Hiroshima Math. J.
**42** (2012), 293–299

# The best constant of $L^p$ Sobolev inequality corresponding to Neumann boundary value problem for $(-1)^M(d/dx)^{2M}$

Yorimasa Oshime, Hiroyuki Yamagishi and Kohtaro Watanabe

(Received February 21, 2011)
(Revised June 14, 2011)

**Abstract.** The best constant of $L^p$ Sobolev inequality for a function with Neumann boundary condition is obtained. The best constant is expressed by $L^q$ norm of $M$-th order Bernoulli polynomial. For $L^p$ Sobolev inequality, the equality holds for a function which is written by Green function with Neumann boundary value problem for $(-1)^M(d/dx)^{2M}$.

## 1. Introduction

Throughout this paper, we assume that $p, q > 1$ and $1/p + 1/q = 1$. Let us introduce $L^p$ norm

$$\|u\|_p = \left(\int_0^1 |u(x)|^p dx\right)^{1/p}$$

and the sequence of Sobolev spaces

$$W(\mathrm{X}, M, p) = \{u \,|\, u^{(M)} \in L^p(0,1),\ u \text{ satisfies } A(\mathrm{X})\} \qquad (M = 1, 2, 3, \dots),$$

where the condition $A(\mathrm{X})$ assumes

$$A(\mathrm{P}) : u^{(i)}(1) - u^{(i)}(0) = 0 \qquad (0 \le i \le M-1), \qquad \int_0^1 u(x)dx = 0,$$

$$A(\mathrm{AP}) : u^{(i)}(1) + u^{(i)}(0) = 0 \qquad (0 \le i \le M-1),$$

$$A(\mathrm{C}) : u^{(i)}(0) = u^{(i)}(1) = 0 \qquad (0 \le i \le M-1),$$

$$A(\mathrm{D}) : u^{(2i)}(0) = u^{(2i)}(1) = 0 \qquad (0 \le i \le [(M-1)/2]),$$

$$A(\mathrm{N}) : u^{(2i+1)}(0) = u^{(2i+1)}(1) = 0 \qquad (0 \le i \le [(M-2)/2]), \qquad \int_0^1 u(x)dx = 0,$$

---

2010 *Mathematics Subject Classification*. Primary 34B27; Secondary 46E35.
*Key words and phrases*. $L^p$ Sobolev inequality, Best constant, Green function, Reproducing kernel, Bernoulli polynomial, Hölder inequality.

$$A(\mathrm{DN}) : u^{(2i)}(0) = 0 \qquad (0 \le i \le [(M-1)/2]),$$
$$u^{(2i+1)}(1) = 0 \qquad (0 \le i \le [(M-2)/2]).$$

It should be noted that if $M = 1$ the boundary conditions for $u$ in $A(\mathrm{N})$ and for $u$ on $x = 1$ in $A(\mathrm{DN})$ are not required. Now, let us consider $L^p$ Sobolev inequality:

$$\sup_{0 \le y \le 1} |u(y)| \le C\|u^{(M)}\|_p \qquad (u \in W(\mathrm{X}, M, p)). \tag{1.1}$$

In our previous work, we have obtained the best constant of $L^p$ Sobolev inequality (1.1) in some boundary conditions as follows:

| boundary condition of Sobolev space | $p = 2$ | $1 < p < \infty$ (general case) |
|---|---|---|
| P (Periodic) | [6] | [1] |
| AP (Anti Periodic) | [6] | — |
| C (Clamped) | [5] | $M = 1, 2, 3$ [4] |
| D (Dirichlet) | [6] | $M = 2m$ [2], $M = 1, 3, 5$ [3] |
| N (Neumann) | [6] | this paper |
| DN (Dirichlet-Neumann) | [6] | [7] |

We note, from this table, the difficulty to obtain the best constant seems to increase for the case $p \ne 2$. Here, we would like to stress that each result for the case $p \ne 2$ was obtained through somewhat different method, since in these cases, unified approach (maximizing the diagonal value of reproducing kernels; see [5, 6]) as in the case $p = 2$ does not exist.

In this paper, we treat the case $W(\mathrm{N}, M, p)$. To state the conclusion, we introduce Bernoulli polynomials $b_j(x)$ defined by

$$\begin{cases} b_0(x) = 1 \\ b_j'(x) = b_{j-1}(x), \qquad \int_0^1 b_j(x)dx = 0 \qquad (j = 1, 2, 3, \dots) \end{cases}$$

and the auxiliary function $b_j(\alpha; x) = b_j(x) - b_j(\alpha)$. Finally, we prepare Green function and its derivative,

$$G(\mathrm{N}, m; x, y) = (-1)^{m+1} 2^{2m-1} \left[ b_{2m}\left(\frac{|x-y|}{2}\right) + b_{2m}\left(\frac{x+y}{2}\right) \right], \tag{1.2}$$

$$\partial_y G(\mathrm{N}, m; x, y)$$
$$= (-1)^{m+1} 2^{2m-2} \left[ -\mathrm{sgn}(x-y) b_{2m-1}\left(\frac{|x-y|}{2}\right) + b_{2m-1}\left(\frac{x+y}{2}\right) \right]. \tag{1.3}$$

Our conclusion is as follows.

Theorem 1.1. *There exists a positive constant C which is independent of u such that $L^p$ Sobolev inequality* (1.1) *holds.*

(1) $M = 2m-1$ $(m = 1, 2, 3, \dots)$*: The best constant is*

$$C(2m-1) = 2^{2m-1}\|b_{2m-1}\|_q = 2^{2m-1}\left(\int_0^1 |b_{2m-1}(x)|^q dx\right)^{1/q}.$$

*If we replace C by* $C(2m-1)$ *in* (1.1)*, then the equality holds for* $u(x) = cU(2m-1; x)$ *where c is an arbitrary constant.* $U(2m-1; x)$ *is given by*

$$U(2m-1; x) = \int_0^1 \partial_y G(\mathrm{N}, m; x, y) F(2m-1; y) dy \qquad (0 < x < 1)$$

*where* $\partial_y G(\mathrm{N}, m; x, y)$ *is given by* (1.3) *and*

$$F(2m-1; y) = \operatorname{sgn}\left(b_{2m-1}\left(\frac{y}{2}\right)\right)\left|b_{2m-1}\left(\frac{y}{2}\right)\right|^{q-1}. \tag{1.4}$$

(2) $M = 2m$ $(m = 1, 2, 3, \dots)$*: The best constant is*

$$C(2m) = 2^{2m}\|b_{2m}(\alpha_0; \cdot)\|_q = 2^{2m}\left(\int_0^1 |b_{2m}(\alpha_0; x)|^q dx\right)^{1/q}.$$

*If we replace C by* $C(2m)$ *in* (1.1)*, then the equality holds for* $u(x) = cU(2m; x)$ *where c is an arbitrary constant.* $U(2m; x)$ *is given by*

$$U(2m; x) = \int_0^1 G(\mathrm{N}, m; x, y) F(2m; y) dy \qquad (0 < x < 1)$$

*where* $G(\mathrm{N}, m; x, y)$ *is given by* (1.2) *and*

$$F(2m; y) = \operatorname{sgn}\left(b_{2m}\left(\alpha_0; \frac{y}{2}\right)\right)\left|b_{2m}\left(\alpha_0; \frac{y}{2}\right)\right|^{q-1}. \tag{1.5}$$

$\alpha_0$ *is the unique solution to the equation,*

$$\int_0^{\alpha} ((-1)^{m+1} b_{2m}(\alpha; x))^{q-1} dx - \int_{\alpha}^{1/2} ((-1)^m b_{2m}(\alpha; x))^{q-1} dx = 0$$

*in the interval* $0 < \alpha < 1/2$.

## 2. Proof of Theorem 1.1

To prove Theorem 1.1, we introduce Sobolev space with periodic boundary condition

$$W(\mathrm{P2}, M, p) = \left\{u \mid u^{(M)} \in L^p(0,2), u^{(i)}(2) - u^{(i)}(0) = 0 \ (0 \le i \le M-1), \int_0^2 u(x) dx = 0\right\}.$$

For $u \in W(\mathrm{P2}, M, p)$, the following conclusion was obtained in [1, Theorem 1.1].

Theorem 2.1. *There exists a positive constant $C$ which is independent of $u$ such that $L^p$ Sobolev inequality*

$$\sup_{0 \le y \le 2} |u(y)| \le C \left( \int_0^2 |u^{(M)}(x)|^p dx \right)^{1/p} \tag{2.1}$$

*holds.*

(1) $M = 2m - 1$ $(m = 1, 2, 3, \dots)$: *The best constant is*

$$C(\mathrm{P2}; 2m-1) = 2^{2m-1-1/p} \|b_{2m-1}\|_q.$$

*If we replace $C$ by $C(\mathrm{P2}; 2m-1)$ in (2.1), then the equality holds for $u(x) = cU(\mathrm{P2}, 2m-1; x)$ where $c$ is an arbitrary constant. $U(\mathrm{P2}, 2m-1; x)$ $(0 < x < 2)$ is given by*

$$U(\mathrm{P2}, 2m-1; x) = \int_0^2 (-1)^{m+1} \operatorname{sgn}(x-y) 2^{2m-2} b_{2m-1}\left( \frac{|x-y|}{2} \right) F(2m-1; y) dy,$$

*where $F(2m-1; y)$ is (1.4).*

(2) $M = 2m$ $(m = 1, 2, 3, \dots)$: *The best constant is*

$$C(\mathrm{P2}; 2m) = 2^{2m-1/p} \|b_{2m}(\alpha_0; \cdot)\|_q.$$

*If we replace $C$ by $C(\mathrm{P2}; 2m)$ in (2.1), then the equality holds for $u(x) = cU(\mathrm{P2}, 2m; x)$ where $c$ is an arbitrary constant. $U(\mathrm{P2}, 2m; x)$ $(0 < x < 2)$ is given by*

$$U(\mathrm{P2}, 2m; x) = \int_0^2 (-1)^{m+1} 2^{2m-1} b_{2m}\left( \frac{|x-y|}{2} \right) F(2m; y) dy$$

*where $F(2m; y)$ is (1.5).*

To prove Theorem 1.1, we prepare the following lemmas.

Lemma 2.1. *For $j = 0, 1, 2, \dots,$ Bernoulli polynomials have the following properties.*

$$(1) \quad b_j(1-x) = (-1)^j b_j(x).$$

$$(2) \quad b_{2j}(\alpha; 1-x) = b_{2j}(\alpha; x).$$

Proof. From the generating function of Bernoulli polynomials,

$$\frac{e^{xt}}{t^{-1}(e^t - 1)} = \sum_{j=0}^{\infty} b_j(x) t^j \qquad (|t| < 2\pi),$$

we have

$$\sum_{j=0}^{\infty} b_j(1-x)t^j = \frac{e^{(1-x)t}}{t^{-1}(e^t-1)} = \frac{e^{x(-t)}}{(-t)^{-1}(e^{-t}-1)} = \sum_{j=0}^{\infty}(-1)^j b_j(x)t^j,$$

that is (1). Since $b_{2j}(\alpha;1-x) = b_{2j}(1-x) - b_{2j}(\alpha) = b_{2j}(x) - b_{2j}(\alpha) = b_{2j}(\alpha;x)$, we have (2).

Lemma 2.2.

$$(1) \quad F(M;2-y) = (-1)^M F(M;y) \qquad (0<y<1)$$

$$(2) \quad U(\mathrm{P2},M;2-x) = U(\mathrm{P2},M;x) \qquad (0<x<1)$$

$$(3) \quad U^{(2i+1)}(\mathrm{P2},M;0) = 0 \qquad (0 \le i \le [(M-2)/2])$$

Proof. Using Lemma 2.1 (1), we have

$$\begin{aligned} F(2m-1;2-y) &= \mathrm{sgn}\left(b_{2m-1}\left(1-\frac{y}{2}\right)\right)\left|b_{2m-1}\left(1-\frac{y}{2}\right)\right|^{q-1} \\ &= -\mathrm{sgn}\left(b_{2m-1}\left(\frac{y}{2}\right)\right)\left|b_{2m-1}\left(\frac{y}{2}\right)\right|^{q-1} = -F(2m-1;y). \end{aligned}$$

Using Lemma 2.1 (2), we have

$$\begin{aligned} F(2m;2-y) &= \mathrm{sgn}\left(b_{2m}\left(\alpha_0;1-\frac{y}{2}\right)\right)\left|b_{2m}\left(\alpha_0;1-\frac{y}{2}\right)\right|^{q-1} \\ &= \mathrm{sgn}\left(b_{2m}\left(\alpha_0;\frac{y}{2}\right)\right)\left|b_{2m}\left(\alpha_0;\frac{y}{2}\right)\right|^{q-1} = F(2m;y). \end{aligned}$$

Thus, (1) is obtained. (2) follows from (1). Differentiating $U(\mathrm{P2},M;x)$ $2i+1$ times, inserting $x=0$ and using Lemma 2.1, 2.2 (1), we have (3). In fact, for the case of $M=2m$,

$$\begin{aligned} U^{(2i+1)}(\mathrm{P2},2m;0) &= -\int_0^2 (-1)^{m+1} 2^{2(m-1-i)} b_{2(m-1-i)+1}\left(\frac{y}{2}\right) F(2m;y)dy \\ &= \int_0^2 (-1)^{m+1} 2^{2(m-1-i)} b_{2(m-1-i)+1}\left(1-\frac{y}{2}\right) F(2m;2-y)dy \\ &= \int_0^2 (-1)^{m+1} 2^{2(m-1-i)} b_{2(m-1-i)+1}\left(\frac{\eta}{2}\right) F(2m;\eta)d\eta \\ &= -U^{(2i+1)}(\mathrm{P2},2m;0), \end{aligned}$$

so we have $U^{(2i+1)}(\mathrm{P2},2m;0) = 0$. The case $M=2m-1$ is shown similarly.

PROOF OF THEOREM 1.1. For any $u \in W(\mathrm{N}, M, p)$, let us define $\tilde{u}(x)$ $(0 \le x \le 2)$ as

$$\tilde{u}(x) = \begin{cases} u(x) & (0 \le x \le 1), \\ u(2-x) & (1 \le x \le 2). \end{cases}$$

Since $u \in W(\mathrm{N}, M, p)$ satisfies Neumann boundary conditions at $x = 0, 1$, it is easy to see $\tilde{u}$ is an element of $W(\mathrm{P2}, M, p)$. So, applying $\tilde{u}$ to (2.1), we have

$$\begin{aligned} \sup_{0 \le y \le 1} |u(y)| &= \sup_{0 \le y \le 2} |\tilde{u}(y)| \\ &\le C(\mathrm{P2}, M) \|\tilde{u}^{(M)}\|_{L^p(0,2)} = 2^{1/p} C(\mathrm{P2}, M) \|u^{(M)}\|_{L^p(0,1)}. \end{aligned} \tag{2.2}$$

This implies the best constant of (1.1) is equal or less than $2^{1/p} C(\mathrm{P2}, M)$. Next, we construct the function which attains the equality of (2.2). Let $\tilde{u}_0(x) = U(\mathrm{P2}, M; x)$ $(0 < x < 2)$. Substituting $\tilde{u} = \tilde{u}_0$ into (2.2), from Theorem 2.1, we have the equality in (2.2). Moreover, from Lemma 2.2 (2), $\tilde{u}_0$ is an even function with respect to $x = 1$. So, $\tilde{u}_0$ satisfies Neumann boundary condition at $x = 1$. In addition, from Lemma 2.2 (3), $\tilde{u}_0$ satisfies Neumann boundary condition at $x = 0$. Let $\tilde{\tilde{u}}_0$ be the restriction of $\tilde{u}_0$ on the interval $[0, 1]$. From the argument above, we see that $\tilde{\tilde{u}}_0 \in W(\mathrm{N}, M, p)$ and satisfies the equality of the following inequality:

$$\sup_{0 \le y \le 1} |u(y)| \le 2^{1/p} C(\mathrm{P2}, M) \|u^{(M)}\|_{L^p(0,1)} \qquad (\forall u \in W(\mathrm{N}, M, p)).$$

Hence, we have proven Theorem 1.1. □

## References

[1] Y. Kametaka, Y. Oshime, K. Watanabe, H. Yamagishi, A. Nagai and K. Takemura, The best constant of $L^p$ Sobolev inequality corresponding to the periodic boundary value problem for $(-1)^M (d/dx)^{2M}$, Sci. Math. Jpn. **e-2007** (2007), 269–281.

[2] Y. Oshime, Y. Kametaka and H. Yamagishi, The best constant of $L^p$ Sobolev inequality corresponding to Dirichlet boundary value problem for $(d/dx)^{4m}$, Sci. Math. Jpn. **e-2008** (2008), 461–469.

[3] Y. Oshime and K. Watanabe, The best constant of $L^p$ Sobolev inequality corresponding to Dirichlet boundary value problem II, Tokyo J. Math. **34** (2011), 115–133.

[4] K. Watanabe, Y. Kametaka, A. Nagai, H. Yamagishi and K. Takemura, Symmetrization of functions and the best constant of 1-dim *Lp* Sobolev inequality, J. Inequal. Appl. Vol. 2009, Article ID 874631, (12pp).

[5] K. Watanabe, Y. Kametaka, H. Yamagishi, A. Nagai and K. Takemura, The best constant of Sobolev inequality corresponding to clamped boundary value problem, Bound. Value Probl. Vol. 2011, Article ID 875057, 17 pages.

[6] H. Yamagishi, Y. Kametaka, A. Nagai, K. Watanabe and K. Takemura, Riemann zeta function and the best constants of five series of Sobolev inequalities, RIMS Kokyuroku Bessatsu **B13** (2009), 125–139.

[7] H. Yamagishi, K. Watanabe and Y. Kametaka, The best constant of $L^p$ Sobolev inequality corresponding to Dirichlet-Neumann boundary value problem, to appear in Math. J. Okayama Univ.

*Yorimasa Oshime*
*Faculty of Science and Engineering, Doshisha University*
*Kyotanabe* 610-0321*, Japan*
*E-mail: yoshime@mail.doshisha.ac.jp*

*Hiroyuki Yamagishi*
*Tokyo Metropolitan College of Industrial Technology*
1-10-40 *Higashi-ooi, Shinagawa Tokyo* 140-0011*, Japan*
*E-mail: yamagisi@s.metro-cit.ac.jp*

*Kohtaro Watanabe*
*Department of Computer Science, National Defense Academy*
1-10-20 *Yokosuka* 239-8686*, Japan*
*E-mail: wata@nda.ac.jp*