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Oscillation theory of third-order nonlinear functional differential equations

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Abstract. In this paper, we are concerned with oscillation of solutions of a certain class of third-order nonlinear delay differential equations of the form $x'''(t) + p(t)x'(t) + q(t)f(x(\tau(t))) = 0$. We establish some new oscillation results that extend and improve some results in the literature in the sense that our results do not require that $\tau'(t) \geq 0$. Some examples are considered to illustrate the main results and some conjectures and open problems are presented.

1. Introduction

In this paper, we are concerned with oscillation of solutions of the third-order nonlinear delay differential equation

$$x'''(t) + p(t)x'(t) + q(t)f(x(\tau(t))) = 0, \qquad \text{for } t \in [t_0, \infty), \tag{E_0}$$

where p, q, and τ are positive real-valued functions, $\tau(t) \leq t$, $\tau(t) \to \infty$ as $t \to \infty$, and $uf(u) > 0$ for $u \neq 0$. By a *solution* of (E_0), we mean a nontrivial real-valued function $x(t) \in C^3[\tau^{-1}(t_0), \infty)$ satisfying (E_0), where $\tau^{-1}(t) = \sup\{s \geq 0 : \tau(s) \leq t\}$ for $t \geq T_0 = \min\{\tau(t) : t \geq 0\}$. Our attention is restricted to those solutions of (E_0) which exist on $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_1\} > 0$ for any $t_1 \geq t_x$. We make a standing hypothesis that (E_0) does possess such solutions. A solution of (E_0) is said to be *oscillatory* if it has arbitrarily large zeros; otherwise, we say it is *nonoscillatory*. Equation (E_0) is *disconjugate* on the interval $I = [t_0, \infty)$ in case no nontrivial solution has more than two zeros on I counting multiplicity. Equation (E_0) is said to be oscillatory in case there exists at least one oscillatory solution. The challenge in the study of the asymptotic behavior of solutions of equations of the form (E_0) or its generalizations is the determination of the signs of the derivatives of the solution $x'(t)$, $x''(t)$, and $x'''(t)$. In the case where $p(t) = 0$, it is known (see [1]) that there are only two cases to consider:

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(I) $x(t) > 0,\ x'(t) > 0,\ x''(t) > 0$, or

(II) $x(t) > 0,\ x'(t) < 0,\ x''(t) > 0$

for $t \geq t_1$ sufficiently large.

When $p(t) \neq 0$, we cannot use the technique employed in [1] to determine the signs of $x'(t)$, $x''(t)$, and $x'''(t)$ and some conditions need to be imposed to determine the signs of these quantities. In fact, there are two different approaches that have been used for this determination, one is due to Lazer [24] and another to Heidel [19]. Lazer's method depends on the sign of functional

$$F(x(t)) = (x'(t))^2 - 2x(t)x''(t) - p(t)x^2(t),$$

whereas Heidel's method depends on the nonoscillatory behavior of the related second-order equation

$$x''(t) + p(t)x(t) = 0.$$

Functionals of this type and its variations have been used by other authors in the study of third-order equations; for example, see Greguš, Graef, and Gera [13].

In the following, for completeness and comparison, we give a detailed discussion of known oscillation results for third-order linear and nonlinear differential equations related to (E_0). Hanan [18] wrote an important paper that was the starting point for many investigations into the asymptotic behavior of solutions of third-order differential equations. He studied the oscillation and nonoscillation of the linear equation

$$x'''(t) + p(t)x'(t) + q(t)x(t) = 0, \qquad t \geq t_0, \tag{E_1}$$

where the coefficients satisfy:

(c_1) $p(t) > 0$ and $2q(t) - p'(t) > 0$;

he proved that if $x(t)$ is a given solution such that

$$F(x(t_1)) = (x'(t_1))^2 - 2x(t_1)x''(t_1) - p(t_1)x^2(t_1) \geq 0,$$

then the zeros of $x(t)$ and $x'(t)$ separate each other in $[t_2, \infty)$, where t_2 is the first zero of $x'(t)$ to the right of $t = t_1$. As for comparison theorems, Hanan [18] considered the equations

$$Lx := x'''(t) + p(t)x'(t) + q(t)x(t) = 0, \qquad t \in [a, b),$$

$$L_1x := x'''(t) + p_1(t)x'(t) + q_1(t)x(t) = 0, \qquad t \in [a, b),$$

where p, q, p_1, and $q_1 \in C[a, b)$, $0 < a < b \leq \infty$. He proved that if the inequalities

(c_2) $p(t) \geq p_1(t)$, $q(t) \geq q_1(t)$, $2q(t) > p'(t)$, and $2q_1(t) > p_1'(t)$

hold for all large t, and if $Lx = 0$ is nonoscillatory, then so is $L_1x = 0$.

A useful comparison equation for third-order linear equations is the Euler equation

$$Ex := x'''(t) + \frac{\alpha}{t^2}x'(t) + \frac{\beta}{t^3}x(t) = 0, \tag{1}$$

where α and β are positive constants. It is known (cf. Swanson [33]) that $Ex = 0$ is disconjugate on $(0, \infty)$ if and only if $\alpha \le 1$ and $|\alpha + \beta| \le 2\left(\frac{1-\alpha}{3}\right)^{3/2}$. In view of this, the following conjecture now arises.

Conjecture 1. *Equation* (1) *is oscillatory if*

$$\alpha < 1 \qquad \textit{and} \qquad |\alpha + \beta| > 2\left(\frac{1-\alpha}{3}\right)^{3/2}.$$

In Section 2, we give an affirmative answer to this conjecture (see Example 2 below).

Using the Euler differential equation as a comparison equation, Hanan [18] proved that if

(c_3) $p_1(t) \ge 0$ and $2q_1(t) > p_1'(t)$,

and there exists a number k, $0 < k < 1$, with

$$\limsup_{t\to\infty} t^2 p_1(t) < k \qquad \text{and} \qquad \limsup_{t\to\infty} t^3 q_1(t) < 2\left(\frac{1-k}{3}\right)^{3/2} - k,$$

then $L_1x = 0$ is nonoscillatory.

Using a different approach, Erbe [7] showed that there are examples of equations L_1x with $\limsup_{t\to\infty} q_1(t) = \infty$ that are nevertheless disconjugate on $[t_0, \infty)$; this follows from some integral forms of his comparison theorems. Erbe proved that if

$$-(t_0/\sqrt{3})(t - t_0) \le P_1(t) + Q_1(t) \le (t - t_0)^2/(3\sqrt{3}) \qquad \text{for } t > t_0,$$

and

$$0 \le t^2 p_1(t) \le k v_0(t) \quad \text{for } t > t_0 \qquad \text{and some} \qquad 0 < k < (1/2) + 1/\sqrt{3} < 1,$$

where

$$P_1(t) = \int_{t_0}^{t} (t - s)s^2 p_1(s)ds, \qquad Q_1(t) = \int_{t_0}^{t} (t - s)s^3 q_1(s)ds,$$

and

$$t^2 v_0(t) = t_0^2(1 - 1/\sqrt{3}) + (t_0(t - t_0)/\sqrt{3}) + P_1(t) + Q_1(t),$$

then $L_1x = 0$ is disconjugate on $[t_0, \infty)$.

Remark 1. As a special case, from the results of Erbe [7], we can deduce the following results that generalize the results of Hanan [18]. *If $q(t)$ has constant sign, then the equation*

$$x'''(t) + q(t)x(t) = 0, \tag{2}$$

is disconjugate on $[a, \infty)$ *if*

$$\int_{t_0}^{t} (t-s)s^3 q(s)ds \leq \frac{(t-t_0)^2}{3\sqrt{3}}, \qquad \text{for } t > t_0. \tag{3}$$

From this remark, we can formulate the following conjecture.

Conjecture 2. *If $q(t) > 0$, then the condition*

$$\liminf_{t\to\infty} \frac{1}{(t-t_0)^2} \int_{t_0}^{t} (t-s)s^3 q(s)ds > \frac{1}{3\sqrt{3}}, \qquad \text{for } t > t_0, \tag{4}$$

is sufficient for the oscillation of (2).

Remark 2. To prove that condition (4) is sharp, we take $q(t) = \frac{2}{3\sqrt{3}t^3}$. Then equation (2) becomes the third-order Euler differential equation

$$x'''(t) + \frac{2}{3\sqrt{3}t^3}x(t) = 0, \qquad t \geq 1, \tag{5}$$

and condition (4) becomes

$$\liminf_{t\to\infty} \frac{1}{(t-t_0)^2} \int_{t_0}^{t} (t-s)s^3 q(s)ds = \frac{2}{3\sqrt{3}} \liminf_{t\to\infty} \frac{1}{(t-1)^2} \int_{1}^{t} (t-s)ds = \frac{1}{3\sqrt{3}}.$$

Thus, condition (4) is not satisfied, and so (5) is not oscillatory. It is known that (5) is disconjugate and the associated characteristic equation has the negative root $\frac{2\sqrt{3}}{3} - 1$ and two equal positive roots given by $1 + \frac{1}{\sqrt{3}}$. On the other hand, it is easy to see that the equation

$$x'''(t) + \frac{1}{\sqrt{3}t^3}x(t) = 0, \qquad t \geq 1, \tag{6}$$

satisfies condition (4), so that it has an oscillatory solution. Note that the roots of the characteristic equation of (6) are given by $m = 1.60732 \pm 0.32634i$ and $m = -0.21463$. This shows that the condition (4) is sharp and cannot be weakened.

Lazer [24] considered equation (E_1) with $p(t) < 0$, and proved that if $q(t)$ has fixed sign and $x(t)$ is a nonoscillatory solution of (E_1), then there exists $t_1 > t_0$ such that either

(III) $x(t)x'(t) < 0$, or

(IV) $x(t)x'(t) > 0$

for $t \geq t_1$ (see Lemmas 1.3 and 2.2 in [24]). He also proved that if (III) holds, then $(-1)^i x(t)x^{(i)}(t) > 0$, for $i = 0, 1, 2, 3$, and he then established some necessary and sufficient conditions for the oscillation of (E_1). Lazer used the Riccati substitution $u(t) = x'(t)/x(t)$ to prove that if $p(t) \leq 0$, $q(t) > 0$, and

$$\int_{t_0}^{\infty} \left[q(t) - \frac{2}{3\sqrt{3}} (-p(t))^{3/2} \right] dt = \infty, \tag{7}$$

then (E_1) is oscillatory (see [24, Theorem 1.3]). On the other hand, in [24], he also considered the case where

(c_4) $p(t) > 0$, $q(t) > 0$, $2q(t) - p'(t) \geq 0$, and $2q(t) - p'(t) \not\equiv 0$ in any subinterval of $[t_0, \infty)$.

Lazer's investigation in this case depends on the value of the function

$$F(x(t)) = (x'(t))^2 - 2x(t)x''(t) - p(t)x^2(t).$$

He proved that if $x(t)$ is a nonoscillatory solution of (E_1) that is eventually nonnegative and there exists $t_1 \in [t_0, \infty)$ such that $F(x(t_1)) \geq 0$, then there exists $t_2 > t_1$ such that

$$x(t) > 0, \qquad x'(t) > 0, \qquad x''(t) > 0, \qquad \text{and} \qquad x'''(t) < 0 \qquad \text{for } t \geq t_2. \tag{8}$$

Waltman [37] considered the equations

$$x'''(t) + p(t)x'(t) + q(t)x^{\gamma}(t) = 0, \qquad t \geq t_0, \tag{E_3}$$

and

$$x'''(t) + p(t)x'(t) + q(t)f(x(t)) = 0, \qquad t \geq t_0, \tag{E_4}$$

where $p(t) \geq 0$ and $q(t) \geq 0$ are continuous functions and γ is the ratio of odd positive integers. He established some oscillation results; in particular, for equation (E_3), he proved that if $p'(t) < 0$ and

$$A + Bt - \int_{t_0}^{t} \int_{0}^{s} q(u)duds < 0, \tag{9}$$

for t sufficiently large, where A, B are any constants, then every solution that has a zero is oscillatory. For equation (E_4), Waltman proved ([37, Theorem 2]) that every solution of (E_4) having a zero is oscillatory if $f(u)/u \geq K > 0$, $Kq(t) - p'(t) > 0$, and

$$\int_{t_0}^{\infty} t[Kq(t) - p'(t)]dt = \infty. \tag{10}$$

REMARK 3. We note that conditions (7) and (10) do not hold for the Euler equation (1) with $p(t) = 0$. This implies that condition (4) in Conjecture 2 improves condition (7) of Lazer [24] and condition (10) of Waltman [37] in this case.

Heidel [19] considered equation (E_3) and investigated the behavior of the nonoscillatory solutions as well as the existence of oscillatory solutions. He considered two different cases, namely,

(c_5) $p(t) \leq 0$ and $q(t) \leq 0$, or

(c_6) $p(t) \geq 0$ and $q(t) \geq 0$.

Most of his results dealt with the behavior of the nonoscillatory solutions. In the case where (c_5) holds, Heidel generalized the results of Lazer [24] and established some sufficient conditions that ensure nonoscillatory solutions of (E_3) satisfy either (III) or (IV). For the existence of oscillatory solutions, Heidel proved that if $\gamma = 1$, $-2/t^2 \leq p(t) \leq 0$, $q(t) \leq 0$, and for some $0 < \alpha < 1$,

$$\int_{t_0}^{\infty} s^{2-\alpha} q(s) ds = -\infty, \qquad \text{and} \qquad \int_{t_0}^{\infty} s^4 [p'(s) - 2q(s)] ds = \infty,$$

then (E_3) has an oscillatory solution. For more results on (E_3), we refer the reader to the paper of Greguš et al. [16] which also improved Waltman's result [37].

Škerlik [31] considered equation (E_1) with $p(t)$ and $q(t)$ being positive functions and improved Lazer's results [24]. He proved that if $t^2 p(t) \leq \frac{1}{4}$ and

$$\int_{t_0}^{\infty} \left[s^2 q(s) + s p(s) - \frac{2}{3\sqrt{3}s} (1 - s^2 p(s))^{3/2} \right] ds = \infty, \tag{11}$$

then (E_1) has the following property.

Property A: *Equation (E_1) has Property A if each solution $x(t)$ is either oscillatory or satisfies* $\lim_{t\to\infty} x^{(i)}(t) = 0$ *for* $i = 0, 1, 2, 3$.

He also showed that the equation is oscillatory if and only if it has Property A. In the special case where $p(t) = 0$, equation (E_1) reduces to (2) and condition (11) becomes

$$\int_{t_0}^{\infty} \left(s^2 q(s) - \frac{2}{3\sqrt{3}s} \right) ds = \infty; \tag{12}$$

this is a sufficient condition for the oscillation of (2). We note here that the results of Škerlik are sharp and cannot be weakened. One of our aims in this paper is to extend Škerlik's results and find similar conditions for the delay equation (E_0).

Tiryaki and Yamen [36] considered equation (E_0) and studied the asymptotic behavior of the nonoscillatory solutions. They asked that $f(u)/u \geq \beta > 0$ and that the delay $\tau(t) \leq t$ satisfies $\tau'(t) \geq 0$. In the case where $p(t) \leq 0$ and $q(t) > 0$, they extended the results of Škerlik [31] and proved that if

$$\int_{t_0}^{\infty} \left[\beta s^2 q(s) + sp(s) - \frac{2}{3\sqrt{3}s}(1 - s^2 p(s))^{3/2} \right] ds = \infty, \tag{13}$$

then every nonoscillatory solution of (E_0) tends to either zero or $\pm\infty$ as $t \to \infty$. They also considered the case where

(c_7) $p(t) \geq 0$, $q(t) > 0$, $2\beta q(t) - p'(t) > 0$, and $t^2 p(t) \leq \frac{1}{4}$.

In this case, they proved that every nonoscillatory solution of (E_0) tends to either zero or $\pm\infty$ as $t \to \infty$ provided (13) holds.

In [34], Tiryaki and Atkas considered the general third-order equation

$$(r_1(t)(r_2(t)x'(t))')' + p(t)x'(t) + q(t)f(x(\tau(t))) = 0, \tag{14}$$

where p, q are positive functions, $uf(u) > 0$ for $u \neq 0$, the delay $\tau(t)$ satisfies $\tau'(t) \geq 0$, and

$$\int_{t_0}^{\infty} \frac{1}{r_1(t)}\, dt = \int_{t_0}^{\infty} \frac{1}{r_2(t)}\, dt = \infty.$$

To prove the main results in [34], the authors applied the Riccati technique used in [26] and established some sufficient conditions to ensure that (14) is either oscillatory or every nonoscillatory solution tends to zero as t tends to infinity. The proofs of the main results in [34] are based on their Lemma 2, which is presented without proof; it is mentioned that its proof is similar to the proof of Lemma 1 in [30]. In fact, to apply the proof of [30, Lemma 1] which is based on the results of Lazer [24] (also see Lemma 1 in [31]) there are some additional conditions that need to be satisfied. These conditions are that $2q(t) - p'(t) \geq 0$ and

$$F(x(t_1)) = (x'(t_1))^2 - 2x(t_1)x''(t_1) - p(t_1)x^2(t_1) \geq 0, \tag{15}$$

where $x_1(t)$ is a solution of (14) with $r_1(t) = r_2(t) = 1$. The authors in [34] did not present a generalization of these conditions to the equation (14), which are important to prove that the solutions have the property

Property V$_2$: $y(t)(r_1(t)y'(t)) > 0$, $y(t)r_2(t)(r_1(t)y'(t))' > 0$, and $y(t)(r_2(t)(r_1(t)y'(t))')' \leq 0$.

This means that Lemma 2 in [34] needs a generalization of condition (15), or to assume that the equation

$$(r_1(t)z'(t))' + \frac{p(t)}{r_2(t)}z(t) = 0$$

is nonoscillatory, in order to prove that solutions have property V_2. On the other hand, their main results are not sharp as will be clear from the following simple example. It is known that the equation

$$x'''(t) + \frac{\beta}{t^3}x(t) = 0, \qquad t \geq 1, \tag{16}$$

is oscillatory if $\beta > 2/(3\sqrt{3})$. But by applying Theorem 1 in [34] shows that (16) is oscillatory if $\beta > 1$; this is different from the oscillation condition for (16). The natural question to ask now is: "Is it possible to find new oscillation results for equation (E_0) without the condition $\tau'(t) \geq 0$ as required in [36] and [34]?" One of our aims in this paper is to give an affirmative answer to this question (see Theorem 7 and Example 3 below).

For additional oscillation results for different forms of third-order equations, we refer the reader to the works of Barrett [2, 3], Bartušek, Cecchi and Martini [4], Cecchi and Marini [5], Džurina [6], Erbe [7], Etgen and Shih [8], Greguš [9], Greguš Jr. [10], Greguš and Graef [11, 12], Greguš, Graef, and Gera [13, 14], Greguš and Greguš Jr. [15, 17], Jones [20, 21], Saker [27], Šoltěs [32], Tiryaki and Çelebi [35], and the references cited therein, as well as the new monograph by Saker [28].

The remainder of this paper is organized as follows: First, we establish some new oscillation criteria for the delay equation (E_0), which in the special case $\tau(t) = t$ include the results established by Hanan [18], Lazer [24], and Waltman [37], and are different from the results of Tiryaki and Yamen [36] and Greguš [10] in the sense that our results will depend on the delay function and do not require any information about an unknown solution to a related equation. Our results also include those of Škerlik [31] when $f(u) = u$ and $\tau(t) = t$, and are different from the results of Džurina [6] in the sense that our results can be easily verified and do not require an additional function $r(t)$ and depend on the solution of a second-order differential equation. They also differ from the results by Tiryaki and Atkas [34] in that we do not require $\tau'(t) \geq 0$ and we give sharp conditions for the oscillation of (1) and (16).

Secondly, we will apply the Riccati technique to establish some new oscillation results of Kamenev-type [22]. Some examples are considered to illustrate the main results.

2. Main results

In this section, we establish some sufficient conditions ensuring that each solution of equation (E_0) is either oscillatory or it tends to zero as t tends to ∞. We will refer to this as:

Property P: *A solution is said to satisfy Property P if it is either oscillatory or it tends to zero as $t \to \infty$. An equation is said to satisfy Property P if each of its nontrivial solutions satisfies Property P.*

We note that if $x(t)$ is a solution of (E_0), then $\hat{x} = -x$ is a solution of an equation of the same form as (E_0). Thus, concerning nonoscillatory solutions of (E_0), we can restrict our attention to only the positive ones in giving proofs.

The following lemma due to Kiguradze and Chanturia [23] will be useful in the remainder of this paper.

Lemma 1. *If the function $f(t)$ satisfies $f^{(i)}(t) > 0$, $i = 0, 1, 2, \ldots, n$, and $f^{(n+1)}(t) < 0$, then $\dfrac{f(t)}{t^n/n!} \geq \dfrac{f'(t)}{t^{n-1}/(n-1)!}$.*

To prove the main oscillation results, we need the following lemma whose proof is similar to that of the proofs of Lemma 3.1 of Lazer [24] and Lemma 2.1 of Tiryaki and Yamen [36], and hence is omitted.

Lemma 2. *Assume that*

(h_1) $f(u)/u \geq k > 0$ *for* $u \neq 0$,

(h_2) $2kq(t) - p'(t) \geq 0$ *for* $t \in [t_0, \infty)$ *and is not identically zero in any subinterval of* $[t_0, \infty)$.

Let $x(t)$ be a nonoscillatory solution of (E_0) that is eventually positive with

$$F(x(t_1)) = (x'(t_1))^2 - 2x(t_1)x''(t_1) - p(t_1)x^2(t_1) \geq 0, \tag{17}$$

for some $t_1 \in [t_0, \infty)$. Then there exists $t_2 \geq t_1$ such that

$$x(t) > 0, \qquad , x'(t) > 0, \qquad x''(t) > 0, \qquad \text{and} \qquad x'''(t) < 0, \tag{18}$$

for $t \geq t_2$.

We are now ready to present our main results in this paper. Recall that we are asking that $p(t)$ and $q(t)$ be positive functions.

Theorem 1. *Assume that (h_1)–(h_2) hold, $p'(t) \leq 0$, and let $x(t)$ be a solution of (E_0) satisfying (17) for some $t_1 \in [t_0, \infty)$. If*

$$\int_{t_0}^{\infty} \frac{\tau^2(s)}{s}(kq(s) - p'(s))ds = \infty, \tag{19}$$

and then $x(t)$ is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (E_0). Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_1$ and that $F(x(t_1)) \geq 0$. By Lemma 2, there exists $t_2 \geq t_1$ such that (18) holds for $t \geq t_2$. Integrating (E_0) from t_2 to t, we have

$$x''(t) - x''(t_2) + p(t)x(t) - p(t_2)x(t_2) = \int_{t_2}^{t} p'(s)x(s)ds - \int_{t_2}^{t} q(s)f(x(\tau(s)))ds$$
$$\leq \int_{t_2}^{t} p'(s)x(s)ds - \int_{t_2}^{t} kq(s)x(\tau(s))ds.$$

Thus,

$$p(t_2)x(t_2) + x''(t_2) \geq x''(t) + p(t)x(t) - \int_{t_2}^{t} p'(s)x(s)ds + \int_{t_2}^{t} kq(s)x(\tau(s))ds$$
$$\geq -\int_{t_2}^{t} p'(s)x(s)ds + \int_{t_2}^{t} kq(s)x(\tau(s))ds$$
$$= \int_{t_2}^{t} \left[kq(s) - p'(s)\frac{x(s)}{x(\tau(s))}\right] x(\tau(s))ds. \tag{20}$$

By Lemma 1, we can easily deduce from (18) that

$$\frac{x'(t)}{x(t)} \leq \frac{2}{t}, \qquad \text{for } t \geq t_2. \tag{21}$$

Integrating (21) from $\tau(t)$ to t, we obtain

$$x(\tau(t)) \geq \frac{\tau^2(t)}{t^2} x(t). \tag{22}$$

Since $x'(t) > 0$ for $t \geq t_2$ and is increasing, we have $x'(t) > A > 0$ for $t \geq t_2$, and since $x(t_2) > 0$, we see that

$$x(t) \geq x(t_2) + A(t - t_2) \geq \frac{A}{2} t, \qquad \text{for } t > t_3 \tag{23}$$

for some $t_3 \geq t_2$. Since $p'(t) \leq 0$ and $x(s) \geq x(\tau(s))$, (20) yields

$$p(t_2)x(t_2) + x''(t_2) \geq \int_{t_3}^{t} [kq(s) - p'(s)] x(\tau(s))ds. \tag{24}$$

Substituting (22) and (23) into (24), we obtain

$$\frac{p(t_2)x(t_2) + x''(t_2)}{A/2} \geq \int_{t_3}^{t} \frac{\tau^2(s)}{s} [kq(s) - p'(s)]ds, \tag{25}$$

so

$$\int_{t_3}^{\infty} \frac{\tau^2(s)}{s} [kq(s) - p'(s)]ds < \infty,$$

which contradicts assumption (19).

Our next theorem is for the case $p'(t) \geq 0$.

THEOREM 2. *Assume that* (h_1) *holds,* $p'(t) \geq 0$,

$$(h_3) \qquad kq(t) - p'(t)\frac{t^2}{\tau^2(t)} \geq 0 \quad \text{for } t \in [t_0, \infty),$$

and let $x(t)$ *be a solution of* (E_0) *satisfying* (17) *for some* $t_1 \in [t_0, \infty)$. *If*

$$\int_{t_0}^{\infty} \frac{\tau^2(s)}{s}\left(kq(s) - p'(s)\frac{s^2}{\tau^2(s)}\right)ds = \infty, \tag{26}$$

then $x(t)$ *is oscillatory.*

PROOF. Proceeding as in the proof of Theorem 1, we again obtain

$$p(t_2)x(t_2) + x''(t_2) \geq \int_{t_2}^{t} kq(s)x(\tau(s))ds - \int_{t_2}^{t} p'(s)x(s)ds.$$

From (22), (23), and condition (h_3), we have

$$p(t_2)x(t_2) + x''(t_2) \geq \frac{A}{2}\int_{t_2}^{t} \frac{\tau^2(s)}{s}\left[kq(s) - p'(s)\frac{s^2}{\tau^2(s)}\right]ds.$$

The resulting contradiction to (26) completes the proof.

REMARK 4. Note that when $\tau(t) = t$, conditions (19) and (26) both become

$$\int_{t_0}^{\infty} s[kq(s) - p'(s)]ds = \infty,$$

which is the condition of Waltman (10) above. In the linear case, $f(u) = u$, this condition becomes that of Hanan [18]. We note that condition (22) which gives a relation between the solution with delay and without delay plays an important role in the proof of the above results. So if a sharper relation can be found, then it would be possible to obtain a better oscillation result. We leave this as an open question for the interested reader.

In the following theorem, we extend the results of Lazer [24] (see Theorem 3.1 in [24]) to the delay equation (E_0).

THEOREM 3. *Assume that* (h_1)–(h_2) *hold. If for some* $m < 1/2$, *the second-order equation*

$$w''(t) + \left[p(t) + q(t)\frac{km\tau^2(t)}{t}\right]w(t) = 0 \tag{27}$$

is oscillatory, then equation (E_0) *is oscillatory, i.e., the equation has an oscillatory solution. In particular, any nontrivial solution* $x(t)$ *of* (E_0) *with* $F(x(t_1)) = (x'(t_1))^2 - 2x(t_1)x''(t_1) - p(t_1)(x(t_1))^2 \geq 0$, *for some* $t_1 \geq t_0$, *is oscillatory.*

PROOF. We follow the proof of Theorem 3.1 of Lazer [24], and suppose that $x(t)$ is a nonoscillatory solution of (E_0). Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t > t_0$ with $F(x(t_1)) \geq 0$ for $t_1 \geq t_0$. By Lemma 2, there exists a number $t_2 \geq t_1$ such that (18) holds for $t \geq t_2$. Hence, as in the proof of Theorem 1, applying Lemma 1, we have $x(t)/x'(t) \geq t/2$ for $t \geq t_2$. Thus, since $m < 1/2$, there exists $t_3 \geq t_2$ such that

$$x(t)/x'(t) \geq mt \qquad \text{for } t \geq t_3. \tag{28}$$

We can write equation (E_0) as the system

$$\begin{cases} x'(t) = w(t) > 0, \\ w'(t) = x''(t) > 0, \\ w''(t) + p(t)w(t) + q(t)(f(x(\tau(t)))) = 0. \end{cases}$$

The third equation can be written in the form

$$w''(t) + \left[p(t) + q(t)\frac{f(x(\tau(t)))}{x'(t)}\right]w(t) = 0. \tag{29}$$

From (h_1), (22), and (28), we have

$$\begin{aligned} p(t) + q(t)\frac{f(x(\tau(t)))}{x'(t)} &\geq p(t) + \frac{q(t)kx(\tau(t))}{x'(t)} \\ &\geq p(t) + \frac{q(t)k\tau^2(t)}{t^2}\frac{x(t)}{x'(t)} \\ &\geq p(t) + \frac{q(t)mk\tau^2(t)}{t}, \end{aligned}$$

for $t \geq t_3$. Since (27) is oscillatory, by the Sturm Comparison Theorem, every nontrivial solution of (29) defined for $t \geq t_3$ is oscillatory. But this contradicts the fact that $w(t) = x'(t) > 0$, and this completes the proof of the theorem.

We need the following lemma whose proof is similar to that of the proof of Theorem 3.6 of Heidel [19] and hence is omitted.

LEMMA 3. *Assume that* (h_1) *and*

(h_4) $p(t) \geq 0$, $q(t) > 0$, *and* $t^2p(t) \leq 1/4$

hold. If $x(t)$ *is a nonoscillatory solution of* (E_0), *then there exists* $t_1 \geq t_0$ *such that either* (i) $x(t)x'(t) > 0$ *or* (ii) $x(t)x'(t) < 0$ *for* $t \geq t_1$.

THEOREM 4. *Let* (h_1) *and* (h_4) *hold and let* $x(t)$ *be a solution of* (E_0). *If*

$$\int_{t_0}^{\infty}\left[k\tau^2(s)q(s) + sp(s) - \frac{2}{3\sqrt{3}s}\left(1 - s^2p(s)\right)^{3/2}\right]ds = \infty, \tag{30}$$

then equation (E_0) *satisfies Property P.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (E_0) on $[t_0, \infty)$. Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t > t_1$. Since $t^2 p(t) \le 1/4$, it follows from Lemma 3 that there exists $t_2 \ge t_1$ such that either

(i) $x(t)x'(t) > 0$, $x''(t) > 0$, and $x'''(t) < 0$ for $t \ge t_2$, or

(ii) $x(t)x'(t) < 0$, $x''(t) > 0$, and $x'''(t) < 0$ for $t \ge t_2$.

First, we consider case (i) and define

$$u(t) := \frac{tx'(t)}{x(t)}. \tag{31}$$

Then $u(t) > 0$ and by using (E_0), we see that $u(t)$ satisfies the second-order Riccati equation

$$\left((tu)' + \frac{3}{2}u^2 - 4u\right)' = -\frac{1}{t}\left(u^3 - 3u^2 + (2 + t^2 p(t))u + t^3 q(t)\frac{f(x(\tau(t)))}{x(t)}\right). \tag{32}$$

Define

$$Q(u) := u^3 - 3u^2 + (2 + t^2 p)u + t^3 q(t)\frac{f(x(\tau(t)))}{x(t)}.$$

Using the fact that the function $Q(u)$ has a minimum value, we can easily see that

$$Q(u) \ge t^3 q(t)\frac{f(x(\tau(t)))}{x(t)} + t^2 p(t) - \frac{2}{3\sqrt{3}}(1 - t^2 p(t))^{3/2}. \tag{33}$$

Substituting the estimate (33) into (32), we have

$$\begin{aligned}&\left((tu)' + \frac{3}{2}u^2 - 4u\right)' \\ &\qquad \le -\frac{1}{t}\left(t^3 q(t)\frac{f(x(\tau(t)))}{x(t)} + t^2 p(t) - \frac{2}{3\sqrt{3}}(1 - t^2 p(t))^{3/2}\right).\end{aligned} \tag{34}$$

When (i) holds, we see that the conditions of Kiguradze's Lemma are satisfied and so we can use the estimate (22). Substituting (22) into (34) and using (h_1), we have for all $t \ge t_2$ that

$$\begin{aligned}&\left((tu)' + \frac{3}{2}u^2 - 4u\right)' \\ &\qquad \le -\frac{1}{t}\left(t^3 q(t)\frac{k\tau^2(t)}{t^2} + t^2 p(t) - \frac{2}{3\sqrt{3}}(1 - t^2 p(t))^{3/2}\right) = -P(t),\end{aligned} \tag{35}$$

where

$$P(t) = k\tau^2(t)q(t) + tp(t) - \frac{2}{3\sqrt{3}t}(1 - t^2p(t))^{3/2}.$$

Integrating (35) from t_2 to t, we have

$$(tu(t))' + \frac{3}{2}u^2(t) - 4u(t) \le K_0 - \int_{t_2}^{t} P(s)ds,$$

where K_0 is a constant. Using the fact that the function $\frac{3}{2}u^2 - 4u$ has a minimum value $-8/3$, we have

$$(tu)' \le K_1 - \int_{t_2}^{t} P(s)ds,$$

where $K_1 = K_0 + \frac{8}{3}$. Integrating again and applying (30) shows that $u(t)$ must eventually become negative, which contradicts the positivity of $u(t)$.

Next, we assume that (ii) holds and suppose that $x'(t) < 0$ for $t \ge t_2$. Hence, $\lim_{t\to\infty} x(t) = L \ge 0$ exists. Let $L > 0$. Then $x(\tau(t)) \ge x(t) \ge L$ for $t \in [t_2, \infty)$. Since $t^2p(t) \le 1/4$, we can easily see that

$$tp(t) - \frac{2}{3\sqrt{3}t}(1 - t^2p(t))^{3/2} \le 0 \qquad \text{for } t^2p(t) \le 1/4,$$

where

$$\begin{aligned}(3\sqrt{3}pt^2)^2 - (2(1 - pt^2)^{3/2})^2 &= 4p^3t^6 + 15p^2t^4 + 12pt^2 - 4 \\ &= (4t^2p - 1)(t^2p + 2)^2.\end{aligned}$$

Consequently, from (30) and the fact that $\tau(t) \le t$, we have

$$\int_{t_0}^{\infty} s^2q(s)ds \ge \int_{t_0}^{\infty} \tau^2(s)q(s)ds = \infty.$$

Multiplying (E_0) by t^2 and integrating from t_2 to t, we have

$$t^2x''(t) - 2tx'(t) + \frac{9}{4}x(t) \le K - Lk\int_{t_2}^{t} s^2q(s)ds,$$

where K is a constant. Since $x(t) > 0$, this implies

$$(t^2x''(t) - 2tx'(t)) < K - Lk\int_{t_2}^{t} s^2q(s)ds.$$

From the last inequality, we see that the right hand side tends to $-\infty$ as $t \to \infty$. However, by Lemma 2.2 in [19], since $\lim_{t\to\infty} x(t) = L$, we have

$\lim_{t\to\infty}(t^2x''(t) - 2tx'(t)) = 0$, which is a contradiction. Hence, $L = 0$ and so $\lim_{t\to\infty} x(t) = 0$. This completes the proof.

From Theorem 4, we have the following Hille-Kneser type oscillation result. It can be considered as the extension of Theorem 5.7 of Hanan [18].

Corollary 1. *Let* (h_1) *and* (h_4) *hold. If*

$$\liminf_{t\to\infty} \frac{k\tau^2(t)tq(t) + t^2p(t)}{(1 - t^2p(t))^{3/2}} > \frac{2}{3\sqrt{3}}, \tag{36}$$

then equation (E_0) *satisfies Property P.*

Remark 5. From Corollary 1, it is clear that when $p(t) = 0$ and $\tau(t) = t$, then condition (36) becomes

$$\liminf_{t\to\infty} t^3q(t) > \frac{2}{3k\sqrt{3}},$$

which is the oscillation condition of Hanan for the equation $x'''(t) + q(t)x(t) = 0$ (see [18, Theorem 5.7]).

Remark 6. Theorem 4 improves Theorem 2.2 in [36] since our results ensure that each solution of (E_0) is either oscillatory or tends to zero as t goes to infinity, but the results in [36] ensure only that the nonoscillatory solutions tend to $\pm\infty$ as $t \to \infty$. To illustrate the results in Theorem 4, we give the following example which shows that condition (10) of Theorem 2.2 in [36] holds, but the solution does not tend to $\pm\infty$ as $t \to \infty$.

Example 1. Consider the third-order linear delay differential equation

$$x'''(t) + \frac{\alpha}{t^3}x\left(\frac{t}{2}\right) = 0, \qquad t \geq 1. \tag{37}$$

Here $\tau(t) = t/2$, $p(t) = 0$, and $q(t) = \alpha/t^3$. We will apply Theorem 4. In this case, we see that

$$\begin{aligned}\int_{t_0}^{\infty}\left[\tau^2(s)q(s) - \frac{2}{3\sqrt{3}s}\right]ds &= \int_1^{\infty}\left[\left(\frac{s}{2}\right)^2\frac{\alpha}{s^3} - \frac{2}{3\sqrt{3}s}\right]ds \\ &= \int_1^{\infty}\left[\frac{1}{36s}(9\alpha - 8\sqrt{3})\right]ds \\ &= (9\alpha - 8\sqrt{3})\int_1^{\infty}\frac{1}{36s}\,ds = \infty,\end{aligned}$$

provided that $\alpha > \frac{8\sqrt{3}}{9}$. Thus (30) is satisfied. Then by Theorem 4, if $x(t)$ is a solution of (37), then $x(t)$ satisfies Property P, i.e., it is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$. Note that when $\tau(t) = t$, the oscillation condition for (37) is $\alpha > \frac{2}{3\sqrt{3}}$ (see Remark 5).

Next, we give an example to show that Theorem 4 can be applied even in the case where Theorems 1 and 3 cannot.

Example 2. Consider the third-order delay differential equation

$$x'''(t) + \frac{\alpha}{t^2}x'(t) + \frac{\beta}{t\tau^2(t)}x(\tau(t)) = 0, \qquad t \geq 1, \tag{38}$$

where $\tau(t) \leq t$ is the delay function and α and β are positive constants such that $\alpha < \frac{1}{4}$. It is easy to see that Theorem 1 cannot be applied to (38). On the other hand, to apply Theorem 3, we note that the equation (27) becomes

$$x''(t) + \left[\frac{\alpha}{t^2} + \frac{m\beta}{t^2}\right]x(t) = 0, \qquad t \geq 1. \tag{39}$$

Applying the Hille-Kneser criterion, we see that equation (39) is oscillatory if $\alpha + m\beta > \frac{1}{4}$ for some $m < 1/2$. That is, $\alpha + \frac{\beta}{2} > \alpha + m\beta > 1/4$. This implies that a sufficient condition for the oscillation of (39) is

$$2\alpha + \beta > 1/2. \tag{40}$$

According to Theorem 4, we see that condition (30) becomes

$$\alpha + \beta > \frac{2}{3\sqrt{3}}(1-\alpha)^{3/2}, \tag{41}$$

which gives us an affirmative answer to Conjecture 1. By choosing $\alpha = 0.06$ and $\beta = 0.3$, we see that the condition (40) is not satisfied. On the other hand, we can easily verify that condition (41) is satisfied in this case since $\alpha + \beta = 0.36 > \frac{2}{3\sqrt{3}}(1-\alpha)^{3/2} = 0.35079$. From this we see that Theorem 4 improves the results of Lazer [24] even for equations without delays.

Remark 7. In Lemma 3, we used the condition $t^2p(t) \leq 1/4$. This condition can be removed and instead make use of the nonoscillatory properties of the equation

$$y''(t) + p(t)y(t) = 0 \tag{42}$$

to prove the results. In this case, the oscillation of equation (E_0) will be close to that of the corresponding second order equation (42) in the sense that equation (E_0) can be written in the form

$$\left(y^2(t)\left(\frac{1}{y(t)}x'(t)\right)'\right)' + y(t)q(t)f(x(\tau(t))) = 0, \tag{43}$$

where $y(t)$ is positive solution of (42). This is easy to see since

$$\frac{1}{y(t)}\left(y^2(t)\left(\frac{1}{y(t)}x'(t)\right)'\right)' = x'''(t) - \frac{y''(t)}{y(t)}x'(t) = x'''(t) + p(t)x'(t).$$

Note that the condition $t^2p(t) \leq 1/4$ implies that equation (42) has a positive nonoscillatory solution $y(t)$ with $y'(t)$ of one sign. We note also that if

$$\int^{\infty} \frac{1}{y^2(s)}\, ds = \int^{\infty} y(s) ds = \infty,$$

then equation (43) is in canonical form and the results in [1, 26] can be applied to obtain several oscillation results. We leave the details to the interested reader.

Remark 8. In case $f(u) = u^{\gamma}$, where $\gamma > 1$ is the quotient of two odd positive integers, we can use the approach taken in the proof of Theorem 4, to obtain (32). Now since $f(u) = u^{\gamma}$, inequality (34) can be written in the form

$$\left((tu)' + \frac{3}{2}u^2 - 4u\right)'$$
$$\leq -\frac{1}{t}\left(t^3q(t)\frac{x(\tau(t))}{x(t)}x^{\gamma-1}(\tau(t)) + t^2p(t) - \frac{2}{3\sqrt{3}}(1 - t^2p(t))^{3/2}\right).$$

Since $x'(t) > 0$ and is increasing for $t \geq T$, we have $x'(t) > A$ for some $A > 0$. Moreover, since $x(T) > 0$, we see that $x(t) \geq x(T) + (t - T)A > A(t - T)$ for $t > T$. Using this and inequality (22), we obtain

$$\left((tu)' + \frac{3}{2}u^2 - 4u\right)'$$
$$\leq -\frac{1}{t}\left(A^{\gamma-1}tq(t)\tau^2(t)(\tau(t) - T)^{\gamma-1} + t^2p(t) - \frac{2}{3\sqrt{3}}(1 - t^2p(t))^{3/2}\right).$$

From this remark and proceeding as in the proof of Theorem 4, we can prove the following result.

Theorem 5. *Let* (h_4) *hold and let* $x(t)$ *be a solution of*

$$x'''(t) + p(t)x'(t) + q(t)x^{\gamma}(\tau(t)) = 0, \tag{44}$$

where $\gamma > 1$ *is the quotient of two odd positive integers. If*

$$\int_{t_0}^{\infty}\left[A^{\gamma-1}sq(s)\tau^2(s)(\tau(s) - T)^{\gamma-1} + s^2p(s) - \frac{2}{3\sqrt{3}s}(1 - s^2p(s))^{3/2}\right]ds = \infty, \tag{45}$$

for every positive constant A, then $x(t)$ is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

REMARK 9. In Theorem 5, it is clear that the condition (45) depends on the additional constant A. So it would be interesting to give a result similar to this theorem without such a constant.

For equation (44), we can use the above calculations and obtain the following extension of Theorem 1 of Waltman [37] (see condition (9) above).

THEOREM 6. *Assume that* (h_2) *holds,* $p'(t) \le 0$, *and* $\gamma > 1$ *is the quotient of two odd positive integers. If*

$$A + Bt - \int_{t_0}^{t}\int_{t_0}^{s}\left(\frac{\tau^2(v)}{v^2}\right)^{\gamma} q(v)dvds < 0,$$

for sufficiently large t, then any continuable solution of (E_0) *that has a zero is oscillatory.*

In the following, we extend some oscillation conditions of Kamenev type. First, we introduce a class of functions $\Re$. Let

$$\mathbf{D}_0 = \{(t,s) : t > s \ge t_0\} \qquad \text{and} \qquad \mathbf{D} = \{(t,s) : t \ge s \ge t_0\}.$$

The function $H \in C(\mathbf{D}, \mathbf{R})$ is said to belong to the class $\Re$ if:

(i) $H(t,t) = 0$ for $t \ge t_0$ and $H(t,s) > 0$ for $(t,s) \in \mathbf{D}_0$;

(ii) $\frac{\partial H(t,s)}{\partial s}$ is continuous on $\mathbf{D}_0$ and

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \ge 0 \tag{46}$$

for a suitable function h.

THEOREM 7. *Assume that* (h_1), (h_2), *and* (h_4) *hold and*

$$\int_{t_0}^{\infty} q(s)\tau^2(s)ds = \infty. \tag{47}$$

If there exist functions H and h satisfying (46) *and a function* $\rho \in C^1([t_0,\infty), [0,\infty))$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\frac{k\rho(s)q(s)\tau^2(s)}{s^2} - \frac{\rho(s)Q^2(t,s)}{4s}\right]ds = \infty, \tag{48}$$

where $Q(t,s) = h(t,s) - \sqrt{H(t,s)}[\rho'(s) - p(s)\rho(s)s]/\rho(s)$, *then equation* (E_0) *satisfies Property P.*

Proof. Let $x(t)$ be a nonoscillatory solution of (E_0) on $[t_0, \infty)$. Without loss of generality, we assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t > t_1$. It follows from Lemma 3 that there exists $t_2 \geq t_1$ such that either (i) $x(t)x'(t) > 0$, $x''(t) > 0$, and $x'''(t) < 0$ for $t \geq t_2$, or (ii) $x(t)x'(t) < 0$, $x''(t) > 0$, and $x'''(t) < 0$ for $t \geq t_2$. First consider case (i) and define

$$\omega(t) = \rho(t)\frac{x''(t)}{x(t)}. \tag{49}$$

Then $\omega(t) > 0$ and after differentiation satisfies the equation

$$\omega'(t) = \frac{\rho'(t)}{\rho(t)}\omega(t) - p(t)\rho(t)\frac{x'(t)}{x(t)} - \rho(t)q(t)\frac{f(x(\tau(t)))}{x(t)} - \rho(t)\frac{x''(t)x'(t)}{x^2(t)}. \tag{50}$$

From (h_1), since $f(u)/u > k > 0$, we see that

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - p(t)\rho(t)\frac{x'(t)}{x(t)} - k\rho(t)q(t)\frac{x(\tau(t))}{x(t)} - \rho(t)\frac{x''(t)x'(t)}{x^2(t)}. \tag{51}$$

As in the proof of Theorem 1, applying Lemma 1, we see that (22) holds. From (22) and (51), we have

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - p(t)\rho(t)\frac{x'(t)}{x(t)} - \frac{k\rho(t)q(t)\tau^2(t)}{t^2} - \rho(t)\frac{x''(t)x'(t)}{x^2(t)}. \tag{52}$$

We will now derive a relation between $x'(t)$ and $x''(t)$ to simplify (52). To do this, we let $U(t) := x'(t) - tx''(t)$ and will show that $U(t) > 0$ eventually. Since $U'(t) = -tx'''(t) > 0$ for $t \in [T, \infty)$, we see that $U(t)$ is strictly increasing on $[T, \infty)$. We claim that there is a $t_1 \in [T, \infty)$ such that $U(t) > 0$ on $[t_1, \infty)$. Assume this is not the case, say $U(t) < 0$ on $[t_1, \infty)$. Then,

$$(x'(t)/t)' = \frac{tx''(t) - x'(t)}{t^2} = -\frac{U(t)}{t^2} > 0, \qquad t \in [t_1, \infty),$$

which implies that $x'(t)/t$ is strictly increasing on $[t_1, \infty)$. Choose $t_2 \in [t_1, \infty)$ so that $\tau(t) \geq \tau(t_2)$ for $t \geq t_2$. Since $x'(t)/t$ is strictly increasing, we have $x'(\tau(t))/\tau(t) \geq x'(\tau(t_2))/\tau(t_2) =: d > 0$, so that $x'(\tau(t)) \geq d\tau(t)$ for $t \geq t_2$. Since $x(t) \geq \frac{t}{2}x'(t) \geq \frac{t}{2}x'(\tau(t))$ (see (21)), we have

$$x(\tau(t)) \geq d\tau(t)\frac{t}{2} \geq \frac{d}{2}\tau^2(t). \tag{53}$$

From (E_0) and the fact that we are in case (i), we see that $x'''(t) + q(t)f(x(\tau(t))) = -p(t)x'(t) < 0$. By (h_1) and (53), we have

$$x'''(t) + \frac{dk}{2}q(t)\tau^2(t) < 0. \tag{54}$$

Integrating both sides of (54) from t_2 to t, we have

$$x''(t) - x''(t_2) + \frac{dk}{2}\int_{t_2}^{t} q(s)\tau^2(s)ds < 0.$$

This implies that

$$x''(t_2) > \frac{dk}{2}\int_{t_2}^{t} q(s)\tau^2(s)ds,$$

which contradicts (47). Hence, there is a $t_1 \in [T, \infty)$ such that $U(t) > 0$ on $[t_1, \infty)$. Consequently,

$$\left(\frac{x'(t)}{t}\right)' = \frac{tx''(t) - x'(t)}{t^2} = -\frac{U(t)}{t^2} < 0, \qquad t \in [t_1, \infty).$$

Then,

$$x'(t) > tx''(t). \tag{55}$$

From (55) and (52), we have

$$\omega'(t) < \frac{1}{\rho(t)}[\rho'(t) - p(t)\rho(t)t]\omega(t) - \frac{k\rho(t)q(t)\tau^2(t)}{t^2} - \frac{t\omega^2(t)}{\rho(t)}. \tag{56}$$

Set $\gamma(s) = [\rho'(t) - p(t)\rho(t)t]/\rho(s)$ and $W(s) = s/\rho(s)$. Then from (56), we have

$$\begin{aligned}
&\int_{t_1}^{t} H(t,s)\frac{k\rho(s)q(s)\tau^2(s)}{s^2}\,ds \\
&\quad \le \int_{t_1}^{t} H(t,s)[-\omega'(s) + \gamma(s)\omega(s) - W(s)\omega^2(s)]ds \\
&\quad = -H(t,s)\omega(s)|_{t_1}^{t} + \int_{t_1}^{t}\left\{\frac{\partial H(t,s)}{\partial s}\omega(s) + H(t,s)[\gamma(s)\omega(s) - W(s)\omega^2(s)]\right\}ds \\
&\quad = H(t,t_1)\omega(t_1) - \int_{t_1}^{t}\left\{\left[\sqrt{H(t,s)W(s)}\omega(s) + \frac{1}{2}\frac{Q(t,s)}{\sqrt{W(s)}}\right]^2 - \frac{Q^2(t,s)}{4W(s)}\right\}ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_{t_1}^{t} H(t,s)\frac{k\rho(s)q(s)\tau^2(s)}{s^2}\,ds \\
&\quad \le H(t,t_1)\omega(t_1) \\
&\qquad - \int_{t_1}^{t}\left\{\left[\sqrt{H(t,s)W(s)}\omega(s) + \frac{1}{2}\frac{Q(t,s)}{\sqrt{W(s)}}\right]^2 - \frac{Q^2(t,s)}{4W(s)}\right\}ds,
\end{aligned} \tag{57}$$

and so

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left(H(t,s)\frac{k\rho(s)q(s)\tau^2(s)}{s^2}-\frac{Q^2(t,s)}{4W(s)}\right)ds \le \omega(t_1),$$

which contradicts (48). If case (ii) holds, then by (47) and the second part of the proof of Theorem 4, we can show that $\lim_{t\to\infty} x(t) = 0$. This completes the proof.

The following result provides an alternative oscillation criteria when (48) is difficult to verify. The notations of Theorem 7 and its proof will be used here.

THEOREM 8. *Let the hypotheses of Theorem 7 hold except for condition* (48) *and assume that*

$$0 < \inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] \le \infty \qquad \textit{and}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{Q^2(t,s)}{W(s)}\,ds < \infty. \tag{58}$$

If there exists $\psi \in C([t_0,\infty),\mathbf{R})$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\psi_+^2(s)W(s)ds = \infty \tag{59}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left\{H(t,s)\frac{k\rho(s)q(s)\tau^2(s)}{s^2}-\frac{Q^2(t,s)}{4W(s)}\right\}ds \ge \sup_{t\ge t_0}\psi(t), \tag{60}$$

where $\psi_+(t) = \max\{\psi(t),0\}$, *then equation* (E_0) *satisfies property* P.

PROOF. As in the proof of Theorem 7, we have (57). It follows that

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\frac{k\rho(s)q(s)\tau^2(s)}{s^2}-\frac{Q^2(t,s)}{4W(s)}\right]ds\\ &\quad\le \omega(t_1)-\liminf_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[\sqrt{H(t,s)W(s)}\,\omega(s)+\frac{Q(t,s)}{2\sqrt{W(s)}}\right]^2 ds.\end{aligned}$$

From (60), we obtain

$$\omega(t_1) \ge \psi(t_1)+\liminf_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[\sqrt{H(t,s)W(s)}\,\omega(s)+\frac{Q(t,s)}{2\sqrt{W(s)}}\right]^2 ds, \tag{61}$$

and hence

$$0 \le \liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)W(s)}\omega(s) + \frac{Q(t,s)}{2\sqrt{W(s)}} \right]^2 ds$$
$$\le \omega(t_1) - \psi(t_1) < \infty. \tag{62}$$

Define the functions α and β by

$$\alpha(t) = \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)W(s)\omega^2(s)ds$$

and

$$\beta(t) = \frac{1}{H(t,t_1)} \int_{t_1}^{t} \sqrt{H(t,s)}Q(t,s)\omega(s)ds.$$

The remainder of the proof is similar to the proof of Theorem 5.2 in [29] and hence is omitted.

Remark 10. For the choice $H(t,s) = (t-s)^n$ and $h(t,s) = n(t-s)^{(n-2)/2}$, Theorem 8 reduces to the Kamenev-type condition. Other possible choices of H include $H(t,s) = \left(\ln \frac{t}{s}\right)^n$ so that $h(t,s) = \frac{n}{s}\left(\ln \frac{t}{s}\right)^{n/2-1}$.

Example 3. Consider the delay differential equation

$$x'''(t) + \frac{\alpha}{t^2}x'(t) + \left(\beta t^3 + \frac{\gamma}{t^4}\right)x(t-4+2\sin t) = 0, \qquad t \ge 6, \tag{63}$$

where $0 < \alpha \le \frac{1}{4}$, $\gamma, \beta > 0$, $f(u) = u$, $k = 1$, and $\tau(t) = t - 4 + 2\sin t$. We note that the results in and [34] and [36] cannot be applied to (63) since $\tau'(t) = 1 + 2\cos t$ oscillates. To apply Theorem 7, first note that $\tau(t) \ge t/2$ for $t \ge 12$. Hence,

$$\int_{12}^{\infty} \left(\beta s^3 + \frac{\gamma}{s^4}\right) \frac{s^2}{4}\, ds \le \int_{6}^{\infty} q(s)\tau^2(s)ds,$$

so (47) holds. Choosing $\rho(s) = 1$ and $H(t,s) = 1$, we have

$$\limsup_{t\to\infty} \int_{12}^{t} \left[\left(\beta s^3 + \frac{\gamma}{s^4}\right) \frac{s^2}{4} - \frac{\alpha^2}{4s^3} \right] ds$$
$$\le \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\frac{k\rho(s)q(s)\tau^2(s)}{s^2} - \frac{\rho(s)Q^2(t,s)}{4s} \right] ds,$$

so (48) holds. By Theorem 7, any solution $x(t)$ of (63) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

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