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A new proof for small cancellation conditions of 2-bridge link groups

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Abstract. The second author and M. Sakuma gave a complete characterization of those essential simple loops on a 2-bridge sphere of a 2-bridge link which are null-homotopic in the link complement. In this paper, we give an alternative proof to this result, by giving a simple proof for the small cancellation conditions of the upper presentations of 2-bridge link groups, which holds the key to the proof the result.

1. Introduction

In [1], the second author and M. Sakuma gave a complete characterization of those essential simple loops in a 2-bridge sphere of a 2-bridge link which are null-homotopic in the link complement, and by using the result, they described all upper-meridian-pair-preserving epimorphisms between 2-bridge link groups. The main purpose of this paper is to give a simple proof for the small cancellation conditions of the upper presentations of 2-bridge link groups, which holds the key to the proof of the main result of [1]. We also give an alternative proof of the main result of [1] using transfinite induction. It is well-known that 2-bridge links, $K(r)$, are parametrized by extended rational numbers, r, and that by Shubert's classification of 2-bridge links [5], it suffices to consider $K(r)$ for $r = \infty$ or $0 < r \le 1$. Here if $r = \infty$ or $r = 1$, then $K(r)$ becomes a trivial 2-component link or a trivial knot, respectively. Since these trivial cases are easy to treat for our purpose (see [1, Section 7]), we may assume $0 < r < 1$. Then such a rational number r is uniquely expressed in the following continued fraction expansion:

$$r = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \ddots + \cfrac{1}{m_k}}} =: [m_1, m_2, \dots, m_k],$$

where $k \ge 1$, $(m_1, \dots, m_k) \in (\mathbf{Z}_+)^k$, and $m_k \ge 2$.

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In [1], the proofs of key lemmas and propositions such as Lemma 7.3 and Propositions 4.3, 4.4 and 4.5 proceed by induction on k, the length of the continued fraction expansion of r, where a rational number $\tilde{r}$ defined by $\tilde{r} = [m_2 - 1, \dots, m_k]$ if $m_2 \geq 2$ and $\tilde{r} = [m_3, \dots, m_k]$ if $m_2 = 1$ plays an important role as a predecessor of $r = [m_1, m_2, \dots, m_k]$ (see [1, Proposition 4.4]).

However, in this paper, we define a well-ordering $\preceq$ on the set of rational numbers greater than 0 and less than 1 (see Definition 3), and then prove key lemmas and propositions such as Lemmas 3 and 4, and Propositions 2 and 3 using transfinite induction with respect to $\preceq$, where a rational number $\tilde{r}$ defined by $\tilde{r} = [m_1 - 1, \dots, m_k]$ if $m_1 \geq 2$ or $\tilde{r} = [m_2 + 1, \dots, m_k]$ if $m_1 = 1$ plays a role as a predecessor of $r = [m_1, m_2, \dots, m_k]$ (see Lemma 2). Note that having a smaller gap between r and $\tilde{r}$ than in [1] makes the proof less complicated.

This paper is organized as follows. In Section 2, we describe the main statement that we are going to re-prove in the present paper. In Section 3, we recall the upper presentation of a 2-bridge link group. In Section 4, we re-prove key lemmas and propositions with some modification, if necessary, to the original statements established in [1]. Finally, Section 5 is devoted to a new proof of the main theorem.

2. Main statement

For a rational number $r \in \hat{\mathbf{Q}} := \mathbf{Q} \cup \{\infty\}$, let $K(r)$ be the 2-bridge link of slope r, which is defined as the sum $(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$ of rational tangles of slope ∞ and r. The common boundary $\partial(B^3, t(\infty)) = \partial(B^3, t(r))$ of the rational tangles is identified with the *Conway sphere* $(\boldsymbol{S}^2, \boldsymbol{P}) := (\mathbf{R}^2, \mathbf{Z}^2)/H$, where H is the group of isometries of the Euclidean plane $\mathbf{R}^2$ generated by the π-rotations around the points in the lattice $\mathbf{Z}^2$. Let $\boldsymbol{S}$ be the 4-punctured sphere $\boldsymbol{S}^2 - \boldsymbol{P}$ in the link complement $S^3 - K(r)$. Any essential simple loop in $\boldsymbol{S}$, up to isotopy, is obtained as the image of a line of slope $s \in \hat{\mathbf{Q}}$ in $\mathbf{R}^2 - \mathbf{Z}^2$ by the covering projection onto $\boldsymbol{S}$. The (unoriented) essential simple loop in $\boldsymbol{S}$ so obtained is denoted by α_s. We also denote by α_s the conjugacy class of an element of $\pi_1(\boldsymbol{S})$ represented by (a suitably oriented) α_s. Then the *link group* $G(K(r)) := \pi_1(S^3 - K(r))$ is identified with $\pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle$, where $\langle\langle \cdot \rangle\rangle$ denotes the normal closure.

Let $\mathscr{D}$ be the *Farey tessellation*, whose ideal vertex set is identified with $\hat{\mathbf{Q}}$. For each $r \in \hat{\mathbf{Q}}$, let Γ_r be the group of automorphisms of $\mathscr{D}$ generated by reflections in the edges of $\mathscr{D}$ with an endpoint r, and let $\hat{\Gamma}_r$ be the group generated by Γ_r and Γ_∞. Then the region, R, bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint r forms a fundamental domain of the action of $\hat{\Gamma}_r$ on $\mathbf{H}^2$ (see Figure 1). Let I_1 and I_2 be the closed intervals in $\hat{\mathbf{R}}$ obtained as the intersection with $\hat{\mathbf{R}}$ of the closure

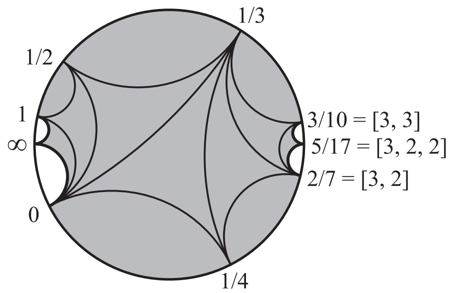

Fig. 1. A fundamental domain of $\hat{\Gamma}_r$ in the Farey tessellation (the shaded domain) for $r = 5/17 = [3, 2, 2]$.

of R. Suppose that r is a rational number with $0 < r < 1$. (We may always assume this except when we treat the trivial knot and the trivial 2-component link.) Write $r = [m_1, m_2, \ldots, m_k]$, where $k \geq 1$, $(m_1, \ldots, m_k) \in (\mathbf{Z}_+)^k$, and $m_k \geq 2$. Then the above intervals are given by $I_1 = [0, r_1]$ and $I_2 = [r_2, 1]$, where

$$r_1 = \begin{cases} [m_1, m_2, \ldots, m_{k-1}] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \ldots, m_{k-1}, m_k - 1] & \text{if } k \text{ is even,} \end{cases}$$

$$r_2 = \begin{cases} [m_1, m_2, \ldots, m_{k-1}, m_k - 1] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \ldots, m_{k-1}] & \text{if } k \text{ is even.} \end{cases}$$

We recall the following fact ([3, Proposition 4.6 and Corollary 4.7] and [1, Lemma 7.1]) which describes the role of $\hat{\Gamma}_r$ in the study of 2-bridge link groups.

Proposition 1. (1) *If two elements s and s' of $\hat{\mathbf{Q}}$ belong to the same orbit $\hat{\Gamma}_r$-orbit, then the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$.*

(2) *For any $s \in \hat{\mathbf{Q}}$, there is a unique rational number $s_0 \in I_1 \cup I_2 \cup \{\infty, r\}$ such that s is contained in the $\hat{\Gamma}_r$-orbit of s_0. In particular, α_s is homotopic to α_{s_0} in $S^3 - K(r)$. Thus if $s_0 \in \{\infty, r\}$, then α_s is null-homotopic in $S^3 - K(r)$.*

The following theorem proved in [1] and to be re-proved in Section 5 of the present paper shows that the converse to Proposition 1(2) also holds.

Theorem 1. *The loop α_s is null-homotopic in $S^3 - K(r)$ if and only if s belongs to the $\hat{\Gamma}_r$-orbit of ∞ or r. In other words, if $s \in I_1 \cup I_2$, then α_s is not null-homotopic in $S^3 - K(r)$.*

3. Upper presentations of 2-bridge link groups

Throughout this paper, the set $\{a, b\}$ denotes the standard meridian-generator of the rank 2 free group $\pi_1(B^3 - t(\infty))$, which is specified as in

[1, Section 3]. For a positive rational number q/p, where p and q are relatively prime positive integers, let u_r be the word in $\{a,b\}$ obtained as follows. (For a geometric description, see [1, Remark 1].) Set $\varepsilon_i = (-1)^{\lfloor iq/p \rfloor}$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

(1) If p is odd, then

$$u_{q/p} = a\hat{u}_{q/p} b^{(-1)^q} \hat{u}_{q/p}^{-1},$$

where $\hat{u}_{q/p} = b^{\varepsilon_1} a^{\varepsilon_2} \dots b^{\varepsilon_{p-2}} a^{\varepsilon_{p-1}}$.

(2) If p is even, then

$$u_{q/p} = a\hat{u}_{q/p} a^{-1} \hat{u}_{q/p}^{-1},$$

where $\hat{u}_{q/p} = b^{\varepsilon_1} a^{\varepsilon_2} \dots a^{\varepsilon_{p-2}} b^{\varepsilon_{p-1}}$.

Then $u_r \in F(a,b) \cong \pi_1(B^3 - t(\infty))$ is represented by the simple loop α_r, and the link group $G(K(r))$ with $r > 0$ has the following presentation, called the *upper presentation*:

$$\begin{aligned} G(K(r)) = \pi_1(S^3 - K(r)) &\cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r \rangle\rangle \\ &\cong F(a,b)/\langle\langle u_r \rangle\rangle \cong \langle a, b \,|\, u_r \rangle. \end{aligned}$$

We recall the definition of the sequence $S(r)$ and the cyclic sequence $CS(r)$ of slope r defined in [1], both of which are read from the single relator u_r of the upper presentation of $G(K(r))$. We first fix some definitions and notation. Let X be a set. By a *word* in X, we mean a finite sequence $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}$ where $x_i \in X$ and $\varepsilon_i = \pm 1$. Here we call $x_i^{\varepsilon_i}$ the *i-th letter* of the word. For two words u, v in X, by $u \equiv v$ we denote the *visual equality* of u and v, meaning that if $u = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ and $v = y_1^{\delta_1} \dots y_m^{\delta_m}$ ($x_i, y_j \in X$; $\varepsilon_i, \delta_j = \pm 1$), then $n = m$ and $x_i = y_i$ and $\varepsilon_i = \delta_i$ for each $i = 1, \dots, n$. The length of a word v is denoted by $|v|$. A word v in X is said to be *reduced* if v does not contain xx^{-1} or $x^{-1}x$ for any $x \in X$. A word is said to be *cyclically reduced* if all its cyclic permutations are reduced. A *cyclic word* is defined to be the set of all cyclic permutations of a cyclically reduced word. By (v) we denote the cyclic word associated with a cyclically reduced word v. Also by $(u) \equiv (v)$ we mean the *visual equality* of two cyclic words (u) and (v). In fact, $(u) \equiv (v)$ if and only if v is visually a cyclic shift of u.

Definition 1. (1) Let v be a nonempty reduced word in $\{a,b\}$. Decompose v into

$$v \equiv v_1 v_2 \dots v_t,$$

where, for each $i = 1, \dots, t-1$, all letters in v_i have positive (resp., negative) exponents, and all letters in v_{i+1} have negative (resp., positive) exponents. Then the sequence of positive integers $S(v) := (|v_1|, |v_2|, \dots, |v_t|)$ is called the *S-sequence of v*.

(2) Let (v) be a nonempty cyclic word in $\{a, b\}$. Decompose (v) into

$$(v) \equiv (v_1 v_2 \dots v_t),$$

where all letters in v_i have positive (resp., negative) exponents, and all letters in v_{i+1} have negative (resp., positive) exponents (taking subindices modulo t). Then the *cyclic* sequence of positive integers $CS(v) := ((|v_1|, |v_2|, \dots, |v_t|))$ is called the *cyclic S-sequence of* (v). Here, the double parentheses denote that the sequence is considered modulo cyclic permutations.

(3) A nonempty reduced word v in $\{a, b\}$ is said to be *alternating* if $a^{\pm 1}$ and $b^{\pm 1}$ appear in v alternately, i.e., neither $a^{\pm 2}$ nor $b^{\pm 2}$ appears in v. A cyclic word (v) is said to be *alternating* if all cyclic permutations of v are alternating. In the latter case, we also say that v is *cyclically alternating*.

Definition 2. For a rational number r with $0 < r \leq 1$, let $G(K(r)) = \langle a, b \,|\, u_r \rangle$ be the upper presentation. Then the symbol $S(r)$ (resp., $CS(r)$) denotes the S-sequence $S(u_r)$ of u_r (resp., cyclic S-sequence $CS(u_r)$ of (u_r)), which is called the *S-sequence of slope r* (resp., the *cyclic S-sequence of slope r*).

The following is cited from [1]. Since its proof in [1] is irrelevant to the modification to be performed in the present paper, we adopt the proof as it is.

Lemma 1 ([1, Proposition 4.2]). *For the positive rational number $r = q/p$, the sequence $S(r)$ has length $2q$, and it represents the cyclic sequence $CS(r)$. Moreover the cyclic sequence $CS(r)$ is invariant by the half-rotation; that is, if $s_j(r)$ denotes the j-th term of $S(r)$ $(1 \leq j \leq 2q)$, then $s_j(r) = s_{q+j}(r)$ for every integer j $(1 \leq j \leq q)$.*

4. New proof for small cancellation conditions of 2-bridge link groups

In this section, we give new proofs to several lemmas and propositions with some modification, if necessary, to the original statements established in [1, Section 4]. These will play crucial roles in the new proof of Theorem 1.

In the remainder of this paper unless specified otherwise, we suppose that r is a rational number with $0 < r \leq 1$, and write r as a continued fraction:

$$r = [m_1, m_2, \dots, m_k],$$

where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbf{Z}_+)^k$ and $m_k \geq 2$ unless $k = 1$.

Lemma 2. *For a rational number $r = [m_1, m_2, \dots, m_k]$ with $0 < r < 1$, let $\tilde{r}$ be a rational number defined as*

$$\tilde{r} = \begin{cases} [m_1 - 1, m_2, m_3, \dots, m_k] & \text{if } m_1 \geq 2; \\ [m_2 + 1, m_3, m_4, \dots, m_k] & \text{if } m_1 = 1. \end{cases}$$

Then we have

$$r = \begin{cases} \tilde{r}/(1+\tilde{r}) & \text{if } m_1 \geq 2; \\ 1-\tilde{r} & \text{if } m_1 = 1. \end{cases}$$

Proof. If $m_1 \geq 2$, then letting $a := 1/\tilde{r} = m_1 - 1 + [m_2, \dots, m_k]$ we have

$$r = [m_1, m_2, \dots, m_k] = 1/(1+a) = 1/(1+1/\tilde{r}) = \tilde{r}/(1+\tilde{r}),$$

as required.

On the other hand, if $m_1 = 1$, then letting $b := 1/\tilde{r} - 1 = m_2 + [m_3, \dots, m_k]$ we have

$$r = [m_1, m_2, \dots, m_k] = 1/(1+1/b) = 1/(1+\tilde{r}/(1-\tilde{r})) = 1-\tilde{r},$$

as required. □

Proposition 2. *For a rational number $r = [m_1, m_2, \dots, m_k]$ with $0 < r < 1$, let $\tilde{r}$ be a rational number defined as in Lemma 2. Put $CS(\tilde{r}) = ((a_1, a_2, \dots, a_t, a_1, a_2, \dots, a_t))$. Then the following hold.*

(1) *If $m_1 \geq 2$, then*

$$CS(r) = ((a_1+1, a_2+1, \dots, a_t+1, a_1+1, a_2+1, \dots, a_t+1)).$$

(2) *If $m_1 = 1$, then every a_i is at least 2, and either*

$$CS(r) = ((2, b_1\langle 1\rangle, 2, b_2\langle 1\rangle, \dots, 2, b_t\langle 1\rangle, 2, b_1\langle 1\rangle, 2, b_2\langle 1\rangle, \dots, 2, b_t\langle 1\rangle))$$

or

$$CS(r) = ((2, b_t\langle 1\rangle, \dots, 2, b_2\langle 1\rangle, 2, b_1\langle 1\rangle, 2, b_t\langle 1\rangle, \dots, 2, b_2\langle 1\rangle, 2, b_1\langle 1\rangle)),$$

where $b_i = a_i - 2$ for every i, and the symbol "$b_i\langle 1\rangle$" represents b_i successive 1's. (Here if $b_i = 0$ for some i, then $b_i\langle 1\rangle$ represents the empty subsequence.)

Proof. (1) Let $m_1 \geq 2$. Write $\tilde{r} = q/p$, where p and q are relatively prime positive integers. By Lemma 2, $r = \tilde{r}/(1+\tilde{r}) = q/(p+q)$. It then follows from Lemma 1 that both the sequences $S(r)$ and $S(\tilde{r})$, and hence both the cyclic sequences $CS(r)$ and $CS(\tilde{r})$, have the same length $2q$. Recall from [1, Lemma 4.8] that if $s_j(r)$ denotes the j-th term of the sequence $S(r)$, then $s_j(r) = \lfloor j(1/r) \rfloor_* - \lfloor (j-1)(1/r) \rfloor_*$, where $\lfloor x \rfloor_*$ is the greatest integer smaller than x. Since $r = \tilde{r}/(1+\tilde{r}) = 1/(1/\tilde{r}+1)$, we have

$$\begin{aligned} s_j(r) &= \lfloor j(1/r) \rfloor_* - \lfloor (j-1)(1/r) \rfloor_* \\ &= \lfloor j(1/\tilde{r}+1) \rfloor_* - \lfloor (j-1)(1/\tilde{r}+1) \rfloor_* \\ &= (\lfloor j(1/\tilde{r}) \rfloor_* + j) - (\lfloor (j-1)(1/\tilde{r}) \rfloor_* + (j-1)) \\ &= 1 + \lfloor j(1/\tilde{r}) \rfloor_* - \lfloor (j-1)(1/\tilde{r}) \rfloor_* \\ &= 1 + s_j(\tilde{r}), \end{aligned}$$

where $s_j(\tilde{r})$ denotes the j-th term of the sequence $S(\tilde{r})$, and hence the assertion follows.

(2) Let $m_1 = 1$. Then $\tilde{r} = [m_2 + 1, m_3, \dots, m_k]$ and $r = 1 - \tilde{r}$ by Lemma 2. Since $m_2 + 1 \geq 2$, (1) implies that every term of $CS(\tilde{r})$ is at least 2, that is, every a_i is at least 2.

To prove the remaining assertion, let f_1 be the reflection of $(B^3, t(\infty))$ in a "horizontal" disk bounded by α_0, and let f_2 be the half Dehn twist of $(B^3, t(\infty))$ along the "vertical" disk bounded by α_∞. Then the automorphisms $(f_i)_*$ of $\pi_1(B^3 - t(\infty)) = F(a,b)$ induced by f_i are given by

$$(f_1)_*(a,b) = (a,b) \qquad (f_2)_*(a,b) = (a,b^{-1})$$

Let f be the composition $f_2 f_1$. Then by the above observation, we have $f_*(a,b) = (a,b^{-1})$. On the other hand, f maps α_r to $f_2(f_1(\alpha_r)) = f_2(\alpha_{-r}) = \alpha_{1-r} = \alpha_{\tilde{r}}$. Thus f_* sends the cyclic word (u_r) to the cyclic word $(u_{\tilde{r}})$ or $(u_{\tilde{r}}^{-1})$. Since $f_*^2 = 1$, this implies that f_* sends the cyclic word $(u_{\tilde{r}})$ to the cyclic word (u_r) or (u_r^{-1}). Thus the cyclic word (u_r) or (u_r^{-1}) is obtained from $(u_{\tilde{r}})$ by replacing b with b^{-1}. In this process, a subword, w, of $(u_{\tilde{r}})$ with $S(w) = (1, a_i, 1)$, say, $w = b^{-1}(abab\dots ab)a^{-1}$ or $b^{-1}(abab\dots a)b^{-1}$ according to whether a_i is even or odd, is transformed to a subword $w' = b(ab^{-1}ab^{-1}\dots ab^{-1})a^{-1}$ or $b(ab^{-1}ab^{-1}\dots a)b$, respectively, of $(u_r^{\pm 1})$ with $S(w') = (2, (a_i - 2)\langle 1 \rangle, 2)$. Since the cyclic sequence $CS(u_r^{-1})$ is the reverse of the cyclic sequence $CS(u_r) = CS(r)$, the assertion now follows. $\square$

Throughout the remainder of this paper, we assume the following well-ordering $\preceq$.

Definition 3. Let $\mathfrak{A}$ be the set of all rational numbers greater than 0 and less than or equal to 1. We define a well-ordering $\preceq$ on $\mathfrak{A}$ by $r_1 \preceq r_2$ if and only if one of the following conditions holds, where $r_1 = [l_1, l_2, \dots, l_h]$ and $r_2 = [n_1, n_2, \dots, n_t]$.

(i) $h < t$.

(ii) $h = t$ and there is a positive integer $j \leq h = t$ such that $l_i = n_i$ for every $i < j$ and $l_j \leq n_j$.

It should be noted that a rational number $\tilde{r}$ defined in Lemma 2 is a predecessor of $r = [m_1, m_2, \dots, m_k]$ with respect to $\preceq$.

Now we are able to give a new proof to the following lemma whose statement is a part of [1, Proposition 4.3]. Note that the remaining part of [1, Proposition 4.3] is not necessary in the present paper.

Lemma 3. *For a rational number $r = [m_1, m_2, \dots, m_k]$, we have the following.*

(1) *Suppose $k = 1$, i.e., $r = 1/m_1$. Then $CS(r) = ((m_1, m_1))$.*

(2) *Suppose $k \geq 2$. Then $CS(r)$ consists of m_1 and $m_1 + 1$.*

Proof. We prove (1) and (2) together by transfinite induction with respect to the well-ordering $\preceq$ defined in Definition 3. The base step is the case when $r = [1]$. In this case, $u_r = ab^{-1}$, and so $CS(r) = ((1, 1))$, as desired. To prove the inductive step, we consider two cases separately.

Case 1. $m_1 \geq 2$.

In this case, put $\tilde{r} = [m_1 - 1, m_2, \dots, m_k]$ as in Lemma 2. Then clearly $\tilde{r} \prec r$. By the inductive hypothesis, $CS(\tilde{r}) = ((m_1 - 1, m_1 - 1))$ provided $k = 1$, and $CS(\tilde{r})$ consists of $m_1 - 1$ and m_1 provided $k \geq 2$. So by Proposition 2(1), $CS(r) = ((m_1, m_1))$ provided $k = 1$, and $CS(r)$ consists of m_1 and $m_1 + 1$ provided $k \geq 2$, as desired.

Case 2. $m_1 = 1$.

In this case, it immediately follows from Proposition 2(2) that $CS(r)$ consists of $1 = m_1$ and $2 = m_1 + 1$, as desired. □

We also give a new proof to the following proposition whose statement is precisely the same as [1, Proposition 4.5].

Proposition 3. *For $r = [m_1, m_2, \dots, m_k]$, the cyclic sequence $CS(r)$ has a decomposition $((S_1, S_2, S_1, S_2))$ which satisfies the following.*

(1) *Each S_i is symmetric, that is, the sequence obtained from S_i by reversing the order is equal to S_i. (Here, S_1 is empty if $k = 1$.)*

(2) *Each S_i occurs only twice on the cyclic sequence $CS(r)$.*

(3) *The subsequence S_1 begins and ends with $m_1 + 1$.*

(4) *The subsequence S_2 begins and ends with m_1.*

Proof. The proof proceeds by transfinite induction with respect to the well-ordering $\preceq$ defined in Definition 3. We take the case when $r = [m_1]$ as the base step. In this case, $CS(r) = ((m_1, m_1))$ by Lemma 3(1). Putting $S_1 = \varnothing$ and $S_2 = (m_1)$, the assertion clearly holds. To prove the inductive step, we consider two cases separately.

Case 1. $m_1 \geq 2$ *and* $k \geq 2$.

Put $\tilde{r} = [m_1 - 1, m_2, \dots, m_k]$ as in Lemma 2. Then clearly $\tilde{r} \prec r$. By the inductive hypothesis, $CS(\tilde{r}) = (\!(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2)\!)$, where $\tilde{S}_1$ and $\tilde{S}_2$ are symmetric subsequences of $CS(\tilde{r})$ such that each $\tilde{S}_i$ occurs only twice in $CS(\tilde{r})$, $\tilde{S}_1$ begins and ends with m_1 (provided that $\tilde{S}_1$ is nonempty), and such that $\tilde{S}_2$ begins and ends with $m_1 - 1$. Write

$$\tilde{S}_1 = (a_1, \dots, a_{t_1}) \qquad \text{and} \qquad \tilde{S}_2 = (a_{t_1+1}, \dots, a_{t_2}),$$

and then take

$$S_1 = (a_1 + 1, \dots, a_{t_1} + 1) \qquad \text{and} \qquad S_2 = (a_{t_1+1} + 1, \dots, a_{t_2} + 1).$$

Clearly S_1 begins and ends with $m_1 + 1$, and S_2 begins and ends with m_1. Also since $\tilde{S}_1$ and $\tilde{S}_2$ are symmetric by the inductive hypothesis, S_1 and S_2 are also symmetric. Moreover, by Proposition 2(1), we have $CS(r) = (\!(S_1, S_2, S_1, S_2)\!)$. It remains to show that each S_i occurs only twice in $CS(r)$. If S_1 occurred more than twice in $(\!(S_1, S_2, S_1, S_2)\!)$, $\tilde{S}_1$ also would occur more than twice in $(\!(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2)\!)$, a contradiction. Similarly, S_2 also occurs only twice in $CS(r)$.

Case 2. $m_1 = 1$ *and* $k \geq 2$.

Put $\tilde{r} = [m_2 + 1, m_3, \dots, m_k]$ as in Lemma 2. Then clearly $\tilde{r} \prec r$. By the inductive hypothesis, $CS(\tilde{r}) = (\!(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2)\!)$, where $\tilde{S}_1$ and $\tilde{S}_2$ are symmetric subsequences of $CS(\tilde{r})$ such that each $\tilde{S}_i$ occurs only twice in $CS(\tilde{r})$, $\tilde{S}_1$ begins and ends with $m_2 + 2$ (provided that $\tilde{S}_1$ is nonempty), and such that $\tilde{S}_2$ begins and ends with $m_2 + 1$. If $k = 2$, then $r = [1, m_2]$ with $m_2 \geq 2$ and $\tilde{r} = [m_2 + 1]$; so $CS(\tilde{r}) = (\!(m_2 + 1, m_2 + 1)\!)$ by Lemma 3(1). Then take

$$S_1 = (2) \qquad \text{and} \qquad S_2 = ((m_2 - 1)\langle 1 \rangle).$$

On the other hand, if $k \geq 3$, then write

$$\tilde{S}_1 = (a_1, \dots, a_{t_1}) \qquad \text{and} \qquad \tilde{S}_2 = (a_{t_1+1}, \dots, a_{t_2}).$$

Here $a_1 = a_{t_1} = m_2 + 2 \geq 3$ and $a_{t_1+1} = a_{t_2} = m_2 + 1 \geq 2$. Now take

$$S_1 = (2, b_{t_1+1}\langle 1 \rangle, 2, \dots, 2, b_{t_2}\langle 1 \rangle, 2) \qquad \text{and} \qquad S_2 = (b_1\langle 1 \rangle, 2, \dots, 2, b_{t_1}\langle 1 \rangle),$$

where $b_i = a_i - 2$ for every i. In either case, we see that S_1 begins and ends with $2 = m_1 + 1$, S_2 begins and ends with $1 = m_1$, and that S_1 and S_2 are symmetric because $\tilde{S}_1$ and $\tilde{S}_2$ are symmetric by the inductive hypothesis. Moreover, Proposition 2(2) implies that either $CS(r) = (\!(S_1, S_2, S_1, S_2)\!)$ or $CS(r) = (\!(\overleftarrow{S_1}, \overleftarrow{S_2}, \overleftarrow{S_1}, \overleftarrow{S_2})\!)$, where the symbol "$\overleftarrow{S_i}$" denotes the reverse of S_i. But since S_1 and S_2 are symmetric, we actually have $CS(r) = (\!(S_1, S_2, S_1, S_2)\!)$ in either case. It remains to show that each S_i occurs only twice in $CS(r)$. If S_1 occurred more than twice in $(\!(S_1, S_2, S_1, S_2)\!)$, $\tilde{S}_2$ also would occur more than twice in $(\!(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2)\!)$, a contradiction. For the assertion for S_2, note that S_2 begins and ends with m_2 successive 1's, and that the maximum number of consecutive occurrences of 1 in $(\!(S_1, S_2, S_1, S_2)\!)$ is m_2. So if S_2 occurred more than twice in $(\!(S_1, S_2, S_1, S_2)\!)$, $\tilde{S}_1$ also would occur more than twice in $(\!(\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2)\!)$, a contradiction. $\square$

In order to prove Theorem 1, we keep the idea of applying small cancellation theory as in [1, Sections 5 and 6]. Briefly speaking, we adopt [1, Section 5] as it is to show that the upper presentation $G(K(r)) = \langle a, b \,|\, u_r \rangle$ with $0 < r < 1$ satisfies the small cancellation conditions $C(4)$ and $T(4)$. And then we investigate properties of van Kampen's diagrams over the presentation $G(K(r)) = \langle a, b \,|\, u_r \rangle$ with boundary label being cyclically alternating as in [1, Section 6]. Sections 5 and 6 in [1] are indeed irrelevant to the modification that we are performing in the present paper. Due to van Kampen's Lemma which is a classical result in combinatorial group theory (see [2]), we obtain the fact that if a cyclically alternating word w equals the identity in $G(K(r))$, then its cyclic word (w) contains a subword z of $(u_r^{\pm 1})$ such that the S-sequence of z is (S_1, S_2, ℓ) or (ℓ, S_2, S_1) for some positive integer ℓ, where $CS(r) = (\!(S_1, S_2, S_1, S_2)\!)$ is as in Proposition 3. In particular, we obtain the following.

Corollary 1 ([1, Corollary 6.4]). *Let* $r = [m_1, m_2, \dots, m_k]$ *with* $0 < r < 1$. *For a rational number* s *with* $0 < s \le 1$, *if* α_s *is null-homotopic in* $S^3 - K(r)$, *then the following hold.*

(1) *If* $k = 1$, *namely* $r = [m_1]$, *then* $CS(s)$ *contains a term bigger than or equal to* m_1.
(2) *If* $k \ge 2$, *then* $CS(s)$ *contains* (S_1, S_2) *or* (S_2, S_1) *as a subsequence, where* $CS(r) = (\!(S_1, S_2, S_1, S_2)\!)$ *is as in Proposition* 3.

Remark 1. In [1, Corollary 6.4], it is mistakenly stated that if α_s is null-homotopic in $S^3 - K(r)$, then $CS(s)$ contains (S_1, S_2) or (S_2, S_1) as a subsequence, regardless of $k \ge 1$. It should be noted that if $k = 1$ and every term of $CS(s)$ is bigger than m_1, then $CS(s)$ does not contain (S_1, S_2) or (S_2, S_1) as a subsequence, because, in this case, S_1 is empty and $S_2 = (m_1)$, i.e., $(S_1, S_2) = (m_1) = (S_2, S_1)$.

5. New proof of Theorem 1

In this section, we prove the only if part of Theorem 1, that is, we prove that for any $s \in I_1 \cup I_2$, α_s is not null-homotopic in $S^3 - K(r)$. The if part is [3, Corollary 4.7].

The following lemma which plays an important role in the proof of Theorem 1 has the same statement as [1, Lemma 7.3], but is re-proved by transfinite induction.

Lemma 4. *Let* $r = [m_1, m_2, \dots, m_k]$ *with* $0 < r < 1$, *and let* $CS(r) = ((S_1, S_2, S_1, S_2))$ *be as in Proposition* 3. *Suppose that a rational number* s *with* $0 < s \le 1$ *has a continued fraction expansion* $s = [l_1, \dots, l_t]$, *where* $t \ge 1$, $(l_1, \dots, l_t) \in (\mathbf{Z}_+)^t$, *and* $l_t \ge 2$ *unless* $t = 1$. *Suppose also that* $CS(s)$ *satisfies the following condition:*

(i) *If* $k = 1$, *namely* $r = [m_1]$, *then* $CS(s)$ *contains a term bigger than or equal to* m_1.

(ii) *If* $k \ge 2$, *then* $CS(s)$ *contains* (S_1, S_2) *or* (S_2, S_1) *as a subsequence.*

Then the following hold.

(1) $t \ge k$.

(2) $l_i = m_i$ *for each* $i = 1, \dots, k-1$.

(3) *Either* $l_k \ge m_k$ *or both* $l_k = m_k - 1$ *and* $t > k$.

Proof. The proof proceeds by transfinite induction with respect to the well-ordering $\preceq$ defined in Definition 3. We take the case when $r = [m_1]$ as the base. By hypothesis (i), $CS(s)$ contains a term bigger than or equal to m_1. Then Lemma 3 implies that either $l_1 \ge m_1$ or both $l_1 = m_1 - 1$ and $t \ge 2$, so that the assertion clearly holds. Now we prove the inductive step. Let $\tilde{r}$ be defined as in Lemma 2. Then clearly $\tilde{r} \prec r$.

Case 1. $m_1 \ge 2$ *and* $k \ge 2$.

In this case, $\tilde{r} = [m_1 - 1, m_2, \dots, m_k]$. By Proposition 3, S_1 begins and ends with $m_1 + 1$, and S_2 begins and ends with m_1. Hence if $CS(s)$ contains (S_1, S_2) or (S_2, S_1) as a subsequence, then $CS(s)$ contains both a term m_1 and a term $m_1 + 1$. By Lemma 3, the only possibility is that $l_1 = m_1$ and $t \ge 2$. Now let $\tilde{s} = [l_1 - 1, \dots, l_t]$. Then we see from Proposition 2(1) that $CS(\tilde{s})$ contains $(\tilde{S}_1, \tilde{S}_2)$ or $(\tilde{S}_2, \tilde{S}_1)$ as a subsequence, where $CS(\tilde{r}) = ((\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2))$. By the inductive hypothesis, we have $t \ge k$, $l_i = m_i$ for each $i = 1, \dots, k-1$, and either $l_k \ge m_k$ or both $l_k = m_k - 1$ and $t > k$, which proves the assertion.

Case 2. $m_1 = 1$ *and* $k \ge 2$.

In this case, $\tilde{r} = [m_2 + 1, m_3, \dots, m_k]$. Arguing as in Case 1, $CS(s)$ contains both a term $m_1 = 1$ and a term $m_1 + 1 = 2$. By Lemma 3, the only possibility is that $l_1 = m_1 = 1$ and $t \ge 2$. Now let $\tilde{s} = [l_2 + 1, \dots, l_t]$. Then we see from Proposition 2(2) that $CS(\tilde{s})$ contains a term greater than or equal to $m_2 + 1$ provided $k = 2$ and that $CS(\tilde{s})$ contains $(\tilde{S}_1, \tilde{S}_2)$ or $(\tilde{S}_2, \tilde{S}_1)$ as a subsequence provided $k \ge 3$, where $CS(\tilde{r}) = ((\tilde{S}_1, \tilde{S}_2, \tilde{S}_1, \tilde{S}_2))$. By the inductive hypothesis, we have $t \geq k$, $l_i = m_i$ for each $i = 2, \dots, k-1$, and either $l_k \geq m_k$ or both $l_k = m_k - 1$ and $t > k$. This together with $l_1 = m_1$ proves the assertion. $\square$

Remark 2. We can easily see that the a rational number s with $0 < s \leq 1$ satisfies the conclusion of Lemma 4 if and only if s lies in the open interval $(r_1, r_2) = (0, 1] - (I_1 \cup I_2)$, where r_1 and r_2 are rational numbers such that $I_1 = [0, r_1]$ and $I_2 = [r_2, 1]$, introduced in Section 2.

We are now in a position to prove the only if part of Theorem 1.

Proof of the only if part of Theorem 1. Since the exceptional cases $r = \infty$ and $r = 1$ can be treated in the same way as in [1, Section 7], we assume $0 < r < 1$. Consider a 2-bridge link $K(r)$, and pick a rational number s from $I_1 \cup I_2$. Suppose on the contrary that α_s is null-homotopic in $S^3 - K(r)$, namely $u_s = 1$ in $G(K(r))$. If $0 < s \leq 1$, then by Corollary 1, $CS(s)$ contains a term greater than or equal to m_1 provided $r = [m_1]$ or otherwise $CS(s)$ contains (S_1, S_2) or (S_2, S_1) as a subsequence, where $CS(r) = ((S_1, S_2, S_1, S_2))$ as in Proposition 3. But then by Lemma 4 together with Remark 2, we have $s \notin I_1 \cup I_2$, a contradiction. So the only possibility is $s = 0$. Then, as mentioned at the end of Section 4 (also see [1, Theorem 6.3]), u_s must contain a subword z of $(u_r^{\pm 1})$ such that the S-sequence of z is (S_1, S_2, ℓ) or (ℓ, S_2, S_1) for some positive integer ℓ. Note that the length of such a subword z is strictly greater than p, half the length of $(u_r^{\pm 1})$, where $r = q/p$. Since $0 < r < 1$, we have $p \geq 2$. So, the word $u_0 = ab$ cannot contain such a subword, a contradiction. This completes the proof of the only if part of Theorem 1. $\square$

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