HIROSHIMA MATH. J.
43 (2013), 1–48

The coarse orbit types of non-Kählerian, symplectic homogeneous spaces whose transformation groups are non-compact simple and isotropy subgroups are compact

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(Received February 21, 2011)
(Revised September 28, 2011)

ABSTRACT. The main purpose of this paper is to determine the coarse orbit types of all non-Kählerian, symplectic homogeneous spaces G/H with G non-compact simple and H compact. Our result, together with the results of Wang (Amer. J. Math., 1954) and Yichao (Sci. China Ser. A, 1986), enables one to know all dual manifolds of every Kähler C-space.

1. Introduction and the main result

The main purpose of this paper is to determine the coarse orbit types of all non-Kählerian, symplectic homogeneous spaces G/H with G non-compact simple and H compact (see Definition 6 (p. 17) for the definition of *coarse orbit type*):

THEOREM 1. *The following is the coarse orbit type of every non-Kählerian, symplectic homogeneous space G/H with G non-compact simple and H compact:*

G	H	No.
$SO_0(2k, 2l-2k+1)$ $l \geq 2$ $2 \leq k \leq l$ BI	$U(i_1) \times U(i_2 - i_1) \times \cdots \times U(i_a - i_{a-1}) \times U(k - i_a)$ $\times U(j_1 - k) \times U(j_2 - j_1) \times \cdots \times U(j_b - j_{b-1})$ $\times SO(2l - 2j_b + 1)$ $0 \leq a \leq k-1,\ i_0 := 0,\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_a \leq k-1$ $0 \leq b \leq l-k,\ j_0 := k,\ k+1 \leq j_1 \leq j_2 \leq \cdots \leq j_b \leq l$	1
$Sp(k, l-k)$ $l \geq 3$ $1 \leq k \leq l-1$	$U(i_1) \times U(i_2 - i_1) \times \cdots \times U(i_a - i_{a-1}) \times U(k - i_a)$ $\times U(j_1 - k) \times U(j_2 - j_1) \times \cdots \times U(j_b - j_{b-1})$ $\times Sp(l - j_b)$ $0 \leq a \leq k-1,\ i_0 := 0,\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_a \leq k-1$ $0 \leq b \leq l-k,\ j_0 := k,\ k+1 \leq j_1 \leq j_2 \leq \cdots \leq j_b \leq l$	1

2010 *Mathematics Subject Classification*. Primary 17B20; Secondary 53C30.
Key words and phrases. symplectic homogeneous space, ISH space, S-element, coadjoint orbit, semisimple Lie group, coarse orbit type.

G	H	No.
	$U(m_1) \times U(m_2 - m_1) \times \cdots \times U(m_c - m_{c-1})$ $\times Sp(k - m_c) \times U(n_1 - k) \times U(n_2 - n_1)$ $\times \cdots \times U(n_d - n_{d-1}) \times U(l - n_d)$ $0 \leq c \leq k,\ m_0 := 0,\ 1 \leq m_1 \leq m_2 \leq \cdots \leq m_c \leq k$ $0 \leq d \leq l - k - 1,\ n_0 := k,$ $k + 1 \leq n_1 \leq n_2 \leq \cdots \leq n_d \leq l - 1$	2
CII	*Remark.* $Sp(k, l-k)/H_1 = Sp(l-k, k)/H_2$, *where we assume* H_2 *to be a subgroup of* $Sp(l-k, k)$ *by identifying* $Sp(k, l-k)$ *with* $Sp(l-k, k)$. *Here* H_{i} *are the same as in* No.i $(\mathrm{i} = 1, 2)$.	
$SO_0(2k, 2l-2k)$ $l \geq 4$ $2 \leq k \leq l-2$	$U(i_1) \times U(i_2 - i_1) \times \cdots \times U(i_a - i_{a-1}) \times U(k - i_a)$ $\times U(j_1 - k) \times U(j_2 - j_1) \times \cdots \times U(j_b - j_{b-1})$ $\times SO(2l - 2j_b)$ $0 \leq a \leq k-1,\ i_0 := 0,\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_a \leq k-1$ $0 \leq b \leq l-k,\ j_0 := k,\ k+1 \leq j_1 \leq j_2 \leq \cdots \leq j_b \leq l$	1
	$U(m_1) \times U(m_2 - m_1) \times \cdots \times U(m_c - m_{c-1})$ $\times SO(2k - 2m_c) \times U(n_1 - k) \times U(n_2 - n_1)$ $\times \cdots \times U(n_d - n_{d-1}) \times U(l - n_d)$ $0 \leq c \leq k,\ m_0 := 0,\ 1 \leq m_1 \leq m_2 \leq \cdots \leq m_c \leq k$ $0 \leq d \leq l - k - 1,\ n_0 := k,$ $k + 1 \leq n_1 \leq n_2 \leq \cdots \leq n_d \leq l - 1$	2
DI	*Remark.* $SO_0(2k, 2l-2k)/H_1 = SO_0(2l-2k, 2k)/H_2$, *where we assume* H_2 *to be a subgroup of* $SO_0(2l-2k, 2k)$ *by identifying* $SO_0(2k, 2l-2k)$ *with* $SO_0(2l-2k, 2k)$. *Here* H_{i} *are the same as in* No.i $(\mathrm{i} = 1, 2)$.	
$E_{6(2)}$	$A_5 \times T,\ A_4 \times A_1 \times T,\ A_2 \times A_2 \times A_1 \times T$	1
	$A_4 \times T^2,\ A_3 \times A_1 \times T^2,\ A_2 \times A_2 \times T^2$ $A_2 \times A_1 \times A_1 \times T^2$	2
	$A_3 \times T^3,\ A_2 \times A_1 \times T^3,\ A_1 \times A_1 \times A_1 \times T^3$	3
	$A_2 \times T^4,\ A_1 \times A_1 \times T^4$	4
	$A_1 \times T^5$	5
EII	T^6	6
$E_{7(7)}$	$A_6 \times T,\ A_4 \times A_2 \times T$	1
	$A_5 \times T^2,\ A_4 \times A_1 \times T^2,\ A_3 \times A_2 \times T^2$ $A_2 \times A_2 \times A_1 \times T^2$	2

G	H	No.
	$A_4 \times T^3,\ A_3 \times A_1 \times T^3,\ A_2 \times A_2 \times T^3$ $A_2 \times A_1 \times A_1 \times T^3$	3
	$A_3 \times T^4,\ A_2 \times A_1 \times T^4,\ A_1 \times A_1 \times A_1 \times T^4$	4
	$A_2 \times T^5,\ A_1 \times A_1 \times T^5$	5
	$A_1 \times T^6$	6
EV	T^7	7
$E_{7(-5)}$	$A_5 \times A_1 \times T,\ A_3 \times A_2 \times A_1 \times T,\ A_1 \times D_5 \times T$ $D_6 \times T$	1
	$A_5 \times T^2,\ A_4 \times A_1 \times T^2,\ A_3 \times A_2 \times T^2$ $A_3 \times A_1 \times A_1 \times T^2$ $A_2 \times A_2 \times A_1 \times T^2,\ A_2 \times A_1 \times A_1 \times A_1 \times T^2$ $A_1 \times D_4 \times T^2,\ D_5 \times T^2$	2
	$A_4 \times T^3,\ A_3 \times A_1 \times T^3,\ A_2 \times A_2 \times T^3$ $A_2 \times A_1 \times A_1 \times T^3$ $A_1 \times A_1 \times A_1 \times A_1 \times T^3,\ D_4 \times T^3$	3
	$A_3 \times T^4,\ A_2 \times A_1 \times T^4,\ A_1 \times A_1 \times A_1 \times T^4$	4
	$A_2 \times T^5,\ A_1 \times A_1 \times T^5$	5
	$A_1 \times T^6$	6
EVI	T^7	7
$E_{8(8)}$	$A_7 \times T,\ A_4 \times A_3 \times T,\ A_2 \times D_5 \times T,\ D_7 \times T$	1
	$A_6 \times T^2,\ A_5 \times A_1 \times T^2,\ A_4 \times A_2 \times T^2$ $A_4 \times A_1 \times A_1 \times T^2$ $A_3 \times A_3 \times T^2,\ A_3 \times A_2 \times A_1 \times T^2$ $A_2 \times A_2 \times A_1 \times A_1 \times T^2$ $A_2 \times D_4 \times T^2,\ A_1 \times D_5 \times T^2,\ D_6 \times T^2$	2
	$A_5 \times T^3,\ A_4 \times A_1 \times T^3,\ A_3 \times A_2 \times T^3$ $A_3 \times A_1 \times A_1 \times T^3$ $A_2 \times A_2 \times A_1 \times T^3,\ A_2 \times A_1 \times A_1 \times A_1 \times T^3$ $A_1 \times D_4 \times T^3,\ D_5 \times T^3$	3
	$A_4 \times T^4,\ A_3 \times A_1 \times T^4,\ A_2 \times A_2 \times T^4$ $A_2 \times A_1 \times A_1 \times T^4$ $A_1 \times A_1 \times A_1 \times A_1 \times T^4,\ D_4 \times T^4$	4
	$A_3 \times T^5,\ A_2 \times A_1 \times T^5,\ A_1 \times A_1 \times A_1 \times T^5$	5

G	H	No.
	$A_2 \times T^6,\ A_1 \times A_1 \times T^6$	6
	$A_1 \times T^7$	7
EVIII	T^8	8
$E_{8(-24)}$	$A_6 \times A_1 \times T,\ A_4 \times A_2 \times A_1 \times T,\ A_1 \times E_6 \times T$ $E_7 \times T$	1
	$A_6 \times T^2,\ A_5 \times A_1 \times T^2,\ A_4 \times A_2 \times T^2$ $A_4 \times A_1 \times A_1 \times T^2$ $A_3 \times A_2 \times A_1 \times T^2,\ A_2 \times A_2 \times A_1 \times A_1 \times T^2$ $A_1 \times D_5 \times T^2$ $D_6 \times T^2,\ E_6 \times T^2$	2
	$A_5 \times T^3,\ A_4 \times A_1 \times T^3,\ A_3 \times A_2 \times T^3$ $A_3 \times A_1 \times A_1 \times T^3$ $A_2 \times A_2 \times A_1 \times T^3,\ A_2 \times A_1 \times A_1 \times A_1 \times T^3$ $A_1 \times D_4 \times T^3,\ D_5 \times T^3$	3
	$A_4 \times T^4,\ A_3 \times A_1 \times T^4,\ A_2 \times A_2 \times T^4$ $A_2 \times A_1 \times A_1 \times T^4$ $A_1 \times A_1 \times A_1 \times A_1 \times T^4,\ D_4 \times T^4$	4
	$A_3 \times T^5,\ A_2 \times A_1 \times T^5,\ A_1 \times A_1 \times A_1 \times T^5$	5
	$A_2 \times T^6,\ A_1 \times A_1 \times T^6$	6
	$A_1 \times T^7$	7
EIX	T^8	8
$F_{4(4)}$	$A_2 \times A_1 \times T,\ C_3 \times T$	1
	$A_2 \times T^2,\ A_1 \times A_1 \times T^2,\ B_2 \times T^2$	2
	$A_1 \times T^3$	3
FI	T^4	4
$F_{4(-20)}$	$A_2 \times A_1 \times T,\ B_3 \times T$	1
	$A_2 \times T^2,\ A_1 \times A_1 \times T^2,\ B_2 \times T^2$	2
	$A_1 \times T^3$	3
FII	T^4	4
$G_{2(2)}$	$A_1 \times T$	1
G	T^2	2

Let G be a connected, absolutely simple Lie group whose center $Z(G)$ is trivial, and let G/H be a symplectic homogeneous space. A fundamental problem on G/H is as follows: "What is a necessary and sufficient condition for G/H to be Kählerian?" The result of Borel [Br] enables us to see the condition (see Matsushima [Ma1] also):

(i) The isotropy subgroup H must be compact when G/H is Kählerian;
(ii) G/H is compact Kählerian if and only if G is compact;
(iii) G/H is non-compact Kählerian if and only if H is compact and G/K is a Hermitian symmetric space of non-compact type. Here K denotes a maximal compact subgroup of G such that $H \subset K$.

Case (ii): In 1954 Wang [Wa] has determined the coarse orbit types of all compact Kählerian homogeneous spaces in Case (ii), where we need to add two items D_6 and $A_1 \times D_5$ into the list of maximal semisimple C-subgroups of E_7 in Theorem VIII [Wa, p. 31] for the sake of completeness (cf. Bordemann-Forger-Römer [Bd-Fo-Rö]). Case (iii): In 1986 Yichao [Yi] has determined the coarse orbit types of all non-compact Kählerian homogeneous spaces in Case (iii), where we need to read $SO_0(n,2)$ instead of $SO_0(2n)$ in the case (4) of Theorem 1 [Yi, p. 450–451]. That is to say, one knows the coarse orbit types of all Kählerian homogeneous spaces in Cases (ii) and (iii).

Symplectic homogeneous space G/H			
	G	H	
(ii)	compact	—	Kähler
(iii)	non-compact	compact	Kähler or non-Kähler

Theorem 1 provides the coarse orbit types of all non-Kählerian, symplectic homogeneous spaces G/H with G non-compact simple and H compact. From this and Yichao's result [Yi] one can know all dual manifolds G/V of Kähler C-spaces $G_{\mathbf{C}}/B$ in the case where $G_{\mathbf{C}}$ are simple (see Griffiths-Schmid [Gr-Sc, p. 260] for the definition of *dual manifold of a Kähler C-space*). Consequently, Theorem 3 (p. 14) and Wang's result [Wa] enable one to know all dual manifolds of every Kähler C-space.

This paper is organized as follows:

§2 Preliminaries

In this section we recall the definition of symplectic homogeneous space. Moreover, we introduce the notion of infinitesimal symplectic homogeneous space (see Definition 2) and we investigate a relation between symplectic homogeneous spaces and infinitesimal ones.

§3 A structure theorem

For a symplectic homogeneous space G/H with G non-compact semisimple, we clarify a necessary and sufficient condition for H to be compact (see Theorem 2) by taking advantage of the notion of S-element (cf. Definition 3).

§4 The coarse orbit types of non-Kählerian, symplectic homogeneous spaces G/H with G non-compact simple and H compact

We determine the coarse orbit type of each non-Kählerian, symplectic homogeneous space G/H with G non-compact simple and H compact by use of Theorem 2.

Notations. Throughout this paper we use the following notations:

(n1) $Z(G)$: the center of a Lie group G,
(n2) $B_{\mathfrak{g}}$: the Killing form of a Lie algebra $\mathfrak{g}$,
(n3) Lie G: the Lie algebra of a Lie group G, i.e., the set of all left-invariant vector fields on G,
(n4) Ad: the adjoint representation of a Lie group,
(n5) ad: the adjoint representation of a Lie algebra,
(n6) $C_G(W)$: the centralizer of an element $W \in \mathrm{Lie}\, G$ in a Lie group G, i.e., $C_G(W) := \{g \in G \mid \mathrm{Ad}(g)W = W\}$,
(n7) $\mathfrak{c}_{\mathfrak{g}}(W)$: the centralizer of an element $W \in \mathfrak{g}$ in a Lie algebra $\mathfrak{g}$,
(n8) G_0: the identity component of a Lie group G,
(n9) A_g: the inner automorphism of a Lie group G determined by an element $g \in G$.

We sometimes denote the Lie algebra of a Lie group by the corresponding German small letter.

2. Preliminaries

2.1. The definition of symplectic homogeneous space. Let us recall the definition of symplectic homogeneous space.

Definition 1 (cf. Chu [Ch, p. 147]). Let G be a connected Lie group, and let H be a closed subgroup of G. Then the coset space G/H is called a *symplectic homogeneous space*, if it admits a G-invariant symplectic form Ω.

Remark 1. *Definition* 1 *is slightly different from the original definition in Chu* [Ch, p. 147]. *Indeed, we do not impose the condition "H is connected" on Definition* 1.

2.2. The definition of infinitesimal symplectic homogeneous space. We first introduce the notion of infinitesimal symplectic homogeneous space, and afterwards investigate a relation between symplectic homogeneous spaces and infinitesimal ones.

Definition 2. (i) Let $\mathfrak{g}$ be a real Lie algebra, and let ω be a skew-symmetric bilinear form on $\mathfrak{g}$ satisfying

$$\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0 \tag{1}$$

for all $X, Y, Z \in \mathfrak{g}$. Then we call the pair $(\mathfrak{g}, \omega)$ or a triplet $(\mathfrak{g}, \mathfrak{g}_\omega, \omega)$ an *infinitesimal symplectic homogeneous space* or an *ISH space*, where $\mathfrak{g}_\omega$ is a subalgebra of $\mathfrak{g}$ defined by

$$\mathfrak{g}_\omega := \{X \in \mathfrak{g} \mid \omega(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}. \tag{2}$$

(ii) We say that an ISH space $(\mathfrak{g}, \omega)$ is *trivial* if $\omega \equiv 0$.

Let us give an example of ISH space:

Example 1. Let $\mathfrak{g}$ be a real Lie algebra. Then $(\mathfrak{g}, \omega^{\mathfrak{g}}_W)$ is an ISH space for any $W \in \mathfrak{g}$, where $\omega^{\mathfrak{g}}_W$ is given by

$$\omega^{\mathfrak{g}}_W(X, Y) := B_{\mathfrak{g}}(W, [X, Y]) \qquad \text{for } X, Y \in \mathfrak{g}. \tag{3}$$

From now on we start studying a relation between symplectic homogeneous spaces and ISH spaces.

Lemma 1. *Let $(G/H, \Omega)$ be a symplectic homogeneous space, and let π denote the projection from G onto G/H. Then $(\mathfrak{g}, \omega)$ is an ISH space, where* $\mathfrak{g} := \mathrm{Lie}\, G$ *and ω is given by*

$$\omega(X, Y) := (\pi^*\Omega)(X, Y) \qquad \textit{for } X, Y \in \mathfrak{g}.$$

Furthermore, $\mathrm{Lie}\, H$ *coincides with $\mathfrak{g}_\omega$ (see* (2) *for $\mathfrak{g}_\omega$).*

Proof. It is natural that ω is a skew-symmetric bilinear form on $\mathfrak{g}$. Let us show that ω satisfies (1). Since Ω is G-invariant, ω is left-invariant. Hence one knows that, for any $X, Y \in \mathfrak{g}$,

$$\omega(X, Y) \text{ is a constant function} \tag{4}$$

(ref. Matsushima [Ma2, p. 193]). Besides, ω is closed (i.e., $d\omega = 0$) because Ω is closed. Consequently, it follows from (4) that for any $X, Y, Z \in \mathfrak{g}$,

$$\begin{aligned} 0 &= (d\omega)(X, Y, Z) \\ &= X(\omega(Y, Z)) - Y(\omega(X, Z)) + Z(\omega(X, Y)) \\ &\quad - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) \\ &= -\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X). \end{aligned}$$

This implies that ω satisfies (1) because ω is skew-symmetric. Now, the rest of proof is to confirm that

$$\mathfrak{h} = \mathfrak{g}_\omega,$$

where $\mathfrak{h} := \mathrm{Lie}\, H$. First, let us show that $\mathfrak{h} \subset \mathfrak{g}_\omega$. Take an element $X \in \mathfrak{h}$. Then $d\pi(X) = 0$, and so one has $\omega(X, Y) = \Omega(d\pi(X), d\pi(Y)) = 0$ for every $Y \in \mathfrak{g}$. Therefore $X \in \mathfrak{g}_\omega$, and $\mathfrak{h} \subset \mathfrak{g}_\omega$. Next, let us deduce that the converse inclusion also holds. Note that Ω_o is a symplectic form on the tangent space $T_o(G/H)$ and $(d\pi)_e : T_eG \to T_o(G/H)$ is surjective, where $o := \pi(e)$. Take an element $Z \in \mathfrak{g}_\omega$. Then (2) implies that for any $Y \in \mathfrak{g}$, one obtains $0 = \omega(Z, Y) = \Omega(d\pi(Z), d\pi(Y))$; in particular, $0 = \Omega_o((d\pi)_e(Z_e), (d\pi)_e(Y_e))$. Hence we see that $(d\pi)_e(Z_e) = 0$, and $d\pi(Z) = 0$ because Z is left-invariant. This shows $Z \in \mathfrak{h}$, and $\mathfrak{g}_\omega \subset \mathfrak{h}$. □

The following proposition will play an important role in Section 3:

Proposition 1 (cf. Matsushima [Ma1, p. 54–55]). *Let $(G/H, \Omega)$ be a symplectic homogeneous space with G semisimple, and let $(\mathfrak{g}, \omega)$ denote the ISH space constructed from $(G/H, \Omega)$ in the way of Lemma* 1. *Then, there exists a unique element $W \in \mathfrak{g}$ such that $\omega = \omega_W^{\mathfrak{g}}$ (see* (3) *for $\omega_W^{\mathfrak{g}}$). In this case, H lies between $C_G(W)_0$ and $C_G(W)$.*

Proof. Théorème 1 in Matsushima [Ma1, p. 54] and its proof enable one to conclude this proposition. □

Definition 3. With the same setting as in Proposition 1; we say that the unique element W is the *symplectic element* or the *S-element* of $(G/H, \Omega)$. Remark that the S-element W of $(G/H, \Omega)$ satisfies two conditions (s1) and (s2):

$$\text{(s1)} \quad \omega_W^{\mathfrak{g}} = \pi^*\Omega, \qquad \text{(s2)} \quad C_G(W)_0 \subset H \subset C_G(W),$$

where π denotes the projection from G onto G/H.

3. A structure theorem

In this section we clarify a necessary and sufficient condition for H to be compact, for a symplectic homogeneous space G/H with G non-compact semisimple (see Theorem 2).

3.1. Let $(G/H, \Omega)$ be a symplectic homogeneous space with G semisimple and H compact, and let W be the S-element of $(G/H, \Omega)$. Then W satisfies $C_G(W)_0 \subset H \subset C_G(W)$. We will prove $H = C_G(W)_0 = C_G(W)$ later (see Proposition 2). For this reason we first recall the definition of elliptic element:

Definition 4 (cf. Kobayashi [Ko]). Let $\mathfrak{g}$ be a real semisimple Lie algebra. An element $X \in \mathfrak{g}$ is called *semisimple*, if the endomorphism $\operatorname{ad} X$ of $\mathfrak{g}$ is semisimple. A semisimple element $Z \in \mathfrak{g}$ is said to be *elliptic*, if all eigenvalues of $\operatorname{ad} Z$ are purely imaginary.

The following lemma is known (e.g. Kobayashi-Ono [Ko-On, Lemma (6.1), p. 83, and its proof, p. 85]), but we prove it for the sake of completeness:

Lemma 2. *Let G be a connected semisimple Lie group, and let Z be an elliptic element of $\mathfrak{g} = \operatorname{Lie} G$. Then, the centralizer $C_G(Z)$ is connected.*

Proof. First, let us prepare notations φ, K and P for proof. Since Z is elliptic, there exists a maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ containing Z. Denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $B_{\mathfrak{g}}$. In this case one gets a Cartan decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \qquad \text{(direct sum) with } Z \in \mathfrak{k},$$

and the Cartan involution θ of $\mathfrak{g}$ by setting $\theta(A + B) := A - B$ for $A \in \mathfrak{k}$, $B \in \mathfrak{p}$. Define an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by $\langle X, Y \rangle := -B_{\mathfrak{g}}(\theta(X), Y)$ for $X, Y \in \mathfrak{g}$; and consider $\mathfrak{g}$ as a Euclidean space with respect to $\langle \cdot, \cdot \rangle$ hereafter. Then, one can obtain a diffeomorphism φ from $K \times P$ onto G defined by

$$\varphi : K \times P \to G, \qquad (k, p) \mapsto k \cdot p,$$

where $K := \{g \in G \mid \operatorname{Ad}(g) \in O(\mathfrak{g})\}$, $P := \exp(\mathfrak{p})$ and we denote by $O(\mathfrak{g})$ (resp. exp) the orthogonal group on $\mathfrak{g}$ (resp. the exponential map of G). In addition, one can see that

$$\operatorname{Lie} K \text{ coincides with } \mathfrak{k} \tag{5}$$

and $\exp|_{\mathfrak{p}} : \mathfrak{p} \to P$ is diffeomorphic, where we denote by $\exp|_{\mathfrak{p}}$ the restriction of exp to $\mathfrak{p}$ (cf. Onishchik-Vinberg [On-Vi, p. 256–257, Theorem 2 and its proof][1]). Now, we will prove that $C_G(Z)$ is connected by taking steps (S1), (S2) and (S3):

- (S1) $C_K(Z) \times C_P(Z)$ is homeomorphic to $C_G(Z)$ via φ, where $C_P(Z)$ is a closed subset of P given by $C_P(Z) := \{p \in P \mid \operatorname{Ad}(p)Z = Z\}$. Here we equip $C_K(Z) \times C_P(Z)$ with the induced topology from $K \times P$;
- (S2) $C_P(Z)$ is connected;
- (S3) $C_K(Z)$ is connected.

[1] There are some minor misprints in [On-Vi]. p. 256, ↑ 11, Read $\operatorname{Ad} g \in \mathrm{O}(\mathfrak{g})$ instead of $\operatorname{Ad} g \in \mathcal{O}(\mathfrak{g})$; p. 256, ↑ 6, Read Problem 7 instead of Problem 9; p. 256, ↑ 3, Read $\hat{P} = \exp \operatorname{ad} \mathfrak{p}$ instead of $\hat{P} = \exp \operatorname{ad} \mathfrak{g}$; p. 257, ↓ 1, Read one-to-one and onto instead of one-to-one; p. 257, ↓ 3, Read $\hat{k} \in \mathrm{O}(\mathfrak{g})$ instead of $\hat{k} \in \mathcal{O}(\mathfrak{g})$; p. 257, ↓ 11, Read bijective instead of injective; p. 257, ↑ 11 and 10, Read $\Psi(g_1, g_2) = \mathrm{id}$ instead of $\psi(g_1, g_2) = \mathrm{id}$; p. 257, ↑ 10, Read $g_1, g_2 \in G$ instead of $g_1, g_2 \in K$.

(S1): Since $\varphi(C_K(Z) \times C_P(Z)) \subset C_G(Z)$ is clear, it suffices to verify that for any $x \in C_G(Z)$, there exist $k \in C_K(Z)$ and $p \in C_P(Z)$ satisfying $k \cdot p = \varphi(k, p) = x$. Take any element $x \in C_G(Z)$. Since $Z = \mathrm{Ad}(x)Z$ one has

$$\exp tZ = \exp t\,\mathrm{Ad}(x)Z = \mathrm{A}_x(\exp tZ) = x \cdot \exp tZ \cdot x^{-1} \tag{6}$$

for every $t \in \mathbf{R}$. Henceforth, we denote $\exp tZ$ by z_t. There exist a unique $k \in K$ and a unique $p \in P$ such that $k \cdot p = x$ (because $\varphi : K \times P \to G$ is diffeomorphic). Let us show that $k \in C_K(Z)$ and $p \in C_P(Z)$. From (6) it is obvious that $z_t = x \cdot z_t \cdot x^{-1}$, so that

$$k \cdot p = x = z_t \cdot x \cdot z_t^{-1} = (z_t \cdot k \cdot z_t^{-1}) \cdot (z_t \cdot p \cdot z_t^{-1}).$$

It follows from $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and (5) that $\mathrm{Ad}\, K_0(\mathfrak{p}) \subset \mathfrak{p}$. Since $z_t = \exp tZ \in K_0$, $P = \exp(\mathfrak{p})$ and $\mathrm{Ad}\, K_0(\mathfrak{p}) \subset \mathfrak{p}$ we deduce that $z_t \cdot k \cdot z_t^{-1} \in K$ and $z_t \cdot p \cdot z_t^{-1} \in P$; and thus $k = z_t \cdot k \cdot z_t^{-1}$ and $p = z_t \cdot p \cdot z_t^{-1}$ by the uniqueness. This yields $\exp tZ = z_t = k \cdot z_t \cdot k^{-1} = \exp t\,\mathrm{Ad}(k)Z$ and $\exp tZ = \exp t\,\mathrm{Ad}(p)Z$ for every $t \in \mathbf{R}$. Consequently we obtain $\mathrm{Ad}(k)Z = Z = \mathrm{Ad}(p)Z$; and $k \in C_K(Z)$, $p \in C_P(Z)$.

(S2): Now, let us demonstrate that $C_P(Z)$ is (arcwise) connected. Take any element $y \in C_P(Z)$ and express it as $y = \exp Y$ $(Y \in \mathfrak{p})$. Then one deduces $z_t = y \cdot z_t \cdot y^{-1}$ by arguments similar to those in (S1). Therefore we have

$$\exp Y = y = \mathrm{A}_{z_t}(y) = \exp \mathrm{Ad}(z_t)Y.$$

Since $\exp : \mathfrak{p} \to P$ is injective and $\mathrm{Ad}(z_t)Y \in \mathfrak{p}$, we perceive that $Y = \mathrm{Ad}(z_t)Y = \mathrm{Ad}(\exp tZ)Y = \exp t\,\mathrm{ad}\, Z(Y)$ for every $t \in \mathbf{R}$; and hence $[Z, Y] = 0$. By $[Z, Y] = 0$ we conclude that for every $t \in \mathbf{R}$,

$$\mathrm{Ad}(\exp tY)Z = \exp t\,\mathrm{ad}\, Y(Z) = \sum_{n \geq 0} \frac{t^n}{n!} (\mathrm{ad}\, Y)^n Z = Z.$$

This assures that the whole 1-parameter subgroup $\{\exp tY \mid t \in \mathbf{R}\}$ lies in $C_P(Z)$, where $\exp tY \in P$ follows from $P = \exp(\mathfrak{p})$ and $Y \in \mathfrak{p}$. So, one can join $y = \exp tY|_{t=1}$ to the unite element $e = \exp tY|_{t=0} \in C_P(Z)$ by an arc in $C_P(Z)$.

(S3): Note that K is connected because $\varphi : K \times P \to G$ is diffeomorphic and both $P = \exp(\mathfrak{p})$ and G are connected. Since $\mathfrak{k}$ is compact one can decompose it as

$$\mathfrak{k} = \mathfrak{k}_{ss} \oplus \mathfrak{z}(\mathfrak{k}) \qquad \text{(direct sum)},$$

where $\mathfrak{k}_{ss}$ (resp. $\mathfrak{z}(\mathfrak{k})$) denotes the semisimple part (resp. center) of $\mathfrak{k}$. This enables us to uniquely express Z as follows:

$$Z = Z_{ss} + Z_{\mathfrak{z}}$$

$(Z_{ss} \in \mathfrak{k}_{ss},\ Z_z \in \mathfrak{z}(\mathfrak{k}))$. Denote by K_{ss} and $Z(K)_0$ the connected Lie subgroups of K corresponding to $\mathfrak{k}_{ss}$ and $\mathfrak{z}(\mathfrak{k})$, respectively. Now, let us conclude that $C_K(Z)$ is connected. Since K is connected, one sees that $K = K_{ss} \cdot Z(K)_0$; so that

$$C_K(Z) = C_{K_{ss}}(Z_{ss}) \cdot Z(K)_0 \tag{7}$$

because $\mathrm{Ad}(k)Z_z = Z_z$ for any $k \in K$, and $\mathrm{Ad}(c)X = X$ for any $c \in Z(K)_0$ and $X \in \mathfrak{k}$. Since K_{ss} is connected and $\mathfrak{k}_{ss}$ is compact semisimple, K_{ss} is compact. This implies that $C_{K_{ss}}(Z_{ss})$ is connected, and it follows from (7) that $C_K(Z)$ is connected. □

Now, let us prove

PROPOSITION 2. *Let $(G/H, \Omega)$ be a symplectic homogeneous space with G semisimple and H compact, and let W be the S-element of $(G/H, \Omega)$. Then, the following three items hold:*
(i) *W is an elliptic element of $\mathfrak{g}$ and $H = C_G(W)_0 = C_G(W)$;*
(ii) *G/H is simply connected;*
(iii) *H contains the center $Z(G)$, and $Z(G)$ is finite.*

PROOF. (i): Proposition 1 assures that W satisfies

$$C_G(W)_0 \subset H \subset C_G(W). \tag{8}$$

Since $\mathrm{Lie}\, C_G(W) = \mathrm{Lie}\, H$ and H is compact, there exists an ad W-invariant inner product on $\mathfrak{g}$. Hence, W is an elliptic element of $\mathfrak{g}$. Therefore Lemma 2 implies that $C_G(W)$ is connected; and hence

$$H = C_G(W)_0 = C_G(W)$$

follows from (8).

(ii): Let $(\tilde{G}, \psi)$ denote the universal covering group of G. Note that $d\psi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is a Lie algebra isomorphism. Let us identify $\tilde{\mathfrak{g}}$ with $\mathfrak{g}$ via $d\psi$. In this case W is an elliptic element of $\tilde{\mathfrak{g}}$ and $C_{\tilde{G}}(W)$ is connected by Lemma 2. Therefore $\tilde{G}/C_{\tilde{G}}(W)$ is simply connected, and hence G/H is also simply connected because $\tilde{G}/C_{\tilde{G}}(W)$ is homeomorphic to $G/C_G(W) = G/H$.

(iii): On the one hand, $Z(G)$ is discrete because G is semisimple. On the other hand, $Z(G) \subset H$ comes from (i) $H = C_G(W)$. Consequently, we deduce (iii) because H is compact. □

LEMMA 3. *For any symplectic homogeneous space G/H with G semisimple and H compact, there exists a maximal compact subgroup of G containing H.*

PROOF. If G is compact, then G in itself can take on the role of the above subgroup. So, let us consider the case where G is non-compact hereafter.

Note that the center $Z(G)$ is finite from Proposition 2. Let $G = K \cdot P$ denote the same decomposition as in the proof of Lemma 2. In this case K is compact because $Z(G)$ is finite (cf. Onishchik-Vinberg [On-Vi, p. 258, Corollary 6]); and so (G, K) is a Riemannian symmetric pair of non-compact type. Hence there exists a maximal compact subgroup K' of G satisfying $H \subset K'$ by Theorem 2.1 in Helgason [He, p. 256]. $\square$

Proposition 2 allows us to deduce

Corollary 1. *Let G/H be a symplectic homogeneous space with G semisimple and H compact, and let K be a maximal compact subgroup of G such that $H \subset K$ (ref. Lemma 3). Then* $\mathrm{rank}(\mathfrak{g})$ *coincides with* $\mathrm{rank}(\mathfrak{k})$*, i.e., every Cartan involution of $\mathfrak{g}$ is inner. This implies that there are no symplectic homogeneous spaces G/H with H compact, in the case where G is one of the following:*

$SL(n, \mathbf{R})$ *with* $n \geq 3$, $SU^*(2n)$ *with* $n \geq 2$,
$SO_0(2n-2k-1, 2k+1)$ *with* $n \geq 4$ *and* $0 \leq k \leq n-1$,
$E_{6(6)}$, $E_{6(-26)}$,
$SL(n, \mathbf{C})$ *with* $n \geq 2$, $SO(n, \mathbf{C})$ *with* $n \geq 3$, $Sp(n, \mathbf{C})$ *with* $n \geq 1$,
$G_2^{\mathbf{C}}$, $F_4^{\mathbf{C}}$,
$E_6^{\mathbf{C}}$, $E_7^{\mathbf{C}}$, $E_8^{\mathbf{C}}$.

Proof. By Proposition 2 there exists an elliptic element W of $\mathfrak{g}$ such that $C_G(W) = H$. Since W is semisimple there exists a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing W. It is obvious that $\mathfrak{t} \subset \mathfrak{c}_{\mathfrak{g}}(W)$; and therefore $C_G(W) = H \subset K$ yields $\mathfrak{t} \subset \mathfrak{k}$. Hence $\mathrm{rank}(\mathfrak{g}) = \mathrm{rank}(\mathfrak{k})$. $\square$

3.2. This subsection is devoted to proving

Theorem 2. *Let $(G/H, \Omega)$ be a symplectic homogeneous space with G non-compact semisimple, and let W be the S-element of $(G/H, \Omega)$. Then the following* (I) *and* (II) *are equivalent:*

(I) *H is compact;*

(II) *the center $Z(G)$ is finite, and there exists a maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ satisfying two conditions* (c1) *$W \in \mathfrak{k}$ and* (c2) $\mathrm{ad}\, W|_{\mathfrak{p}}$ *is a linear isomorphism of $\mathfrak{p}$. Here $\mathfrak{p}$ denotes the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $B_{\mathfrak{g}}$.*

Remark 2. *The condition* (c2) *in Theorem* 2 *does not make sense when G is compact, because G is compact if and only if $\mathfrak{p} = \{0\}$.*

Proof (Proof of Theorem 2). (I) $\to$ (II): Suppose that H is compact. By Proposition 2 we see that $Z(G)$ is finite and $C_G(W) = H$. Moreover, Lemma

3 assures that there exists a maximal compact subgroup K' of G satisfying $C_G(W) = H \subset K'$. Therefore it suffices to prove that

$$\mathrm{ad}\, W|_{\mathfrak{p}'} : \mathfrak{p}' \to \mathfrak{p}' \text{ is bijective,} \tag{9}$$

where $\mathfrak{p}'$ denotes the orthogonal complement of $\mathfrak{k}' = \mathrm{Lie}\, K'$ in $\mathfrak{g}$ with respect to $B_{\mathfrak{g}}$. Remark that $\mathfrak{g} = \mathfrak{k}' \oplus \mathfrak{p}'$ (direct sum). Take an element $Y' \in \mathfrak{p}'$ and suppose that $\mathrm{ad}\, W(Y') = [W, Y'] = 0$. Then $Y' \in \mathfrak{c}_{\mathfrak{g}}(W) \subset \mathfrak{k}'$, and $Y' \in \mathfrak{k}' \cap \mathfrak{p}' = \{0\}$. So we deduce that $\mathrm{ad}\, W|_{\mathfrak{p}'}$ is injective, and hence it is bijective.

(II) $\to$ (I): Suppose that $Z(G)$ is finite, and that a maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ satisfies the conditions (c1) and (c2). Recall that W satisfies $C_G(W)_0 \subset H \subset C_G(W)$ because G is semisimple (cf. Proposition 1). From (c1) one confirms that W is an elliptic element of $\mathfrak{g}$. Consequently Lemma 2 implies that $C_G(W) = C_G(W)_0$, and so

$$H = C_G(W)_0 = C_G(W). \tag{10}$$

Since $Z(G)$ is finite, there exists a maximal compact subgroup K of G such that Lie K coincides with $\mathfrak{k}$ (see Onishchik-Vinberg [On-Vi, Theorem 2 (p. 256) and Corollary 6 (p. 258)] again). In order to prove that H is compact, we will only verify

$$\mathfrak{c}_{\mathfrak{g}}(W) = \mathfrak{c}_{\mathfrak{k}}(W) \tag{11}$$

because it follows from (10) and (11) that $H = C_G(W)_0 = C_K(W)_0$ is compact. Note that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (direct sum). Take an element $C \in \mathfrak{c}_{\mathfrak{g}}(W)$ and express it as $C = A + B$ $(A \in \mathfrak{k}, B \in \mathfrak{p})$. Then $0 = [C, W] = [A, W] + [B, W]$. It follows from (c1) that $[A, W] \in \mathfrak{k}$ and $[B, W] \in \mathfrak{p}$; so that $[A, W] = 0 = [B, W]$. This, together with (c2), implies that $A \in \mathfrak{c}_{\mathfrak{k}}(W)$ and $B = 0$. Therefore $C = A \in \mathfrak{c}_{\mathfrak{k}}(W)$, and $\mathfrak{c}_{\mathfrak{g}}(W) \subset \mathfrak{c}_{\mathfrak{k}}(W)$. The converse inclusion is clear. □

3.3. Direct product. Our aim in this subsection is to assert that a symplectic homogeneous space $(G/H, \Omega)$ with G non-compact semisimple and H compact is the *direct product* of simply connected, symplectic homogeneous spaces $(G_k/H_k, \Omega_k)$ with G_k simple and H_k compact (see Theorem 3). Here, the word "direct product" means

Definition 5. For a finite number of symplectic homogeneous spaces $(G_k/H_k, \Omega_k)$, $1 \leq k \leq b$, the *direct product* $(G/H, \Omega) = (G_1/H_1, \Omega_1) \times \cdots \times (G_b/H_b, \Omega_b)$, which is also a symplectic homogeneous space, is defined by $G := G_1 \times \cdots \times G_b$, $H := H_1 \times \cdots \times H_b$ and $\Omega := \Omega_1 \times \cdots \times \Omega_b$.

To accomplish the aim, we first prove Lemmas 4 and 5.

Lemma 4. *Let G be a connected semisimple Lie group, let X be a non-zero element of $\mathfrak{g}$, and let H be a closed subgroup of G such that $C_G(X)_0 \subset H \subset$*

$C_G(X)$. *Then, there exists a unique* G*-invariant symplectic form* Ω_X *on* G/H *such that* $\omega_X^{\mathfrak{g}} = \pi^*\Omega_X$, *where* π *denotes the projection from* G *onto* G/H *(see* (3) *for* $\omega_X^{\mathfrak{g}}$*). In this case,* X *becomes the* S*-element of* $(G/H, \Omega_X)$.

Proof. (Uniqueness): Suppose that there exists a G-invariant symplectic form Ω' on G/H satisfying $\omega_X^{\mathfrak{g}} = \pi^*\Omega'$. Since $\pi^*\Omega_X = \omega_X^{\mathfrak{g}} = \pi^*\Omega'$ one sees that $(\Omega_X)_o = (\Omega')_o$ at the origin $o \in G/H$. Therefore $\Omega_X = \Omega'$ follows from both Ω_X and Ω' being G-invariant.

(Existence): Let α_X denote a 1-form on $\mathfrak{g}$ given by

$$\alpha_X(Y) := B_{\mathfrak{g}}(X, Y) \qquad \text{for } Y \in \mathfrak{g}.$$

It is known that the coadjoint orbit $G/C_G(X)$ of G through α_X admits a G-invariant symplectic form $\bar{\Omega}_X$ which satisfies

$$\alpha_X([Y, Z]) = (\bar{\pi}^*\bar{\Omega}_X)(Y, Z) \qquad \text{for any } Y, Z \in \mathfrak{g}$$

(e.g. Guillemin-Sternberg [Gu-St, p. 178]), where $\bar{\pi}$ denotes the projection from G onto $G/C_G(X)$. Denote by Pr the projection from G/H onto $G/C_G(X)$, $gH \mapsto gC_G(X)$. Then $\Omega_X := \mathrm{Pr}^*\,\bar{\Omega}_X$ is a G-invariant symplectic form on G/H, where we should remark that $(d\,\mathrm{Pr})_o : T_o(G/H) \to T_o(G/C_G(X))$ is linearly isomorphic by virtue of $C_G(X)_0 \subset H \subset C_G(X)$. Furthermore, Ω_X satisfies $\omega_X^{\mathfrak{g}} = \bar{\pi}^*\bar{\Omega}_X = \pi^*\Omega_X$ because $\bar{\pi}^*\bar{\Omega}_X = \pi^*\Omega_X$ follows from $\mathrm{Pr} \circ \pi = \bar{\pi}$ and $\omega_X^{\mathfrak{g}}(Y, Z) = B_{\mathfrak{g}}(X, [Y, Z]) = \alpha_X([Y, Z]) = (\bar{\pi}^*\bar{\Omega}_X)(Y, Z)$ for all $Y, Z \in \mathfrak{g}$. □

Lemma 5. *Let* L *be a connected simple Lie group with the trivial center, let* $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}$ *denote a Cartan decomposition, and let* T *be a non-zero element of* $\mathfrak{k}$ $(\subset \mathfrak{l})$. *Then, the adjoint orbit* $L/C_L(T)$ *of* L *through* T *satisfies conditions* (i) *and* (ii)*:*

(i) $L/C_L(T)$ *is simply connected;*

(ii) L *acts on* $L/C_L(T)$ *effectively.*

In addition; if L *is non-compact and* T *satisfies the condition* (c2) *in Theorem* 2*:*

$$\text{(c2)} \quad \mathrm{ad}\, T|_{\mathfrak{p}} \ \textit{is a linear isomorphism of } \mathfrak{p},$$

then $L/C_L(T)$ *satisfies a further condition* (iii)*:* $C_L(T)$ *is compact.*

Proof. (i): Lemma 2 and $T \in \mathfrak{k}$ allow us to conclude that $L/C_L(T)$ is simply connected (ref. the proof of Proposition 2-(ii)).

(ii): (ii) comes from L being simple and the center $Z(L) = \{e\}$.

(iii): We deduce (iii) by Theorem 2 and $Z(L) = \{e\}$. □

Now, we are in a position to demonstrate

Theorem 3. *Let* $(G/H, \Omega)$ *be a symplectic homogeneous space with* G *non-compact semisimple and* H *compact, where* G *acts on* G/H *effectively. Then,*

$(G/H, \Omega)$ is the direct product of simply connected, symplectic homogeneous spaces $(G_k/H_k, \Omega_k)$ with G_k simple and H_k compact. Furthermore, each G_k acts on G_k/H_k effectively.

Remark 3. *Though the above G is non-compact, the product of $(G_k/H_k, \Omega_k)$ may include compact factors $(G_j/H_j, \Omega_j)$.*

Proof (Proof of Theorem 3). First, we are going to consider relation a between an infinitesimal decomposition of $(G/H, \Omega)$ and a Cartan decomposition of $\mathfrak{g} = \mathrm{Lie}\, G$. Let W be the S-element of $(G/H, \Omega)$. By Proposition 2 we see that W satisfies

$$\text{(s1)} \quad \omega_W^{\mathfrak{g}} = \pi^* \Omega, \qquad \text{(s2)} \quad H = C_G(W), \tag{12}$$

where π denotes the projection from G onto G/H. Moreover, by Theorem 2 there exists a maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ such that

$$\text{(c1)} \quad W \in \mathfrak{k}, \qquad \text{(c2)} \quad \mathrm{ad}\, W|_{\mathfrak{p}} \text{ is a linear isomorphism of } \mathfrak{p}, \tag{13}$$

where $\mathfrak{p}$ denotes the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $B_{\mathfrak{g}}$. Here one has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let us denote by θ the Cartan involution of $\mathfrak{g}$ with respect to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Express $\mathfrak{g}$ as $\mathfrak{g} = \bigoplus_{i=1}^a \mathfrak{g}_i \oplus \bigoplus_{j=a+1}^b \mathfrak{g}_j$ (direct sum), where all $\mathfrak{g}_i$ (resp. $\mathfrak{g}_j$) are non-compact (resp. compact) simple ideals of $\mathfrak{g}$. Then Corollary 1 means that $\theta(\mathfrak{g}_k) \subset \mathfrak{g}_k$ for any $1 \le k \le b$. Therefore we deduce that

$$\begin{cases} \mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i \text{ is a Cartan decomposition of } \mathfrak{g}_i & \text{for } 1 \le i \le a; \\ \mathfrak{g}_j = \mathfrak{k}_j & \text{for } a+1 \le j \le b, \end{cases} \tag{14}$$

where $\mathfrak{k}_k := \mathfrak{k} \cap \mathfrak{g}_k$ and $\mathfrak{p}_k := \mathfrak{p} \cap \mathfrak{g}_k$ for $1 \le k \le b$. By considering the decomposition $\mathfrak{g} = \bigoplus_{i=1}^a \mathfrak{g}_i \oplus \bigoplus_{j=a+1}^b \mathfrak{g}_j$, we express W as $W = \sum_{i=1}^a W_i + \sum_{j=a+1}^b W_j$. In order to complete the proof of Theorem 3 we need the following:

Lemma 6. *With the above setting, the following five items hold for the S-element $W = \sum_{i=1}^a W_i + \sum_{j=a+1}^b W_j$:*

(i) $W_k \neq 0$ *for any* $1 \le k \le b$*;*
(ii) $W_k \in \mathfrak{k}_k$ *for any* $1 \le k \le b$*;*
(iii) $\mathrm{ad}\, W_i|_{\mathfrak{p}_i}$ *is a linear isomorphism of* $\mathfrak{p}_i$ *for every* $1 \le i \le a$*;*
(iv) $\omega_W^{\mathfrak{g}} = \omega_{W_k}^{\mathfrak{g}_k}$ *on* $\mathfrak{g}_k \times \mathfrak{g}_k$ *for any* $1 \le k \le b$*;*
(v) $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(W) = \bigoplus_{k=1}^b \mathfrak{c}_{\mathfrak{g}_k}(W_k)$ *(direct sum).*

Proof. Since all $\mathfrak{g}_k$ are simple ideals of $\mathfrak{g} = \bigoplus_{t=1}^b \mathfrak{g}_t$, one sees that

$$[\mathfrak{g}_n, \mathfrak{g}_m] = \{0\} \qquad \text{if } n \neq m. \tag{15}$$

(i): Suppose that there exists $1 \le c \le b$ such that $W_c = 0$. By (15) we deduce $[W, \mathfrak{g}_c] = [\sum_{k=1}^{b} W_k, \mathfrak{g}_c] = [W_c, \mathfrak{g}_c] = \{0\}$, and therefore it follows from (12)-(s2) that $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(W)$ contains the non-trivial ideal $\mathfrak{g}_c$ of $\mathfrak{g}$. That is a contradiction because G is effective on G/H. For this reason (i) holds.

(ii): $\theta(W_k) = W_k$, for each k, is immediate from (13)-(c1), $\sum_{t=1}^{b} \theta(W_t) = \theta(W) = W = \sum_{t=1}^{b} W_t$, $\theta(\mathfrak{g}_t) \subset \mathfrak{g}_t$ and $\mathfrak{g} = \bigoplus_{t=1}^{b} \mathfrak{g}_t$ (direct sum). Thus (ii) follows.

(iii): Take an element $Y_i \in \mathfrak{p}_i$ and suppose that $[W_i, Y_i] = 0$. From (15) one obtains $[W, Y_i] = [W_i, Y_i] = 0$, and $Y_i = 0$ because of (13)-(c2). This implies that $\operatorname{ad} W_i|_{\mathfrak{p}_i}$ is injective; and so (iii) holds.

(iv): Since $\mathfrak{g}_k$ is an ideal of $\mathfrak{g}$, the Killing form $B_{\mathfrak{g}_k}$ of $\mathfrak{g}_k$ coincides with the restriction of $B_{\mathfrak{g}}$ to $\mathfrak{g}_k \times \mathfrak{g}_k$. Hence (iv) follows from (3) and (15).

(v): $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(W)$ is immediate from (12)-(s2). $\mathfrak{c}_{\mathfrak{g}}(W) = \bigoplus_{k=1}^{b} \mathfrak{c}_{\mathfrak{g}_k}(W_k)$ follows from $W = \sum_{k=1}^{b} W_k$, (15) and $\mathfrak{g} = \bigoplus_{k=1}^{b} \mathfrak{g}_k$ (direct sum). □

Let us continue proving Theorem 3. From now on, we are going to consider a global decomposition of $(G/H, \Omega)$. Note that the center $Z(G)$ is trivial because (12)-(s2) and G acts on G/H effectively. Accordingly one may assume that G is the adjoint group of $\mathfrak{g}$. First let us prove that

$$G = G_1 \times \cdots \times G_b, \tag{16}$$

where G_k denotes the adjoint group of $\mathfrak{g}_k$ for $1 \le k \le b$. Let $\bar{G}$ be a simply connected Lie group with $\operatorname{Lie} \bar{G} = \mathfrak{g}$, and let $\bar{G} = \bar{G}_1 \times \cdots \times \bar{G}_b$ denote the decomposition of $\bar{G}$ corresponding to $\mathfrak{g} = \bigoplus_{k=1}^{b} \mathfrak{g}_k$, where $\mathfrak{g}_k = \operatorname{Lie} \bar{G}_k$ for $1 \le k \le b$. Then it follows that $Z(\bar{G}) = Z(\bar{G}_1 \times \cdots \times \bar{G}_b)$ and $Z(\bar{G}_1 \times \cdots \times \bar{G}_b) = Z(\bar{G}_1) \times \cdots \times Z(\bar{G}_b)$. Therefore one has $\bar{G}/Z(\bar{G}) = \bar{G}_1/Z(\bar{G}_1) \times \cdots \times \bar{G}_b/Z(\bar{G}_b)$. This implies (16) because $G = \bar{G}/Z(\bar{G})$ and $G_k = \bar{G}_k/Z(\bar{G}_k)$ for all $1 \le k \le b$. Now, let us assume that $\mathfrak{g}_k = \operatorname{Lie} G_k$ for $1 \le k \le b$. Lemma 6-(v) enables us to have $C_G(W) = C_{G_1}(W_1) \times \cdots \times C_{G_b}(W_b)$ because it follows from Lemmas 2 and 6-(ii) that all $C_G(W)$ and $C_{G_k}(W_k)$ are connected. Hence (12)-(s2) yields

$$H = C_{G_1}(W_1) \times \cdots \times C_{G_b}(W_b). \tag{17}$$

For each $1 \le k \le b$, Lemma 4 allows us to get a unique G_k-invariant symplectic form Ω_k on $G_k/C_{G_k}(W_k)$ satisfying $\omega_{W_k}^{\mathfrak{g}_k} = (\pi_k)^* \Omega_k$, where π_k denotes the projection from G_k onto $G_k/C_{G_k}(W_k)$. By (12) and Lemma 6-(iv) we conclude

$$\Omega = \Omega_1 \times \cdots \times \Omega_b.$$

Therefore it is immediate from (16) and (17) that $(G/H, \Omega)$ is the direct product of $(G_k/C_{G_k}(W_k), \Omega_k)$, $k = 1, \ldots, b$. Consequently, the rest of proof is to demonstrate that (A) $G_k/C_{G_k}(W_k)$ is simply connected, (B) G_k is simple, (C) $C_{G_k}(W_k)$ is compact and (D) G_k acts on $G_k/C_{G_k}(W_k)$ effectively. However, (B) is clear and both (A) and (D) come from Lemma 5 and Lemma 6-(i), (ii). So, it suffices to show that (C) $C_{G_k}(W_k)$ is compact. On the one hand, if $1 \leq k \leq a$, then G_k is non-compact in view of (14), and so Lemmas 5 and 6-(iii) allow us to conclude that $C_{G_k}(W_k)$ is compact. One the other hand, if $a+1 \leq k \leq b$, then G_k is compact in view of (14), and hence $C_{G_k}(W_k)$ is compact. □

4. The coarse orbit types of non-Kählerian, symplectic homogeneous spaces G/H with G non-compact simple and H compact

The main purpose of this section is to determine the *coarse orbit type* of each non-Kählerian, symplectic homogeneous space G/H with G non-compact simple and H compact (cf. Subsections 4.4 through 4.14). Here, the word "coarse orbit type" means

Definition 6. Let G/H and G/H' be two homogeneous spaces of a connected Lie group G. Then we say that they are of the *same coarse orbit type*, if H is isomorphic to H' as a Lie group.

Remark 4. (i) *G/H and G/H' are of the same coarse orbit type in the case where* Lie H *is Lie algebra isomorphic to* Lie H' *and both H and H' are connected.* (ii) *The sense of coarse orbit type is different from that of orbit type. Indeed, for compact Kählerian homogeneous space G_2/H there are two classes $G_2/(A_1 \times T)$, G_2/T^2 in the sense of coarse orbit type, but three classes in the sense of orbit type (ref. Bordemann-Forger-Römer* [Bd-Fo-Rö, p. 643]*).*

4.1. Reduction. Borel's result [Br] (cf. Section 1) and Corollary 1 enable us to accomplish the main purpose by only considering the case where G is one of the following Lie groups:

<table>
<tr><th colspan="4">List I</th></tr>
<tr><td colspan="4">BI: $SO_0(2k, 2l-2k+1)$ with $l \geq 2$ and $2 \leq k \leq l$</td></tr>
<tr><td colspan="4">CII: $Sp(k, l-k)$ with $l \geq 3$ and $1 \leq k \leq l-1$</td></tr>
<tr><td colspan="4">DI: $SO_0(2k, 2l-2k)$ with $l \geq 4$ and $2 \leq k \leq l-2$</td></tr>
<tr><td>EII: $E_{6(2)}$</td><td>EV: $E_{7(7)}$</td><td>EVI: $E_{7(-5)}$</td><td>EVIII: $E_{8(8)}$</td></tr>
<tr><td>EIX: $E_{8(-24)}$</td><td>FI: $F_{4(4)}$</td><td>FII: $F_{4(-20)}$</td><td>G: $G_{2(2)}$</td></tr>
</table>

It has been shown that for a non-Kählerian, symplectic homogeneous space G/H with G non-compact simple and H compact, the isotropy subgroup H is connected (cf. Proposition 2). Consequently one can achieve the main purpose by arguments in the Lie algebra level. For this reason we will review root systems and Cartan decompositions of non-compact, absolutely simple Lie algebras.

4.2. Root system and Cartan decomposition. Let $\mathfrak{g}_{\mathbf{C}}$ be a complex simple Lie algebra, let $\mathfrak{t}_{\mathbf{C}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$, and let $\triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ denote the set of all non-zero roots of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{t}_{\mathbf{C}}$. Then there exists a basis (so-called, *Weyl basis*) $\{X_\alpha \mid \alpha \in \triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})\}$ of $\mathfrak{g}_{\mathbf{C}}$ such that

$$
\begin{aligned}
&[X_\alpha, X_{-\alpha}] = H_\alpha, \qquad [H, X_\alpha] = \alpha(H) \cdot X_\alpha \qquad \text{for } H \in \mathfrak{t}_{\mathbf{C}};\\
&[X_\alpha, X_\beta] = 0 \qquad \text{if } \alpha + \beta \neq 0 \text{ and } \alpha + \beta \notin \triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}});\\
&[X_\alpha, X_\beta] = N_{\alpha,\beta} \cdot X_{\alpha+\beta} \qquad \text{if } \alpha + \beta \in \triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}}),
\end{aligned}
$$

where the real constants $N_{\alpha,\beta}$ satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ (cf. Helgason [He, Theorem 5.5, p. 176]). Here H_α is an element of $\mathfrak{t}_{\mathbf{C}}$ given by $B_{\mathfrak{g}_{\mathbf{C}}}(H, H_\alpha) = \alpha(H)$ for $H \in \mathfrak{t}_{\mathbf{C}}$. The Weyl basis gives rise to a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}_{\mathbf{C}}$ as follows:

$$
\mathfrak{g}_u := i\mathfrak{t}_{\mathbf{R}} \oplus \bigoplus_{\alpha \in \triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})} \operatorname{span}_{\mathbf{R}}\{X_\alpha - X_{-\alpha}\} \oplus \operatorname{span}_{\mathbf{R}}\{i(X_\alpha + X_{-\alpha})\} \tag{18}
$$

(see the proof of Theorem 6.3 in Helgason [He, p. 181]), where $\mathfrak{t}_{\mathbf{R}}$ denotes a real vector subspace of $\mathfrak{t}_{\mathbf{C}}$ defined by $\mathfrak{t}_{\mathbf{R}} := \operatorname{span}_{\mathbf{R}}\{H_\alpha \mid \alpha \in \triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})\}$ $(= \{H \in \mathfrak{t}_{\mathbf{C}} \mid \alpha(H) \in \mathbf{R} \text{ for all } \alpha \in \triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})\})$. Now, let $\Pi_{\triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$ denote the set of all simple roots in $\triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$, and let θ be an involution of $\mathfrak{g}_{\mathbf{C}}$ satisfying four conditions

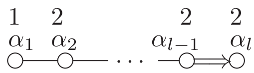
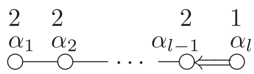
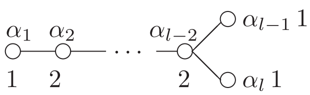
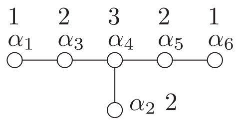
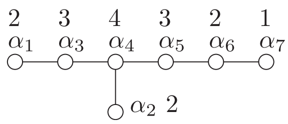
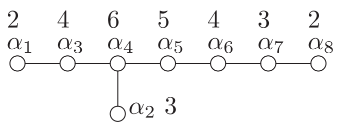
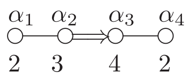
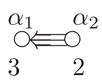
$$
\begin{aligned}
&(\mathrm{v1}) \quad \theta(\mathfrak{g}_u) \subset \mathfrak{g}_u, \qquad (\mathrm{v2}) \quad \theta \neq \mathrm{id} \text{ on } \mathfrak{g}_u, \qquad (\mathrm{v3}) \quad \theta(\mathfrak{t}_{\mathbf{C}}) \subset \mathfrak{t}_{\mathbf{C}},\\
&(\mathrm{v4}) \quad {}^t\theta(\Pi_{\triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}) = \Pi_{\triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}.
\end{aligned}
$$

Denote by $\mathfrak{k}$ and $\mathfrak{p}^*$ the $(+1)$-eigenspace and (-1)-eigenspace of θ in $\mathfrak{g}_u$, respectively. Then one gets a non-compact real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbf{C}}$ by setting

$$
\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}, \qquad \mathfrak{p} := i\mathfrak{p}^*.
$$

Remark 5. (i) *θ becomes a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is its Cartan decomposition.* (ii) *$\mathfrak{k} \cap i\mathfrak{t}_{\mathbf{R}}$ is a maximal abelian subalgebra of $\mathfrak{k}$, because it follows from* (v4) *that θ leaves fixed a regular element of $\mathfrak{g}_{\mathbf{C}}$ contained in $\mathfrak{t}_{\mathbf{C}}$ (see Murakami* [Mu2, Proposition 1, p. 106]*).* (iii) *Every non-compact, absolutely simple Lie algebra can be, up to isomorphism, given by the above fashion* *(cf. Murakami* [Mu3]*). Henceforth, we assume that each non-compact, absolutely simple Lie algebra* $\mathfrak{g}$ *is given by the above fashion, and we identify* $\mathrm{Aut}(\mathfrak{g})$ *and* $\mathrm{Aut}(\mathfrak{g}_u)$ *with* $\{\phi \in \mathrm{Aut}(\mathfrak{g}_{\mathbf{C}}) \mid \phi(\mathfrak{g}) \subset \mathfrak{g}\}$ *and* $\{\psi \in \mathrm{Aut}(\mathfrak{g}_{\mathbf{C}}) \mid \psi(\mathfrak{g}_u) \subset \mathfrak{g}_u\}$*, respectively. Here we denote by* $\mathrm{Aut}(\mathfrak{u})$ *the group of automorphisms of a Lie algebra* $\mathfrak{u}$.

Define an element $Z_a \in \mathfrak{t}_{\mathbf{R}}$ by $\alpha_b(Z_a) := \delta_{a,b}$ for $\{\alpha_b\}_{b=1}^{l} = \Pi_{\triangle(\mathfrak{g}_{\mathbf{C}},\mathfrak{t}_{\mathbf{C}})}$ $(a = 1, \dots, l)$. Then Murakami [Mu3, p. 297] provides us with the following inner involutions θ of $\mathfrak{g}_{\mathbf{C}}$ which satisfy the above conditions (v1), (v2), (v3) and (v4) (see Borel-de Siebenthal [Br-dS] also):

List II: the inner and non-Hermitian types (ref. Murakami [Mu3])				
$\mathfrak{g}_u$	Dynkin diagram	θ	$\mathfrak{k}$	$\mathfrak{g}$
B_l $l \geq 2$		$\exp \pi\, \mathrm{ad}(iZ_k)$ $2 \leq k \leq l$	$D_k \times B_{l-k}$	$\mathfrak{so}(2k, 2l-2k+1)$ BI
C_l $l \geq 3$		$\exp \pi\, \mathrm{ad}(iZ_k)$ $1 \leq k \leq l-1$	$C_k \times C_{l-k}$	$\mathfrak{sp}(k, l-k)$ CII
D_l $l \geq 4$		$\exp \pi\, \mathrm{ad}(iZ_k)$ $2 \leq k \leq l-2$	$D_k \times D_{l-k}$	$\mathfrak{so}(2k, 2l-2k)$ DI
E_6		$\exp \pi\, \mathrm{ad}(iZ_3)$	$A_5 \times A_1$	EII: $\mathfrak{e}_{6(2)}$
E_7		$\exp \pi\, \mathrm{ad}(iZ_2)$	A_7	EV: $\mathfrak{e}_{7(7)}$
		$\exp \pi\, \mathrm{ad}(iZ_1)$	$A_1 \times D_6$	EVI: $\mathfrak{e}_{7(-5)}$
E_8		$\exp \pi\, \mathrm{ad}(iZ_1)$	D_8	EVIII: $\mathfrak{e}_{8(8)}$
		$\exp \pi\, \mathrm{ad}(iZ_8)$	$A_1 \times E_7$	EIX: $\mathfrak{e}_{8(-24)}$
F_4		$\exp \pi\, \mathrm{ad}(iZ_1)$	$A_1 \times C_3$	FI: $\mathfrak{f}_{4(4)}$
		$\exp \pi\, \mathrm{ad}(iZ_4)$	B_4	FII: $\mathfrak{f}_{4(-20)}$
G_2		$\exp \pi\, \mathrm{ad}(iZ_2)$	$A_1 \times A_1$	G: $\mathfrak{g}_{2(2)}$

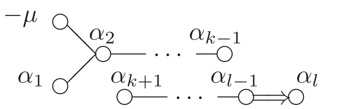
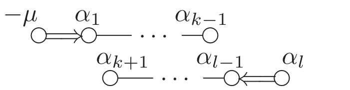
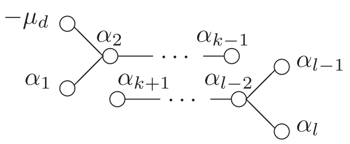
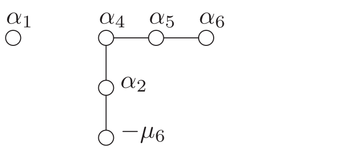
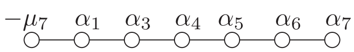
Remark 6. *There are differences with respect to numbering of simple roots in the Dynkin diagrams of type E_6, E_7, E_8 and G_2, between Murakami* [Mu3] *and Bourbaki* [Bu].[2] *Throughout this paper, we apply the numbering in Bourbaki* [Bu] *to our arguments.*

For every involution θ listed above, one has

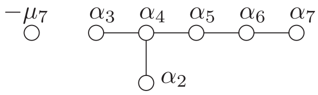
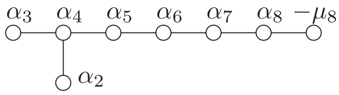
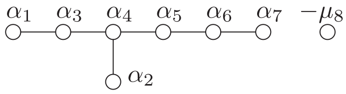
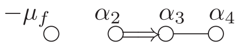
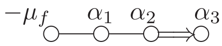
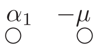
$$\mathfrak{k} \cap i\mathfrak{t}_{\mathbf{R}} = i\mathfrak{t}_{\mathbf{R}}. \tag{19}$$

Hence, $i\mathfrak{t}_{\mathbf{R}}$ is a maximal abelian subalgebra of $\mathfrak{k}$ in List II. Taking List II into consideration, we settle a simple root system $\Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$ for $\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$:

List III

	$\mathfrak{g}$	$\mathfrak{k}$	$\Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$
BI	$\mathfrak{so}(2k, 2l-2k+1)$ $l \geq 2,\ 2 \leq k \leq l$	 $D_k \times B_{l-k},\ \theta = \exp \pi\, \mathrm{ad}(iZ_k)$	$\{\alpha_a, -\mu\}_{a=1}^{k-1} \cup \{\alpha_b\}_{b=k+1}^{l}$ $\mu := \alpha_1 + 2\sum_{j=2}^{l} \alpha_j$
CII	$\mathfrak{sp}(k, l-k)$ $l \geq 3,\ 1 \leq k \leq l-1$	 $C_k \times C_{l-k},\ \theta = \exp \pi\, \mathrm{ad}(iZ_k)$	$\{\alpha_a, -\mu\}_{a=1}^{k-1} \cup \{\alpha_b\}_{b=k+1}^{l}$ $\mu := 2\sum_{j=1}^{l-1} \alpha_j + \alpha_l$
DI	$\mathfrak{so}(2k, 2l-2k)$ $l \geq 4,\ 2 \leq k \leq l-2$	 $D_k \times D_{l-k},\ \theta = \exp \pi\, \mathrm{ad}(iZ_k)$	$\{\alpha_a, -\mu_d\}_{a=1}^{k-1} \cup \{\alpha_b\}_{b=k+1}^{l}$
EII	$\mathfrak{e}_{6(2)}$	 $A_5 \times A_1,\ \theta = \exp \pi\, \mathrm{ad}(iZ_3)$	$\{-\mu_6, \alpha_2, \alpha_b\}_{b=4}^{6} \cup \{\alpha_1\}$
EV	$\mathfrak{e}_{7(7)}$	 $A_7,\ \theta = \exp \pi\, \mathrm{ad}(iZ_2)$	$\{-\mu_7, \alpha_1, \alpha_b\}_{b=3}^{7}$

[2] There are some minor misprints in [Bu]. p. 269, ↓ 8, Read $\varepsilon_i - \varepsilon_j =$ instead of $\varepsilon_i = \varepsilon_i - \varepsilon_j =$; p. 269, ↓ 10, Read $2\sum_{i \leq k < l} \alpha_k + \alpha_l$ instead of $\sum_{i \leq k < l} \alpha_k + \alpha_l$; p. 271, ↓ 8, Read $\sum_{i \leq k < j} \alpha_k$ instead of $\sum_{i < k < j} \alpha_k$; p. 289, ↓ 9, Read $3\alpha_1 + 2\alpha_2,\ \alpha_2$ instead of $3\alpha_1 + 2\alpha_2$.

	$\mathfrak{g}$	$\mathfrak{k}$	$\Pi_{\triangle(\mathfrak{k}_{\mathbf{C}},\mathfrak{t}_{\mathbf{C}})}$
EVI	$\mathfrak{e}_{7(-5)}$	 $A_1 \times D_6,\ \theta = \exp \pi\, \mathrm{ad}(iZ_1)$	$\{-\mu_7\} \cup \{\alpha_b\}_{b=2}^{7}$
EVIII	$\mathfrak{e}_{8(8)}$	 $D_8,\ \theta = \exp \pi\, \mathrm{ad}(iZ_1)$	$\{-\mu_8, \alpha_b\}_{b=2}^{8}$
EIX	$\mathfrak{e}_{8(-24)}$	 $A_1 \times E_7,\ \theta = \exp \pi\, \mathrm{ad}(iZ_8)$	$\{-\mu_8\} \cup \{\alpha_a\}_{a=1}^{7}$
FI	$\mathfrak{f}_{4(4)}$	 $A_1 \times C_3,\ \theta = \exp \pi\, \mathrm{ad}(iZ_1)$	$\{-\mu_f\} \cup \{\alpha_b\}_{b=2}^{4}$
FII	$\mathfrak{f}_{4(-20)}$	 $B_4,\ \theta = \exp \pi\, \mathrm{ad}(iZ_4)$	$\{-\mu_f, \alpha_a\}_{a=1}^{3}$
G	$\mathfrak{g}_{2(2)}$	 $A_1 \times A_1,\ \theta = \exp \pi\, \mathrm{ad}(iZ_2)$	$\{-\mu\} \cup \{\alpha_1\}$ $\mu := 3\alpha_1 + 2\alpha_2$

Here $\mu_d := \alpha_1 + 2\sum_{j=2}^{l-2} \alpha_j + \alpha_{l-1} + \alpha_l$, $\mu_f := 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, $\mu_6 := \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, $\mu_7 := 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ and $\mu_8 := 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$.

With the above setting we will determine the coarse orbit types of non-Kählerian, symplectic homogeneous spaces G/H with G non-compact simple and H compact later (cf. Subsections 4.4 through 4.14).

4.3. A condition for centralizers to be compact. Let $\mathfrak{g}$ be a non-compact semisimple Lie algebra, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, and let W be an element of $\mathfrak{k} \subset \mathfrak{g}$. Then the proof of Theorem 2 implies that the centralizer $\mathfrak{c}_{\mathfrak{g}}(W)$ is compact if W satisfies the condition

(c2) $\mathrm{ad}\, W|_{\mathfrak{p}}$ is a linear isomorphism of $\mathfrak{p}$.

This condition is not suitable for us to apply the root theory to $\mathfrak{c}_{\mathfrak{g}}(W)$. For this reason we rewrite it as follows:

Lemma 7. *With the above setting; the following items* (i), (ii) *and* (iii) *are equivalent:*

(i) *the centralizer* $\mathfrak{c}_{\mathfrak{g}}(W)$ *is compact;*

(ii) ad $W|_{\mathfrak{p}}$ *is a linear isomorphism of* $\mathfrak{p}$*;*

(iii) $\mathfrak{c}_{\mathfrak{g}}(W) = \mathfrak{c}_{\mathfrak{k}}(W)$.

Proof. (i) $\to$ (ii): Suppose that (i) $\mathfrak{c}_{\mathfrak{g}}(W)$ is compact. Take an element $Y \in \mathfrak{p}$ and suppose that $[W, Y] = 0$. Then $Y \in \mathfrak{c}_{\mathfrak{g}}(W)$. On the one hand, by virtue of (i) there exists an ad Y-invariant inner product on $\mathfrak{g}$, and thus all eigenvalues of ad Y are purely imaginary. On the other hand, ad Y is symmetric with respect to the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ because $Y \in \mathfrak{p}$ (see the proof of Lemma 2 for $\langle \cdot, \cdot \rangle$), and thus all eigenvalues of ad Y are real. These imply that $Y = 0$ and ad $W|_{\mathfrak{p}}$ is injective. Hence it is a linear isomorphism of $\mathfrak{p}$.

(ii) $\to$ (iii): Suppose that (ii) holds. Then one can obtain $\mathfrak{c}_{\mathfrak{g}}(W) = \mathfrak{c}_{\mathfrak{k}}(W)$ from $W \in \mathfrak{k}$ and (ii) (see the proof of (11)).

(iii) $\to$ (i): Clear. □

4.4. An outline of argument for determining the coarse orbit types, and the coarse orbit type of type BI. In this subsection we will determine the coarse orbit type of symplectic homogeneous spaces G/H such that $G = SO_0(2k, 2l - 2k + 1)$ and H is compact (see Proposition 4). The arguments in this subsection will be helpful for the reader to understand the arguments in the coming Subsections 4.5 through 4.13.

Let G/H be a non-Kählerian, symplectic homogeneous space with G non-compact simple and H compact. Recall that we assume Lie $G = \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ to be given by a compact simple Lie algebra $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ (cf. Subsection 4.2). The main purpose of this paper is to determine the coarse orbit type of G/H. By virtue of Proposition 2, it suffices to determine, up to isomorphism, the centralizers $\mathfrak{c}_{\mathfrak{g}}(W)$ of elliptic elements $W \in \mathfrak{g}$ such that $\mathfrak{c}_{\mathfrak{g}}(W)$ are compact. For this reason we need to search elliptic elements $W \in \mathfrak{g}$ whose centralizers $\mathfrak{c}_{\mathfrak{g}}(W)$ are compact. One may assume that such an element $W \in \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ belongs to a fixed positive Weyl chamber $\mathscr{W}_{\mathfrak{k}}$ of $\mathfrak{k}$. Let us explain the reason why one may assume W to belong to $\mathscr{W}_{\mathfrak{k}}$, from now on. Let W be an elliptic element of $\mathfrak{g}$ such that $\mathfrak{c}_{\mathfrak{g}}(W)$ is compact. Since W is elliptic there exists a maximal compact subalgebra $\mathfrak{k}'$ of $\mathfrak{g}$ containing W. In this case we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k}' \oplus \mathfrak{p}'$ with $W \in \mathfrak{k}'$, where $\mathfrak{p}'$ denotes the orthogonal complement of $\mathfrak{k}'$ in $\mathfrak{g}$ with respect to $B_{\mathfrak{g}}$. According to Helgason [He, p. 183, Theorem 7.2], there exists an inner automorphism ψ of $\mathfrak{g}$ satisfying $\psi(\mathfrak{k}') = \mathfrak{k}$ and $\psi(\mathfrak{p}') = \mathfrak{p}$. Then $\psi(W) \in \mathfrak{k}$ and $\psi(\mathfrak{c}_{\mathfrak{g}}(W)) = \mathfrak{c}_{\mathfrak{g}}(\psi(W))$ is compact. This implies that one may assume W to belong to the fixed maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ from the beginning, as far as determining the centralizer $\mathfrak{c}_{\mathfrak{g}}(W)$ which is compact (up to inner automorphism). So, one may assume $W \in \mathscr{W}_{\mathfrak{k}}$ because any element of $\mathfrak{k}$ can be mapped into $\mathscr{W}_{\mathfrak{k}}$ by an inner automorphism of $\mathfrak{k}$ ($\subset \mathfrak{g}$).

Remark 7. *The above arguments assure that one can determine the centralizers $\mathfrak{c}_{\mathfrak{g}}(W)$ which are compact (up to isomorphism) by means of determining the centralizers $\mathfrak{c}_{\mathfrak{g}}(T)$ which are compact with $T \in \mathscr{W}_{\mathfrak{k}}$.*

This subsection consists of three paragraphs.

§4.4.1 A dual basis of $\Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$ and a positive Weyl chamber $\mathscr{W}_{\mathfrak{k}}$

Following List III (ref. p. 20) we define a dual basis $\{T_i\}_{i=1}^{l}$ of $\Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$ (cf. (b1.1)) and fix a positive Weyl chamber $\mathscr{W}_{\mathfrak{k}}$ of $\mathfrak{k}$ in $\mathfrak{t}_{\mathbf{R}}$ (see (b1.2) and Remark 8).

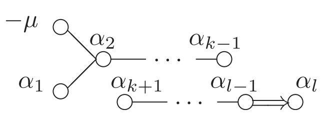
§4.4.2 A condition for the centralizer $\mathfrak{c}_{\mathfrak{g}}(T)$ of an element $T \in \mathscr{W}_{\mathfrak{k}}$ to be compact

We first read off a vector space structure of $\mathfrak{p}_{\mathbf{C}}$ from List III (see (b1.3)) and afterwards investigate a condition for the centralizer $\mathfrak{c}_{\mathfrak{g}}(T)$ of an element $T \in \mathscr{W}_{\mathfrak{k}}$ to be compact by taking (b1.3) into consideration (cf. Lemma 8). Moreover, we obtain (b1.9) from Lemmas 8, 9 and 10. This (b1.9) assures that one can determine the centralizers $\mathfrak{c}_{\mathfrak{g}}(T')$ which are compact with $T' \in \mathscr{W}_{\mathfrak{k}}$ (up to isomorphism) by means of determining all elements $\mathfrak{c}_{\mathfrak{k}}(T) \in C_K^{\mathrm{BI}}(k)$ (see (b1.8) for $C_K^{\mathrm{BI}}(k)$).

§4.4.3 A result for type BI

We determine all elements of $C_K^{\mathrm{BI}}(k)$ and assert Proposition 4.

4.4.1. *A dual basis of $\Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$ and a positive Weyl chamber $\mathscr{W}_{\mathfrak{k}}$.*

	$\mathfrak{g}$	$\mathfrak{k}$	$\Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$
BI	$\mathfrak{so}(2k, 2l-2k+1)$		$\{\alpha_a, -\mu\}_{a=1}^{k-1} \cup \{\alpha_b\}_{b=k+1}^{l}$
	$l \geq 2,\ 2 \leq k \leq l$	$D_k \times B_{l-k},\ \theta = \exp \pi\, \mathrm{ad}(iZ_k)$	$\mu := \alpha_1 + 2\sum_{j=2}^{l} \alpha_j$

Let $\beta_a := \alpha_{k-a}$ for $1 \leq a \leq k-1$, $\beta_k := -\mu$, and $\beta_b := \alpha_b$ for $k+1 \leq b \leq l$. Denote by $\{T_i\}_{i=1}^{l}$ the dual basis of $\{\beta_i\}_{i=1}^{l} = \Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$. Then one can express it as follows:

$$\begin{aligned} &T_c = Z_{k-c} - Z_k \quad \text{for } 1 \leq c \leq k-2, \qquad T_{k-1} = Z_1 - Z_k/2, \\ &T_k = -Z_k/2, \qquad T_b = -Z_k + Z_b \quad \text{for } k+1 \leq b \leq l \end{aligned} \tag{b1.1}$$

by means of the dual basis $\{Z_i\}_{i=1}^{l}$ of $\{\alpha_i\}_{i=1}^{l} = \Pi_{\triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$. Now, let us fix a positive Weyl chamber $\mathscr{W}_{\mathfrak{k}}$ of $\mathfrak{k}$ in $\mathfrak{t}_{\mathbf{R}}$:

$$\mathscr{W}_{\mathfrak{k}} := \{T \in \mathfrak{t}_{\mathbf{R}} \mid \beta_i(T) \geq 0 \text{ for all } 1 \leq i \leq l\} \tag{b1.2}$$

(see Subsection 4.2 for $\mathfrak{t}_{\mathbf{R}}$). Needless to say, $T = \sum_{i=1}^{l} \lambda_i T_i$ belongs to $\mathscr{W}_{\mathfrak{k}}$ if and only if $\lambda_i \geq 0$ for all $1 \leq i \leq l$. We will consider a condition for the centralizer $\mathfrak{c}_{\mathfrak{g}}(T)$ of an element $T \in \mathscr{W}_{\mathfrak{k}}$ to be compact in the next paragraph.

Remark 8. *Throughout this paper, we regard each element of* $\mathscr{W}_{\mathfrak{k}}$ *as an element of* $\mathfrak{k}$ *by identifying* $T \in \mathfrak{t}_{\mathbf{R}}$ *with* $iT \in i\mathfrak{t}_{\mathbf{R}} \subset \mathfrak{k}$ *(cf.* (19)*).*

Now, let us state Proposition 3 which will streamline the procedure for computing $\mathfrak{c}_{\mathfrak{g}}(T)$ later:

Proposition 3 (cf. Wolf-Gray [Wo-Gr, p. 83–84]). *Let* $T = \sum_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$. *Suppose that* $\lambda_p \neq 0$ *for all* $p \in \{i_a\}_{a=1}^{k}$ *and* $\lambda_q = 0$ *for all* $q \in \{1, \dots, l\} \setminus \{i_a\}_{a=1}^{k}$, *where* $1 \leq i_1 \leq \cdots \leq i_k \leq l$. *Then,* $\{\beta_q\}_{q \in \{1,\dots,l\} \setminus \{i_a\}_{a=1}^{k}}$ *is a simple root system for* $\triangle(\mathfrak{c}_{\mathfrak{k}}(T), \mathfrak{t}_{\mathbf{R}})$.

Proof. One can get the conclusion by the proof of Proposition 2.8 in Wolf-Gray [Wo-Gr, p. 83]. □

Remark 9. *Proposition* 3 *implies the following* (i) *and* (ii)*:*

(i) *For* $T = \sum_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$, *the structure of* $\mathfrak{c}_{\mathfrak{k}}(T)$ *depends only on whether each* λ_i *is zero or not.*

(ii) *For any* $1 \leq k \leq l$ *and real number* $\zeta > 0$, *both* $T = \sum_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$ *and* $T' := \sum_{a=1}^{k-1} \lambda_a T_a + \zeta \lambda_k T_k + \sum_{b=k+1}^{l} \lambda_b T_b$ *give rise to the same centralizer* $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{c}_{\mathfrak{k}}(T')$ *in* $\mathfrak{k}$.

Note that $\mathfrak{c}_{\mathfrak{g}}(T)$ *does not always coincide with* $\mathfrak{c}_{\mathfrak{g}}(T')$ *even if* $\mathfrak{c}_{\mathfrak{k}}(T) = \mathfrak{c}_{\mathfrak{k}}(T')$.

4.4.2. *A condition for the centralizer* $\mathfrak{c}_{\mathfrak{g}}(T)$ *of an element* $T \in \mathscr{W}_{\mathfrak{k}}$ *to be compact.* Recall that we assume $\mathfrak{so}(2k, 2l-2k+1) = \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ to be given by compact simple Lie algebra $\mathfrak{so}(2l+1) = \mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ with involution $\theta = \exp \pi \operatorname{ad}(iZ_k)$ (cf. Subsection 4.2). Since $i\mathfrak{p}$ is the (-1)-eigenspace of θ in $\mathfrak{g}_u$ we confirm that

$$\mathfrak{p}_{\mathbf{C}} = \bigoplus_{\alpha \in \triangle^+(\alpha_k, 1)} \operatorname{span}_{\mathbf{C}}\{X_\alpha\} \oplus \operatorname{span}_{\mathbf{C}}\{X_{-\alpha}\} \qquad \text{(direct sum)},$$

$$\begin{aligned} \triangle^+(\alpha_k, 1) &:= \left\{ \sum\nolimits_{i=1}^{l} n_i \alpha_i \in \triangle^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}}) \mid n_k = 1 \right\} \\ &= \left\{ \begin{array}{ll} \sum_{p \leq s \leq q} \alpha_s & (1 \leq p \leq k \leq q \leq l), \\ \sum_{p \leq t < r} \alpha_t + 2 \sum_{r \leq t \leq l} \alpha_t & (1 \leq p \leq k < r \leq l) \end{array} \right\} \end{aligned} \tag{b1.3}$$

(see Subsection 4.2 for $X_{\pm\alpha}$; Plate II in Bourbaki [Bu, p. 265]), where $\mathfrak{p}_{\mathbf{C}}$ denotes the (-1)-eigenspace of θ in $\mathfrak{g}_{\mathbf{C}} = \mathfrak{so}(2l+1, \mathbf{C})$. Now, we want to clarify a condition for the centralizer $\mathfrak{c}_{\mathfrak{g}}(T)$ of an element $T \in \mathscr{W}_{\mathfrak{k}}$ to be compact. Lemma 7 enables us to see that $\mathfrak{c}_{\mathfrak{g}}(T)$ is compact if and only if

$$\alpha(T) \neq 0 \qquad \text{for all } \alpha \in \triangle^+(\alpha_k, 1) \tag{b1.4}$$

because of (b1.3) and $[T, X_\alpha] = \alpha(T) \cdot X_\alpha$. Taking (b1.4) into consideration we are going to clarify the condition. Take an element $T = \sum_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$ and express it as follows:

$$T = \sum_{i=1}^{l} \lambda_i T_i = \sum_{a=1}^{k-1} \lambda_{k-a} Z_a - \left(\sum_{c=1}^{k-2} \lambda_c + \lambda_{k-1}/2 + \lambda_k/2 + \sum_{b=k+1}^{l} \lambda_b \right) Z_k + \sum_{b=k+1}^{l} \lambda_b Z_b \tag{b1.5}$$

by means of (b1.1). A direct computation, combined with $\alpha_a(Z_b) = \delta_{a,b}$, gives

$$\sum\nolimits_{p \leq s \leq q} \alpha_s(T) = -\sum\nolimits_{i=k-p+1}^{k-1} \lambda_i + \lambda_{k-1}/2 - \lambda_k/2 - \sum\nolimits_{j=q+1}^{l} \lambda_j \quad (1 \leq p \leq k \leq q \leq l); \tag{b1.6}$$

$$\left(\sum\nolimits_{p \leq t < r} \alpha_t + 2\sum\nolimits_{r \leq t \leq l} \alpha_t\right)(T) = -\sum\nolimits_{i=k-p+1}^{k-1} \lambda_i + \lambda_{k-1}/2 - \lambda_k/2 + \sum\nolimits_{h=r}^{l} \lambda_h \quad (1 \leq p \leq k < r \leq l). \tag{b1.7}$$

Let us clarify a necessary and sufficient condition for $\mathfrak{c}_{\mathfrak{g}}(T)$ to be compact:

Lemma 8 (BI). *With the above setting; for* $T = \sum_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$, *the centralizer* $\mathfrak{c}_{\mathfrak{g}}(T)$ *is compact if and only if* (i) *"*$\lambda_{k-1} > 0$ *or* $\lambda_k > 0$*" and* (ii) $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{c}_{\mathfrak{k}}(T)$.

Proof. ($\Rightarrow$): Suppose that $\mathfrak{c}_{\mathfrak{g}}(T)$ is compact. Then it follows from (b1.4) and $(\text{b1.6})_{p=2,q=l}$ that $\lambda_{k-1} + \lambda_k \neq 0$. Therefore $\lambda_{k-1} > 0$ or $\lambda_k > 0$ because $\lambda_{k-1}, \lambda_k \geq 0$. Besides, the supposition and Lemma 7 allow us to have $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{c}_{\mathfrak{k}}(T)$. ($\Leftarrow$): It is clear, since $\mathfrak{c}_{\mathfrak{k}}(T)$ is compact. □

Now, let $C_K^{\mathrm{BI}}(k-1)$, $C_K^{\mathrm{BI}}(k)$ and C_G^{BI} denote the following sets defined by

$$\begin{aligned} C_K^{\mathrm{BI}}(k-1) &:= \left\{ \mathfrak{c}_{\mathfrak{k}}(T) \,|\, T = \sum\nolimits_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_{k-1} > 0 \right\}, \\ C_K^{\mathrm{BI}}(k) &:= \left\{ \mathfrak{c}_{\mathfrak{k}}(T) \,|\, T = \sum\nolimits_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_k > 0 \right\}, \\ C_G^{\mathrm{BI}} &:= \{ \mathfrak{c}_{\mathfrak{g}}(T') \,|\, \mathfrak{c}_{\mathfrak{g}}(T') \text{ is compact with } T' \in \mathscr{W}_{\mathfrak{k}} \}, \end{aligned} \tag{b1.8}$$

respectively. Our aim is to determine all elements of C_G^{BI} up to isomorphism. Lemma 8 implies that

$$C_G^{\mathrm{BI}} \subset C_K^{\mathrm{BI}}(k-1) \cup C_K^{\mathrm{BI}}(k).$$

We will prove Lemmas 9 and 10 later. By Lemma 9 one can deduce $C_K^{\mathrm{BI}}(k) \subset C_G^{\mathrm{BI}}$. Lemma 10 provides us with an outer involution ψ of $\mathfrak{g}$ satisfying $\psi(C_K^{\mathrm{BI}}(k)) = C_K^{\mathrm{BI}}(k-1)$ and $\psi(C_G^{\mathrm{BI}}) = C_G^{\mathrm{BI}}$. Accordingly Lemmas 9 and 10 allow us to conclude

$$C_G^{\mathrm{BI}} = C_K^{\mathrm{BI}}(k) \cup \psi(C_K^{\mathrm{BI}}(k)). \tag{b1.9}$$

This means that for our aim, it is enough to determine all elements of $C_K^{\mathrm{BI}}(k)$. Let us prove Lemmas 9 and 10.

Lemma 9 (BI). *With the above setting; for any $\bar{T} = \sum_{i=1}^{l} \bar{\lambda}_i T_i \in \mathscr{W}_{\mathfrak{k}}$ with $\bar{\lambda}_k > 0$, there exists an element $\bar{T}' \in \mathscr{W}_{\mathfrak{k}}$ such that* (i) *$\mathfrak{c}_{\mathfrak{g}}(\bar{T}')$ is compact and* (ii) *$\mathfrak{c}_{\mathfrak{g}}(\bar{T}') = \mathfrak{c}_{\mathfrak{k}}(\bar{T})$.*

Proof. It is easy to get a real number $\zeta > 0$ satisfying

$$\bar{\lambda}_{k-1}/2 - \zeta\bar{\lambda}_k/2 + \sum\nolimits_{b=k+1}^{l} \bar{\lambda}_b < 0$$

since $\bar{\lambda}_k > 0$. By use of ζ we define an element $\bar{T}' \in \mathscr{W}_{\mathfrak{k}}$ as follows: $\bar{T}' := \sum_{a=1}^{k-1} \bar{\lambda}_a T_a + \zeta\bar{\lambda}_k T_k + \sum_{b=k+1}^{l} \bar{\lambda}_b T_b$. Since $\bar{\lambda}_c \geq 0$ for all $1 \leq c \leq l$ we see that $\bar{T}'$ satisfies

$$\begin{aligned}\sum\nolimits_{p \leq s \leq q} \alpha_s(\bar{T}') &= -\sum\nolimits_{i=k-p+1}^{k-1} \bar{\lambda}_i + \bar{\lambda}_{k-1}/2 - \zeta\bar{\lambda}_k/2 - \sum\nolimits_{j=q+1}^{l} \bar{\lambda}_j \\ &\leq \bar{\lambda}_{k-1}/2 - \zeta\bar{\lambda}_k/2 + \sum\nolimits_{b=k+1}^{l} \bar{\lambda}_b < 0\end{aligned}$$

in Case (b1.6); and $(\sum_{p \leq t < r} \alpha_t + 2\sum_{r \leq t \leq l} \alpha_t)(\bar{T}') < 0$ in Case (b1.7). So it follows from (b1.3) that $\alpha(\bar{T}') < 0$ for all $\alpha \in \triangle^+(\alpha_k, 1)$. Hence Lemma 7 and (b1.4) imply that $\mathfrak{c}_{\mathfrak{g}}(\bar{T}')$ is compact and $\mathfrak{c}_{\mathfrak{g}}(\bar{T}') = \mathfrak{c}_{\mathfrak{k}}(\bar{T}')$. Remark 9-(ii) allows us to have $\mathfrak{c}_{\mathfrak{k}}(\bar{T}') = \mathfrak{c}_{\mathfrak{k}}(\bar{T})$. This yields $\mathfrak{c}_{\mathfrak{g}}(\bar{T}') = \mathfrak{c}_{\mathfrak{k}}(\bar{T})$. □

Lemma 10 (BI). *There exists an outer involution ψ of $\mathfrak{g} = \mathfrak{so}(2k, 2l-2k+1)$, $l \geq 2$ and $2 \leq k \leq l$, satisfying $\psi(\mathfrak{k}) \subset \mathfrak{k}$, $\psi(T_{k-1}) = T_k$ and $\psi(T_h) = T_h$ for any $1 \leq h \leq l$ with $h \neq k-1, k$.*

Proof. Define an involutive linear map ψ of $\mathfrak{t}_{\mathbf{R}}$ by setting

$${}^t\psi(\alpha_1) := -\mu\Big(= -\alpha_1 - 2\sum\nolimits_{t=2}^{l} \alpha_t\Big), \qquad {}^t\psi(\alpha_j) := \alpha_j \qquad \text{for } 2 \leq j \leq l. \tag{b1.10}$$

Then one has

$$B_{\mathfrak{g}_{\mathbf{C}}}(H_{\alpha_i}, H_{\alpha_j})/B_{\mathfrak{g}_{\mathbf{C}}}(H_{\alpha_j}, H_{\alpha_j}) = B_{\mathfrak{g}_{\mathbf{C}}}(H_{{}^t\psi(\alpha_i)}, H_{{}^t\psi(\alpha_j)})/B_{\mathfrak{g}_{\mathbf{C}}}(H_{{}^t\psi(\alpha_j)}, H_{{}^t\psi(\alpha_j)})$$

for all $1 \leq i, j \leq l$ (see Subsection 4.2 for H_α). Hence Lemme 1 in Murakami [Mu3, p. 295] enables us to extend ψ to $\mathfrak{g}_{\mathbf{C}}$ as involution because both $\{\alpha_i\}_{i=1}^{l}$

and $\{-\mu, \alpha_j\}_{j=2}^{l}$ are simple root systems for $\triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$. We denote this involution by the same notation ψ. Note that (i) $\psi(\mathfrak{g}_u) \subset \mathfrak{g}_u$ follows from (18) and (ii) ψ satisfies

$$\psi(Z_1) = -Z_1, \qquad \psi(Z_j) = -2Z_1 + Z_j \qquad \text{for } 2 \le j \le l \qquad \text{(b1.11)}$$

because of $\alpha_a(Z_b) = \delta_{a,b}$ and (b1.10). This and (b1.1) yield $\psi(T_{k-1}) = T_k$ and $\psi(T_h) = T_h$ for any $1 \le h \le l$ with $h \ne k-1, k$. We can obtain $\psi \circ \theta = \theta \circ \psi$ from $\psi(Z_k) = -2Z_1 + Z_k$, $[Z_1, Z_k] = 0$, $\theta = \exp \pi\, \mathrm{ad}(iZ_k)$ and $\mathrm{id} = \exp \pi\, \mathrm{ad}(2iZ_1)$. It follows from $\psi \circ \theta = \theta \circ \psi$ that $\psi(\mathfrak{g}) \subset \mathfrak{g}$ and $\psi(\mathfrak{k}) \subset \mathfrak{k}$. Thus we have constructed an involution ψ of $\mathfrak{g}$ satisfying $\psi(\mathfrak{k}) \subset \mathfrak{k}$, $\psi(T_{k-1}) = T_k$ and $\psi(T_h) = T_h$ for any $1 \le h \le l$ with $h \ne k-1, k$. Remark that ψ is an outer automorphism of $\mathfrak{g}$ (because of Corollary 2 in Murakami [Mu1, p. 108] and the fixed point set $\mathrm{Fix}(\mathfrak{k}, \psi) = B_{k-1} \times B_{l-k}$). □

4.4.3. *A result for type BI.* Now, let us demonstrate

Proposition 4. *The following is the coarse orbit type of symplectic homogeneous space G/H with $G = SO_0(2k, 2l-2k+1)$, $l \ge 2$ and $2 \le k \le l$, and H compact:*

G	H	No.
$SO_0(2k, 2l-2k+1)$ $l \ge 2$ $2 \le k \le l$ BI	$U(i_1) \times U(i_2 - i_1) \times \cdots \times U(i_a - i_{a-1}) \times U(k - i_a)$ $\times U(j_1 - k) \times U(j_2 - j_1) \times \cdots \times U(j_b - j_{b-1})$ $\times SO(2l - 2j_b + 1)$ $0 \le a \le k-1,\ i_0 := 0,\ 1 \le i_1 \le i_2 \le \cdots \le i_a \le k-1$ $0 \le b \le l-k,\ j_0 := k,\ k+1 \le j_1 \le j_2 \le \cdots \le j_b \le l$	1

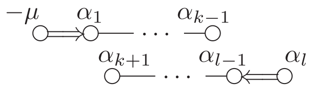
Proof. Our goal is to determine the isotropy subgroups H up to isomorphism. Proposition 2 means that it is enough to determine the centralizers $\mathfrak{c}_{\mathfrak{g}}(W)$ which are compact (up to isomorphism). Remark 7 and (b1.9) enable one to get the conclusion, if we determine all elements $\mathfrak{c}_{\mathfrak{k}}(T) \in C_K^{\mathrm{BI}}(k)$ (see (b1.8) for $C_K^{\mathrm{BI}}(k)$). For this reason we are going to determine each element of $C_K^{\mathrm{BI}}(k)$. Let $T = \sum_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$ with $\lambda_k > 0$. Without loss of generality, one may assume that $\lambda_p \ne 0$ for all $p \in \{i_x\}_{x=1}^{a} \cup \{j_y\}_{y=1}^{b}$ and $\lambda_q = 0$ for all $q \in \{1, \ldots, k-1, k+1, \ldots, l\} \backslash (\{i_x\}_{x=1}^{a} \cup \{j_y\}_{y=1}^{b})$, where $1 \le i_1 \le \cdots \le i_a \le k-1$ and $k+1 \le j_1 \le \cdots \le j_b \le l$. Then Proposition 3 assures that $\{\beta_q\}_{q \in \{1,\ldots,k-1,k+1,\ldots,l\} \backslash (\{i_x\}_{x=1}^{a} \cup \{j_y\}_{y=1}^{b})}$ is a simple root system for $\triangle(\mathfrak{c}_{\mathfrak{k}}(T), \mathfrak{t}_{\mathbf{R}})$. Therefore we have

$$\mathfrak{c}_{\mathfrak{k}}(T) = \bigoplus_{x=1}^{a} \mathfrak{u}(i_x - i_{x-1}) \oplus \mathfrak{u}(k - i_a) \oplus \bigoplus_{y=1}^{b} \mathfrak{u}(j_y - j_{y-1}) \oplus \mathfrak{so}(2l - 2j_b + 1),$$

where $i_0 := 0$ and $j_0 := k$. □

4.5. The coarse orbit type of type CII. In this subsection, we will determine the coarse orbit type of symplectic homogeneous spaces G/H with $G = Sp(k, l-k)$ and H compact (see Proposition 5) by arguments similar to those in Subsection 4.4.

4.5.1. *A dual basis of $\Pi_{\triangle(\mathfrak{t}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$ and a positive Weyl chamber $\mathscr{W}_{\mathfrak{k}}$.*

	$\mathfrak{g}$	$\mathfrak{k}$	$\Pi_{\triangle(\mathfrak{t}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$
CII	$\mathfrak{sp}(k, l-k)$		$\{\alpha_a, -\mu\}_{a=1}^{k-1} \cup \{\alpha_b\}_{b=k+1}^{l}$
	$l \geq 3,\ 1 \leq k \leq l-1$	$C_k \times C_{l-k},\ \theta = \exp \pi\, \mathrm{ad}(iZ_k)$	$\mu := 2\sum_{j=1}^{l-1} \alpha_j + \alpha_l$

Let $\beta_a := \alpha_{k-a}$ for $1 \leq a \leq k-1$, $\beta_k := -\mu$ and $\beta_b := \alpha_b$ for $k+1 \leq b \leq l$. We denote by $\{T_i\}_{i=1}^{l}$ the dual basis of $\{\beta_i\}_{i=1}^{l} = \Pi_{\triangle(\mathfrak{t}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$. Then it follows from $\alpha_a(Z_b) = \delta_{a,b}$ that

$$\begin{aligned} &T_a = Z_{k-a} - Z_k && \text{for } 1 \leq a \leq k-1, && T_k = -Z_k/2, \\ &T_c = -Z_k + Z_c && \text{for } k+1 \leq c \leq l-1, && T_l = -Z_k/2 + Z_l. \end{aligned} \tag{c2.1}$$

Throughout Subsection 4.5, we fix a positive Weyl chamber $\mathscr{W}_{\mathfrak{k}}$ of $\mathfrak{k}$ in $\mathfrak{t}_{\mathbf{R}}$ as follows: $\mathscr{W}_{\mathfrak{k}} := \{T \in \mathfrak{t}_{\mathbf{R}} \mid \beta_i(T) \geq 0 \text{ for all } 1 \leq i \leq l\}$ $(= \{\sum_{i=1}^{l} \lambda_i T_i \mid \lambda_i \geq 0$ for all $1 \leq i \leq l\})$.

4.5.2. *A condition for the centralizer $\mathfrak{c}_{\mathfrak{g}}(T)$ of an element $T \in \mathscr{W}_{\mathfrak{k}}$ to be compact.* Arguments in this paragraph will be similar to those in Paragraph 4.4.2. Recall that we assume $\mathfrak{sp}(k, l-k) = \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ to be given by compact simple Lie algebra $\mathfrak{sp}(l) = \mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ with involution $\theta = \exp \pi\, \mathrm{ad}(iZ_k)$ (cf. Subsection 4.2). We will clarify a necessary and sufficient condition for the centralizer $\mathfrak{c}_{\mathfrak{g}}(T)$ of an element $T \in \mathscr{W}_{\mathfrak{k}}$ to be compact (cf. Lemma 11 below). Since $\theta = \exp \pi\, \mathrm{ad}(iZ_k)$ one perceives that the (-1)-eigenspace $\mathfrak{p}_{\mathbf{C}}$ of θ in $\mathfrak{g}_{\mathbf{C}} = \mathfrak{sp}(l, \mathbf{C})$ is given by

$$\mathfrak{p}_{\mathbf{C}} = \bigoplus_{\alpha \in \triangle^+(\alpha_k, 1)} \mathrm{span}_{\mathbf{C}}\{X_\alpha\} \oplus \mathrm{span}_{\mathbf{C}}\{X_{-\alpha}\} \qquad (\text{direct sum}),$$

$$\begin{aligned} \triangle^+(\alpha_k, 1) &:= \left\{ \sum\nolimits_{i=1}^{l} n_i \alpha_i \in \triangle^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}}) \mid n_k = 1 \right\} \\ &= \left\{ \begin{matrix} \sum_{p \leq s < q} \alpha_s & (1 \leq p \leq k < q \leq l), \\ \sum_{p \leq t < r} \alpha_t + 2\sum_{r \leq t < l} \alpha_t + \alpha_l & (1 \leq p \leq k < r \leq l) \end{matrix} \right\} \end{aligned} \tag{c2.2}$$

(cf. Plate III in Bourbaki [Bu, p. 269][3]). In this case Lemma 7 implies that for $T \in \mathscr{W}_{\mathfrak{k}}$, the centralizer $\mathfrak{c}_{\mathfrak{g}}(T)$ is compact if and only if T satisfies

$$\alpha(T) \neq 0 \qquad \text{for all } \alpha \in \triangle^+(\alpha_k, 1). \tag{c2.3}$$

Take an element $T = \sum_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$ and express it as follows:

$$T = \sum_{i=1}^{l} \lambda_i T_i = \sum_{a=1}^{k-1} \lambda_{k-a} Z_a - \left(\sum_{a=1}^{k-1} \lambda_a + \lambda_k/2 + \sum_{c=k+1}^{l-1} \lambda_c + \lambda_l/2 \right) Z_k + \sum_{b=k+1}^{l} \lambda_b Z_b \tag{c2.4}$$

by means of (c2.1). Then direct computations enable us to have

$$\sum\nolimits_{p \leq s < q} \alpha_s(T) = -\sum\nolimits_{i=k-p+1}^{k-1} \lambda_i - \lambda_k/2 - \sum\nolimits_{j=q}^{l-1} \lambda_j - \lambda_l/2 \qquad (1 \leq p \leq k < q \leq l); \tag{c2.5}$$

$$\left(\sum\nolimits_{p \leq t < r} \alpha_t + 2\sum\nolimits_{r \leq t < l} \alpha_t + \alpha_l\right)(T) = -\sum\nolimits_{i=k-p+1}^{k-1} \lambda_i - \lambda_k/2 + \sum\nolimits_{h=r}^{l-1} \lambda_h + \lambda_l/2 \qquad (1 \leq p \leq k < r \leq l) \tag{c2.6}$$

because of $\alpha_a(Z_b) = \delta_{a,b}$. Now, let us clarify the necessary and sufficient condition:

LEMMA 11 (CII). *With the setting above; for* $T = \sum_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$, $\mathfrak{c}_{\mathfrak{g}}(T)$ *is compact if and only if* (i) "$\lambda_k > 0$ *or* $\lambda_l > 0$" *and* (ii) $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{c}_{\mathfrak{k}}(T)$.

PROOF. ($\Rightarrow$): Suppose that $\mathfrak{c}_{\mathfrak{g}}(T)$ is compact. From $(\text{c2.5})_{p=1,q=l}$ and (c2.3) we obtain $\lambda_k + \lambda_l \neq 0$, and so $\lambda_k > 0$ or $\lambda_l > 0$ because $\lambda_k, \lambda_l \geq 0$. The rest of proof follows from the proof of Lemma 8. □

Let us consider three sets $C_K^{\text{CII}}(k)$, $C_K^{\text{CII}}(l)$ and C_G^{CII}:

$$\begin{aligned} C_K^{\text{CII}}(k) &:= \left\{ \mathfrak{c}_{\mathfrak{k}}(T) \mid T = \sum\nolimits_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_k > 0 \right\}, \\ C_K^{\text{CII}}(l) &:= \left\{ \mathfrak{c}_{\mathfrak{k}}(T) \mid T = \sum\nolimits_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_l > 0 \right\}, \\ C_G^{\text{CII}} &:= \{ \mathfrak{c}_{\mathfrak{g}}(T') \mid \mathfrak{c}_{\mathfrak{g}}(T') \text{ is compact with } T' \in \mathscr{W}_{\mathfrak{k}} \}. \end{aligned} \tag{c2.7}$$

[3] See the footnote 2 (p. 20) again.

By Lemma 11 one deduces $C_G^{\mathrm{CII}} \subset C_K^{\mathrm{CII}}(k) \cup C_K^{\mathrm{CII}}(l)$. The following Lemma 12 allows us to have

$$C_G^{\mathrm{CII}} = C_K^{\mathrm{CII}}(k) \cup C_K^{\mathrm{CII}}(l). \tag{c2.8}$$

Lemma 12 (CII). *With the above setting; for any* $T = \sum_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$ *with* $\lambda_k > 0$ *or* $\lambda_l > 0$, *there exists an element* $T' \in \mathscr{W}_{\mathfrak{k}}$ *such that* (i) $\mathfrak{c}_{\mathfrak{g}}(T')$ *is compact and* (ii) $\mathfrak{c}_{\mathfrak{g}}(T') = \mathfrak{c}_{\mathfrak{k}}(T)$.

Proof. Case $\lambda_k > 0$: We will consider the case $\lambda_k > 0$ first. Since $\lambda_k > 0$ there exists a real number $\xi > 0$ satisfying

$$-\xi\lambda_k/2 + \sum\nolimits_{c=k+1}^{l-1} \lambda_c + \lambda_l/2 < 0.$$

Define an element $T' \in \mathscr{W}_{\mathfrak{k}}$ by $T' := \sum_{a=1}^{k-1} \lambda_a T_a + \xi\lambda_k T_k + \sum_{b=k+1}^{l} \lambda_b T_b$. Then (c2.5) yields $\sum_{p \le s < q} \alpha_s(T') = -\sum_{i=k-p+1}^{k-1} \lambda_i - \xi\lambda_k/2 - \sum_{j=q}^{l-1} \lambda_j - \lambda_l/2 < 0$ because $\lambda_i \ge 0$ for all $1 \le i \le l$ and $\xi\lambda_k > 0$. In addition, (c2.6) yields

$$\begin{aligned}
&\Big(\sum\nolimits_{p \le t < r} \alpha_t + 2\sum\nolimits_{r \le t < l} \alpha_t + \alpha_l\Big)(T') \\
&\quad = -\sum\nolimits_{i=k-p+1}^{k-1} \lambda_i - \xi\lambda_k/2 + \sum\nolimits_{h=r}^{l-1} \lambda_h + \lambda_l/2 \\
&\quad \le -\xi\lambda_k/2 + \sum\nolimits_{c=k+1}^{l-1} \lambda_c + \lambda_l/2 < 0.
\end{aligned}$$

These, together with (c2.2), mean that $\alpha(T') < 0$ for every $\alpha \in \triangle^+(\alpha_k, 1)$. Consequently Lemma 7 and (c2.3) enable us to verify that $\mathfrak{c}_{\mathfrak{g}}(T')$ is compact and $\mathfrak{c}_{\mathfrak{g}}(T') = \mathfrak{c}_{\mathfrak{k}}(T')$. So Remark 9 gives us $\mathfrak{c}_{\mathfrak{g}}(T') = \mathfrak{c}_{\mathfrak{k}}(T') = \mathfrak{c}_{\mathfrak{k}}(T)$.

Case $\lambda_l > 0$: Since $\lambda_l > 0$ there exists a real number $\zeta > 0$ such that

$$-\sum\nolimits_{a=1}^{k-1} \lambda_a - \lambda_k/2 + \zeta\lambda_l/2 > 0.$$

By use of ζ, we define an element $T'' \in \mathscr{W}_{\mathfrak{k}}$ by $T'' := \sum_{j=1}^{l-1} \lambda_j T_j + \zeta\lambda_l T_l$. By arguments similar to those stated above, one can conclude that $\sum_{p \le s < q} \alpha_s(T'') < 0$ and $(\sum_{p \le t < r} \alpha_t + 2\sum_{r \le t < l} \alpha_t + \alpha_l)(T'') > 0$, so that $\alpha(T'') \ne 0$ for every $\alpha \in \triangle^+(\alpha_k, 1)$. This and (c2.3) assure that $\mathfrak{c}_{\mathfrak{g}}(T'')$ is compact. Hence, the rest of proof is to confirm that $\mathfrak{c}_{\mathfrak{g}}(T'') = \mathfrak{c}_{\mathfrak{k}}(T)$, but that is immediate (ref. Case $\lambda_k > 0$). □

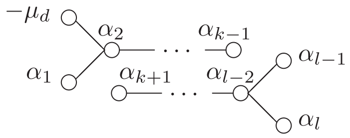
4.5.3. *A result for type CII.*

Proposition 5. *The following is the coarse orbit type of symplectic homogeneous space* G/H *with* $G = Sp(k, l-k)$, $l \ge 3$ *and* $1 \le k \le l-1$, *and* H *compact:*

G	H	No.
$Sp(k, l-k)$ $l \geq 3$ $1 \leq k \leq l-1$	$U(i_1) \times U(i_2 - i_1) \times \cdots \times U(i_a - i_{a-1}) \times U(k - i_a)$ $\times U(j_1 - k) \times U(j_2 - j_1) \times \cdots \times U(j_b - j_{b-1}) \times Sp(l - j_b)$ $0 \leq a \leq k-1,\ i_0 := 0,\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_a \leq k-1$ $0 \leq b \leq l-k,\ j_0 := k,\ k+1 \leq j_1 \leq j_2 \leq \cdots \leq j_b \leq l$	1
	$U(m_1) \times U(m_2 - m_1) \times \cdots \times U(m_c - m_{c-1}) \times Sp(k - m_c)$ $\times U(n_1 - k) \times U(n_2 - n_1) \times \cdots \times U(n_d - n_{d-1}) \times U(l - n_d)$ $0 \leq c \leq k,\ m_0 := 0,\ 1 \leq m_1 \leq m_2 \leq \cdots \leq m_c \leq k$ $0 \leq d \leq l-k-1,\ n_0 := k,\ k+1 \leq n_1 \leq n_2 \leq \cdots \leq n_d \leq l-1$	2
CII	*Remark.* $Sp(k, l-k)/H_1 = Sp(l-k, k)/H_2$, *where we assume* H_2 *to be a subgroup of* $Sp(l-k, k)$ *by identifying* $Sp(k, l-k)$ *with* $Sp(l-k, k)$. *Here* H_i *are the same as in* No.i $(i = 1, 2)$.	

PROOF. Refer to the proof of Proposition 4 and (c2.8). Here we remark that H_1 and H_2 come from elements of $C_K^{\mathrm{CII}}(k)$ and $C_K^{\mathrm{CII}}(l)$, respectively (see (c2.7) for $C_K^{\mathrm{CII}}(k)$ and $C_K^{\mathrm{CII}}(l)$). □

4.6. The coarse orbit type of type DI. Our aim in this subsection is to determine the coarse orbit type of symplectic homogeneous spaces G/H with $G = SO_0(2k, 2l-2k)$ and H compact (see Proposition 6).

	$\mathfrak{g}$	$\mathfrak{k}$	$\Pi_{\triangle(\mathfrak{k}_\mathbf{C}, \mathfrak{t}_\mathbf{C})}$
DI	$\mathfrak{so}(2k, 2l-2k)$ $l \geq 4,$ $2 \leq k \leq l-2$	 $D_k \times D_{l-k},\ \theta = \exp \pi\, \mathrm{ad}(iZ_k)$	$\{\alpha_a, -\mu_d\}_{a=1}^{k-1} \cup \{\alpha_b\}_{b=k+1}^{l}$

Here $\mu_d = \alpha_1 + 2\sum_{j=2}^{l-2} \alpha_j + \alpha_{l-1} + \alpha_l$.

Arguments stated below will be similar to those in Subsection 4.4. Let $\beta_a := \alpha_{k-a}$ for $1 \leq a \leq k-1$, $\beta_k := -\mu_d$ and $\beta_b := \alpha_b$ for $k+1 \leq b \leq l$. Denote by $\{T_i\}_{i=1}^{l}$ the dual basis of $\{\beta_i\}_{i=1}^{l} = \Pi_{\triangle(\mathfrak{k}_\mathbf{C}, \mathfrak{t}_\mathbf{C})}$. In this case it follows from $\alpha_a(Z_b) = \delta_{a,b}$ that

$$
\begin{aligned}
&T_p = Z_{k-p} - Z_k && \text{for } 1 \leq p \leq k-2, && T_{k-1} = Z_1 - Z_k/2, \\
&T_k = -Z_k/2, && T_q = -Z_k + Z_q && \text{for } k+1 \leq q \leq l-2, \\
&T_r = -Z_k/2 + Z_r && \text{for } r = l-1, l.
\end{aligned} \tag{d1.1}
$$

Take an element $T = \sum_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} := \{\sum_{i=1}^{l} \bar{\lambda}_i T_i \mid \bar{\lambda}_i \geq 0 \text{ for all } 1 \leq i \leq l\}$. Then a direct computation, together with (d1.1), gives us

$$T = \sum_{i=1}^{l} \lambda_i T_i = \sum_{a=1}^{k-1} \lambda_{k-a} Z_a - \left(\sum_{p=1}^{k-2} \lambda_p + \lambda_{k-1}/2 + \lambda_k/2 + \sum_{q=k+1}^{l-2} \lambda_q + \lambda_{l-1}/2 + \lambda_l/2 \right) Z_k + \sum_{b=k+1}^{l} \lambda_b Z_b. \tag{d1.2}$$

On can deduce the following (d1.3) by (d1.2) and arguments similar to those in Paragraph 4.4.2:

$$C_G^{\mathrm{DI}} = C_K^{\mathrm{DI}}(k-1) \cup C_K^{\mathrm{DI}}(k) \cup C_K^{\mathrm{DI}}(l-1) \cup C_K^{\mathrm{DI}}(l), \tag{d1.3}$$

where C_G^{DI}, $C_K^{\mathrm{DI}}(k-1)$, $C_K^{\mathrm{DI}}(k)$, $C_K^{\mathrm{DI}}(l-1)$ and $C_K^{\mathrm{DI}}(l)$ are defined as follows:

$$\begin{aligned} C_K^{\mathrm{DI}}(x) &:= \left\{ \mathfrak{c}_{\mathfrak{k}}(T) \mid T = \textstyle\sum_{i=1}^{l} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_x > 0 \right\} \\ &\qquad \text{for } x = k-1, k, l-1, l; \\ C_G^{\mathrm{DI}} &:= \{ \mathfrak{c}_{\mathfrak{g}}(T') \mid \mathfrak{c}_{\mathfrak{g}}(T') \text{ is compact with } T' \in \mathscr{W}_{\mathfrak{k}} \}. \end{aligned} \tag{d1.4}$$

Now, let us demonstrate Lemmas 13 and 14 which provide outer involutions ϕ and ψ of $\mathfrak{g}$ satisfying $\phi(C_G^{\mathrm{DI}}) = C_G^{\mathrm{DI}}$, $\phi(C_K^{\mathrm{DI}}(k)) = C_K^{\mathrm{DI}}(k-1)$ and $\phi(C_K^{\mathrm{DI}}(r)) = C_K^{\mathrm{DI}}(r)$ for $r = l-1, l$, and $\psi(C_G^{\mathrm{DI}}) = C_G^{\mathrm{DI}}$, $\psi(C_K^{\mathrm{DI}}(s)) = C_K^{\mathrm{DI}}(s)$ for $s = k-1, k$ and $\psi(C_K^{\mathrm{DI}}(l)) = C_K^{\mathrm{DI}}(l-1)$, respectively. In terms of ϕ and ψ we can reduce (d1.3) to

$$C_G^{\mathrm{DI}} = C_K^{\mathrm{DI}}(k) \cup C_K^{\mathrm{DI}}(l) \cup \phi(C_K^{\mathrm{DI}}(k)) \cup \psi(C_K^{\mathrm{DI}}(l)). \tag{d1.5}$$

Lemma 13 (DI). *There exists an outer involution ϕ of $\mathfrak{g} = \mathfrak{so}(2k, 2l-2k)$, $l \geq 4$ and $2 \leq k \leq l-2$, satisfying $\phi(\mathfrak{k}) \subset \mathfrak{k}$, $\phi(T_{k-1}) = T_k$ and $\phi(T_h) = T_h$ for any $1 \leq h \leq l$ with $h \neq k-1, k$.*

Proof. Define an involutive linear map ϕ of $\mathfrak{t}_{\mathbf{R}}$ by

$${}^t\phi(\alpha_1) := -\mu_d \left(= -\alpha_1 - 2\textstyle\sum_{t=2}^{l-2} \alpha_t - \alpha_{l-1} - \alpha_l \right), \quad {}^t\phi(\alpha_j) := \alpha_j \qquad \text{for } 2 \leq j \leq l.$$

Then it follows from $\alpha_a(Z_b) = \delta_{a,b}$ that

$$\begin{aligned} &\phi(Z_1) = -Z_1, && \phi(Z_c) = -2Z_1 + Z_c \qquad \text{for } 2 \leq c \leq l-2, \\ &\phi(Z_r) = -Z_1 + Z_r \qquad \text{for } r = l-1, l. \end{aligned}$$

This and (d1.1) imply that $\phi(T_{k-1}) = T_k$ and $\phi(T_h) = T_h$ for any $1 \leq h \leq l$ with $h \neq k-1, k$. Accordingly we can get the conclusion because one can extend

ϕ to $\mathfrak{g}$ as involution (ref. the proof of Lemma 10), where we remark that $[\theta, \phi] = 0$ and $\phi(\mathfrak{k}) \subset \mathfrak{k}$ follow from $\phi(Z_k) = -2Z_1 + Z_k$ and $\theta = \exp \pi \,\mathrm{ad}(iZ_k)$. □

Lemma 14 (DI). *There exists an outer involution ψ of $\mathfrak{g} = \mathfrak{so}(2k, 2l-2k)$, $l \geq 4$ and $2 \leq k \leq l-2$, satisfying $\psi(\mathfrak{k}) \subset \mathfrak{k}$, $\psi(T_{l-1}) = T_l$ and $\psi(T_j) = T_j$ for any $1 \leq j \leq l-2$.*

Proof. Define an involutive linear map ψ of $\mathfrak{t}_{\mathbf{R}}$ by

$$^t\psi(\alpha_j) := \alpha_j \quad \text{for } 1 \leq j \leq l-2, \qquad ^t\psi(\alpha_{l-1}) := \alpha_l, \qquad ^t\psi(\alpha_l) := \alpha_{l-1}.$$

One can complete this proof by arguments similar to those in the proof of Lemma 13. □

Now, we are in a position to state

Proposition 6. *The following is the coarse orbit type of symplectic homogeneous space G/H with $G = SO_0(2k, 2l-2k)$, $l \geq 4$ and $2 \leq k \leq l-2$, and H compact:*

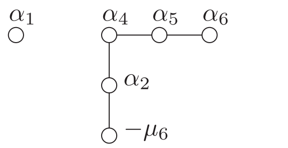
G	H	No.
$SO_0(2k, 2l-2k)$ $l \geq 4$ $2 \leq k \leq l-2$	$U(i_1) \times U(i_2 - i_1) \times \cdots \times U(i_a - i_{a-1}) \times U(k - i_a)$ $\times U(j_1 - k) \times U(j_2 - j_1) \times \cdots \times U(j_b - j_{b-1})$ $\times SO(2l - 2j_b)$ $0 \leq a \leq k-1,\ i_0 := 0,\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_a \leq k-1$ $0 \leq b \leq l-k,\ j_0 := k,\ k+1 \leq j_1 \leq j_2 \leq \cdots \leq j_b \leq l$	1
	$U(m_1) \times U(m_2 - m_1) \times \cdots \times U(m_c - m_{c-1})$ $\times SO(2k - 2m_c) \times U(n_1 - k) \times U(n_2 - n_1)$ $\times \cdots \times U(n_d - n_{d-1}) \times U(l - n_d)$ $0 \leq c \leq k,\ m_0 := 0,\ 1 \leq m_1 \leq m_2 \leq \cdots \leq m_c \leq k$ $0 \leq d \leq l-k-1,\ n_0 := k,$ $k+1 \leq n_1 \leq n_2 \leq \cdots \leq n_d \leq l-1$	2
DI	*Remark.* $SO_0(2k, 2l-2k)/H_1 = SO_0(2l-2k, 2k)/H_2$, *where we assume H_2 to be a subgroup of $SO_0(2l-2k, 2k)$ by identifying $SO_0(2k, 2l-2k)$ with $SO_0(2l-2k, 2k)$. Here H_{i} are the same as in* No.i (i = 1, 2).	

Proof. Refer to the proof of Proposition 4 and (d1.5). Here we remark that H_1 and H_2 come from elements of $C_K^{\mathrm{DI}}(k)$ and $C_K^{\mathrm{DI}}(l)$, respectively (see (d1.4) for $C_K^{\mathrm{DI}}(k)$ and $C_K^{\mathrm{DI}}(l)$). □

4.7. The coarse orbit type of type EII. Our goal in this subsection is to determine the coarse orbit type of symplectic homogeneous spaces G/H with

$G = E_{6(2)}$ and H compact (see Proposition 7). We need to treat an exceptional Lie group in this subsection. For this reason we are going to construct arguments in detail. Notice the proofs of Lemmas 15 and 16.

4.7.1. *A dual basis of $\Pi_{\triangle(\mathfrak{t}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$ and a positive Weyl chamber $\mathscr{W}_{\mathfrak{k}}$.*

	$\mathfrak{g}$	$\mathfrak{k}$	$\Pi_{\triangle(\mathfrak{t}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$
EII	$\mathfrak{e}_{6(2)}$	 $A_5 \times A_1$, $\theta = \exp \pi\, \mathrm{ad}(iZ_3)$	$\{-\mu_6, \alpha_2, \alpha_b\}_{b=4}^{6} \cup \{\alpha_1\}$

Here $\mu_6 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$.

Let $\beta_1 := \alpha_1$, $\beta_2 := -\mu_6$, $\beta_3 := \alpha_2$ and $\beta_b := \alpha_b$ for $4 \leq b \leq 6$. Denote by $\{T_i\}_{i=1}^{6}$ the dual basis of $\{\beta_i\}_{i=1}^{6} = \Pi_{\triangle(\mathfrak{t}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$. Then $\alpha_a(Z_b) = \delta_{a,b}$ yields

$$\begin{aligned} &T_1 = Z_1 - Z_3/2, \qquad T_2 = -Z_3/2, \qquad T_3 = Z_2 - Z_3, \\ &T_4 = -3Z_3/2 + Z_4, \qquad T_5 = -Z_3 + Z_5, \qquad T_6 = -Z_3/2 + Z_6. \end{aligned} \tag{e2.1}$$

In view of (e2.1) we can express an element $T = \sum_{i=1}^{6} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$ as follows:

$$\begin{aligned} T = \sum_{i=1}^{6} \lambda_i T_i = \lambda_1 Z_1 + \lambda_3 Z_2 - (\lambda_1/2 + \lambda_2/2 + \lambda_3 + 3\lambda_4/2 \\ + \lambda_5 + \lambda_6/2) Z_3 + \sum_{b=4}^{6} \lambda_b Z_b. \end{aligned} \tag{e2.2}$$

Here $\mathscr{W}_{\mathfrak{k}} := \{\sum_{i=1}^{6} \lambda_i T_i \mid \lambda_i \geq 0 \text{ for all } 1 \leq i \leq 6\}$. From (e2.2) we will obtain a necessary and sufficient condition for the centralizer $\mathfrak{c}_{\mathfrak{g}}(T)$ of an element $T \in \mathscr{W}_{\mathfrak{k}}$ to be compact in the next paragraph.

4.7.2. *A condition for the centralizer $\mathfrak{c}_{\mathfrak{g}}(T)$ of an element $T \in \mathscr{W}_{\mathfrak{k}}$ to be compact.* Recall that we assume $\mathfrak{e}_{6(2)} = \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ to be given by compact Lie algebra $\mathfrak{e}_6 = \mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ with involution $\theta = \exp \pi\, \mathrm{ad}(iZ_3)$. Since $\mathfrak{p}$ is the (-1)-eigenspace of θ in $\mathfrak{g}$, we have

$$\begin{aligned} &\mathfrak{p}_{\mathbf{C}} = \bigoplus_{\alpha \in \triangle^+(\alpha_3, 1)} \mathrm{span}_{\mathbf{C}}\{X_\alpha\} \oplus \mathrm{span}_{\mathbf{C}}\{X_{-\alpha}\} \qquad \text{(direct sum)}, \\ &\triangle^+(\alpha_3, 1) := \left\{ \sum_{i=1}^{6} n_i \alpha_i \in \triangle^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}}) \,\middle|\, n_3 = 1 \right\} \end{aligned} \tag{e2.3}$$

(see Subsection 4.2). Here $\mathfrak{p}_{\mathbf{C}}$ is the (-1)-eigenspace of θ in $\mathfrak{g}_{\mathbf{C}} = \mathfrak{e}_6^{\mathbf{C}}$. We want to clarify a necessary and sufficient condition for the centralizer $\mathfrak{c}_{\mathfrak{g}}(T)$ of an element $T \in \mathscr{W}_{\mathfrak{k}}$ to be compact. Lemma 7 implies that for $T \in \mathscr{W}_{\mathfrak{k}}$, $\mathfrak{c}_{\mathfrak{g}}(T)$ is compact if and only if T satisfies

$$\alpha(T) \neq 0 \qquad \text{for all } \alpha \in \triangle^+(\alpha_3, 1) \tag{e2.4}$$

because of $[T, X_\alpha] = \alpha(T) \cdot X_\alpha$ and (e2.3). Now, let us clarify the necessary and sufficient condition:

Lemma 15 (EII). *With the above setting; for $T \in \mathscr{W}_{\mathfrak{k}}$, the centralizer $\mathfrak{c}_{\mathfrak{g}}(T)$ is compact if and only if* (i) *"$\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_4 > 0$ or $\lambda_6 > 0$" and* (ii) $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{c}_{\mathfrak{k}}(T)$.

Proof. ($\Rightarrow$): Suppose that $\mathfrak{c}_{\mathfrak{g}}(T)$ is compact. Note that $\beta := \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ $(= \varepsilon_2 + \varepsilon_4)$ belongs to $\triangle^+(\alpha_3, 1)$ (cf. Plate V in Bourbaki [Bu, p. 275]). A direct computation, combined with (e2.2) and $\alpha_a(Z_b) = \delta_{a,b}$, enables us to have

$$\beta(T) = -(\lambda_1 + \lambda_2 + \lambda_4 + \lambda_6)/2.$$

Therefore (e2.4) assure that $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_6 \neq 0$, so that "$\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_4 > 0$ or $\lambda_6 > 0$" because $\lambda_i \geq 0$ for every $1 \leq i \leq 6$. Besides, $\mathfrak{c}_{\mathfrak{g}}(T) = \mathfrak{c}_{\mathfrak{k}}(T)$ follows from Lemma 7. ($\Leftarrow$): Clear. □

Define five sets $C_K^{\mathrm{EII}}(1)$, $C_K^{\mathrm{EII}}(2)$, $C_K^{\mathrm{EII}}(4)$, $C_K^{\mathrm{EII}}(6)$ and C_G^{EII} by

$$\begin{aligned} C_K^{\mathrm{EII}}(x) &:= \left\{ \mathfrak{c}_{\mathfrak{k}}(T) \mid T = \sum\nolimits_{i=1}^{6} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_x > 0 \right\} \\ &\quad \text{for } x = 1, 2, 4, 6; \\ C_G^{\mathrm{EII}} &:= \{ \mathfrak{c}_{\mathfrak{g}}(T') \mid \mathfrak{c}_{\mathfrak{g}}(T') \text{ is compact with } T' \in \mathscr{W}_{\mathfrak{k}} \}. \end{aligned} \tag{e2.5}$$

Our aim is to determine all elements of C_G^{EII} up to isomorphism. Lemma 15 means that $C_G^{\mathrm{EII}} \subset C_K^{\mathrm{EII}}(1) \cup C_K^{\mathrm{EII}}(2) \cup C_K^{\mathrm{EII}}(4) \cup C_K^{\mathrm{EII}}(6)$. The converse inclusion also holds by virtue of the following Lemma 16; and hence

$$C_G^{\mathrm{EII}} = C_K^{\mathrm{EII}}(1) \cup C_K^{\mathrm{EII}}(2) \cup C_K^{\mathrm{EII}}(4) \cup C_K^{\mathrm{EII}}(6). \tag{e2.6}$$

This implies that for the aim, it is enough to determine all elements of $C_K^{\mathrm{EII}}(1)$, $C_K^{\mathrm{EII}}(2)$, $C_K^{\mathrm{EII}}(4)$ and $C_K^{\mathrm{EII}}(6)$.

Lemma 16 (EII). *With the above setting; let $T = \sum_{i=1}^{6} \lambda_i T_i$ be an element of $\mathscr{W}_{\mathfrak{k}}$. Suppose that $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_4 > 0$ or $\lambda_6 > 0$. Then, there exists an element $T' \in \mathscr{W}_{\mathfrak{k}}$ satisfying* (i) *$\mathfrak{c}_{\mathfrak{g}}(T')$ is compact and* (ii) $\mathfrak{c}_{\mathfrak{g}}(T') = \mathfrak{c}_{\mathfrak{k}}(T)$.

Proof. Case $\lambda_4 > 0$: First, let us consider the case $\lambda_4 > 0$. Divide $\triangle^+(\alpha_3, 1)$ into two subsets $\triangle^+(\alpha_3, 1 : \alpha_4, \leq 1)$ and $\triangle^+(\alpha_3, 1 : \alpha_4, \geq 2)$:

$$\triangle^+(\alpha_3, 1) = \triangle^+(\alpha_3, 1 : \alpha_4, \leq 1) \sqcup \triangle^+(\alpha_3, 1 : \alpha_4, \geq 2) \qquad \text{(direct sum)},$$

$$\triangle^+(\alpha_3, 1 : \alpha_4, \leq 1) := \Big\{ \sum\nolimits_{i=1}^{6} n_i\alpha_i \in \triangle^+(\alpha_3, 1) \,|\, n_4 \leq 1 \Big\},$$

$$\triangle^+(\alpha_3, 1 : \alpha_4, \geq 2) := \Big\{ \sum\nolimits_{i=1}^{6} m_i\alpha_i \in \triangle^+(\alpha_3, 1) \,|\, m_4 \geq 2 \Big\}.$$

For $\beta = \sum_{a=1}^{2} n_a\alpha_a + \alpha_3 + \sum_{b=4}^{6} n_b\alpha_b \in \triangle^+(\alpha_3, 1 : \alpha_4, \leq 1)$ we have

$$\beta(T) = \lambda_1 n_1 + \lambda_3 n_2 - (\lambda_1/2 + \lambda_2/2 + \lambda_3 + 3\lambda_4/2 + \lambda_5 + \lambda_6/2) + \sum_{b=4}^{6} \lambda_b n_b \tag{e2.7}$$

by virtue of (e2.2) and $\alpha_a(Z_b) = \delta_{a,b}$. Similarly one has

$$\gamma(T) = \lambda_1 m_1 + \lambda_3 m_2 - (\lambda_1/2 + \lambda_2/2 + \lambda_3 + 3\lambda_4/2 + \lambda_5 + \lambda_6/2) + \sum_{b=4}^{6} \lambda_b m_b \tag{e2.8}$$

for $\gamma = \sum_{a=1}^{2} m_a\alpha_a + \alpha_3 + \sum_{b=4}^{6} m_b\alpha_b \in \triangle^+(\alpha_3, 1 : \alpha_4, \geq 2)$. Notice that the coefficient of λ_4 is negative in (e2.7) and is positive in (e2.8), indeed it is $(n_4 - 3/2)$ in (e2.7) and is $(m_4 - 3/2)$ in (e2.8). This assures that if λ_4 is sufficiently large value, then one can assert that $\beta(T) < 0$ for all $\beta \in \triangle^+(\alpha_3, 1 : \alpha_4, \leq 1)$ and $\gamma(T) > 0$ for all $\gamma \in \triangle^+(\alpha_3, 1 : \alpha_4, \geq 2)$—that is, $\alpha(T) \neq 0$ for every $\alpha \in \triangle^+(\alpha_3, 1)$, where we remark that $\triangle^+(\alpha_3, 1) = \triangle^+(\alpha_3, 1 : \alpha_4, \leq 1) \sqcup \triangle^+(\alpha_3, 1 : \alpha_4, \geq 2)$ is a finite set. Accordingly $T' := \sum_{c=1}^{3} \lambda_c T_c + \zeta\lambda_4 T_4 + \sum_{d=5}^{6} \lambda_d T_d$ is an element of $\mathscr{W}_{\mathfrak{k}}$ and satisfies $\alpha(T') \neq 0$ for every $\alpha \in \triangle^+(\alpha_3, 1)$, if we take a sufficiently large number $\zeta > 0$. Hence Lemma 7 and (e2.4) imply that $\mathfrak{c}_{\mathfrak{g}}(T')$ is compact and $\mathfrak{c}_{\mathfrak{g}}(T') = \mathfrak{c}_{\mathfrak{k}}(T')$. In addition, Remark 9 tells us that $\mathfrak{c}_{\mathfrak{g}}(T') = \mathfrak{c}_{\mathfrak{k}}(T') = \mathfrak{c}_{\mathfrak{k}}(T)$.

Case $\lambda_1 > 0$, $\lambda_2 > 0$ or $\lambda_6 > 0$: One can get the conclusion in Cases $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_6 > 0$ by taking $\triangle^+(\alpha_3, 1 : \alpha_1, 0) \sqcup \triangle^+(\alpha_3, 1 : \alpha_1, 1)$, $\triangle^+(\alpha_3, 1)$, and $\triangle^+(\alpha_3, 1 : \alpha_6, 0) \sqcup \triangle^+(\alpha_3, 1 : \alpha_6, 1)$ into consideration, respectively. Here $\triangle^+(\alpha_3, 1 : \alpha_j, p) := \{\sum_{i=1}^{6} n_i\alpha_i \in \triangle^+(\alpha_3, 1) \,|\, n_j = p\}$ for $j = 1, 6$ and $p = 0, 1$. $\square$

We can reduce (e2.6) to

$$C_G^{\mathrm{EII}} = C_K^{\mathrm{EII}}(1) \cup C_K^{\mathrm{EII}}(2) \cup C_K^{\mathrm{EII}}(4) \cup \phi(C_K^{\mathrm{EII}}(2)) \tag{e2.9}$$

by proving

Lemma 17 (EII). *There exists an outer involution ϕ of $\mathfrak{g} = \mathfrak{e}_{6(2)}$ satisfying $\phi(\mathfrak{k}) \subset \mathfrak{k}$, $\phi(T_1) = T_1$, $\phi(T_2) = T_6$, $\phi(T_3) = T_5$ and $\phi(T_4) = T_4$.*

Proof. Define an involutive linear map ϕ of $\mathfrak{t}_{\mathbf{R}}$ by

$$ {}^t\phi(\alpha_1) := \alpha_1, \quad {}^t\phi(\alpha_2) := \alpha_5, \quad {}^t\phi(\alpha_3) := \alpha_3, \quad {}^t\phi(\alpha_4) := \alpha_4, $$
$$ {}^t\phi(\alpha_5) := \alpha_2, \quad {}^t\phi(\alpha_6) := -\mu_6(= -\alpha_1 - 2\alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6). $$

See the proof of Lemma 10. □

4.7.3. *A result for type EII.* The proof of Proposition 4, together with (e2.9), allows us to assert

Proposition 7. *The following is the coarse orbit type of symplectic homogeneous space G/H with $G = E_{6(2)}$ and H compact:*

G	H	No.
$E_{6(2)}$	$A_5 \times T,\ A_4 \times A_1 \times T,\ A_2 \times A_2 \times A_1 \times T$	1
	$A_4 \times T^2,\ A_3 \times A_1 \times T^2,\ A_2 \times A_2 \times T^2,\ A_2 \times A_1 \times A_1 \times T^2$	2
	$A_3 \times T^3,\ A_2 \times A_1 \times T^3,\ A_1 \times A_1 \times A_1 \times T^3$	3
	$A_2 \times T^4,\ A_1 \times A_1 \times T^4$	4
	$A_1 \times T^5$	5
EII	T^6	6

4.8. The coarse orbit type of type EV. In this subsection we determine the coarse orbit type of symplectic homogeneous spaces G/H with $G = E_{7(7)}$ and H compact (see Proposition 8).

	$\mathfrak{g}$	$\mathfrak{k}$	$\Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$
EV	$\mathfrak{e}_{7(7)}$	$-\mu_7$ ○—○ α_1 —○ α_3 —○ α_4 —○ α_5 —○ α_6 —○ α_7 $A_7,\ \theta = \exp \pi\, \mathrm{ad}(iZ_2)$	$\{-\mu_7, \alpha_1, \alpha_b\}_{b=3}^{7}$

Here $\mu_7 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$.

Our arguments will be similar to those in Subsection 4.7. Let $\beta_1 := -\mu_7$, $\beta_2 := \alpha_1$ and $\beta_b := \alpha_b$ for $3 \leq b \leq 7$. Denote by $\{T_i\}_{i=1}^{7}$ the dual basis of $\{\beta_i\}_{i=1}^{7} = \Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$. Then one can express T_i as follows:

$$\begin{aligned} &T_1 = -Z_2/2, && T_2 = Z_1 - Z_2, && T_3 = -3Z_2/2 + Z_3, \\ &T_4 = -2Z_2 + Z_4, && T_5 = -3Z_2/2 + Z_5, && T_6 = -Z_2 + Z_6, \\ &T_7 = -Z_2/2 + Z_7 \end{aligned} \tag{e5.1}$$

by means of the dual basis $\{Z_i\}_{i=1}^7$ of $\{\alpha_i\}_{i=1}^7 = \Pi_{\triangle(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$. It follows from (e5.1) that an element $T = \sum_{i=1}^7 \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$ can be rewritten as

$$\begin{aligned} T = \sum_{i=1}^{7} \lambda_i T_i = \lambda_2 Z_1 - (\lambda_1/2 + \lambda_2 + 3\lambda_3/2 + 2\lambda_4 \\ + 3\lambda_5/2 + \lambda_6 + \lambda_7/2) Z_2 + \sum_{b=3}^{7} \lambda_b T_b. \end{aligned} \tag{e5.2}$$

Here $\mathscr{W}_{\mathfrak{k}} := \{\sum_{i=1}^7 \lambda_i T_i \,|\, \lambda_i \geq 0 \text{ for all } 1 \leq i \leq 7\}$. By (e5.2) and arguments similar to those in Paragraph 4.7.2 one can conclude that

$$C_G^{\mathrm{EV}} = C_K^{\mathrm{EV}}(1) \cup C_K^{\mathrm{EV}}(3) \cup C_K^{\mathrm{EV}}(5) \cup C_K^{\mathrm{EV}}(7), \tag{e5.3}$$

where $C_K^{\mathrm{EV}}(1)$, $C_K^{\mathrm{EV}}(3)$, $C_K^{\mathrm{EV}}(5)$, $C_K^{\mathrm{EV}}(7)$ and C_G^{EV} are defined as follows:

$$\begin{aligned} C_K^{\mathrm{EV}}(x) := \left\{ c_{\mathfrak{k}}(T) \,\middle|\, T = \textstyle\sum_{i=1}^7 \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_x > 0 \right\} \\ \text{for } x = 1, 3, 5, 7; \\ C_G^{\mathrm{EV}} := \{ c_{\mathfrak{g}}(T') \,|\, c_{\mathfrak{g}}(T') \text{ is compact with } T' \in \mathscr{W}_{\mathfrak{k}} \}. \end{aligned} \tag{e5.4}$$

The following Lemma 18 allows us to reduce (e5.3) to

$$C_G^{\mathrm{EV}} = C_K^{\mathrm{EV}}(1) \cup C_K^{\mathrm{EV}}(3) \cup \phi(C_K^{\mathrm{EV}}(1)) \cup \phi(C_K^{\mathrm{EV}}(3)). \tag{e5.5}$$

Lemma 18 (EV). *There exists an outer involution ϕ of $\mathfrak{g} = \mathfrak{e}_{7(7)}$ satisfying $\phi(\mathfrak{k}) \subset \mathfrak{k}$, $\phi(T_1) = T_7$, $\phi(T_2) = T_6$, $\phi(T_3) = T_5$ and $\phi(T_4) = T_4$.*

Proof. Define an involutive linear map ϕ of $\mathfrak{t}_{\mathbf{R}}$ by

$$\begin{aligned} &{}^t\phi(\alpha_1) := \alpha_6, \quad {}^t\phi(\alpha_2) := \alpha_2, \quad {}^t\phi(\alpha_3) := \alpha_5, \quad {}^t\phi(\alpha_4) := \alpha_4, \quad {}^t\phi(\alpha_5) := \alpha_3, \\ &{}^t\phi(\alpha_6) := \alpha_1, \quad {}^t\phi(\alpha_7) := -\mu_7 (= -2\alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7). \end{aligned}$$

See the proof of Lemma 10 for the rest of proof. □

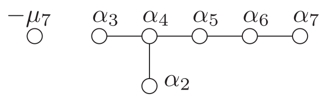
Now, let us state

PROPOSITION 8. *The following is the coarse orbit type of symplectic homogeneous space G/H with $G = E_{7(7)}$ and H compact:*

G	H	No.
$E_{7(7)}$	$A_6 \times T,\ A_4 \times A_2 \times T$	1
	$A_5 \times T^2,\ A_4 \times A_1 \times T^2,\ A_3 \times A_2 \times T^2,\ A_2 \times A_2 \times A_1 \times T^2$	2
	$A_4 \times T^3,\ A_3 \times A_1 \times T^3,\ A_2 \times A_2 \times T^3,\ A_2 \times A_1 \times A_1 \times T^3$	3
	$A_3 \times T^4,\ A_2 \times A_1 \times T^4,\ A_1 \times A_1 \times A_1 \times T^4$	4
	$A_2 \times T^5,\ A_1 \times A_1 \times T^5$	5
	$A_1 \times T^6$	6
EV	T^7	7

PROOF. The proof of Proposition 4 and (e5.5) imply that it is enough to determine all elements of $C_K^{\mathrm{EV}}(1)$ and $C_K^{\mathrm{EV}}(3)$. We determine the elements by Proposition 3 and direct computations. □

4.9. The coarse orbit type of type EVI. This subsection is devoted to determining the coarse orbit type of symplectic homogeneous spaces G/H with $G = E_{7(-5)}$ and H compact (see Proposition 9).

	$\mathfrak{g}$	$\mathfrak{k}$	$\Pi_{\triangle(\mathfrak{k}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})}$
EVI	$\mathfrak{e}_{7(-5)}$	 $A_1 \times D_6,\ \theta = \exp \pi\, \mathrm{ad}(iZ_1)$	$\{-\mu_7\} \cup \{\alpha_b\}_{b=2}^{7}$

Here $\mu_7 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$.

Arguments stated below are similar to those in Subsection 4.7. Let $\beta_1 := -\mu_7$, $\beta_b := \alpha_b$ for $2 \le b \le 7$. Denote by $\{T_i\}_{i=1}^{7}$ the dual basis of $\{\beta_i\}_{i=1}^{7} = \Pi_{\triangle(\mathfrak{k}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})}$. Then one can express T_i as follows:

$$
\begin{aligned}
&T_1 = -Z_1/2, && T_2 = -Z_1 + Z_2, && T_3 = -3Z_1/2 + Z_3,\\
&T_4 = -2Z_1 + Z_4, && T_5 = -3Z_1/2 + Z_5, && T_6 = -Z_1 + Z_6,\\
&T_7 = -Z_1/2 + Z_7
\end{aligned}
\tag{e6.1}
$$

by means of the dual basis $\{Z_i\}_{i=1}^{7}$ of $\{\alpha_i\}_{i=1}^{7} = \Pi_{\triangle(\mathfrak{g}_{\mathbf{C}},\mathfrak{t}_{\mathbf{C}})}$. Hence, an element $T = \sum_{i=1}^{7} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$ is expressed as

$$T = \sum_{i=1}^{7} \lambda_i T_i = -(\lambda_1/2 + \lambda_2 + 3\lambda_3/2 + 2\lambda_4 + 3\lambda_5/2 + \lambda_6 + \lambda_7/2)Z_1 + \sum_{b=2}^{7} \lambda_b Z_b, \tag{e6.2}$$

where $\mathscr{W}_{\mathfrak{k}} := \{\sum_{i=1}^{7} \lambda_i T_i \mid \lambda_i \geq 0 \text{ for all } 1 \leq i \leq 7\}$. We can see that

$$C_G^{\mathrm{EVI}} = C_K^{\mathrm{EVI}}(1) \cup C_K^{\mathrm{EVI}}(3) \cup C_K^{\mathrm{EVI}}(5) \cup C_K^{\mathrm{EVI}}(7) \tag{e6.3}$$

by (e6.2) and arguments similar to those in Paragraph 4.7.2. Here $C_K^{\mathrm{EVI}}(1)$, $C_K^{\mathrm{EVI}}(3)$, $C_K^{\mathrm{EVI}}(5)$, $C_K^{\mathrm{EVI}}(7)$ and C_G^{EVI} are defined as follows:

$$C_K^{\mathrm{EVI}}(x) := \left\{\mathfrak{c}_{\mathfrak{k}}(T) \mid T = \sum_{i=1}^{7} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_x > 0\right\} \qquad \text{for } x = 1, 3, 5, 7;$$

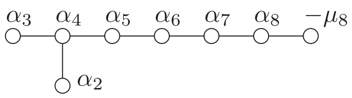
$$C_G^{\mathrm{EVI}} := \{\mathfrak{c}_{\mathfrak{g}}(T') \mid \mathfrak{c}_{\mathfrak{g}}(T') \text{ is compact with } T' \in \mathscr{W}_{\mathfrak{k}}\}.$$

One can determine all elements of $C_K^{\mathrm{EVI}}(1)$, $C_K^{\mathrm{EVI}}(3)$, $C_K^{\mathrm{EVI}}(5)$ and $C_K^{\mathrm{EVI}}(7)$ by Proposition 3 and direct computations. Therefore, the proof of Proposition 4 and (e6.3) imply

PROPOSITION 9. *The following is the coarse orbit type of symplectic homogeneous space G/H with $G = E_{7(-5)}$ and H compact:*

G	H	No.
$E_{7(-5)}$	$A_5 \times A_1 \times T,\ A_3 \times A_2 \times A_1 \times T,\ A_1 \times D_5 \times T,\ D_6 \times T$	1
	$A_5 \times T^2,\ A_4 \times A_1 \times T^2,\ A_3 \times A_2 \times T^2,\ A_3 \times A_1 \times A_1 \times T^2$ $A_2 \times A_2 \times A_1 \times T^2,\ A_2 \times A_1 \times A_1 \times A_1 \times T^2$ $A_1 \times D_4 \times T^2,\ D_5 \times T^2$	2
	$A_4 \times T^3,\ A_3 \times A_1 \times T^3,\ A_2 \times A_2 \times T^3,\ A_2 \times A_1 \times A_1 \times T^3$ $A_1 \times A_1 \times A_1 \times A_1 \times T^3,\ D_4 \times T^3$	3
	$A_3 \times T^4,\ A_2 \times A_1 \times T^4,\ A_1 \times A_1 \times A_1 \times T^4$	4
	$A_2 \times T^5,\ A_1 \times A_1 \times T^5$	5
	$A_1 \times T^6$	6
EVI	T^7	7

4.10. The coarse orbit type of type EVIII. In this subsection we determine the coarse orbit type of symplectic homogeneous spaces G/H with $G = E_{8(8)}$ and H compact (see Proposition 10).

	$\mathfrak{g}$	$\mathfrak{k}$	$\Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$
EVIII	$\mathfrak{e}_{8(8)}$	 $D_8,\ \theta = \exp \pi\, \mathrm{ad}(iZ_1)$	$\{-\mu_8, \alpha_b\}_{b=2}^{8}$

Here $\mu_8 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$.

Our arguments in this subsection will be similar to those in Subsection 4.7. Let $\beta_1 := -\mu_8$ and $\beta_b := \alpha_{10-b}$ for $2 \le b \le 8$. We denote by $\{T_i\}_{i=1}^{8}$ the dual basis of $\{\beta_i\}_{i=1}^{8} = \Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$. Then $\alpha_a(Z_b) = \delta_{a,b}$ yields

$$\begin{aligned} &T_1 = -Z_1/2, && T_2 = -Z_1 + Z_8, && T_3 = -3Z_1/2 + Z_7,\\ &T_4 = -2Z_1 + Z_6, && T_5 = -5Z_1/2 + Z_5, && T_6 = -3Z_1 + Z_4,\\ &T_7 = -2Z_1 + Z_3, && T_8 = -3Z_1/2 + Z_2. \end{aligned} \tag{e8.1}$$

So an element $T = \sum_{i=1}^{8} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$ can be expressed as follows:

$$\begin{aligned} T = \sum_{i=1}^{8} \lambda_i T_i = -(&\lambda_1/2 + \lambda_2 + 3\lambda_3/2 + 2\lambda_4 + 5\lambda_5/2 + 3\lambda_6 \\ &+ 2\lambda_7 + 3\lambda_8/2)Z_1 + \sum_{b=2}^{8} \lambda_{10-b} Z_b, \end{aligned} \tag{e8.2}$$

where $\mathscr{W}_{\mathfrak{k}} := \{\sum_{i=1}^{8} \lambda_i T_i \mid \lambda_i \ge 0 \text{ for all } 1 \le i \le 8\}$. Therefore one can confirm

$$C_G^{\mathrm{EVIII}} = C_K^{\mathrm{EVIII}}(1) \cup C_K^{\mathrm{EVIII}}(3) \cup C_K^{\mathrm{EVIII}}(5) \cup C_K^{\mathrm{EVIII}}(8) \tag{e8.3}$$

by arguments similar to those in Paragraph 4.7.2. Here $C_K^{\mathrm{EVIII}}(1)$, $C_K^{\mathrm{EVIII}}(3)$, $C_K^{\mathrm{EVIII}}(5)$, $C_K^{\mathrm{EVIII}}(8)$ and C_G^{EVIII} are defined as follows:

$$C_K^{\mathrm{EVIII}}(x) := \left\{ \mathfrak{c}_{\mathfrak{k}}(T) \mid T = \sum_{i=1}^{8} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_x > 0 \right\} \qquad \text{for } x = 1, 3, 5, 8;$$

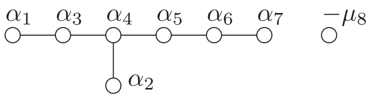
$$C_G^{\mathrm{EVIII}} := \{\mathfrak{c}_{\mathfrak{g}}(T') \mid \mathfrak{c}_{\mathfrak{g}}(T') \text{ is compact with } T' \in \mathscr{W}_{\mathfrak{k}}\}.$$

Proposition 3 enables us to determine all elements of $C_K^{\mathrm{EVIII}}(1)$, $C_K^{\mathrm{EVIII}}(3)$, $C_K^{\mathrm{EVIII}}(5)$ and $C_K^{\mathrm{EVIII}}(8)$; and hence we can conclude the following proposition by the proof of Proposition 4:

Proposition 10. *The following is the coarse orbit type of symplectic homogeneous space G/H with $G = E_{8(8)}$ and H compact:*

G	H	No.
$E_{8(8)}$	$A_7 \times T,\ A_4 \times A_3 \times T,\ A_2 \times D_5 \times T,\ D_7 \times T$	1
	$A_6 \times T^2,\ A_5 \times A_1 \times T^2,\ A_4 \times A_2 \times T^2,\ A_4 \times A_1 \times A_1 \times T^2$ $A_3 \times A_3 \times T^2,\ A_3 \times A_2 \times A_1 \times T^2,\ A_2 \times A_2 \times A_1 \times A_1 \times T^2$ $A_2 \times D_4 \times T^2,\ A_1 \times D_5 \times T^2,\ D_6 \times T^2$	2
	$A_5 \times T^3,\ A_4 \times A_1 \times T^3,\ A_3 \times A_2 \times T^3,\ A_3 \times A_1 \times A_1 \times T^3$ $A_2 \times A_2 \times A_1 \times T^3,\ A_2 \times A_1 \times A_1 \times A_1 \times T^3,\ A_1 \times D_4 \times T^3$ $D_5 \times T^3$	3
	$A_4 \times T^4,\ A_3 \times A_1 \times T^4,\ A_2 \times A_2 \times T^4,\ A_2 \times A_1 \times A_1 \times T^4$ $A_1 \times A_1 \times A_1 \times A_1 \times T^4,\ D_4 \times T^4$	4
	$A_3 \times T^5,\ A_2 \times A_1 \times T^5,\ A_1 \times A_1 \times A_1 \times T^5$	5
	$A_2 \times T^6,\ A_1 \times A_1 \times T^6$	6
	$A_1 \times T^7$	7
EVIII	T^8	8

4.11. The coarse orbit type of type EIX. In this subsection we determine the coarse orbit type of symplectic homogeneous spaces G/H with $E_{8(-24)}$ and H compact (see Proposition 11).

	$\mathfrak{g}$	$\mathfrak{k}$	$\Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$
EIX	$\mathfrak{e}_{8(-24)}$	 $A_1 \times E_7,\ \theta = \exp \pi\, \mathrm{ad}(iZ_8)$	$\{-\mu_8\} \cup \{\alpha_a\}_{a=1}^7$

Here $\mu_8 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$.

Our arguments are similar to those in Subsection 4.7. Let $\beta_a := \alpha_a$ for $1 \le a \le 7$ and $\beta_8 := -\mu_8$. Denote by $\{T_i\}_{i=1}^8$ the dual basis of $\{\beta_i\}_{i=1}^8 = \Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$. Then it follows from $\alpha_a(Z_b) = \delta_{a,b}$ that

$$\begin{aligned} &T_1 = Z_1 - Z_8, && T_2 = Z_2 - 3Z_8/2, && T_3 = Z_3 - 2Z_8, \\ &T_4 = Z_4 - 3Z_8, && T_5 = Z_5 - 5Z_8/2, && T_6 = Z_6 - 2Z_8, \\ &T_7 = Z_7 - 3Z_8/2, && T_8 = -Z_8/2. \end{aligned} \tag{e9.1}$$

This enables us to rewrite an element $T = \sum_{i=1}^{8} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$ as follows:

$$T = \sum_{i=1}^{8} \lambda_i T_i = \sum_{a=1}^{7} \lambda_a Z_a - (\lambda_1 + 3\lambda_2/2 + 2\lambda_3 + 3\lambda_4 + 5\lambda_5/2 + 2\lambda_6 + 3\lambda_7/2 + \lambda_8/2) Z_8. \qquad \text{(e9.2)}$$

Here $\mathscr{W}_{\mathfrak{k}} := \{\sum_{i=1}^{8} \lambda_i T_i \mid \lambda_i \geq 0 \text{ for all } 1 \leq i \leq 8\}$. Accordingly we deduce

$$C_G^{\mathrm{EIX}} = C_K^{\mathrm{EIX}}(2) \cup C_K^{\mathrm{EIX}}(5) \cup C_K^{\mathrm{EIX}}(7) \cup C_K^{\mathrm{EIX}}(8) \qquad \text{(e9.3)}$$

by arguments similar to those in Paragraph 4.7.2. Here $C_K^{\mathrm{EIX}}(2)$, $C_K^{\mathrm{EIX}}(5)$, $C_K^{\mathrm{EIX}}(7)$, $C_K^{\mathrm{EIX}}(8)$ and C_G^{EIX} are give by

$$C_K^{\mathrm{EIX}}(x) := \left\{ \mathfrak{c}_{\mathfrak{k}}(T) \mid T = \sum_{i=1}^{8} \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_x > 0 \right\} \qquad \text{for } x = 2, 5, 7, 8;$$

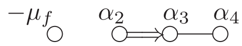
$$C_G^{\mathrm{EIX}} := \{ \mathfrak{c}_{\mathfrak{g}}(T') \mid \mathfrak{c}_{\mathfrak{g}}(T') \text{ is compact with } T' \in \mathscr{W}_{\mathfrak{k}} \}.$$

One can determine all elements of $C_K^{\mathrm{EIX}}(2)$, $C_K^{\mathrm{EIX}}(5)$, $C_K^{\mathrm{EIX}}(7)$ and $C_K^{\mathrm{EIX}}(8)$ by Proposition 3 and direct computations; and therefore the proof of Proposition 4 and (e9.3) allow us to conclude

Proposition 11. *The following is the coarse orbit type of symplectic homogeneous space G/H with $G = E_{8(-24)}$ and H compact:*

G	H	No.
$E_{8(-24)}$	$A_6 \times A_1 \times T$, $A_4 \times A_2 \times A_1 \times T$, $A_1 \times E_6 \times T$, $E_7 \times T$	1
	$A_6 \times T^2$, $A_5 \times A_1 \times T^2$, $A_4 \times A_2 \times T^2$, $A_4 \times A_1 \times A_1 \times T^2$ $A_3 \times A_2 \times A_1 \times T^2$, $A_2 \times A_2 \times A_1 \times A_1 \times T^2$, $A_1 \times D_5 \times T^2$ $D_6 \times T^2$, $E_6 \times T^2$	2
	$A_5 \times T^3$, $A_4 \times A_1 \times T^3$, $A_3 \times A_2 \times T^3$, $A_3 \times A_1 \times A_1 \times T^3$ $A_2 \times A_2 \times A_1 \times T^3$, $A_2 \times A_1 \times A_1 \times A_1 \times T^3$, $A_1 \times D_4 \times T^3$ $D_5 \times T^3$	3
	$A_4 \times T^4$, $A_3 \times A_1 \times T^4$, $A_2 \times A_2 \times T^4$, $A_2 \times A_1 \times A_1 \times T^4$ $A_1 \times A_1 \times A_1 \times A_1 \times T^4$, $D_4 \times T^4$	4
	$A_3 \times T^5$, $A_2 \times A_1 \times T^5$, $A_1 \times A_1 \times A_1 \times T^5$	5
	$A_2 \times T^6$, $A_1 \times A_1 \times T^6$	6
	$A_1 \times T^7$	7
EIX	T^8	8

4.12. The coarse orbit type of type FI. Our goal in this subsection is to determine the coarse orbit type of symplectic homogeneous spaces G/H with $G = F_{4(4)}$ and H compact (see Proposition 12).

	$\mathfrak{g}$	$\mathfrak{k}$	$\Pi_{\triangle(\mathfrak{k}_{\mathbf{C}},\mathfrak{t}_{\mathbf{C}})}$
FI	$\mathfrak{f}_{4(4)}$	 $A_1 \times C_3,\ \theta = \exp \pi\, \mathrm{ad}(iZ_1)$	$\{-\mu_f\} \cup \{\alpha_b\}_{b=2}^4$ $\mu_f = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$

Our arguments will be similar to those in Subsection 4.7. Let $\beta_1 := -\mu_f$ and $\beta_b := \alpha_b$ for $2 \le b \le 4$. Denote by $\{T_i\}_{i=1}^4$ the dual basis of $\{\beta_i\}_{i=1}^4 = \Pi_{\triangle(\mathfrak{k}_{\mathbf{C}},\mathfrak{t}_{\mathbf{C}})}$. Then one has

$$\begin{aligned} &T_1 = -Z_1/2, \qquad T_2 = -3Z_1/2 + Z_2, \qquad T_3 = -2Z_1 + Z_3, \\ &T_4 = -Z_1 + Z_4 \end{aligned} \tag{f1.1}$$

in terms of $\alpha_a(Z_b) = \delta_{a,b}$. It follows from (f1.1) that an element $T = \sum_{i=1}^4 \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$ can be rewritten as

$$T = \sum_{i=1}^4 \lambda_i T_i = -(\lambda_1/2 + 3\lambda_2/2 + 2\lambda_3 + \lambda_4)Z_1 + \sum_{b=2}^4 \lambda_b Z_b, \tag{f1.2}$$

where $\mathscr{W}_{\mathfrak{k}} := \{\sum_{i=1}^4 \lambda_i T_i \mid \lambda_i \ge 0 \text{ for all } 1 \le i \le 4\}$. Hence one can confirm that

$$C_G^{\mathrm{FI}} = C_K^{\mathrm{FI}}(1) \cup C_K^{\mathrm{FI}}(2) \tag{f1.3}$$

by arguments similar to those in Paragraph 4.7.2. Here we define sets $C_K^{\mathrm{FI}}(1)$, $C_K^{\mathrm{FI}}(2)$ and C_G^{FI} by

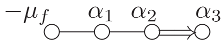
$$\begin{aligned} &C_K^{\mathrm{FI}}(1) := \Big\{ \mathfrak{c}_{\mathfrak{k}}(T) \mid T = \textstyle\sum_{i=1}^4 \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_1 > 0 \Big\}, \\ &C_K^{\mathrm{FI}}(2) := \Big\{ \mathfrak{c}_{\mathfrak{k}}(T) \mid T = \textstyle\sum_{i=1}^4 \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_2 > 0 \Big\}, \\ &C_G^{\mathrm{FI}} := \{ \mathfrak{c}_{\mathfrak{g}}(T') \mid \mathfrak{c}_{\mathfrak{g}}(T') \text{ is compact with } T' \in \mathscr{W}_{\mathfrak{k}} \}, \end{aligned}$$

respectively. This (f1.3) gives us

Proposition 12. *The following is the coarse orbit type of symplectic homogeneous space G/H with $G = F_{4(4)}$ and H compact:*

G	H	No.
$F_{4(4)}$	$A_2 \times A_1 \times T,\ C_3 \times T$	1
	$A_2 \times T^2,\ A_1 \times A_1 \times T^2,\ B_2 \times T^2$	2
	$A_1 \times T^3$	3
FI	T^4	4

4.13. The coarse orbit type of type FII. In this subsection we determine the coarse orbit type of symplectic homogeneous spaces G/H with $G = F_{4(-20)}$ and H compact (see Proposition 13).

	$\mathfrak{g}$	$\mathfrak{k}$	$\Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$
FII	$\mathfrak{f}_{4(-20)}$	 $B_4,\ \theta = \exp \pi\, \mathrm{ad}(iZ_4)$	$\{-\mu_f, \alpha_a\}_{a=1}^3$ $\mu_f = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$

Arguments stated below are similar to those in Subsection 4.7. Let $\beta_1 := -\mu_f$ and $\beta_b := \alpha_{b-1}$ for $2 \le b \le 4$. We denote by $\{T_i\}_{i=1}^4$ the dual basis of $\{\beta_i\}_{i=1}^4 = \Pi_{\triangle(\mathfrak{k}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})}$. Then it follows from $\alpha_b(Z_a) = \delta_{b,a}$ that

$$\begin{aligned} &T_1 = -Z_4/2, \qquad T_2 = Z_1 - Z_4, \qquad T_3 = Z_2 - 3Z_4/2, \\ &T_4 = Z_3 - 2Z_4. \end{aligned} \tag{f2.1}$$

So an element $T = \sum_{i=1}^4 \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}}$ can be expressed as follows:

$$T = \sum_{i=1}^4 \lambda_i T_i = \sum_{a=1}^3 \lambda_{a+1} Z_a - (\lambda_1/2 + \lambda_2 + 3\lambda_3/2 + 2\lambda_4) Z_4, \tag{f2.2}$$

where $\mathscr{W}_{\mathfrak{k}} := \{\sum_{i=1}^4 \lambda_i T_i \mid \lambda_i \ge 0 \text{ for all } 1 \le i \le 4\}$. Therefore one can deduce

$$C_G^{\mathrm{FII}} = C_K^{\mathrm{FII}}(1) \cup C_K^{\mathrm{FII}}(3) \tag{f2.3}$$

by arguments similar to those in Paragraph 4.7.2. Here $C_K^{\mathrm{FII}}(1)$, $C_K^{\mathrm{FII}}(3)$ and C_G^{FII} are given by

$$C_K^{\mathrm{FII}}(1) := \left\{ \mathfrak{c}_{\mathfrak{k}}(T) \mid T = \sum\nolimits_{i=1}^4 \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_1 > 0 \right\},$$

$$C_K^{\mathrm{FII}}(3) := \left\{ \mathfrak{c}_{\mathfrak{k}}(T) \mid T = \sum\nolimits_{i=1}^4 \lambda_i T_i \in \mathscr{W}_{\mathfrak{k}} \text{ with } \lambda_3 > 0 \right\},$$

$$C_G^{\mathrm{FII}} := \{ \mathfrak{c}_{\mathfrak{g}}(T') \mid \mathfrak{c}_{\mathfrak{g}}(T') \text{ is compact with } T' \in \mathscr{W}_{\mathfrak{k}} \},$$

respectively. The proof of Proposition 4, together with (f2.3), allows us to assert

Proposition 13. *The following is the coarse orbit type of symplectic homogeneous space* G/H *with* $G = F_{4(-20)}$ *and* H *compact:*

G	H	No.
$F_{4(-20)}$	$A_2 \times A_1 \times T,\ B_3 \times T$	1
	$A_2 \times T^2,\ A_1 \times A_1 \times T^2,\ B_2 \times T^2$	2
	$A_1 \times T^3$	3
FII	T^4	4

4.14. The coarse orbit type of type G. We know the coarse orbit type of symplectic homogeneous spaces G/H with $G = G_{2(2)}$ and H compact from Proposition 5.5 in Boumuki [Bm, p. 1157]:

Proposition 14. *The following is the coarse orbit type of symplectic homogeneous space* G/H *with* $G = G_{2(2)}$ *and* H *compact:*

G	H	No.
$G_{2(2)}$	$A_1 \times T$	1
G	T^2	2

We conclude Theorem 1 by collecting eleven Propositions 4 through 14.

Concluding Remark. Symplectic homogeneous spaces in Theorem 1 cannot admit any invariant Kählerian structures (ref. Section 1), but they admit invariant pseudo-Kählerian structures (see Theorem in Dorfmeister-Guan [Do-Gu1, p. 330] and our Proposition 2).

Acknowledgement

The author would like to express his sincere gratitude to Professors Yoshihiro Ohnita and Tomonori Noda for their encouragement. Many thanks are due to the editor and the referee for their instructive suggestions to an earlier version of this paper.

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