^2$$

for any $X \in \mathfrak{m}_1$. By Lemma 2

$$\begin{aligned} B_{\mathfrak{g}}(X, X) &= B_{\mathfrak{k}}(X, X) + \sum_i ([X, [X, Y_i]_{\mathfrak{m}}], Y_i) = B_{\mathfrak{k}}(X, X) - \sum_i |[X, Y_i]_{\mathfrak{m}_2}|^2, \\ ([X_i, Y_j]_{\mathfrak{m}}, X)^2 &= ([Y_j, X_i]_{\mathfrak{m}_1}, X)^2 = ([Y_j, X]_{\mathfrak{m}_1}, X_i)^2, \\ \sum_{i,j} ([X_i, Y_j]_{\mathfrak{m}}, X)^2 &= \sum_i |[X, Y_i]_{\mathfrak{m}_1}|^2, \end{aligned}$$

then we get

PROPOSITION 3 ([10]). *For any $X \in \mathfrak{m}_1$ the equality*

$$\text{Ric}^*(X, X) = \text{Ric}(X, X) - \frac{1}{2}\sum_{1 \leq i < j \leq r} ([Y_i, Y_j]_{\mathfrak{m}_1}, X)^2$$

holds.

Since $(M = G/H, g)$ has nonpositive Ricci curvature, then from Proposition 3 we get $\text{Ric}^*(X, X) \leq 0$ for any $X \in \mathfrak{m}_1$. Since $(M^* = K^*/H^*, g^*)$ is a compact homogeneous Riemannian manifold with nonpositive Ricci curvature, then it is a Euclidean torus by Corollary 1. Therefore, $\text{Ric}^*(X, X) = 0$ for all $X \in \mathfrak{m}_1$ and \mathfrak{m}_1 lies in the center of \mathfrak{k} . Now, from Proposition 3 we get $\text{Ric}(X, X) = 0$ and $([\mathfrak{m}_2, \mathfrak{m}_2]_{\mathfrak{m}_1}, X) = 0$ for all $X \in \mathfrak{m}_1$. Hence, $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h} \oplus \mathfrak{m}_2$.

Let \mathfrak{p} be a $B_{\mathfrak{g}}$ -orthogonal compliment to \mathfrak{k} in \mathfrak{g} . It is well known that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$; if $\mathfrak{h}_1 := [\mathfrak{p}, \mathfrak{p}]$, then $\mathfrak{g}_1 := \mathfrak{h}_1 \oplus \mathfrak{p}$ is a maximal semisimple ideal of noncompact type in the Lie algebra \mathfrak{g} .

Now we will prove that $\mathfrak{p} = \mathfrak{m}_2$. By Lemma 2 we get that for any $U \in \mathfrak{m}_1$ the operator $\text{ad}_U^{\mathfrak{m}}$ is skew-symmetric. The same is true for any $U \in \mathfrak{h}$, and, therefore, for any $U \in \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_1$. From the relations $\mathfrak{p} \subset \mathfrak{m}$, $\mathfrak{h}_1 \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}_1] = 0$, and $[\mathfrak{k}, \mathfrak{p}] = \mathfrak{p}$ we get

$$(\mathfrak{m}_1, \mathfrak{p}) = (\mathfrak{m}_1, [\mathfrak{k}, \mathfrak{p}]_{\mathfrak{m}}) = -([\mathfrak{k}, \mathfrak{m}_1]_{\mathfrak{m}}, \mathfrak{p}) = 0,$$

since the operators $\text{ad}_U^{\mathfrak{m}}$ are skew-symmetric for all $U \in \mathfrak{k}$. This proves $\mathfrak{p} = \mathfrak{m}_2$. Therefore, $\mathfrak{h}_1 := [\mathfrak{p}, \mathfrak{p}] = [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h} \oplus \mathfrak{m}_2$, and we get $\mathfrak{h}_1 \subset \mathfrak{h}$.

Let \mathfrak{g}_2 be a $B_{\mathfrak{g}}$ -orthogonal compliment to \mathfrak{g}_1 in \mathfrak{g} . Then \mathfrak{g}_2 is a maximal compact semisimple ideal in the Lie algebra \mathfrak{g} . If \mathfrak{h}_2 is a $B_{\mathfrak{g}}$ -orthogonal compliment to \mathfrak{h}_1 in \mathfrak{h} , then $\mathfrak{g}_2 = \mathfrak{h}_2 \oplus \mathfrak{m}_1$. Recall that $[\mathfrak{h}_2, \mathfrak{m}_1] \subset [\mathfrak{h}, \mathfrak{m}_1] = 0$. Hence, \mathfrak{h}_2 is an ideal in the Lie algebra \mathfrak{g} . Since the homogeneous space G/H is effective, then \mathfrak{h}_2 is trivial. Since $\mathfrak{m}_1 = \mathfrak{g}_2$ is commutative (\mathfrak{m}_1 lies in the center of \mathfrak{k}), then it is also trivial (otherwise, \mathfrak{g}_2 is not semisimple). Hence, $\mathfrak{h} = \mathfrak{k}$, and $(M = G/H, g)$ is a symmetric space.

Therefore, Theorem 2 is completely proved. \square

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