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# Oscillation and nonoscillation of certain second order quasilinear dynamic equations

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**Abstract.** We establish new oscillation and nonoscillation theorems for the second order quasilinear dynamic equation

$$(r(t)|y^{\Delta}(t)|^{\alpha-1}y^{\Delta}(t))^{\Delta}+f(t,y^{\sigma}(t))=0$$

on a time scale $\mathbf{T}$. Our results not only extend the results given in [J. Wang, On second order quasilinear oscillations, Funkcialaj Ekvacioj, **41** (1998), 25–54], but also unify the oscillation and nonoscillation criteria for second order quasilinear differential equations and difference equations.

## 1. Introduction and preliminary

The theory of time scales, which has recently received much attention, was introduced by Stefan Hilger [13] in order to unify continuous and discrete analysis. For completeness, we recall the following concepts related to the notions of time scales; see [4] and [5] for more details. A time scale $\mathbf{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbf{R}$. Since the oscillation of solutions near infinity is our primary concern, throughout this paper we assume that $\sup \mathbf{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbf{T}}$ by $[t_0,\infty)_{\mathbf{T}}:=[t_0,\infty)\cap\mathbf{T}$. On any time scale $\mathbf{T}$ we define the forward and backward jump operators by

$$\sigma(t):=\inf\{s\in\mathbf{T}:s>t\}\qquad\text{and}\qquad\rho(t):=\sup\{s\in\mathbf{T},s<t\},$$

where $\inf\varnothing:=\sup\mathbf{T}$ and $\sup\varnothing:=\inf\mathbf{T}$; here $\varnothing$ denotes the empty set. A point $t\in\mathbf{T}$ with $t>\inf\mathbf{T}$, is said to be left-dense if $\rho(t)=t$, right-dense if $t<\sup\mathbf{T}$ with $\sigma(t)=t$, left-scattered if $\rho(t)<t$ and right-scattered if $\sigma(t)>t$. The graininess function $\mu$ for a time scale $\mathbf{T}$ is defined by $\mu(t):=\sigma(t)-t$, and

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for any function $f:\mathbf{T}\to\mathbf{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. We say that $f:\mathbf{T}\to\mathbf{R}$ is (delta) differentiable at $t\in\mathbf{T}$ provided

$$f^\Delta(t):=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists when $\sigma(t)=t$ (here by $s\to t$ it is understood that $s$ approaches $t$ in the time scale), and when $f$ is continuous at $t$ and $\sigma(t)>t$,

$$f^\Delta(t):=\frac{f^\sigma(t)-f(t)}{\mu(t)}.$$

We say the function $f:[t_0,\infty)_{\mathbf{T}}\to\mathbf{R}$ is $rd$-continuous and write $f\in C_{rd}([t_0,\infty)_{\mathbf{T}},\mathbf{R})$ provided $f$ is continuous at right-dense points in $[t_0,\infty)_{\mathbf{T}}$ and $f$ has finite left-hand limits at left-dense points in $[t_0,\infty)_{\mathbf{T}}$. The set of functions $f:[t_0,\infty)_{\mathbf{T}}\to\mathbf{R}$ which are differentiable and whose derivative is $rd$-continuous function is denoted by $C^1_{rd}([t_0,\infty)_{\mathbf{T}},\mathbf{R})$. In addition, if $f^\Delta\geq 0$, then $f$ is nondecreasing. A useful formula is

$$f^\sigma(t)=f(t)+\mu(t)f^\Delta(t).$$

For $a,b\in\mathbf{T}$, and a differentiable function $f$, the Cauchy integral of $f^\Delta$ is defined by

$$\int_a^b f^\Delta(s)\Delta s:=f(b)-f(a),$$

and the improper integral defined as

$$\int_a^\infty f(s)\Delta s:=\lim_{t\to\infty}\int_a^t f(s)\Delta s.$$

Note that in case $\mathbf{T}=\mathbf{R}$, we have

$$\sigma(t)=t,\qquad \mu(t)=0,\qquad f^\triangle(t)=f'(t),\qquad \int_a^b f(t)\triangle t=\int_a^b f(t)dt.$$

When $\mathbf{T}=\mathbf{N}$, we have $\sigma(t)=t+1$, $\mu(t)=1$, and

$$f^\triangle(t)=\triangle f(t):=f(t+1)-f(t),\qquad \int_a^b f(t)\triangle t=\sum_{t=a}^{b-1}f(t).$$

When $\mathbf{T}=h\mathbf{N}:=\{hk:k\in\mathbf{N},h>0\}$, we have $\sigma(t)=t+h$, $\mu(t)=h$, and

$$x^\triangle(t)=\triangle_h x(t)=\frac{x(t+h)-x(t)}{h},\qquad \int_a^b f(t)\triangle t=\sum_{k=0}^{(b-a-h)/h}f(a+kh)h,$$

and when $\mathbf{T} = q^{\mathbf{N}_0} := \{t : t = q^n, n \in \mathbf{N}_0, q > 1\}$, we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$, and

$$x^{\triangle}(t) = \triangle_q x(t) = \frac{x(qt) - x(t)}{(q-1)t}, \qquad \int_a^b f(t)\triangle t = \sum_{t \in [a,b)} f(t)\mu(t).$$

In the present paper, we are concerned with the oscillation and nonoscillation of the second order quasilinear dynamic equation

$$(r(t)|y^{\Delta}(t)|^{\alpha-1}y^{\Delta}(t))^{\Delta} + f(t, y^{\sigma}(t)) = 0, \tag{1.1}$$

where $t \in [t_0, \infty)_{\mathbf{T}}$. In this paper, we consider situations described by the following conditions:

(A1) $\alpha > 0$ is a fixed constant;

(A2) $r \in C_{rd}([t_0, \infty)_{\mathbf{T}}, \mathbf{R}^+)$ with $\int_{t_0}^{\infty} r^{-1/\alpha}(s)\Delta s = \infty$, $\mathbf{R}^+ = (0, \infty)$;

(A3) $f \in C([t_0, \infty)_{\mathbf{T}} \times \mathbf{R}, \mathbf{R})$ with $yf(t, y) > 0$ for all $y \neq 0$ and each fixed $t \in [t_0, \infty)_{\mathbf{T}}$;

(A4) $f(t, y)$ is nondecreasing with respect to $y$ for each fixed $t \in [t_0, \infty)_{\mathbf{T}}$;

(A5) $f(t, y)$ is nonincreasing with respect to $y$ for each fixed $t \in [t_0, \infty)_{\mathbf{T}}$, when $|y|$ is sufficiently large.

By a solution of Eq. (1.1) we mean a nontrivial real-valued function $y \in C_{rd}^1([t_0, \infty)_{\mathbf{T}}, \mathbf{R})$ which has the properties $r|y^{\Delta}|^{\alpha-1}y^{\Delta} \in C_{rd}^1([t_0, \infty)_{\mathbf{T}}, \mathbf{R})$. Our attention is restricted to those solutions $y(t)$ of Eq. (1.1) which exist on some half-linear $[t_y, \infty)_{\mathbf{T}} \subseteq [t_0, \infty)_{\mathbf{T}}$ and satisfy $\sup\{|y(t)| : t \in [T, \infty)_{\mathbf{T}}\} > 0$ for any $T \in [t_y, \infty)_{\mathbf{T}}$. A solution $y(t)$ of Eq. (1.1) is called to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called to be nonoscillatory. The equation itself is called to be oscillatory if all its solutions are oscillatory.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of linear, nonlinear, half-linear, quasilinear dynamic equations on time scales; see for example [1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20] and the references cited therein. However, to the best of our knowledge, very little is known about the case of general $f(t, y)$ in which $t$ and $y$ are not necessarily separable; see [10]. Our purpose here is to develop oscillation theory for such a general case of Eq. (1.1) on time scales. This work was motivated by the paper of Wang [20] in which a detailed analysis of oscillation properties was given for the second order ordinary quasilinear differential equation

$$(|y'|^{\alpha-1}y')' + f(t, y) = 0, \qquad t \geq 0. \tag{1.2}$$

We will follow closely the presentation of Wang [20] and show that almost all of his results can be generalized to Eq. (1.1). Our main results are stated and proved in sections 2, 3 and 4. Examples of discrete systems illustrating the results are also given.

To prove our main results, we need the following known result [4, Theorem 1.90].

Lemma 1.1 (Keller's Chain Rule). *Let $f : \mathbf{R} \to \mathbf{R}$ be continuously differentiable and suppose $g : \mathbf{T} \to \mathbf{R}$ is delta differentiable. Then $f \circ g : \mathbf{T} \to \mathbf{R}$ is delta differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t) g^{\Delta}(t)) dh \right\} g^{\Delta}(t)$$

*holds.*

## 2. Existence of nonoscillatory solutions

For the simplicity, define

$$R(t) = \int_{t_0}^{t} \frac{\Delta s}{r^{1/\alpha}(s)}, \qquad t \in [t_0, \infty)_{\mathbf{T}}.$$

We begin with the following lemmas.

Lemma 2.1. *Assume that* (A1)–(A3) *hold. Let $y(t)$ be a nonoscillatory solution of Eq.* (1.1). *Then there exists $t_1 \in [t_0, \infty)_{\mathbf{T}}$ such that $y(t) y^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$.*

Proof. Without loss of generality we may suppose that $y(t) > 0$ and $y^{\sigma}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}} \subseteq [t_0, \infty)_{\mathbf{T}}$. From (1.1), by (A3), $(r(t)|y^{\Delta}(t)|^{\alpha-1} y^{\Delta}(t))^{\Delta} = -f(t, y^{\sigma}(t)) < 0$, $t \in [t_1, \infty)_{\mathbf{T}}$, and so $r(t)|y^{\Delta}(t)|^{\alpha-1} y^{\Delta}(t)$ is decreasing for $t \in [t_1, \infty)_{\mathbf{T}}$. We claim that $r(t)|y^{\Delta}(t)|^{\alpha-1} y^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$, so that $y(t) y^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$. In fact, if $r(t^*)|y^{\Delta}(t^*)|^{\alpha-1} y^{\Delta}(t^*) = -k < 0$ for some $t^* \in [t_1, \infty)_{\mathbf{T}}$, then $r(t)|y^{\Delta}(t)|^{\alpha-1} y^{\Delta}(t) \le -k < 0$ for $t \in [t^*, \infty)_{\mathbf{T}}$, which is equivalent to $y^{\Delta}(t) \le -\left(\frac{k}{r(t)}\right)^{1/\alpha}$ for $t \in [t^*, \infty)_{\mathbf{T}}$. Integrating the last inequality from $t^*$ to $t$ and letting $t \to \infty$, we see, in view of (A1), that $y(t) \to -\infty$ as $t \to \infty$. But this contradicts the assumed positivity of $y(t)$. Therefore, $r(t)|y^{\Delta}(t)|^{\alpha-1} y^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$ as claimed. □

Next, we classify all possible nonoscillatory solutions of Eq. (1.1) according to their asymptotic behavior as $t \to \infty$.

Lemma 2.2. *Under the conditions* (A1)–(A3), *any nonoscillatory solution $y(t)$ of Eq.* (1.1) *is of one of the following three types:*

( I ) $\lim_{t\to\infty} \frac{y(t)}{R(t)} = const \neq 0$;

( II ) $\lim_{t\to\infty} \frac{y(t)}{R(t)} = 0$, *and* $\lim_{t\to\infty} y(t) = \infty$ *or* $-\infty$;

(III) $\lim_{t\to\infty} \frac{y(t)}{R(t)} = 0$, *and* $\lim_{t\to\infty} y(t) = const \neq 0$.

Proof. Let $y(t)$ be a nonoscillatory solution of Eq. (1.1). Without loss of generality we may assume $y(t) > 0$ and $y^\sigma(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}} \subseteq [t_0, \infty)_{\mathbf{T}}$. By Lemma 2.1, one has $y^\Delta(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$. From (1.1), $(r(t)(y^\Delta(t))^\alpha)^\Delta < 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$, and so $r(t)(y^\Delta(t))^\alpha$ is an eventually positive decreasing function, so that $\lim_{t\to\infty} r(t)(y^\Delta(t))^\alpha$ is either positive or zero, i.e., $\lim_{t\to\infty} r^{1/\alpha}(t) y^\Delta(t)$ is either positive or zero. In the first case, by L'Hôpital's Rule [4, Theorem 1.120],

$$\lim_{t\to\infty} \frac{y(t)}{R(t)} = \lim_{t\to\infty} r^{1/\alpha}(t) y^\Delta(t) = const \neq 0.$$

In the second case, since $y(t)$ is increasing, $y(t)$ tends to a positive limit, finite or infinite, as $t \to \infty$. $\square$

Now we give criteria for the existence of nonoscillatory solutions of Eq. (1.1) of type (I), (II) and (III).

Theorem 2.1. *Assume that* (A1)–(A4) *hold. Then Eq.* (1.1) *has a nonoscillatory solution of type* (I) *if and only if there exists a constant* $c \neq 0$ *such that*

$$\int^\infty |f(t, cR^\sigma(t))| \triangle t < \infty. \tag{2.1}$$

Proof. (The "only if" part). Let $y(t)$ be a nonoscillatory solution of type (I) of Eq. (1.1). Without loss of generality we may assume that $y(t)$ and $y^\sigma(t)$ are eventually positive. Furthermore, there exist a $t_1 \in [t_0, \infty)_{\mathbf{T}}$ and positive constants $\ell$ and $L$ such that

$$y(t) > 0, \qquad y^\Delta(t) > 0, \qquad \ell R(t) < y(t) < LR(t) \qquad \text{for } t \in [t_1, \infty)_{\mathbf{T}}.$$

Consequently,

$$\ell R^\sigma(t) < y^\sigma(t) < LR^\sigma(t) \qquad \text{for } t \in [t_1, \infty)_{\mathbf{T}}. \tag{2.2}$$

Integrating (1.1) from $t_1$ to $t$ and noting that $y^\Delta(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$, we have

$$\int_{t_1}^\infty f(s, y^\sigma(s)) \triangle s < \infty. \tag{2.3}$$

By using the monotonicity of $f$ in (A4), we see from (2.2) and (2.3) that

$$\int^{\infty} f(s, \ell R^{\sigma}(s)) \triangle s < \infty,$$

which is nothing but (2.1) with $c = \ell$.

(The "if" part). Suppose that (2.1) holds for some $c = 2k \neq 0$, in which we assume $k > 0$ without loss of generality. By (2.1), let $t_1 \in [t_0, \infty)_{\mathbf{T}}$ so large that

$$\int_{t_1}^{\infty} f(t, 2kR^{\sigma}(t)) \triangle t \leq (2^{\alpha} - 1)k^{\alpha}.$$

Consider the convex set $Y \subset C[t_1, \infty)_{\mathbf{T}}$ and the mapping $\Phi : Y \to C[t_1, \infty)_{\mathbf{T}}$ defined by

$$Y := \{ y \in C[t_1, \infty)_{\mathbf{T}} : kR(t) \leq y(t) \leq 2kR(t), t \in [t_1, \infty)_{\mathbf{T}} \}$$

and

$$(\Phi y)(t) := 2kR(t_1) + \int_{t_1}^{t} \left( \frac{1}{r(s)} \left( k^{\alpha} + \int_{s}^{\infty} f(u, y^{\sigma}(u)) \Delta u \right) \right)^{1/\alpha} \Delta s, \qquad t \in [t_1, \infty)_{\mathbf{T}}.$$

It is routinely verified that $\Phi$ map $Y$ into itself and that the map $\Phi : Y \to Y$ is compact. The Schauder-Tychonoff fixed point theorem therefore ensures the existence of a function $y \in Y$ such that $y = \Phi y$, that is,

$$y(t) = 2kR(t_1) + \int_{t_1}^{t} \left( \frac{1}{r(s)} \left( k^{\alpha} + \int_{s}^{\infty} f(u, y^{\sigma}(u)) \Delta u \right) \right)^{1/\alpha} \Delta s, \qquad t \in [t_1, \infty)_{\mathbf{T}}.$$

It is easy to see that $y(t)$ is a solution of Eq. (1.1) on $[t_1, \infty)_{\mathbf{T}}$ with the desired property $\lim_{t \to \infty} \frac{y(t)}{R(t)} = k$. □

Theorem 2.2. *Assume that* (A1)–(A4) *hold. Then Eq.* (1.1) *has a nonoscillatory solution of type* (III) *if and only if there exists a constant* $c \neq 0$ *such that*

$$\int^{\infty} \left( \frac{1}{r(t)} \int_{t}^{\infty} |f(s, c)| \Delta s \right)^{1/\alpha} \Delta t < \infty. \tag{2.4}$$

Proof. (The "only if" part). Let $y(t)$ be a nonoscillatory solution of type (III) of Eq. (1.1). We may assume that $y(t)$ and $y^{\sigma}(t)$ are eventually positive. Furthermore, there exist a $t_1 \in [t_0, \infty)_{\mathbf{T}}$ and positive constants $\ell$ and $L$ such that

$$y(t) > 0, \qquad y^{\Delta}(t) > 0, \qquad \ell < y(t) < L \qquad \text{for } t \in [t_1, \infty)_{\mathbf{T}}.$$

The monotonicity condition of $f$ in (A4) implies

$$f(t, y^{\sigma}(t)) \geq f(t, y(t)) \geq f(t, \ell), \qquad t \in [t_1, \infty)_{\mathbf{T}}. \tag{2.5}$$

Integrating (1.1) from $s$ to $t$ and noting that $y^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$, we have

$$\int_s^t f(r, y^{\sigma}(r)) \Delta r = r(s)(y^{\Delta}(s))^{\alpha} - r(t)(y^{\Delta}(t))^{\alpha} < r(s)(y^{\Delta}(s))^{\alpha}$$

for $t \geq s \geq t_1$, which gives

$$\left(\frac{1}{r(s)} \int_s^{\infty} f(r, y^{\sigma}(r)) \Delta r\right)^{1/\alpha} \leq y^{\Delta}(s), \qquad s \in [t_1, \infty)_{\mathbf{T}}.$$

Then,

$$\int_{t_1}^t \left(\frac{1}{r(s)} \int_s^{\infty} f(r, y^{\sigma}(r)) \Delta r\right)^{1/\alpha} \Delta s \leq y(t) - y(t_1), \qquad t \in [t_1, \infty)_{\mathbf{T}},$$

which combined with (2.5) yields

$$\int_{t_1}^{\infty} \left(\frac{1}{r(s)} \int_s^{\infty} f(r, \ell) \Delta r\right)^{1/\alpha} \Delta s \leq L - l < \infty.$$

This means that (2.4) holds with $c = \ell$.

(The "if" part). Let (2.4) hold for some $c \neq 0$, in which we may choose $c > 0$ without loss of generality. Thanks to (2.4), we choose $t_1 \in [t_0, \infty)_{\mathbf{T}}$ large enough so that

$$\int_{t_1}^{\infty} \left(\frac{1}{r(t)} \int_t^{\infty} f(s, c) \Delta s\right)^{1/\alpha} \Delta t \leq \frac{c}{2}.$$

Consider the convex set $Y \subset C[t_1, \infty)_{\mathbf{T}}$ and the mapping $\Phi : Y \to C[t_1, \infty)_{\mathbf{T}}$ defined by

$$Y := \left\{ y \in C[t_1, \infty)_{\mathbf{T}} : \frac{c}{2} \leq y(t) \leq c, t \in [t_1, \infty)_{\mathbf{T}} \right\}$$

and

$$(\Phi y)(t) := c - \int_t^{\infty} \left(\frac{1}{r(s)} \int_s^{\infty} f(u, y^{\sigma}(u)) \Delta u\right)^{1/\alpha} \Delta s, \qquad t \in [t_1, \infty)_{T}.$$

It is easily verified that $\Phi$ has a fixed element $y \in Y$ by the Schauder-Tychonoff fixed point theorem, i.e., $y = \Phi y$. So

$$y(t) = c - \int_t^{\infty} \left(\frac{1}{r(s)} \int_s^{\infty} f(u, y^{\sigma}(u)) \Delta u\right)^{1/\alpha} \Delta s, \qquad t \in [t_1, \infty)_{\mathbf{T}}.$$

This shows that $y(t)$ is a solution of Eq. (1.1) on $[t_1, \infty)_{\mathbf{T}}$ and satisfies $\lim_{t\to\infty} y(t) = c$. □

A sufficient condition for the existence of a nonoscillatory solution of type (II) of Eq. (1.1) is given in the next theorem.

Theorem 2.3. *Assume that* (A1)–(A4) *hold. Suppose that* (2.1) *holds for some* $c \neq 0$, *and in addition that*

$$\int^{\infty} \left( \frac{1}{r(t)} \int_t^{\infty} |f(s,d)| \Delta s \right)^{1/\alpha} \Delta t = \infty \tag{2.6}$$

*for all* $d \neq 0$ *with* $cd > 0$. *Then Eq.* (1.1) *has a nonoscillatory solution of type* (II).

Proof. We may suppose that $c > 0$, and take a $k \in (0, c)$. The condition (2.1) allows us to choose $t_1 \in [t_0, \infty)_{\mathbf{T}}$ so large that

$$\int_{t_1}^{\infty} f(t, k(R^{\sigma}(t) + 1)) \Delta t \leq k^{\alpha}.$$

Consider the convex set $Y \subset C[t_1, \infty)_{\mathbf{T}}$ and the mapping $\Phi : Y \to C[t_1, \infty)_{\mathbf{T}}$ defined by

$$Y := \{ y \in C[t_1, \infty)_{\mathbf{T}} : k \leq y(t) \leq k(R^{\sigma}(t) + 1), t \in [t_1, \infty)_{\mathbf{T}} \}$$

and

$$(\Phi y)(t) := k + \int_{t_1}^{t} \left( \frac{1}{r(s)} \int_s^{\infty} f(u, y^{\sigma}(u)) \Delta u \right)^{1/\alpha} \Delta s, \qquad t \in [t_1, \infty)_{\mathbf{T}}.$$

Then, applying the Schauder-Tychonoff fixed point theorem, we see that there exists an element $y \in Y$ such that $y = \Phi y$. This function $y = y(t)$ satisfies

$$y(t) = k + \int_{t_1}^{t} \left( \frac{1}{r(s)} \int_s^{\infty} f(u, y^{\sigma}(u)) \Delta u \right)^{1/\alpha} \Delta s, \qquad t \in [t_1, \infty)_{\mathbf{T}}, \tag{2.7}$$

which implies that $y(t)$ is a positive solution of Eq. (1.1). From (2.7), we also see that

$$\lim_{t\to\infty} \frac{y(t)}{R(t)} = \lim_{t\to\infty} r^{1/\alpha}(t) y^{\Delta}(t) = \lim_{t\to\infty} \left( \int_t^{\infty} f(u, y^{\sigma}(u)) \Delta u \right)^{1/\alpha} \to 0,$$

and, by (2.6),

$$\lim_{t\to\infty} y(t) \geq \lim_{t\to\infty} \left( k + \int_{t_1}^{t} \left( \frac{1}{r(s)} \int_s^{\infty} f(u,k) \Delta u \right)^{1/\alpha} \Delta s \right) = \infty.$$

It follows therefore that $y(t)$ is a solution of type (II). □

EXAMPLE 2.1. Consider the dynamic equation

$$(|y^{\Delta}(t)|^{\alpha-1}y^{\Delta}(t))^{\Delta} + p(t)|y^{\sigma}(t)|^{\beta-1}y^{\sigma}(t) = 0, \tag{2.8}$$

where $\alpha$ and $\beta$ are positive constants and $p \in C_{rd}([t_0, \infty)_{\mathbf{T}}, \mathbf{R}^+)$.

Clearly, the conditions (A1)–(A4) are satisfied for Eq. (2.8). It is easy to see that the conditions (2.1) and (2.4) respectively reduce to

$$\int^{\infty} \sigma^{\beta}(t)p(t)\Delta t < \infty \tag{2.9}$$

and

$$\int^{\infty} \left(\int_t^{\infty} p(s)\Delta s\right)^{1/\alpha} \Delta t < \infty. \tag{2.10}$$

Hence, by Theorems 2.1–2.3, we have

(1) Eq. (2.8) has a nonoscillatory solution such that $\lim_{t\to\infty} \frac{y(t)}{t} = const \neq 0$ if and only if (2.9) holds.

(2) Eq. (2.8) has a nonoscillatory solution such that $\lim_{t\to\infty} y(t) = const \neq 0$ if and only if (2.10) holds.

(3) Eq. (2.8) has a nonoscillatory solution such that

$$\lim_{t\to\infty} \frac{y(t)}{t} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} y(t) = \infty \ \text{ or } \ -\infty,$$

if

$$\int^{\infty} \sigma^{\beta}(t)p(t)\Delta t < \infty \qquad \text{and} \qquad \int^{\infty} \left(\int_t^{\infty} p(s)\Delta s\right)^{1/\alpha} \Delta t = \infty.$$

In particular, let

$$\mathbf{T} = q^{\mathbf{N}} = \{t : t = q^k, k \in \mathbf{N}, q > 1\} \qquad \text{and} \qquad p(t) = \frac{1}{t\sigma(t)}.$$

Noting that $\sigma(t) = qt$, we have

(4) Eq. (2.8) has a nonoscillatory solution such that $\lim_{t\to\infty} \frac{y(t)}{t} = const \neq 0$ if and only if $\beta < 1$ holds.

(5) Eq. (2.8) has a nonoscillatory solution such that $\lim_{t\to\infty} y(t) = const \neq 0$ if and only if $\alpha < 1$ holds.

(6) Eq. (2.8) has a nonoscillatory solution such that $\lim_{t\to\infty} \frac{y(t)}{t} = 0$, and $\lim_{t\to\infty} y(t) = \infty$ *or* $-\infty$ if $\beta < 1 \leq \alpha$.

## 3. Oscillation criteria—sufficient conditions

In this section, we will establish new oscillation criteria for Eq. (1.1). We now start with the following three lemmas.

Lemma 3.1. *Let $x \in C_{rd}^1([t_0, \infty)_{\mathbf{T}}, \mathbf{R}^+)$ with $x^{\Delta}(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbf{T}}$ and $\gamma > 0$. Then the following inequalities are valid, for $t \in [t_0, \infty)_{\mathbf{T}}$,*

(1) $\dfrac{(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)} \geq \gamma \dfrac{x^{\Delta}(t)}{x^{\sigma}(t)}$;

(2) $(x^{\gamma}(t))^{\Delta} \leq \begin{cases} \gamma x^{\gamma-1}(t) x^{\Delta}(t), & 0 < \gamma \leq 1, \\ \gamma (x^{\sigma}(t))^{\gamma-1} x^{\Delta}(t), & \gamma > 1. \end{cases}$

Proof. (1) Since $x(t) > 0$ and $x^{\Delta}(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbf{T}}$, Lemma 1.1 applies to give

$$\begin{aligned}
(x^{\gamma}(t))^{\Delta} &= \gamma \left\{ \int_0^1 [x(t) + h\mu(t) x^{\Delta}(t)]^{\gamma-1} dh \right\} x^{\Delta}(t) \\
&= \gamma \left\{ \int_0^1 [(1-h)x(t) + h x^{\sigma}(t)]^{\gamma-1} dh \right\} x^{\Delta}(t) \\
&\geq \begin{cases} \gamma (x(t))^{\gamma-1} x^{\Delta}(t), & \gamma > 1, \\ \gamma (x^{\sigma}(t))^{\gamma-1} x^{\Delta}(t), & 0 < \gamma \leq 1. \end{cases}
\end{aligned}$$

Thus,

$$\frac{(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)} \geq \begin{cases} \gamma \frac{x^{\Delta}(t)}{x(t)}, & \gamma > 1, \\ \gamma \frac{(x^{\sigma}(t))^{\gamma-1}}{x^{\gamma}(t)} x^{\Delta}(t), & 0 < \gamma \leq 1, \end{cases}$$

since $x^{\Delta}(t) \geq 0$ implies that $x(t) \leq x^{\sigma}(t)$ for $t \in [t_0, \infty)_{\mathbf{T}}$. Thus, Case (1) holds for both $\gamma > 1$ and $0 < \gamma \leq 1$.

(2) Case (2) similarly follows, by Lemma 1.1, from

$$\begin{aligned}
(x^{\gamma}(t))^{\Delta} &= \gamma \left\{ \int_0^1 [(1-h)x(t) + h x^{\sigma}(t)]^{\gamma-1} dh \right\} x^{\Delta}(t) \\
&\leq \begin{cases} \gamma x^{\gamma-1}(t) x^{\Delta}(t), & 0 < \gamma \leq 1, \\ \gamma (x^{\sigma}(t))^{\gamma-1} x^{\Delta}(t), & \gamma > 1, \end{cases}
\end{aligned}$$

since $x(t) > 0$ and $x^{\Delta}(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbf{T}}$. ∎

Lemma 3.2. *Let $v \in C_{rd}^1([t_0, \infty)_{\mathbf{T}}, \mathbf{R})$ be a nonoscillatory solution of the second order dynamic equation*

$$(r(t)|v^{\Delta}(t)|^{\alpha-1} v^{\Delta}(t))^{\Delta} + q(t)|v^{\sigma}(t)|^{\alpha-1} v^{\sigma}(t) = 0, \qquad t \in [t_0, \infty)_{\mathbf{T}}, \tag{3.1}$$

*where $\alpha > 0$, $q(t) \in C_{rd}([t_0, \infty)_{\mathbf{T}}, \mathbf{R}^+)$ and $r(t)$ satisfies the condition* (A2). *Then the function $w(t)$ defined by*

$$w(t) := r(t) \left| \frac{v^{\Delta}(t)}{v(t)} \right|^{\alpha-1} \frac{v^{\Delta}(t)}{v(t)}$$

*satisfies the first order differential inequality*

$$w^{\Delta}(t) + q(t) + \frac{\alpha}{r^{1/\alpha}(t)} |w^{\sigma}(t)|^{(1+\alpha)/\alpha} \le 0, \qquad t \in [t_0, \infty)_{\mathbf{T}}. \tag{3.2}$$

Proof. Without loss of generality we may assume that $v(t) > 0$ and $v^{\sigma}(t) > 0$ for $t \in [t_0, \infty)_{\mathbf{T}}$. Then, by Lemma 2.1, $v^{\Delta}(t) > 0$ for $t \in [t_0, \infty)_{\mathbf{T}}$. Further, by [4, Theorem 1.2 (iv)] and (3.1),

$$w^{\Delta}(t) = \left(\frac{r(t)(v^{\Delta}(t))^{\alpha}}{v^{\alpha}(t)}\right)^{\Delta} = -q(t) - \frac{r(t)(v^{\Delta}(t))^{\alpha}(v^{\alpha}(t))^{\Delta}}{(v^{\sigma}(t))^{\alpha} v^{\alpha}(t)}. \tag{3.3}$$

In view of Lemma 3.1 (1), we get

$$\frac{(v^{\alpha}(t))^{\Delta}}{v^{\alpha}(t)} \ge \alpha \frac{v^{\Delta}(t)}{v^{\sigma}(t)} \qquad \text{for } [t_0, \infty)_{\mathbf{T}}. \tag{3.4}$$

Recall that $r(t)(v^{\Delta}(t))^{\alpha} \le 0$ for $[t_0, \infty)_{\mathbf{T}}$. Then, by (3.4),

$$\begin{aligned} \frac{r(t)(v^{\Delta}(t))^{\alpha}(v^{\alpha}(t))^{\Delta}}{(v^{\sigma}(t))^{\alpha} v^{\alpha}(t)} &\ge \alpha \frac{r(t)(v^{\Delta}(t))^{\alpha+1}}{(v^{\sigma}(t))^{\alpha+1}} = \alpha \frac{(r(t)(v^{\Delta}(t))^{\alpha})^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)(v^{\sigma}(t))^{\alpha+1}} \\ &\ge \alpha \frac{((r(t)(v^{\Delta}(t))^{\alpha})^{\sigma})^{(\alpha+1)/\alpha}}{r^{1/\alpha}(t)(v^{\sigma}(t))^{\alpha+1}} = \frac{\alpha}{r^{1/\alpha}(t)} (w^{\sigma}(t))^{(1+\alpha)/\alpha}. \end{aligned} \tag{3.5}$$

Combining (3.3) and (3.5), one obtain (3.2). □

Lemma 3.3. *If* $u \in C_{rd}^1([t_0, \infty)_{\mathbf{T}}, \mathbf{R}^+)$, *then*

$$(u^{\alpha+1}(t))^{\Delta} \le \frac{\alpha+1}{\alpha} (u^{\alpha}(t))^{\Delta} u^{\sigma}(t), \qquad \alpha > 0,\ t \in [t_0, \infty)_{\mathbf{T}}. \tag{3.6}$$

Proof. By Lemma 1.1, we have

$$(u^{\alpha+1}(t))^{\Delta} = (\alpha+1) \int_0^1 [hu^{\sigma}(t) + (1-h)u(t)]^{\alpha} u^{\Delta}(t) dh$$

and

$$(u^{\alpha}(t))^{\Delta} u^{\sigma}(t) = \alpha \int_0^1 [hu^{\sigma}(t) + (1-h)u(t)]^{\alpha-1} u^{\Delta}(t) u^{\sigma}(t) dh.$$

Thus, we have

$$\begin{aligned}
&\frac{\alpha}{\alpha+1}(u^{\alpha+1}(t))^{\Delta} - (u^{\alpha}(t))^{\Delta}u^{\sigma}(t) \\
&\qquad = \alpha\int_0^1 (1-h)[u(t)-u^{\sigma}(t)][hu^{\sigma}(t)+(1-h)u(t)]^{\alpha-1}u^{\Delta}(t)dh \\
&\qquad = -\alpha\int_0^1 (1-h)[hu^{\sigma}(t)+(1-h)u(t)]^{\alpha-1}(u^{\Delta}(t))^2\mu(t)dh \le 0.
\end{aligned}$$

Hence, (3.6) holds. □

Lemma 3.4. *Assume that* (A1)–(A3) *hold. Let $y(t)$ be a nonoscillatory solution of Eq.* (1.1), *then there exists a constant $k > 0$ such that $y(t)$ eventually satisfies $|y(t)| \le kR(t)$.*

Proof. Without loss of generality we assume $y(t) > 0$ and $y^{\sigma}(t) > 0$ eventually. Then, by Lemma 2.1, there exists a $t_1 \in [t_0, \infty)_{\mathbf{T}}$ such that $y^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$. It follows from (1.1) that $(r(t)(y^{\Delta}(t))^{\alpha})^{\Delta} < 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$. Namely, $r(t)(y^{\Delta}(t))^{\alpha}$ is an eventually positive decreasing function, and hence there exist constants $m > 0$ and $t_2 \in [t_1, \infty)_{\mathbf{T}}$ such that $r(t)(y^{\Delta}(t))^{\alpha} \le m^{\alpha}$ and $R(t) \ge 1$ for $t \in [t_2, \infty)_T$. This gives $y^{\Delta}(t) \le \frac{m}{r^{1/\alpha}(t)}$. Integrating this inequality from $t_2$ to $t$, we obtain, for $t \in [t_2, \infty)_{\mathbf{T}}$,

$$y(t) \le y(t_2) + m\int_{t_2}^{t}\frac{\Delta s}{r^{1/\alpha}(s)} \le y(t_2)R(t) + mR(t) \le kR(t)$$

for some $k > 0$. □

Now, we are in a position to give and show our main results.

Theorem 3.1. *Assume that* (A1)–(A3) *hold. If for all $\delta > 0$,*

$$\int^{\infty} \inf_{\delta \le |y| < \infty} |f(t, y)|\Delta t = \infty, \tag{3.7}$$

*then* (1.1) *is oscillatory.*

Proof. Assume that (1.1) has a nonoscillatory solution $y(t)$, we may suppose that $y(t) > 0$ and $y^{\sigma}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}} \subseteq [t_0, \infty)_{\mathbf{T}}$, since a parallel argument holds for the case $y(t) < 0$. Then, by Lemma 2.1, $y^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$. Integration of (1.1) from $t_1$ to $t$ gives

$$\int_{t_1}^{\infty} f(s, y^{\sigma}(s))\Delta s \le r(t_1)(y^{\Delta}(t_1))^{\alpha} < \infty. \tag{3.8}$$

On the other hand, since $y^{\Delta}(t) > 0$ implies that $0 < y^{\sigma}(t_1) \le y^{\sigma}(t)$ for $t \in [t_1, \infty)_{\mathbf{T}}$, and putting $\delta := y^{\sigma}(t_1)$, by (3.7), we find

$$\int_{t_1}^{\infty} f(t, y^{\sigma}(t))\Delta t \ge \int_{t_1}^{\infty} \inf_{\delta \le |y^{\sigma}(t)| < \infty} |f(t, y^{\sigma}(t))| \Delta t = \infty,$$

which contradicts (3.8). □

Theorem 3.2. *Assume that* (A1)–(A3) *hold. If for some* $0 < \lambda < \alpha$ *and all* $\delta > 0$*,*

$$\int^{\infty} R^{\lambda}(t) \inf_{\delta \le |y| < \infty} \frac{|f(t, y)|}{|y|^{\alpha}} \Delta t = \infty, \tag{3.9}$$

*and* $R(t)/R^{\sigma}(t) \ge c > 0$*, then Eq.* (1.1) *is oscillatory.*

Proof. Assume that (1.1) has a nonoscillatory solution $y(t)$. Without lose of generality we may suppose that $y(t) > 0$ and $y^{\sigma}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}} \subseteq [t_0, \infty)_{\mathbf{T}}$, Then, Lemma 2.1 implies that $y^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$. Put

$$w(t) := r(t)\left(\frac{y^{\Delta}(t)}{y(t)}\right)^{\alpha}.$$

Applying Lemma 3.2 to (1.1), we then get

$$(r(t)|y^{\Delta}(t)|^{\alpha-1}y^{\Delta}(t))^{\Delta} + \frac{f(t, y^{\sigma}(t))}{(y^{\sigma}(t))^{\alpha}}|y^{\sigma}(t)|^{\alpha-1}y^{\sigma}(t) = 0, \qquad t \in [t_1, \infty)_{\mathbf{T}}.$$

Consequently,

$$w^{\Delta}(t) + \frac{\alpha}{r^{1/\alpha}(t)}(w^{\sigma}(t))^{(\alpha+1)/\alpha} + \frac{f(t, y^{\sigma}(t))}{(y^{\sigma}(t))^{\alpha}} \le 0, \qquad t \in [t_1, \infty)_{\mathbf{T}}. \tag{3.10}$$

Multiplying (3.10) by $R^{\lambda}(t)$ and integrating it over $[t_1, t]_{\mathbf{T}}$, we then have, for $t \in [t_1, \infty)_{\mathbf{T}}$,

$$\int_{t_1}^{t} R^{\lambda}(s)w^{\Delta}(s)\Delta s + \alpha\int_{t_1}^{t} R^{\lambda}(s)R^{\Delta}(s)(w^{\sigma}(s))^{(\alpha+1)/\alpha}\Delta s + \int_{t_1}^{t} R^{\lambda}(s)\frac{f(s, y^{\sigma}(s))}{(y^{\sigma}(s))^{\alpha}}\Delta s \le 0.$$

By using the integration by parts formula [4, Theorem 1.77 (vi)] for the first term of the last inequality, we get, for $t \in [t_1, \infty)_{\mathbf{T}}$,

$$R^{\lambda}(t)w(t) - \int_{t_1}^{t} (R^{\lambda}(s))^{\Delta} w^{\sigma}(s)\Delta s + \alpha\int_{t_1}^{t} R^{\lambda}(s)R^{\Delta}(s)(w^{\sigma}(s))^{(1+\alpha)/\alpha}\Delta s + \int_{t_1}^{t} R^{\lambda}(s)\frac{f(s, y^{\sigma}(s))}{(y^{\sigma}(s))^{\alpha}}\Delta s \le c_1, \tag{3.11}$$

where $c_1 > 0$ is a constant. We now consider two cases: $0 < \lambda \le 1$ and $\lambda > 1$.

Case 1. $0 < \lambda \le 1$. By Lemma 3.1 (2), we get

$$(R^{\lambda}(s))^{\Delta} \le \lambda R^{\lambda-1}(s) R^{\Delta}(s), \qquad 0 < \lambda \le 1. \tag{3.12}$$

Substituting (3.12) into (3.11), we get

$$\begin{aligned} R^{\lambda}(t)w(t) &- \lambda \int_{t_1}^{t} R^{\lambda-1}(s)R^{\Delta}(s)w^{\sigma}(s)\Delta s \\ &+ \alpha \int_{t_1}^{t} R^{\lambda}(s)R^{\Delta}(s)(w^{\sigma}(s))^{(\alpha+1)/\alpha}\Delta s + \int_{t_1}^{t} R^{\lambda}(s)\frac{f(s, y^{\sigma}(s))}{(y^{\sigma}(s))^{\alpha}}\Delta s \le c_1. \end{aligned} \tag{3.13}$$

Suppose first that

$$\int_{t_1}^{\infty} R^{\lambda-1}(s)R^{\Delta}(s)w^{\sigma}(s)\Delta s < \infty.$$

It then follows from (3.13) that

$$\int_{t_1}^{t} R^{\lambda}(s)\frac{f(s, y^{\sigma}(s))}{(y^{\sigma}(s))^{\alpha}}\Delta s \le c_1 + \lambda \int_{t_1}^{t} R^{\lambda-1}(s)R^{\Delta}(s)w^{\sigma}(s)\Delta s, \qquad t \in [t_1, \infty)_{\mathbf{T}},$$

which, in the limit as $t \to \infty$, shows that

$$\int_{t_1}^{\infty} R^{\lambda}(s)\frac{f(s, y^{\sigma}(s))}{(y^{\sigma}(s))^{\alpha}}\Delta s < \infty.$$

But this is impossible, because (3.9) implies that

$$\int_{t_1}^{\infty} R^{\lambda}(s)\frac{f(s, y^{\sigma}(s))}{(y^{\sigma}(s))^{\alpha}}\Delta s \ge \int_{t_1}^{\infty} R^{\lambda}(s) \inf_{\delta \le |y^{\sigma}(s)| < \infty} \frac{|f(s, y^{\sigma}(s))|}{|y^{\sigma}(s)|^{\alpha}}\Delta s = \infty, \tag{3.14}$$

where $\delta = y^{\sigma}(t_1) > 0$.

Suppose next that

$$\int_{t_1}^{\infty} R^{\lambda-1}(s)R^{\Delta}(s)w^{\sigma}(s)\Delta s = \infty. \tag{3.15}$$

Then, by (3.13),

$$\begin{aligned} \int_{t_1}^{t} R^{\lambda}(s)\frac{f(s, y^{\sigma}(s))}{(y^{\sigma}(s))^{\alpha}}\Delta s &\le c_1 + \lambda \int_{t_1}^{t} R^{\lambda-1}(s)R^{\Delta}(s)w^{\sigma}(s)\Delta s \\ &\quad - \alpha \int_{t_1}^{t} R^{\lambda}(s)R^{\Delta}(s)(w^{\sigma}(s))^{(1+\alpha)/\alpha}\Delta s. \end{aligned} \tag{3.16}$$

By means of Hölder's inequality [4, Theorem 6.13], one then have

$$\begin{aligned}
&\int_{t_1}^{t} R^{\lambda-1}(s)R^{\Delta}(s)w^{\sigma}(s)\Delta s \\
&\quad = \int_{t_1}^{t} (R^{\lambda}(s)R^{\Delta}(s))^{\alpha/(\alpha+1)} w^{\sigma}(s)\left(\frac{R^{\Delta}(s)}{R^{\alpha+1-\lambda}(s)}\right)^{1/(\alpha+1)}(s)\Delta s \\
&\quad \le \left(\int_{t_1}^{t} R^{\lambda}(s)R^{\Delta}(s)(w^{\sigma}(s))^{(\alpha+1)/\alpha}\Delta s\right)^{\alpha/(\alpha+1)} \left(\int_{t_1}^{t} \frac{R^{\Delta}(s)}{R^{\alpha+1-\lambda}(s)}\Delta s\right)^{1/(\alpha+1)}. \qquad (3.17)
\end{aligned}$$

By Lemma 3.1 (1), we get, for $\alpha - \lambda > 0$,

$$\frac{(R^{\alpha-\lambda}(t))^{\Delta}}{R^{\alpha-\lambda}(t)} \ge (\alpha-\lambda)\frac{R^{\Delta}(t)}{R^{\sigma}(t)}. \qquad (3.18)$$

Then, by [4, Theorem 1.20 (iv)] and (3.18),

$$\begin{aligned}
-\left(\frac{1}{R^{\alpha-\lambda}(t)}\right)^{\Delta} &= \frac{(R^{\alpha-\lambda}(t))^{\Delta}}{(R^{\sigma}(t))^{\alpha-\lambda}R^{\alpha-\lambda}(t)} \ge (\alpha-\lambda)\frac{R^{\Delta}(t)}{(R^{\sigma}(t))^{\alpha+1-\lambda}} \\
&= (\alpha-\lambda)\frac{R^{\Delta}(t)}{R^{\alpha+1-\lambda}(t)}\left(\frac{R(t)}{R^{\sigma}(t)}\right)^{\alpha+1-\lambda} \\
&\ge (\alpha-\lambda)c_1^{\alpha+1-\lambda}\frac{R^{\Delta}(t)}{R^{\alpha+1-\lambda}(t)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\int_{t_1}^{t} \frac{R^{\Delta}(s)}{R^{\alpha+1-\lambda}(s)}\Delta s &\le -\frac{c_1^{\lambda-\alpha-1}}{\alpha-\lambda}\int_{t_1}^{t}\left(\frac{1}{R^{\alpha-\lambda}(s)}\right)^{\Delta}\Delta s \\
&\le \frac{c_1^{\lambda-\alpha-1}}{\alpha-\lambda}\,\frac{1}{R^{\alpha-\lambda}(t_1)} =: c_2. \qquad (3.19)
\end{aligned}$$

Combining (3.17) with (3.19), we have

$$\begin{aligned}
&\int_{t_1}^{t} R^{\lambda-1}(s)R^{\Delta}(s)w^{\sigma}(s)\Delta s \\
&\quad \le c_2^{1/(\alpha+1)}\left(\int_{t_1}^{t} R^{\lambda}(s)R^{\Delta}(s)(w^{\sigma}(s))^{(\alpha+1)/\alpha}\Delta s\right)^{\alpha/(\alpha+1)} \\
&\quad = c_2^{1/(\alpha+1)}\frac{\int_{t_1}^{t} R^{\lambda}(s)R^{\Delta}(s)(w^{\sigma}(s))^{(\alpha+1)/\alpha}\Delta s}{(\int_{t_1}^{t} R^{\lambda}(s)R^{\Delta}(s)(w^{\sigma}(s))^{(\alpha+1)/\alpha}\Delta s)^{1/\alpha}}. \qquad (3.20)
\end{aligned}$$

On the other hand, since it from (3.15) that

$$\int_{t_1}^{t} R^{\lambda}(s)R^{\Delta}(s)(w^{\sigma}(s))^{(\alpha+1)/\alpha}\Delta s \to \infty \qquad \text{as } t \to \infty,$$

we see from (3.20) that there exists $t_2 \in [t_1, \infty)_{\mathbf{T}}$ such that

$$\int_{t_1}^{t} R^{\lambda-1}(s)R^{\Delta}(s)w^{\sigma}(s)\Delta s \le \frac{\alpha}{\lambda}\int_{t_1}^{t} R^{\lambda}(s)R^{\Delta}(s)(w^{\sigma}(s))^{(\alpha+1)/\alpha}\Delta s$$

for all $t \in [t_2, \infty)_{\mathbf{T}}$. Using the above inequality in (3.16), we conclude that

$$\int_{t_0}^{\infty} R^{\lambda}(s)\frac{f(s, y^{\sigma}(s))}{(y^{\sigma}(s))^{\alpha}}\Delta s \le c_1,$$

in contradiction to (3.14) which also holds in the present situation.

Case 2. $\lambda > 1$. By Lemma 3.1 (2), we get

$$(R^{\lambda}(s))^{\Delta} \le \lambda(R^{\sigma}(s))^{\lambda-1}R^{\Delta}(s), \qquad \lambda > 1. \tag{3.21}$$

Substituting (3.21) into (3.11), we get

$$\begin{aligned} R^{\lambda}(t)w(t) - \lambda\int_{t_1}^{t}(R^{\sigma}(s))^{\lambda-1}R^{\Delta}(s)w^{\sigma}(s)\Delta s \\ + \alpha\int_{t_1}^{t} R^{\lambda}(s)R^{\Delta}(s)(w^{\sigma}(s))^{(\alpha+1)/\alpha}\Delta s + \int_{t_1}^{t} R^{\lambda}(s)\frac{f(s, y^{\sigma}(s))}{(y^{\sigma}(s))^{\alpha}}\Delta s \le c_1. \end{aligned} \tag{3.22}$$

Similarly to the proof of Case 1, supposing that

$$\int_{t_1}^{\infty} R^{\lambda-1}(s)(R^{\sigma}(s))^{\lambda-1}w^{\sigma}(s)\Delta s < \infty,$$

we also get a contradiction to (3.14). Hence, we suppose next that

$$\int_{t_1}^{\infty}(R^{\sigma}(s))^{\lambda-1}R^{\Delta}(s)w^{\sigma}(s)\Delta s = \infty.$$

Then, by (3.22),

$$\begin{aligned} \int_{t_1}^{t} R^{\lambda}(s)\frac{f(s, y^{\sigma}(s))}{(y^{\sigma}(s))^{\alpha}}\Delta s \le c_1 + \lambda\int_{t_1}^{t}(R^{\sigma}(s))^{\lambda-1}R^{\Delta}(s)w^{\sigma}(s)\Delta s \\ - \alpha\int_{t_1}^{t} R^{\lambda}(s)R^{\Delta}(s)(w^{\sigma}(s))^{(1+\alpha)/\alpha}\Delta s. \end{aligned}$$

By means of Hölder's inequality [4, Theorem 6.13] again,

$$\int_{t_1}^{t} (R^\sigma(s))^{\lambda-1} R^\Delta(s) w^\sigma(s) \Delta s \le \left( \int_{t_1}^{t} R^\lambda(s) R^\Delta(s) (w^\sigma(s))^{(\alpha+1)/\alpha} \Delta s \right)^{\alpha/(\alpha+1)}$$
$$\times \left( \int_{t_1}^{t} \frac{R^\Delta(s)}{(R^\sigma(s))^{\alpha+1-\lambda}} \left( \frac{R^\sigma(s)}{R(s)} \right)^{\alpha\lambda} \Delta s \right)^{1/(\alpha+1)}$$
$$\le \frac{1}{c_1^{\alpha\lambda/(\alpha+1)}} \left( \int_{t_1}^{t} R^\lambda(s) R^\Delta(s) (w^\sigma(s))^{(\alpha+1)/\alpha} \Delta s \right)^{\alpha/(\alpha+1)}$$
$$\times \left( \int_{t_1}^{t} \frac{R^\Delta(s)}{(R^\sigma(s))^{\alpha+1-\lambda}} \Delta s \right)^{1/(\alpha+1)}.$$

By [4, Theorem 1.20 (iv)], and note that (3.18),

$$-\left( \frac{1}{R^{\alpha-\lambda}(t)} \right)^\Delta = \frac{(R^{\alpha-\lambda}(t))^\Delta}{(R^\sigma(t))^{\alpha-\lambda} R^{\alpha-\lambda}(t)} \ge (\alpha-\lambda) \frac{R^\Delta(t)}{(R^\sigma(t))^{\alpha+1-\lambda}}.$$

Using the same argument as the proof of Case 1, we finish the proof. □

Theorem 3.3. *Assume that* (A1)–(A4) *hold. If for all* $\delta$, $\delta'$ *with* $\delta' > \delta > 0$,

$$\int^\infty \inf_{\delta \le |y| \le \delta'} |f(t, y)| \Delta t = \infty, \tag{3.23}$$

*and there is a continuous function* $\varphi : [y_0, \infty) \to \mathbf{R}^+$, $y_0 > 0$, *and a constant* $t_2 \in [t_1, \infty)_{\mathbf{T}}$ *such that for* $t \in [t_2, \infty)_{\mathbf{T}}$,

$$\inf_{|y| \ge y_0} \frac{f(t, y)}{\varphi(|y|)} \ge \frac{1}{r^{1/\alpha}(t)} \tag{3.24}$$

*and*

$$\int_{y_0}^\infty \varphi(y) dy = \infty, \tag{3.25}$$

*then* (1.1) *is oscillatory.*

Proof. Assume that (1.1) has a nonoscillatory solution $y(t)$, we may suppose that $y(t) > 0$ and $y^\sigma(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}} \subseteq [t_0, \infty)_{\mathbf{T}}$, then $y^\Delta(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$ and (3.8) holds by Lemma 2.1. The solution $y(t)$ is either bounded or unbounded.

If $y(t)$ is bounded, then $\delta \leq y(t) \leq \delta'$ for $t \in [t_1, \infty)_{\mathbf{T}}$, for some positive constant $\delta$ and $\delta'$, and by (3.23), we have

$$\int_{t_1}^{\infty} f(t, y^{\sigma}(t))\Delta t \geq \int_{t_1}^{\infty} \inf_{\delta \leq |y^{\sigma}(t)| \leq \delta'} |f(t, y^{\sigma}(t)|\Delta t = \infty.$$

But this contradicts (3.8).

If $y(t)$ is unbounded, and let $u(t) = r^{1/\alpha}(t)y^{\Delta}(t)$, then $u(t) > 0$ and $u(t)$ is nonincreasing for $t \in [t_1, \infty)_{\mathbf{T}}$. In view of Lemma 3.3, we have

$$\begin{aligned}\frac{\alpha}{\alpha+1}((r^{1/\alpha}(t)y^{\Delta}(t))^{\alpha+1})^{\Delta} &\leq ((r^{1/\alpha}(t)y^{\Delta}(t))^{\alpha})^{\Delta}(r^{1/\alpha}(t)y^{\Delta}(t))^{\sigma} \\ &\leq (r(t)(y^{\Delta}(t))^{\alpha})^{\Delta} r^{1/\alpha}(t)y^{\Delta}(t).\end{aligned}$$

Multiplying (1.1) by $r^{1/\alpha}(t)y^{\Delta}(t)$ and integrating it from $t_1$ to $t$, and using the above inequality, we arrive at

$$\begin{aligned}\frac{\alpha}{\alpha+1}&[(r^{1/\alpha}(t)y^{\Delta}(t))^{\alpha+1} - (r^{1/\alpha}(t_1)y^{\Delta}(t_1))^{\alpha+1}] \\ &+ \int_{t_1}^{t} r^{1/\alpha}(s)f(s, y^{\sigma}(s))y^{\Delta}(s)\Delta s \leq 0,\end{aligned}$$

from which follows that

$$\int_{t_1}^{\infty} r^{1/\alpha}(s)f(s, y^{\sigma}(s))y^{\Delta}(s)\Delta s < \infty.$$

Then we have, by (A4),

$$\int^{\infty} r^{1/\alpha}(s)f(s, y(s))y^{\Delta}(s)\Delta s < \infty. \tag{3.26}$$

On the other hand, (3.24) implies that

$$r^{1/\alpha}(t)f(t, y) \geq \varphi(y) \qquad \text{for } t \in [t_2, \infty)_{\mathbf{T}},\ y \in [y_0, \infty),$$

and $t_2$ can be chosen so that $y(t) \geq y_0$ for $t \in [t_2, \infty)_{\mathbf{T}}$. Therefore, it follows from (3.25) that

$$\begin{aligned}\int_{t_2}^{\infty} r^{1/\alpha}(s)f(s, y(s))y^{\Delta}(s)\Delta s &\geq \int_{t_2}^{\infty} \varphi(y(s))y^{\Delta}(s)\Delta s \\ &= \int_{y(t_2)}^{\infty} \varphi(u)du = \infty,\end{aligned}$$

which contradicts (3.26). $\square$

Theorem 3.4. *Assume that* (A1)–(A3) *and* (A5) *hold. If for all* $\delta, \delta'$ *with* $\delta' \geq \delta \geq 0$, (3.23) *holds, and for all* $c \neq 0$,

$$\int^{\infty} |f(t, cR^{\sigma}(t))| \Delta t = \infty, \tag{3.27}$$

*then Eq.* (1.1) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of Eq. (1.1), and we suppose that $y(t) > 0$ and $y^{\sigma}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}} \subseteq [t_0, \infty)_{\mathbf{T}}$. Then $y^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$ by Lemma 2.1. Note that $y(t)$ may be either bounded or unbounded. If $y(t)$ is bounded, following the proof of Theorem 3.3, by (3.23), we get a contraction. So $y(t)$ is unbounded. Let $y_2 > 0$ be such that $f(t, y)$ is nonincreasing in $y$ for $y > y_2$. Since $y(t) \to \infty$ as $t \to \infty$, there exist, by Lemma 3.4, positive constants $k$ and $t_2 \in [t_1, \infty)_{\mathbf{T}}$ such that $y_2 \leq y^{\sigma}(t) \leq kR^{\sigma}(t)$ for $t \in [t_2, \infty)_{\mathbf{T}}$, which implies $f(t, y^{\sigma}(t)) \geq f(t, kR^{\sigma}(t))$ for $t \in [t_2, \infty)_{\mathbf{T}}$ by (A5). It follows from (3.8) and (3.27) that

$$\infty > \int_{t_2}^{\infty} f(s, y^{\sigma}(s)) \Delta s \geq \int_{t_2}^{\infty} f(s, kR^{\sigma}(s)) \Delta s = \infty,$$

which is a contradiction. □

Remark 3.1. If we choose $\mathbf{T} = \mathbf{R}$ and $r(t) \equiv 1$, then Theorems 3.1–3.4 reduce to [20, Theorems 1.1 and 1.2] for Eq. (1.2).

Now we consider the dynamic equation

$$(|y^{\Delta}(t)|^{\alpha-1} y^{\Delta}(t))^{\Delta} + q(t) g(y^{\sigma}(t)) = 0 \qquad \text{for } t \in [t_0, \infty)_{\mathbf{T}}, \tag{3.28}$$

where $q \in C_{rd}([t_0, \infty)_{\mathbf{T}}, \mathbf{R}^+)$, $g \in C(\mathbf{R}, \mathbf{R})$ with $yg(y) > 0$ for all $y \neq 0$.

By Theorems 3.1–3.4, we have

Corollary 3.1. *Eq.* (3.28) *is oscillatory if one of the following conditions is satisfied*

(1) $\liminf_{|y| \to \infty} |f(y)| > 0$ *and* $\int^{\infty} q(t) \Delta t = \infty$;

(2) $\liminf_{|y| \to \infty} \frac{|f(y)|}{|y|^{\alpha}} > 0$, $\frac{t}{\sigma(t)} \geq c > 0$ *and* $\int^{\infty} t^{\lambda} q(t) \Delta t = \infty$ *for some* $0 < \lambda < \alpha$;

(3) $\liminf_{|t| \to \infty} |q(t)| > 0$ *and* $g(y)$ *is nondecreasing;*

(4) $\liminf_{|t| \to \infty} |q(t)| > 0$ *and* $g(y)$ *is nonincreasing with* $\int^{\infty} |g(cR^{\sigma}(t))| \Delta t = \infty$ *for all* $c \neq 0$.

## 4. Oscillation criteria—necessary and sufficient conditions

In this section, we give the necessary and sufficient conditions for all solution of Eq. (1.1) to be oscillatory.

Theorem 4.1. *Assume that* (A1)–(A4) *hold. Suppose in addition that there exists a continuous nondecreasing function* $\varphi : \mathbf{R} \to \mathbf{R}$ *with the properties that*

$$sgn\ \varphi(u) = sgn\ u, \qquad \int^{\pm\infty} \frac{du}{\varphi(u)} < \infty, \tag{4.1}$$

*and for some constants* $k > 0$, $y_1 > 0$ *and* $\ell \neq 0$ *with* $sgn\ \ell = sgn\ y$,

$$|f(t, y)| \geq k|f(t, \ell)|\,|\varphi(y)|^{\alpha}, \qquad t \in [t_1, \infty)_{\mathbf{T}},\ |y| \geq y_1. \tag{4.2}$$

*Then Eq.* (1.1) *is oscillatory if and only if*

$$\int^{\infty} \left( \frac{1}{r(t)} \int_t^{\infty} |f(s, c)| \Delta s \right)^{1/\alpha} \Delta t = \infty \qquad \text{for all } c \neq 0. \tag{4.3}$$

Proof. (The "only if" part). If (4.3) is violated, then, by Theorem 2.2, Eq. (1.1) has a nonoscillatory solution $y(t)$ such that $\lim_{t\to\infty} y(t) = const \neq 0$.

(The "if" part). Let (4.3) hold and suppose that Eq. (1.1) has a nonoscillatory solution $y(t)$. We may assume that $y(t) > 0$ and $y^{\sigma}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}} \subseteq [t_0, \infty)_T$. Then, by Lemma 2.1, $y^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$. It follows from Theorem 2.2 that $\lim_{t\to\infty} y(t) = +\infty$. Integration of (1.1) gives, for $s \in [t_1, \infty)_{\mathbf{T}}$,

$$r^{1/\alpha}(s) y^{\Delta}(s) \geq \left( \int_s^{\infty} f(u, y^{\sigma}(u)) \Delta u \right)^{1/\alpha}.$$

This together with (A4) and the fact $\varphi$ is nondecreasing on $\mathbf{R}$ imply that, for $s \in [t_1, \infty)_{\mathbf{T}}$,

$$\begin{aligned} \frac{y^{\Delta}(s)}{\varphi(y(s))} &\geq \left( \frac{1}{r(s)} \int_s^{\infty} \frac{f(t, y^{\sigma}(u))}{\varphi^{\alpha}(y(s))} \Delta u \right)^{1/\alpha} \\ &\geq \left( \frac{1}{r(s)} \int_s^{\infty} \frac{f(t, y(u))}{\varphi^{\alpha}(y(u))} \Delta u \right)^{1/\alpha}. \end{aligned} \tag{4.4}$$

Note that $y(t) \to +\infty$ as $t \to \infty$, and in view of (4.2), there is $t_2 \in [t_1, \infty)_{\mathbf{T}}$ such that

$$\frac{f(t, y(t))}{\varphi^{\alpha}(y(t))} \geq kf(t, \ell), \qquad t \in [t_2, \infty)_{\mathbf{T}} \tag{4.5}$$

for some $k > 0$ and $\ell > 0$. Substituting (4.5) into (4.4), and integrating it from $t_2$ to $t$, we obtain

$$\int_{y(t_2)}^{y(t)} \frac{du}{\varphi(u)} \geq k^{1/\alpha} \int_{t_2}^{t} \left( \frac{1}{r(s)} \int_{s}^{\infty} f(u, \ell) \Delta u \right)^{1/\alpha} \Delta s,$$

which, in the limit as $t \to \infty$, gives

$$k^{1/\alpha} \int_{t_2}^{t} \left( \frac{1}{r(s)} \int_{s}^{\infty} f(u, \ell) \Delta u \right)^{1/\alpha} \Delta s < \infty.$$

This contradicts (4.3). $\square$

THEOREM 4.2. *Assume that* (A1)–(A4) *hold. Suppose in addition that there exists a continuous nonincreasing function* $\psi : [-M, M] \to \mathbf{R}$, $M > 0$, *such that*

$$sgn\ \psi(v) = sgn\ v, \qquad \int_{\pm 0}^{\pm M} \frac{dv}{\psi(|v|^{(1-\alpha)/\alpha} v)} < \infty, \tag{4.6}$$

*and for some constants* $k > 0$, $v_1 > 0$,

$$|f(t, uv)| \geq k |f(t, u)|\, |\psi(v)|, \qquad t \in [t_1, \infty)_{\mathbf{T}},\ u \neq 0,\ 0 < |v| \leq v_1. \tag{4.7}$$

*Then Eq.* (1.1) *is oscillatory if and only if*

$$\int^{\infty} |f(t, cR^{\sigma}(t))| \Delta t = \infty \qquad \textit{for all } c \neq 0. \tag{4.8}$$

PROOF. (The "only if" part). If (4.8) is violated, then, by Theorem 2.1, Eq. (1.1) has a nonoscillatory solution $y(t)$ such that $\lim_{t\to\infty} \frac{y(t)}{R(t)} = const \neq 0$.

(The "if" part). Let (4.8) hold and suppose that Eq. (1.1) has a nonoscillatory solution $y(t)$. We may assume that $y(t) > 0$ and $y^{\sigma}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}} \subseteq [t_0, \infty)_{\mathbf{T}}$. Then, by Lemma 2.1, $y^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$. Let $\xi(t) := r^{1/\alpha}(t) y^{\Delta}(t)$, $t \in [t_1, \infty)_{\mathbf{T}}$. Because of Theorem 2.1 and (4.8), the nonoscillatory solution $y(t)$ has to be either of type (II) and type (III), and hence $\xi(t) = r^{1/\alpha}(t) y^{\Delta}(t)$ decreases to 0 as $t \to \infty$. Observe that, for $t \in [t_1, \infty)_{\mathbf{T}}$,

$$\begin{aligned} y(t) - y(t_1) &= \int_{t_1}^{t} y^{\Delta}(s) \Delta s = \int_{t_1}^{t} \frac{r^{1/\alpha}(s) y^{\Delta}(s)}{r^{1/\alpha}(s)} \Delta s \geq \int_{t_1}^{t} \frac{r^{1/\alpha}(t) y^{\Delta}(t)}{r^{1/\alpha}(s)} \Delta s \\ &= r^{1/\alpha}(t) y^{\Delta}(t) [R(t) - R(t_1)] = \xi(t) [R(t) - R(t_1)], \end{aligned}$$

which implies that there are positive constants $t_2 \in [t_1, \infty)_{\mathbf{T}}$ and $0 < c < 1$ such that

$$0 < \xi(t) \leq M \qquad \text{and} \qquad y(t) \geq c\xi(t) R(t), \qquad t \in [t_2, \infty)_{\mathbf{T}}. \tag{4.9}$$

For a fixed $0 < u_0 \le M$, define

$$G(u) := \int_u^{u_0} \frac{dv}{\psi(v^{1/\alpha})}, \qquad u \in (0, u_0]$$

and

$$\kappa(t,h) := r(t)(y^{\Delta}(t))^{\alpha} + h\mu(t)(r(t)(y^{\Delta}(t))^{\alpha})^{\Delta}, \qquad t \in [t_2, \infty)_{\mathbf{T}},\ h \in [0,1].$$

Then,

$$\kappa(t,h) = (1-h)\xi^{\alpha}(t) + h(\xi^{\sigma}(t))^{\alpha} \ge (\xi^{\sigma}(t))^{\alpha}. \tag{4.10}$$

By Lemma 1.1, the fact $\psi(u)$ is nonincreasing and (4.10), we have, for $t \in [t_2, \infty)_{\mathbf{T}}$,

$$\begin{aligned} [G(r(t)(y^{\Delta}(t))^{\alpha})]^{\Delta} &= \left(\int_0^1 G'(\kappa(t,h))dh\right)(r(t)(y^{\Delta}(t))^{\alpha})^{\Delta} \\ &= \left(\int_0^1 \frac{dh}{\psi(\kappa^{1/\alpha}(t,h))}\right) f(t, y^{\sigma}(t)) \\ &\ge \left(\int_0^1 \frac{dh}{\psi(\xi^{\sigma}(t))}\right) f(t, y^{\sigma}(t)) = \frac{f(t, y^{\sigma}(t))}{\psi(\xi^{\sigma}(t))}. \end{aligned} \tag{4.11}$$

In view of (4.9) and (A4), we see that

$$\frac{f(t, y^{\sigma}(t))}{\psi(\xi^{\sigma}(t))} \ge \frac{f(t, c\xi^{\sigma}(t)R^{\sigma}(t))}{\psi(\xi^{\sigma}(t))} \ge kf(t, cR^{\sigma}(t)), \qquad t \in [t_3, \infty)_{\mathbf{T}}, \tag{4.12}$$

where the second inequality in (4.9) and the nondecreasing property of $f(t, y)$ in $y$ have been used. Combining (4.11) with (4.12), we have

$$[G(r(t)(y^{\Delta}(t))^{\alpha})]^{\Delta} \ge kf(t, cR^{\sigma}(t)) \qquad \text{for } t \in [t_3, \infty)_{\mathbf{T}},$$

from which it follows that, by (4.6),

$$k\int_{t_3}^{t} f(s, cR^{\sigma}(s))\Delta s \le G(r(t)(y^{\Delta}(t))^{\alpha}) \le \int_0^M \frac{dv}{\psi(v^{1/\alpha})}.$$

This shows that $k\int_{t_3}^{\infty} f(s, cR^{\sigma}(s))\Delta s < \infty$, a contradiction to (4.8). □

Remark 4.1. If we choose $\mathbf{T} = \mathbf{R}$ and $r(t) \equiv 1$, then Theorems 4.1 and 4.2 reduce to [20, Theorems 2.3 and 2.4] for Eq. (1.2).

Let $\mathbf{T} = \mathbf{N}$, Eq. (1.1) becomes the difference equation

$$\Delta(r_n|\Delta y_n|^{\alpha-1}\Delta y_n) + f(n, y_{n+1}) = 0, \qquad n \in \mathbf{N}, \tag{4.13}$$

where $\Delta y_n = y_{n+1} - y_n$.

By Theorems 4.1 and 4.2, we have

Corollary 4.1. *Assume that* (A1)–(A4) *hold.*

(1) *Suppose in addition there exists a continuous nondecreasing function* $\varphi(u) : \mathbf{R} \to \mathbf{R}$ *such that* (4.1) *and* (4.2) *hold. Then* (4.13) *is oscillatory if and only if*

$$\sum_{n=1}^{\infty} \left( \frac{1}{r_n} \sum_{k=n}^{\infty} |f(k,c)| \right)^{1/\alpha} = \infty \qquad \text{for all } c \neq 0.$$

(2) *Suppose in addition there exists a continuous nonincreasing function* $\psi(v) : [-M, M] \to \mathbf{R}$, $M > 0$, *such that* (4.6) *and* (4.7) *hold. Then* (4.13) *is oscillatory if and only if*

$$\sum_{n=1}^{\infty} |f(n, cR_{n+1})| = \infty \qquad \text{for all } c \neq 0.$$

Remark 4.1. Theorems 4.1 and 4.2 extend and improve Theorems 3.1 and 3.2 in Grace et al. [10], respectively.

Finally, we give some examples to illustrate our main results.

Example 4.1. Let $\mathbf{T} = \mathbf{N}$, and consider the difference equation

$$\Delta(|\Delta y_n|^{\alpha-1} \Delta y_n) + \frac{n^{\nu} |y_{n+1}|^{p-1} y_{n+1}}{1 + n^{\mu} |y_{n+1}|^{q}} = 0, \qquad n \in \mathbf{N}, \tag{4.14}$$

where $\Delta y_n = y_{n+1} - y_n$, $p > 0$, $q > 0$, $\mu > 0$, and $\nu$ are constants. Equation (4.14) is oscillatory provided that one of the following is satisfies.

(1) $p \geq q + \alpha$, and $\mu - \nu \leq \lambda + 1$ for some $0 < \lambda < \alpha$;
(2) $q + \alpha > p \geq q$, and $\mu - \nu \leq 1$;
(3) $q > p \geq q - 1$ and $\mu - \nu \leq 0$;
(4) $q > p$ and $\mu - \nu + q - p \leq 1$.

Proof. Case (1). Noting that

$$|f(n,y)| = \frac{n^{\nu} |y|^{p}}{1 + n^{\mu} |y|^{q}} \geq n^{\nu-\mu} \frac{|y|^{p}}{1 + |y|^{q}},$$

and $g(y) = \frac{y^{p-\alpha}}{1+y^{q}}$ is an increasing function, then we have

$$\begin{aligned} \int^{\infty} t^{\lambda} \inf_{\delta \leq |y| < \infty} \frac{|f(t,y)|}{|y|^{\alpha}} \Delta t &\geq \inf_{\delta \leq |y| < \infty} \frac{|y|^{p-\alpha}}{1 + |y|^{q}} \sum_{n=1}^{\infty} n^{\lambda+\nu-\mu} \\ &= \frac{\delta^{p-\alpha}}{1 + \delta^{q}} \sum_{n=1}^{\infty} n^{\lambda+\nu-\mu} = \infty. \end{aligned}$$

Hence, by Theorem 3.2, Eq. (4.14) is oscillatory.

Similarly, Cases (2), (3) and (4) can be prove by Theorems 3.1, 3.3 and 3.4, respectively. $\square$

EXAMPLE 4.2. Let $\mathbf{T} = q^{\mathbf{N}} := \{q^k : k \in \mathbf{N}, q > 1\}$, and consider the $q$-difference equation

$$\Delta_q(|\Delta_q y(t)|^{\alpha-1}\Delta_q y(t)) + f(t, y(qt)) = 0, \qquad t \in \mathbf{T}, \tag{4.15}$$

where

$$\Delta_q y(t) = \frac{y(qt) - y(t)}{(q-1)t}, \qquad f(t, y) = \frac{t^\beta}{|y|^{\gamma-1}y}.$$

If $\beta \geq -1$, $\gamma \geq 0$, and $\gamma - \beta \leq 2$, then Eq. (4.15) is oscillatory.

PROOF. Clearly, $f(t, y)$ is a nonincreasing function with respect to $y$ for each fixed $t \in [t_0, \infty)_{\mathbf{T}}$. Note that for all $\delta$, $\delta'$ with $\delta' \geq \delta \geq 0$ and for all $c \neq 0$,

$$\int^\infty \inf_{\delta \leq |y| \leq \delta'} |f(t, y)| \Delta t = \frac{1}{(\delta')^\gamma} \int^\infty t^\beta \Delta t = \infty,$$

and

$$\int^\infty |f(t, cR^\sigma(t))| \Delta t = \frac{q-1}{q^\gamma} \sum_{k=1}^\infty q^{(\beta+1-\gamma)k} = \infty.$$

Hence, by Theorem 3.4, Eq. (4.15) is oscillatory. $\square$

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