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Invariant Teichmüller disks under hyperbolic mapping classes

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Abstract. Let $\tilde{S}$ be an analytically finite Riemann surface of type (p,n) with $3p+n>3$. Let $x \in \tilde{S}$ and $S = \tilde{S} \backslash \{x\}$. Let Mod_S^x denote the x-pointed mapping class group of S and $\mathrm{Mod}_{\tilde{S}}$ the mapping class group of $\tilde{S}$. Then the natural projection $J : T(S) \to T(\tilde{S})$ between Teichmüller spaces induces a group epimorphism $I : \mathrm{Mod}_S^x \to \mathrm{Mod}_{\tilde{S}}$. It is well known that for a given Teichmüller disk $\tilde{\Delta}$ in $T(\tilde{S})$, there is a family $\mathscr{F}(\tilde{\Delta})$ of Teichmüller disks $\Delta(z)$ in $T(S)$ parametrized by a hyperbolic plane. If $\tilde{\Delta}$ is invariant under a hyperbolic mapping class $\tilde{\theta}$, then all known hyperbolic mapping classes $\theta \in \mathrm{Mod}_S^x$ for which $I(\theta) = \tilde{\theta}$ stem from the construction of $\mathscr{F}(\tilde{\Delta})$. We show that if $\tilde{\theta}$ is represented by a product of Dehn twists along two filling simple closed geodesics, then there exist infinitely many hyperbolic mapping classes $\gamma \in \mathrm{Mod}_S^x$ with $I(\gamma) = \tilde{\theta}$ so that their invariant Teichmüller disks are not members of $\mathscr{F}(\tilde{\Delta})$. The result contrasts with the original pattern established by I. Kra.

1. Introduction

Let $\tilde{S}$ be an analytically finite Riemann surface of type (p,n) with $3p-3+n>0$, where p is the genus and n is the number of punctures of $\tilde{S}$. Let $x \in \tilde{S}$ and $S = \tilde{S} \backslash \{x\}$. Let $\tilde{\Delta}$ be a Teichmüller disk in the Teichmüller space $T(\tilde{S})$. In [10] Kra obtained a family $\mathscr{F}(\tilde{\Delta})$ of Teichmüller disks $\Delta(z)$ in the Teichmüller space $T(S)$ that is parametrized by the hyperbolic plane $\mathbf{H} = \{z \in \mathbf{C} : \mathrm{Im}\, z > 0\}$ such that (i) the natural projection $J : T(S) \to T(\tilde{S})$, defined by ignoring the puncture x, realizes an isometric embedding with respect to the Teichmüller metrics on $T(S)$ and $T(\tilde{S})$ when restricted to each member of $\mathscr{F}(\tilde{\Delta})$, and (ii) $J(\Delta(z)) = \tilde{\Delta}$ for each $z \in \mathbf{H}$.

Assume that $\tilde{\Delta}$ is an invariant disk under a hyperbolic mapping class $\tilde{\theta}$ of $\tilde{S}$. Let Mod_S^x be the x-pointed mapping class group of S. In [17] we characterized an open dense subset $\mathscr{U} \subset \mathbf{H}$ such that for every $z \in \mathscr{U}$ the stabilizer of $\Delta(z) \in \mathscr{F}(\tilde{\Delta})$ in Mod_S^x is trivial. Let

$$I : \mathrm{Mod}_S^x \to \mathrm{Mod}_{\tilde{S}} \tag{1.1}$$

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denote the natural group epimorphism of Mod_S^x onto the mapping class group $\mathrm{Mod}_{\tilde{S}}$. From Proposition 3 of Kra [10], for a pair $(\tilde{\Delta}, \tilde{\theta})$ which satisfies certain condition (see Section 2), there exists a discrete set $\mathscr{U}_0 \subset \mathbf{H} \backslash \mathscr{U}$ such that for each $z_i \in \mathscr{U}_0$, $\Delta(z_i) \in \mathscr{F}(\tilde{\Delta})$ is an invariant disk under a mapping class $\theta_i \in \mathrm{Mod}_S^x$, where θ_i acts on $\Delta(z_i)$ as a hyperbolic Möbius transformation and satisfies $I(\theta_i) = \tilde{\theta}$, which leads to that θ_i are all hyperbolic mapping classes. The main purpose of this article is to prove the following result.

Theorem 1.1. (1) *The only hyperbolic mapping classes $\theta \in \mathrm{Mod}_S^x$ for which $I(\theta) = \tilde{\theta}$ and $\theta(\Delta) = \Delta$ for some $\Delta \in \mathscr{F}(\tilde{\Delta})$ are those mapping classes obtained from Kra's construction.*

(2) *If in addition $\tilde{\theta}$ is represented by a finite product of Dehn twists along two filling simple closed geodesics (Thurston's construction* [14, 15]*), then there exist infinitely many hyperbolic mapping classes $\gamma \in \mathrm{Mod}_S^x$ such that $I(\gamma) = \tilde{\theta}$ while their associated Teichmüller disks $\Delta(\gamma)$ are not members of $\mathscr{F}(\tilde{\Delta})$.*

Every hyperbolic element $\gamma \in \mathrm{Mod}_S^x$ is represented by a pseudo-Anosov map $f : S \to S$ (see Thurston [14] for the definition of a pseudo-Anosov map) with the associated dilatation $\lambda(f)$ which is also denoted by $\lambda(\gamma)$. The number $\log \lambda(\gamma)$ is the translation length of γ as an isometry of $T(S)$ with respect to the Teichmüller metric on $T(S)$ (see Bers [4]). It is well-known (see Arnoux–Yoccoz [1] and Ivanov [9]) that

$$\mathrm{Spec}(\mathrm{Mod}_S) = \{\log \lambda(\gamma) : \gamma \text{ are hyperbolic mapping classes on } S\}$$

is an unbounded discrete subset of $\mathbf{R}$. The construction of hyperbolic mapping classes in Theorem 1.1 (2) also yields the following corollary.

Corollary 1.1. *Let $\tilde{\theta}$ be as in Theorem* 1.1 (2). *Then*

$$\mathscr{T} = \{\log \lambda(\gamma) : \gamma \in \mathrm{Mod}_S^x \text{ are hyperbolic mapping classes with } I(\gamma) = \tilde{\theta}\}$$

is an unbounded discrete subset of $\mathbf{R}$.

This article is organized as follows. In Section 2, we present some background materials. In Section 3, we discuss some properties of general hyperbolic mapping classes of S that project to a given hyperbolic mapping class $\tilde{\theta}$ on $\tilde{S}$, and prove Theorem 1.1 (1). In Section 4, we study Dehn twists and their relationship with geometric intersection numbers of simple closed geodesics. In Section 5, we consider some special hyperbolic mapping classes $\tilde{\theta}$ and investigate Teichmüller disks invariant under those hyperbolic mapping classes γ of S with $I(\gamma) = \tilde{\theta}$. In Sections 6, we describe certain lifts of a given Dehn twist along a simple curve on $\tilde{S}$. Section 7 is devoted to the proofs of Theorem 1.1 (2) and Corollary 1.1.

2. Notation and background

We first review some basic facts in Teichmüller theory. For more information, see Gardiner [7], Imayoshi–Taniguchi [8] and Nag [13].

Let $\tilde{S}$ be as in Section 1. In what follows, a conformal structure on $\tilde{S}$ is identified with a Beltrami differential. The Teichmüller space $T(\tilde{S})$ is defined as a space of equivalence classes $[v]$ of all conformal structures $v = v(\tilde{S})$ on $\tilde{S}$, where two conformal structures $v_1(\tilde{S})$ and $v_2(\tilde{S})$ are in the same equivalence class if there is a conformal map $h : v_1(\tilde{S}) \to v_2(\tilde{S})$ isotopic to the identity. The Teichmüller distance between two points $[v_1], [v_2] \in T(\tilde{S})$ is defined by

$$d([v_1], [v_2]) = \frac{1}{2} \inf \log K[f], \tag{2.2}$$

where the infimum is taken over all quasiconformal maps $f : v_1(\tilde{S}) \to v_2(\tilde{S})$ isotopic to $v_2 \circ v_1^{-1}$ and $K[f]$ is the maximal dilatation of f.

According to Ahlfors and Bers [2], for each conformal structure v on $\tilde{S}$, there is a quasiconformal map $w^v : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ that fixes $0, 1, \infty$, and satisfies

$$\frac{\partial_{\bar{z}} w^v(z)}{\partial_z w^v(z)} = \begin{cases} v(z) & \text{if } z \in \mathbf{H}, \\ 0 & \text{if } z \in \hat{\mathbf{C}} \backslash \overline{\mathbf{H}}. \end{cases}$$

The domain $w^v(\mathbf{H})$ depends only on $[v]$. We may therefore form the Bers fiber space via

$$F(\tilde{S}) = \{([v], z) \in T(\tilde{S}) \times \mathbf{C}; z \in w^v(\mathbf{H})\}.$$

Let $\pi : F(\tilde{S}) \to T(\tilde{S})$ denote the natural projection defined by sending a point $([v], z)$ to $[v]$. Then π is holomorphic.

The group of isotopy classes of self-homeomorphisms of $\tilde{S}$ is the mapping class group $\mathrm{Mod}_{\tilde{S}}$, which naturally acts as isometries with respect to the Teichmüller metric on $T(\tilde{S})$. Let $\mathrm{mod}(\tilde{S})$ be the group consisting of equivalence classes $[\hat{w}]$ of self-maps $\hat{w}$ of $\mathbf{H}$ descending to self-maps $\tilde{w}$ of $\tilde{S}$ under the universal covering map $\varrho : \mathbf{H} \to \tilde{S}$, where two such maps $\hat{w}_1$, $\hat{w}_2$ are in the same equivalence class if $\hat{w}_1 = \hat{w}_2$ on $\mathbf{R}$. The group $\mathrm{mod}(\tilde{S})$ naturally acts on $F(\tilde{S})$ as a group of fiber-preserving holomorphic automorphisms (see Bers [3]). More precisely, for each $[\hat{w}] \in \mathrm{mod}(\tilde{S})$ and each point $([v], z) \in F(\tilde{S})$, we have

$$[\hat{w}]([v], \hat{z}) = ([\sigma], \hat{y}), \tag{2.3}$$

where σ is the Beltrami coefficient of $w^v \circ \hat{w}^{-1}$ and $\hat{y} = w^\sigma \circ \hat{w} \circ (w^v)^{-1}(\hat{z})$. With the aid of the Bers isomorphism $\varphi : F(\tilde{S}) \to T(S)$ (Theorem 9 of [3]), the group $\mathrm{mod}(\tilde{S})$ is isomorphic to the x-pointed mapping class group Mod_S^x by a conjugation φ^*:

$$\mathrm{mod}(\tilde{S}) \ni [\hat{w}] \xrightarrow{\varphi^*} \varphi \circ [\hat{w}] \circ \varphi^{-1} \in \mathrm{Mod}_S^x. \tag{2.4}$$

Let G be the covering group of the universal covering map $\varrho : \mathbf{H} \to \tilde{S}$. Then G can be regarded as a normal subgroup of $\mathrm{mod}(\tilde{S})$. In this way, G acts on $F(\tilde{S})$ and keeps each fiber of $F(\tilde{S})$ invariant. Under the isomorphism (2.4), the group G is isomorphic to the subgroup $I^{-1}(\mathrm{id})$ of Mod_S^x, where I is as defined in (1.1). For simplicity, throughout the article we write $[\hat{w}]^* = \varphi^*([\hat{w}])$ for $[\hat{w}] \in \mathrm{mod}(\tilde{S})$. In particular, for an element $h \in G$, we simply use the symbol h^* to denote the corresponding mapping class in $I^{-1}(\mathrm{id})$ as well as a representative in the mapping class.

Following Bers [4], a mapping class $\tilde{\theta}$ is hyperbolic if $\inf d([v], \tilde{\theta}([v]))$ is achieved and is positive, where d is the Teichmüller distance on $T(\tilde{S})$ (see (2.2)) and the infimum is taken over all points $[v] \in T(\tilde{S})$. Fix a hyperbolic mapping class $\tilde{\theta} \in \mathrm{Mod}_{\tilde{S}}$. Then $\tilde{\theta}$ is represented by an absolutely extremal Teichmüller map $\tilde{\omega}$ on a surface (call it $\tilde{S}$ again). Associated to $\tilde{\omega}$ there is a pair of transverse trajectories on $\tilde{S}$, and thus $\tilde{\omega}$ in turn determines a holomorphic quadratic differential $\phi_{\tilde{\omega}}$ on $\tilde{S}$ which may have simple poles at punctures of $\tilde{S}$ and satisfies

$$\iint_{\tilde{S}} |\phi_{\tilde{\omega}}(z)| dxdy = 1. \tag{2.5}$$

Now $\mu = \bar{\phi}_{\tilde{\omega}}/|\phi_{\tilde{\omega}}|$ is a $(-1,1)$-form on $\tilde{S}$. Let $\mathbf{D}$ be the unit disk $\{t : |t| < 1\}$ equipped with the hyperbolic metric $|dt|/(1-|t|^2)$. Also let

$$\tilde{\Delta} = \{[t\mu] : t \in \mathbf{D}\} \qquad \text{and} \qquad \tilde{L} = \{[t\mu] : t \in (-1,1)\}.$$

Then $\tilde{\Delta}$ is a Teichmüller disk and $\tilde{L}$ is a Teichmüller geodesic. It is trivial that $\tilde{L} \subset \tilde{\Delta} \subset T(\tilde{S})$, and that both $\tilde{L}$ and $\tilde{\Delta}$ are invariant under the action of $\tilde{\theta}$. For each $\hat{z} \in \mathbf{H}$, one constructs

$$\mathscr{D}_{\tilde{\omega}}(\hat{z}) = \{([t\mu], w^{t\mu}(\hat{z})) : t \in \mathbf{D}\} \subset F(\tilde{S}). \tag{2.6}$$

It is well known [10] that $\Delta_{\tilde{\omega}}(\hat{z}) := \varphi(\mathscr{D}_{\tilde{\omega}}(\hat{z}))$ is a Teichmüller disk in $T(S)$. We thus obtain a parametric family

$$\mathscr{F}(\tilde{\Delta}) = \{\Delta_{\tilde{\omega}}(\hat{z}) : \hat{z} \in \mathbf{H}\} \tag{2.7}$$

of Teichmüller disks in $T(S)$. The natural projection $J : T(S) \to T(\tilde{S})$ realizes an isometric embedding of each $\Delta_{\tilde{\omega}}(\hat{z})$ into $T(\tilde{S})$ with $J(\Delta_{\tilde{\omega}}(\hat{z})) = \tilde{\Delta}$ with respect to the Teichmüller metrics on $T(S)$ and $T(\tilde{S})$.

Assume that $\phi_{\tilde{\omega}}$ has distinct non-puncture zeros $\tilde{z}_1, \ldots, \tilde{z}_m$ on $\tilde{S}$ and that $\tilde{\omega}$ fixes these zeros. Fix a fundamental region $\Sigma \subset \mathbf{H}$ for G and let $\hat{z}_1, \ldots, \hat{z}_m \subset \Sigma$ be such that $\varrho(\hat{z}_i) = \tilde{z}_i$ for $i = 1, 2, \ldots, m$.

Let $\hat{\omega}_i : \mathbf{H} \to \mathbf{H}$ be a lift of $\tilde{\omega}$ such that $\hat{\omega}_i(\hat{z}_i) = \hat{z}_i$. From (2.3), we see that the element $[\hat{\omega}_i]$ of $\mathrm{mod}(\tilde{S})$, represented by $\hat{\omega}_i$, acts on $F(\tilde{S})$ via the formula

$$[\hat{\omega}_i]([t\mu], w^{t\mu}(\hat{z}_i)) = ([v(t)], \hat{y}_i), \tag{2.8}$$

where $v(t)$ is the Beltrami coefficient of $w^{t\mu} \circ \hat{\omega}_i^{-1}$ and

$$\hat{y}_i = w^{v(t)} \circ \hat{\omega}_i \circ (w^{t\mu})^{-1}(w^{t\mu}(\hat{z}_i)) = w^{v(t)}(\hat{z}_i). \tag{2.9}$$

Note that $\mu = \bar{\phi}_{\tilde{\omega}}/|\phi_{\tilde{\omega}}|$ and that the Beltrami coefficient of $\hat{\omega}_i$ is $k\mu$ for some $k \in (-1, 1)$. Easy calculations show that the Beltrami coefficient of $\hat{\omega}_i^{-1}$ is $k_1\mu$ for $k_1 = -k$. The chain rule (see, for example, Gardiner [7]) then shows that the Beltrami coefficient of $w^{t\mu} \circ \hat{\omega}_i^{-1}$ is

$$v(t) = \frac{t\mu - k\mu}{1 - kt} = \left(\frac{t - k}{1 - kt}\right)\mu \qquad \text{for } t \in \mathbf{D}. \tag{2.10}$$

Write $M(t) = \frac{t-k}{1-kt}$. Then clearly, $M : \mathbf{D} \to \mathbf{D}$ is a Möbius transformation. Recall that k is real. We see that $M(t) = \frac{t-k}{1-kt}$ is real if and only if t is real, which says that M keeps the interval $(-1, 1)$ invariant. Hence by (2.9) and (2.10), we can write (2.8) as

$$[\hat{\omega}_i]([t\mu], w^{t\mu}(\hat{z}_i)) = ([M(t)\mu], w^{M(t)\mu}(\hat{z}_i)). \tag{2.11}$$

It follows from (2.6) and (2.11) that

$$[\hat{\omega}_i](\mathscr{D}_{\tilde{\omega}}(\hat{z}_i)) = \mathscr{D}_{\tilde{\omega}}(\hat{z}_i).$$

Denote by $\theta_i = [\hat{\omega}_i]^*$. Then $\theta_i \in \mathrm{Mod}_S^x$. From (2.11), we know that $[\hat{\omega}_i]$ leaves invariant the line

$$\mathscr{L}_i = \{([t\mu], w^{t\mu}(\hat{z}_i)); t \in (-1, 1)\} \subset \mathscr{D}_{\tilde{\omega}}(\hat{z}_i).$$

Hence the mapping class θ_i leaves invariant the Teichmüller geodesic $\varphi(\mathscr{L}_i) = L_i$ as well as the Teichmüller disk $\Delta_{\tilde{\omega}}(\hat{z}_i)$. From Corollary 1 to Theorem 5 of Bers [4], θ_i is hyperbolic. Since $I(\theta_i)$ is the mapping class induced by $\tilde{\omega}$ that is the projection of $\hat{\omega}_i$, we have $I(\theta_i) = \tilde{\theta}$.

Notice that $J|_{\Delta_{\tilde{\omega}}(\hat{z}_i)} : \Delta_{\tilde{\omega}}(\hat{z}_i) \to \tilde{\Delta} \subset T(\tilde{S})$ is an isometric embedding with respect to the Teichmüller metrics on $T(S)$ and $T(\tilde{S})$. This implies that the projection $J : T(S) \to T(\tilde{S})$ sends L_i onto a Teichmüller geodesic $J(L_i)$ that is invariant under $\tilde{\theta}$. This shows that $\tilde{L} = J(L_i)$.

Let $s : \tilde{\Delta} \to F(\tilde{S})$ denote the holomorphic map that sends $[t\mu] \in \tilde{\Delta}$ to $([t\mu], w^{t\mu}(\hat{z}_i))$. Then $s(\tilde{\Delta}) = \mathscr{D}_{\tilde{\omega}}(\hat{z}_i)$ and $\pi \circ s = \mathrm{id}$. Let $\imath : \mathbf{D} \to \tilde{\Delta}$ denote the isometry that sends $t \in \mathbf{D}$ to $[t\mu] \in \tilde{\Delta}$. By (2.11) and the above discussion, we obtain

$$\imath \circ M \circ \imath^{-1}|_{(\tilde{\Delta},\tilde{L})} = \tilde{\theta}|_{(\tilde{\Delta},\tilde{L})}$$

and

$$(s \circ \imath) \circ M \circ (s \circ \imath)^{-1}|_{(\mathscr{D}_{\tilde{\omega}(\hat{z}_i)},\mathscr{L}_i)} = [\hat{\omega}_i]|_{(\mathscr{D}_{\tilde{\omega}(\hat{z}_i)},\mathscr{L}_i)}.$$

The result is summarized in the following lemma.

Lemma 2.1 (Kra [10]). *Let $\tilde{\theta} \in \mathrm{Mod}_{\tilde{S}}$ be hyperbolic and be represented by an absolutely extremal Teichmüller map $\tilde{\omega} : \tilde{S} \to \tilde{S}$. Let $\phi_{\tilde{\omega}}$ be the corresponding quadratic differential. Assume that $\phi_{\tilde{\omega}}$ has non-puncture zeros $\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_m$, and that $\tilde{\omega}(\tilde{z}_i) = \tilde{z}_i$. Let $\Sigma \subset \mathbf{H}$ be a fundamental region for the covering map ϱ, and let $\hat{z}_i \in \Sigma$ be such that $\varrho(\hat{z}_i) = \tilde{z}_i$. Then for each i with $1 \le i \le m$, there is a hyperbolic mapping class $\theta_i \in \mathrm{Mod}_S^x$ such that* (i) *$I(\theta_i) = \tilde{\theta}$;* (ii) *$\theta_i$ keeps the Teichmüller disk $\Delta_{\tilde{\omega}}(\hat{z}_i)$ invariant; and* (iii) *if we denote by L_i the invariant Teichmüller geodesic under θ_i, then $L_i \subset \Delta_{\tilde{\omega}}(\hat{z}_i)$, $\theta_i(L_i) = L_i$ and $J(L_i) = \tilde{L}$.*

3. Elements of $\mathscr{F}(\tilde{\Delta})$ invariant under hyperbolic mapping classes

In this section, we discuss more properties of the members of $\mathscr{F}(\tilde{\Delta})$ invariant under the hyperbolic mapping class θ_i. Let

$$A_i = \{g^*(\Delta_{\tilde{\omega}}(\hat{z}_i)) \text{ for } g \in G\}, \qquad 1 \le i \le m. \tag{3.12}$$

Lemma 3.1. *A_i and A_j are disjoint for $i \neq j$.*

Proof. Suppose $g^*(\Delta_{\tilde{\omega}}(\hat{z}_i)) = h^*(\Delta_{\tilde{\omega}}(\hat{z}_j))$ for some $g, h \in G$. Then clearly, $g(\mathscr{D}_{\tilde{\omega}}(\hat{z}_i)) = h(\mathscr{D}_{\tilde{\omega}}(\hat{z}_j))$. Note that the action of $g \in G$ on $F(\tilde{S})$ is

$$g(\mathscr{D}_{\tilde{\omega}}(\hat{z}_i)) = \{[t\mu], g^{t\mu} w^{t\mu}(\hat{z}_i)\}, \tag{3.13}$$

where $g^{t\mu} = w^{t\mu} g (w^{t\mu})^{-1}$. So (3.13) becomes

$$g(\mathscr{D}_{\tilde{\omega}}(\hat{z}_i)) = \{[t\mu], w^{t\mu} g(\hat{z}_i)\}. \tag{3.14}$$

This tells us that $g(\mathscr{D}_{\tilde{\omega}}(\hat{z}_i)) = \mathscr{D}_{\tilde{\omega}}(g(\hat{z}_i))$. The same is true for h. Therefore, $\mathscr{D}_{\tilde{\omega}}(g(\hat{z}_i)) = \mathscr{D}_{\tilde{\omega}}(h(\hat{z}_j))$. It follows from Lemma 3.5 of [16] that $g(\hat{z}_i) = h(\hat{z}_j)$, which says that $\hat{z}_i$ and $\hat{z}_j$ project to the same zero $\tilde{z}_i = \tilde{z}_j$ of $\phi_{\tilde{\omega}}$. This is a contradiction. So A_i and A_j are disjoint for $i \neq j$. □

Let

$$A = \bigcup_i A_i. \tag{3.15}$$

Then by Lemma 3.1, A is a disjoint union of $A_1, A_2, \dots, A_m$. Obviously, $A \subset \mathscr{F}(\tilde{\Delta})$.

Lemma 3.2. *Let $\gamma \in \mathrm{Mod}_S^x$ be hyperbolic that keeps an element $\Delta \in A$ invariant. Suppose $I(\gamma) = \tilde{\theta}$. Then there are $g \in G$ and $i \in \{1, 2, \dots, m\}$ such that $\gamma = g^* \circ \theta_i \circ (g^*)^{-1}$, where θ_i is as defined in Lemma* 2.1.

Proof. We may assume that $\Delta \in A_1$. Write $\mathscr{D} = \varphi^{-1}(\Delta)$. By definition, for some $g \in G$, $\mathscr{D} = g(\mathscr{D}_{\tilde{\omega}}(\hat{z}_1))$. Recall that $\mathscr{D}_{\tilde{\omega}}(\hat{z}_1) = \{[t\mu], w^{t\mu}(\hat{z}_1)\}$ for $\mu = \bar{\phi}_{\tilde{\omega}}/|\phi_{\tilde{\omega}}|$ and $t \in \mathbf{D}\backslash\{0\}$. By the same argument as in Lemma 3.1, we have $g(\mathscr{D}_{\tilde{\omega}}(\hat{z}_1)) = \mathscr{D}_{\tilde{\omega}}(g(\hat{z}_1))$ and thus $\varphi(\mathscr{D}_{\tilde{\omega}}(g(\hat{z}_1))) \in A_1$. By Lemma 2.1 and (3.14), the hyperbolic element $g^* \circ \theta_1 \circ (g^*)^{-1}$ keeps Δ and a Teichmüller geodesic L invariant, where $L \subset \Delta$. Let $\mathscr{L} = \varphi^{-1}(L)$. By (3.13), $\pi(\mathscr{L}) = J(L) = \tilde{L}$, where we recall that $\pi : F(\tilde{S}) \to T(\tilde{S})$ is the (holomorphic) natural projection which sends $([v], z)$ to $[v]$, and $\tilde{L} \subset T(\tilde{S})$ is the Teichmüller geodesic invariant under the hyperbolic $\tilde{\theta}$.

By hypothesis, γ also keeps Δ invariant. Let $L' \subset \Delta$ be the invariant Teichmüller geodesic under γ. Suppose $L' \neq L$. Note that $J|_{\Delta} : \Delta \to T(\tilde{S})$ is an isometric embedding, $J(L')$ is also a Teichmüller geodesic in $T(\tilde{S})$. Thus $J(L') \neq \tilde{L}$. By Lemma 3.2 of [17], $J(L')$ is an invariant Teichmüller geodesic under the action of $I(\gamma) = \tilde{\theta}$. We see that $\tilde{\theta}$ keeps both $\tilde{L}$ and $J(L')$ invariant. By the uniqueness part of Theorem 4 of Bers [4], $\tilde{L} = J(L')$. So we must have $L' = L$. In other words, γ and $g^* \circ \theta_1 \circ (g^*)^{-1}$ share a common invariant Teichmüller geodesic. Whence, it follows that there is an integer α such that

$$\gamma^{\alpha} = g^* \circ \theta_1 \circ (g^*)^{-1}. \tag{3.16}$$

To see that $\alpha = 1$, we note that $I(\gamma^{\alpha}) = I(\gamma)^{\alpha} = \tilde{\theta}^{\alpha}$. From (3.16) and Lemma 2.1 we obtain $I(\gamma^{\alpha}) = I(g^* \circ \theta_1 \circ (g^*)^{-1}) = I(\theta_1) = \tilde{\theta}$. So $\tilde{\theta} = \tilde{\theta}^{\alpha}$ and thus that $\alpha = 1$. Therefore $\gamma = g^* \circ \theta_1 \circ (g^*)^{-1}$. This completes the proof of Lemma 3.2. □

Remark 1. *In* [3], *Bers proved that for any hyperbolic mapping class $\tilde{\theta}$, there is an invariant Teichmüller geodesic under the mapping class. For the uniqueness part of the result, we refer to Bestvina–Feighn* [5]. *The idea is to use the so-called "flaring condition" to construct ending laminations λ^+ and λ^- for the hyperbolic (pseudo-Anosov) mapping class $\tilde{\theta}$ which determine the desired Teichmüller geodesic.*

The following lemma says that there is no element in $\mathscr{F}(\tilde{\Delta})\backslash A$ that is invariant under any hyperbolic mapping class γ that projects to $\tilde{\theta}$, which together with Lemma 3.2 completes the proof of (1) of Theorem 1.1.

Lemma 3.3. *Let $\gamma \in \mathrm{Mod}_S^x$ be a hyperbolic mapping class that keeps a Teichmüller disk $\Delta \in \mathscr{F}(\tilde{\Delta})$ invariant. Assume that $I(\gamma) = \tilde{\theta}$. Then $\Delta \in A$.*

Proof. Let $[\hat{\omega}] \in \operatorname{mod}(\tilde{S})$ be such that $[\hat{\omega}]^* = \gamma$ and $[\hat{\omega}](\mathscr{D}) = \mathscr{D}$ for $\mathscr{D} = \varphi^{-1}(\varDelta)$. Since $\hat{\omega}$ is a lift of $\tilde{\omega}$, the Beltrami coefficient of $\hat{\omega}$ is also $k\mu$. Now by assumption, $\mathscr{D}$ is of the form (2.6). By Lemma 2.1, $\pi|_{\mathscr{D}} : \mathscr{D} \to T(\tilde{S})$ is am embedding, $\mathscr{D}$ crosses the central fiber $\mathbf{H} \subset F(\tilde{S})$ exactly once. Set $\hat{z} = \mathscr{D} \cap \mathbf{H}$. Then $\mathscr{D} = \mathscr{D}_{\tilde{\omega}}(\hat{z})$.

From the same computation as in the proof of Lemma 2.1, the action of $[\hat{\omega}]$ on $F(\tilde{S})$ can be written as

$$[\hat{\omega}]([t\mu], w^{t\mu}(\hat{z})) = ([M(t)\mu], w^{M(t)\mu}(\hat{z}')) \qquad \text{for } \hat{z}' = \hat{\omega}(\hat{z}), \tag{3.17}$$

where $M : \mathbf{D} \to \mathbf{D}$, as defined in the proof of Lemma 2.1, is a Möbius transformation that sends $(-1, 1)$ to $(-1, 1)$.

Now from (3.17), we know that $[\hat{\omega}](\mathscr{D}_{\tilde{\omega}}(\hat{z}))$ is of form (2.6) that passes through $([0], \hat{z}')$. If $\hat{z}' \neq \hat{z}$, then by Lemma 3.5 of [17], $[\hat{\omega}](\mathscr{D}_{\tilde{\omega}}(\hat{z}))$ would be disjoint from $\mathscr{D}_{\tilde{\omega}}(\hat{z})$, and this would imply that $[\hat{\omega}](\mathscr{D}_{\tilde{\omega}}(\hat{z})) \neq \mathscr{D}_{\tilde{\omega}}(\hat{z})$, and hence $\gamma(\varDelta) \neq \varDelta$. This is a contradiction. Thus we conclude that $\hat{z}' = \hat{z}$. That is, $\hat{\omega}(\hat{z}) = \hat{z}$. Letting $\tilde{z} = \varrho(\hat{z})$, it follows that

$$\tilde{\omega}(\tilde{z}) = \tilde{\omega}(\varrho(\hat{z})) = \varrho(\hat{\omega}(\hat{z})) = \varrho(\hat{z}) = \tilde{z}.$$

Observe that $\tilde{\omega}$ is an absolutely extremal map; it does not fix any point away from zeros of $\phi_{\tilde{\omega}}$ and punctures of $\tilde{S}$. We see that $\tilde{z}$ is one of the non-puncture zeros of $\phi_{\tilde{\omega}}$, i.e., $\tilde{z} \in \{\tilde{z}_1, \ldots, \tilde{z}_m\}$. Assume that $\tilde{z} = \tilde{z}_1$. Then there exists an element $g \in G$ such that $g(\hat{z}_1) = \hat{z}$, which tells us that

$$g(\mathscr{D}_{\tilde{\omega}}(\hat{z}_1)) = \mathscr{D}.$$

It follows that

$$\varDelta = \varphi(\mathscr{D}) = \varphi(g(\mathscr{D}_{\tilde{\omega}}(\hat{z}_1))) = g^*(\varDelta_{\tilde{\omega}}(\hat{z}_1)).$$

By (3.12), $\varDelta \in A_1$. This proves Lemma 3.3. □

4. Dehn twists and intersection numbers of simple closed geodesics

Let $\mathscr{S}$ be the set of oriented simple closed geodesics on S. For any $c \in \mathscr{S}$, we define the positive Dehn twist along c as a self-map of S obtained by cutting S along c, rotating one of the copies of c by 360 degrees in the counterclockwise direction and gluing the two copies back together. If we regard a cylinder on S as the annulus $\mathscr{A} = (\mathbf{R}/r\mathbf{Z}) \times [0, s]$ so that $(\mathbf{R}/r\mathbf{Z}) \times \{s\}$ is identified with c, then with respect to the coordinates (x, y) on $\mathscr{A}$, the Dehn twist can also be expressed as $(x, y) \mapsto (x + yr/s, y)$. In what follows, the Dehn twist and the mapping class represented by the Dehn twist are both denoted by the same symbol t_c.

For a map $f : S \to S$, we use the symbol $f(c)$ to denote the geodesics homotopic to the image curve of c under the map f. The image curve is denoted by $c(f)$. For any $a, b \in \mathscr{S}$, let $\#\{a, b\}$ denote the set of points of intersections between a and b. $\#\{a, b\}$ is empty if and only if $a = b$ or a and b are disjoint. Let $i(a, b)$ denote the cardinality of $\#\{a, b\}$. Assume that c intersects both a and b.

Lemma 4.1. *With the above conditions,* $i(a, t_c^k(b)) \to +\infty$ *as* $k \to \pm\infty$.

Remark 2. *The author is grateful to the referee for pointing out that a related result can be found in* [6]. *For completeness and reference purpose, we outline the proof below.*

Proof. We use similar notations $\#\{b(t_c^k), a\}$ and $i(b(t_c^k), a)$ to denote the set of points of intersection between $b(t_c^k)$ and a and the cardinality of $\#\{b(t_c^k), a\}$, respectively. There are two types of points in $\#\{b(t_c^k), a\}$: (I) points of intersection arising from the Dehn twist t_c^k; and (II) existing points of intersection between a and b.

We first assume that c does not contain any points of intersection between a and b. Thus a small annular neighborhood $A(c)$ of c can be chosen so that $A(c)$ does not contain any points of intersection between a and b. Let t_c be performed within $A(c)$. This means that $t_c|_{S \setminus A(c)} = \mathrm{id}$. By assumption, all type (I) points lie in $A(c)$ and all type (II) points lie outside of $A(c)$.

By the definition of the Dehn twists, as $k \to \pm\infty$, $b(t_c^k)$ can intersect a as many times as possible, and all these type (I) points lie in $A(c)$ and stay in one side of c. This tells us that the number of type (I) points goes to infinity as $k \to \pm\infty$. Observe that in the deformation process from $b(t_c^k)$ to $t_c^k(b)$, a type (I) point cannot cancel any type (I) point. However, it is possible for a type (I) point to cancel a type (II) point. But since there are only finitely many type (II) points, we see that there are at most finitely many type (I) points that could possibly cancel some type (II) points. We conclude that $i(a, t_c^k(b)) \to +\infty$ as $k \to \pm\infty$.

If c contains some points of intersection between a and b, then these points stay in $\#\{b(t_c^k), a\}$ and during the deformation from $b(t_c^k)$ to $t_c^k(b)$, these intersection points do not cancel with any type (I) points or any other type (II) points. □

As a simple example, we consider a special case where a and b are disjoint. In this case, $i(a, t_c^k(b)) \geq |k|$ for any integer k; and $i(a, t_c^k(b)) = |k|$ if and only if $i(a, c) = i(b, c) = 1$.

Let $c_1, c_0 \in \mathscr{S}$ be disjoint, and homotopic to each other as x is filled in. That is, c_1, c_0 are boundary components of an x-punctured cylinder P. Then there is a primitive simple hyperbolic element $h \in G$ such that $h^* \in \mathrm{Mod}_S^x$

is represented by $t_{c_0}^{-1} \circ t_{c_1}$. Conversely, for any primitive hyperbolic element $h \in G$, there are $c_1, c_0 \in \mathscr{S}$ so that $h^* = t_{c_0}^{-1} \circ t_{c_1}$ and $\{c_0, c_1\}$ forms the boundary of an x-punctured cylinder P on S (Theorem 2 of [10] and Theorem 2 of [12]). Since c_1 is disjoint from c_0, t_{c_0} and t_{c_1} commute. Let c be the central curve of such an x-punctured cylinder P, by which we mean that c is a simple curve that passes through x and is disjoint from c_0 and c_1.

Let $a, b \in \mathscr{S}$ be as before. Let P be an x-punctured cylinder with $\partial P = \{c_0, c_1\}$ so that its central curve c intersects both a and b. This means that both a and b go through P. Let $h \in G$ be the primitive simple hyperbolic element so that h^* is represented by $t_{c_0}^{-1} \circ t_{c_1}$.

Lemma 4.2. *With the above conditions, we have* $i(a, (h^*)^k(b)) \to +\infty$ *as* $k \to \pm\infty$.

Proof. Let $A(c_0)$, $A(c_1)$ be small annular neighborhoods of c_0 and c_1, respectively. Make P a little bit larger so that P contains $A(c_0)$ and $A(c_1)$. By the construction, we obtain

$$\#\{a, b(t_{c_0}^{-k} \circ t_{c_1}^k)\} = \#\{a, b(t_{c_0}^{-k})\} \cup \#\{a, b(t_{c_1}^k)\}. \tag{4.18}$$

It follows that

$$i(a, b(t_{c_0}^{-k} \circ t_{c_1}^k)) \geq i(a, b(t_{c_0}^{-k})) + i(a, b(t_{c_1}^k)) - i(a, b). \tag{4.19}$$

Observe that $i(a, b(t_{c_0}^{-k})) \to +\infty$ and $i(a, b(t_{c_1}^k)) \to +\infty$ as $k \to \pm\infty$ (see also Lemma 4.1). From (4.19) one can conclude that

$$i(a, b(t_{c_0}^{-k} \circ t_{c_1}^k)) \to +\infty$$

as $k \to \pm\infty$. By the same argument of Lemma 4.1, we know that in order to prove $i(a, (h^*)^k(b)) \to +\infty$, we need to show (i) during the deformation from $b(t_{c_0}^{-k} \circ t_{c_1}^k)$ to the geodesic $(h^*)^k(b)$, any type (I) point in $\#\{a, b(t_{c_0}^{-k})\}$ does not cancel a type (I) point in $\#\{a, b(t_{c_1}^k)\}$, and (ii) during the deformation any two type (I) points in $\#\{a, b(t_{c_0}^{-k})\}$ or in $\#\{a, b(t_{c_1}^k)\}$ cannot cancel each other.

Note that points in $\#\{a, b(t_{c_0}^{-k} \circ t_{c_1}^k)\}$ are canceled in pairs, and two intersection points z_1 and z_2 between a and $b(t_{c_0}^{-k} \circ t_{c_1}^k)$ are canceled each other if and only if $\{z_1, z_2\}$ are vertices of a bigon. So the proof will be completed once we show that all the components of $S \backslash \{a, b(t_{c_0}^{-k} \circ t_{c_1}^k)\}$ lying in P are not bigons.

Observe that the spin map $t_{c_0}^{-k} \circ t_{c_1}^k$ can also be obtained by the following procedure: fill in the point x, push the point x along c k times, and when x returns to its original position, remove x from the surface. Figure 1 below depicts the portion of the curve $b(t_{c_0}^{-k} \circ t_{c_1}^k)$ in P in the case of $k = 3$.

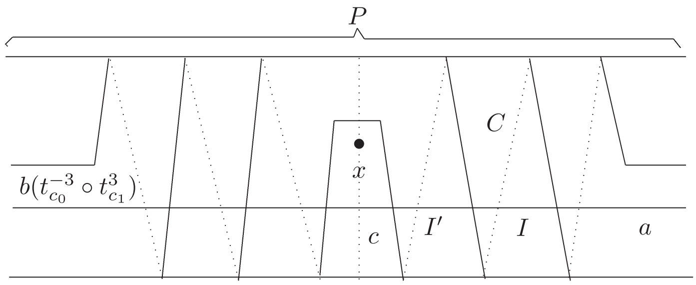

Fig. 1

Let $\mathscr{I}$ denote the collection of intervals of $\{a \backslash \#\{a, b(t_{c_0}^{-k} \circ t_{c_1}^{k})\}$ contained in P. For any component C of $S \backslash \{a, b(t_{c_0}^{-k} \circ t_{c_1}^{k})\}$ lying in P, let $I \in \mathscr{I}$ be one of boundary components of C (shown in Figure 1). Let $z \in I$. From Figure 1, we find that there is a path in C that connects from z to a point $z' \in I'$ for $I \neq I'$. This shows that any component of $S \backslash \{a, b(t_{c_0}^{-k} \circ t_{c_1}^{k})\}$ that lies in P is not a bigon. This completes the proof of Lemma 4.2. $\square$

5. Lifts of Dehn twists

From Section 4, the Dehn twist $t_{\tilde{c}}$ for a simple closed geodesic $\tilde{c}$ on $\tilde{S}$ can be similarly defined. In this section, we review the construction of some lifts of a positive Dehn twist on $\tilde{S}$. Let $\tilde{c}$ be a simple closed geodesic on $\tilde{S}$. Let $\tau : \mathbf{H} \to \mathbf{H}$ be any lift of $t_{\tilde{c}}$ under the universal covering map $S : \mathbf{H} \to \tilde{S}$. That is, τ satisfies

$$\text{(i)}\ \tau G \tau^{-1} = G \qquad \text{and} \qquad \text{(ii)}\ \varrho \circ \tau = t_{\tilde{c}} \circ \varrho,$$

where (i) says that τ descends a map $\tilde{w}$ on $\tilde{S}$ and (ii) says that $\tilde{w} = t_{\tilde{c}}$. Observe that the set

$$\{\varrho^{-1}(\tilde{c})\} = \{\text{geodesics } \hat{c} \subset \mathbf{H} : \varrho(\hat{c}) = \tilde{c}\}$$

is a disjoint union of geodesics in $\mathbf{H}$ and divides $\mathbf{H}$ into infinitely many simply connected and convex regions. Let Ω be one of such regions. Figure 2 (a) depicts such a region with boundary components lying in $\{\varrho^{-1}(\tilde{c})\}$. That is, $\partial\Omega \subset \{\varrho^{-1}(\tilde{c})\}$.

For each $\hat{c} \in \{\varrho^{-1}(\tilde{c})\}$, let $K_{\hat{c}}$ denote a small "crescent" neighborhood of $\hat{c}$ so that $\varrho(K_{\hat{c}})$ is an annular neighborhood $A(\tilde{c})$ of $\tilde{c}$. Let $\Omega_0 \subset \Omega$ be a smaller region obtained from Ω by removing $K_{\hat{c}} \cap \Omega$ from Ω for all $\hat{c} \in \partial\Omega$. Let $\hat{z} \in \Omega_0$, and $z = \varrho(\hat{z}) \in \tilde{S}$. Then z stays outside of $A(\tilde{c})$. Note that the Dehn twist $t_{\tilde{c}}$ can be performed so that $t_{\tilde{c}}|_{\tilde{S} \backslash A(\tilde{c})} = \mathrm{id}$. Hence $t_{\tilde{c}}(z) = z$. From (ii) above,

$$\varrho \circ \tau(\hat{z}) = t_{\tilde{c}} \circ \varrho(\hat{z}) = t_{\tilde{c}}(z) = z.$$

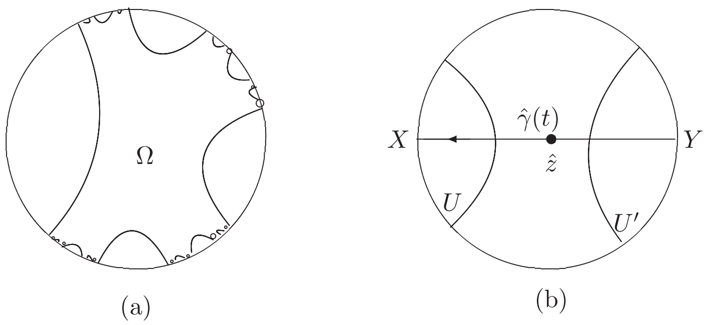

Fig. 2

It turns out that $\hat{z}$ and $\tau(\hat{z})$ are G-equivalent, i.e., there is an element $g_{\hat{z}} \in G$ such that $\tau(\hat{z}) = g_{\hat{z}}(\hat{z})$. Since G is discrete and Ω_0 is connected, $g_{\hat{z}}$ does not depends on $\hat{z} \in \Omega_0$. Write $g = g_{\hat{z}}$ and define

$$\tau_\Omega = g^{-1}\tau. \tag{5.1}$$

One can easily check that τ_Ω also satisfies the above conditions (i) and (ii). In addition, the restriction $\tau_\Omega|_{\Omega_0} = \mathrm{id}$.

Now the complement $\mathbf{H}\backslash\Omega$ is a disjoint union of half-planes. In what follows, we denote by $\mathscr{U}_\Omega$ the collection of all components of $\mathbf{H}\backslash\Omega$.

Let $\tilde{\gamma}$ be a simple closed geodesic on $\tilde{S}$ that intersects $\tilde{c}$ and $z \in \tilde{\gamma}$. Then $\tilde{\gamma}$ can be lifted to a geodesic $\hat{\gamma}$ passing through $\hat{z}$. Parameterize $\hat{\gamma} = \hat{\gamma}(t)$, $-\infty < t < +\infty$, so that $\hat{\gamma}(0) = \hat{z}$. Since $\tilde{\gamma}$ intersects $\tilde{c}$, $\hat{\gamma}(t)$ crosses a component U of $\mathbf{H}\backslash\Omega$ for some $t_0 > 0$. Also, $\hat{\gamma}(t)$ crosses a different component U' of $\mathbf{H}\backslash\Omega$ for some $t_1 < 0$. See Figure 2 (b).

We claim that $\tau_\Omega|_U \neq \mathrm{id}$. Suppose to the contrary that $\tau_\Omega|_U = \mathrm{id}$. If $\tau_\Omega|_{U'} = \mathrm{id}$, then $\tau_\Omega(\hat{\gamma})$ and $\hat{\gamma}$ share the same endpoints X and Y. So $\tau_\Omega(\hat{\gamma})$ is homotopic to $\hat{\gamma}$ (rel the endpoints), which leads to that $t_{\tilde{c}}(\tilde{\gamma})$ is homotopic to $\tilde{\gamma}$. This is a contradiction. If $\tau_\Omega|_{U'} \neq \mathrm{id}$, then since $\tau_\Omega(\hat{\gamma})$ projects to $t_{\tilde{c}}(\tilde{\gamma})$, the geodesic connecting X and $\tau_\Omega(Y)$ is invariant under the action of a hyperbolic element of G. It follows that there are two hyperbolic elements of G that share one fixed point X. This contradicts that G is discrete.

Since U is arbitrary, we conclude that $\tau_\Omega|_{U_k} \neq \mathrm{id}$ for any component U_k of $\mathbf{H}\backslash\Omega$. To understand the action of τ_Ω on each component U_k of $\mathbf{H}\backslash\Omega$, we observe that $\hat{\gamma}(t)$ projects to $\tilde{\gamma}(t)$. By examining the action of $t_{\tilde{c}}$ on $\tilde{\gamma}(t)$, we find that the point $\tilde{\gamma}(t_0^-)$ travels along the circle $\tilde{c}$ once in the counterclockwise direction, returns to its original position, and glues with $\tilde{\gamma}(t_0^+)$. Thus τ_Ω sends $\hat{\gamma}(t_0^-)$ to a G-equivalent point $\tau_\Omega(\hat{\gamma}(t_0^-))$. It turns out that τ_Ω sends $\hat{\gamma}(t)$ to a "broken" geodesic.

Observe that geodesics in $\{\varrho^{-1}(\tilde{c})\}$ lying in U_k divides U_k into infinitely many (mutually disjoint) regions Ω_{ki}, $i = 1, 2, \ldots$. As we mentioned before, the restriction of τ_Ω to each $\Omega_{ki} \backslash \{K_{\hat{c}} : \hat{c} \in \partial\Omega_{ki}\}$ is realized by a non-trivial hyperbolic element of G. We also notice that geodesics in $\{\varrho^{-1}(\tilde{c})\}$ that lie in U_k determines infinitely many half-planes contained in U_k. These half-planes form a partially ordered set Λ_k defined by inclusion, and elements of $\mathscr{U}_\Omega$ are considered maximal elements in $\bigcup_i \Lambda_i$. For any point $\hat{z} \in U_k$ that lies outside of $\{K_{\hat{c}} : \hat{c} \in \varrho^{-1}(\tilde{c})\}$, there are only finitely many elements $U_{k0}, U_{k1}, \ldots, U_{km} \in \Lambda_k$ such that

$$\hat{z} \in U_{km} \subset \cdots \subset U_{k1} \subset U_{k0} = U_k.$$

For $i = 0, \ldots, m$, we let g_{ki} be the primitive hyperbolic element of G that keeps U_{ki} invariant and takes the same orientation as that of the Dehn twist $t_{\tilde{c}}$. Then

$$\tau_\Omega(\hat{z}) = g_{k0} g_{k1} \ldots g_{km}(\hat{z}). \tag{5.2}$$

By construction, we also see that τ_Ω is a quasiconformal map. From (5.2), the map τ_Ω acts like a Möbius transformation on each component of $\mathbf{H} \backslash \bigcup K_{\hat{c}}$, but for a different component, the Möbius transformation is different. This particularly implies that the Beltrami coefficient of τ_Ω is supported on the union

$$\bigcup \{K_{\hat{c}} : \hat{c} \in \{\varrho^{-1}(\tilde{c})\}\}$$

Thus the map τ_Ω extends to a quasiconformal homeomorphism of $\overline{\mathbf{H}}$ onto itself, and the restriction $\tau_\Omega|_{\hat{\mathbf{R}}}$ is quasisymmetric. Hence $[\tau_\Omega] \in \mathrm{mod}(\tilde{S})$. By Lemma 3.2 of [16], we see that

$$[\tau_\Omega]^* = t_c, \tag{5.3}$$

where $c \in \mathscr{S}$ is homotopic to $\tilde{c}$ on $\tilde{S}$ as x is filled in.

Recall that all boundary components of Ω are geodesics in $\{\varrho^{-1}(\tilde{c})\}$. For each $\hat{c} \in \partial\Omega$, there is a primitive simple hyperbolic element $h \in G$ that keeps $\hat{c}$ invariant and takes the same orientation as that of $t_{\tilde{c}}$. By Theorem 2 of [10, 12], we can write $h^* = t_{c_0}^{-1} \circ t_c$, where c_0, c are the boundary geodesics of an x-punctured cylinder P on S and c is also determined by (5.3), which means that c_0 and c are disjoint and are homotopic to each other when x is filled in. From the above construction, the geodesic $c \in \mathscr{S}$ depends only on Ω and not on any particular boundary component of Ω. The following result is interesting in its own right.

Proposition 5.1. *There is a bijection between the set of elements of $\mathscr{U}_\Omega$ and the set of x-punctured cylinders on S all of which share the common boundary geodesic c.*

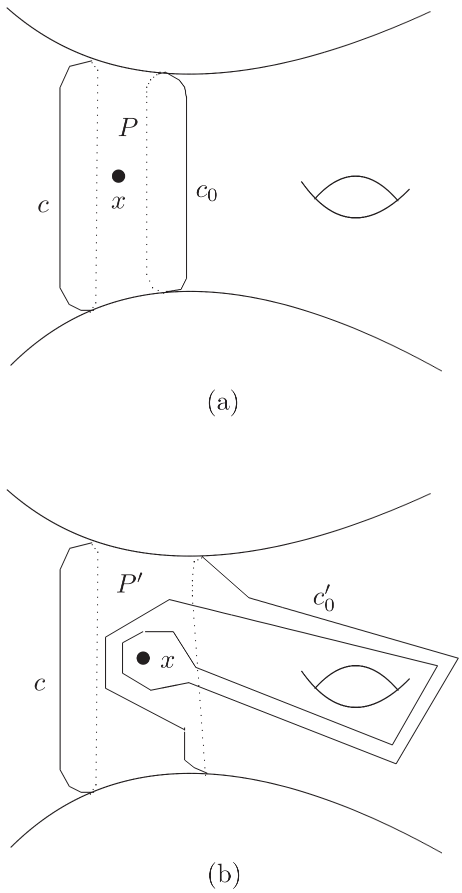

Fig. 3

Figure 3 above exhibits two x-punctured cylinders P and P' on S that share the common boundary component c.

6. Constructions of hyperbolic mapping classes through lifts of Dehn twists

In the rest of this article, we assume that the hyperbolic mapping class $\tilde{\theta}$ (introduced in Section 1) is also represented by a finite product of Dehn twists:

$$\tilde{\lambda} = \prod_i t_{\tilde{a}}^{n_i} \circ t_{\tilde{b}}^{m_i}, \tag{6.4}$$

where m_i, n_i are integers and $(\tilde{a}, \tilde{b})$ is a pair of filling simple closed geodesics on $\tilde{S}$ (in the sense that each component of $\tilde{S} \backslash \{\tilde{a} \cup \tilde{b}\}$ is either a topological disk or a topological once punctured disk). This means that $\tilde{\lambda}$ and $\tilde{\omega}$ both represent $\tilde{\theta}$. That is to say, $\tilde{\lambda}$ is isotopic to $\tilde{\omega}$ via an isotopy $F(\cdot,t)$, $0 \le t \le 1$. Note that for $i = 1,2,\dots,m$, $\tilde{\lambda}(\tilde{z}_i) = \tilde{\omega}(\tilde{z}_i)$. Thus the curves $c_i = F(\tilde{z}_i, t)$, $0 \le t \le 1$, are closed curves which may or may not be trivial.

Let $\tilde{\Omega}$ be a component of $\tilde{S}\backslash\{\tilde{a},\tilde{b}\}$. Let Ω_1 and Ω_2 respectively be the components of $\mathbf{H}\backslash\{\varrho^{-1}(\tilde{a})\}$ and $\mathbf{H}\backslash\{\varrho^{-1}(\tilde{b})\}$, such that $\Omega_1 \cap \Omega_2 \neq \varnothing$ and $\varrho(\Omega_1 \cap \Omega_2) = \tilde{\Omega}$. By the same construction as in Section 5, for the geodesics $\tilde{a}$ and $\tilde{b}$, we can obtain the two lifts τ_{Ω_1} and τ_{Ω_2} of $t_{\tilde{a}}$ and $t_{\tilde{b}}$, respectively. Let

$$\lambda_0 = \prod_i \tau_{\Omega_1}^{n_i} \tau_{\Omega_2}^{m_i},$$

where n_i and m_i are as given in (6.4). By Lemma 3.2 of [16], we have

$$[\tau_{\Omega_1}]^* = t_a \qquad \text{and} \qquad [\tau_{\Omega_2}]^* = t_b \tag{6.5}$$

for some $a, b \in \mathscr{S}$ that are homotopic to $\tilde{a}$ and $\tilde{b}$ as x is filled in. It is obvious that $I([\lambda_0]^*) = \tilde{\theta}$. Unfortunately, due to lack of evidence, we do not know whether $[\lambda_0]^*$ is a hyperbolic mapping class.

To find a way around, we fix the region Ω_1, and make various selections for Ω_2. Note that different choices of Ω_2 give rise to different lifts of $t_{\tilde{b}}$. Our aim is to choose a sequence $\{\Omega_{2,k}\}$ of regions in $\mathbf{H}$ so that the corresponding lifts $\tau_{\Omega_{2,k}}$ of $t_{\tilde{b}}$ satisfy the following additional condition: for each k, the product

$$\lambda_k = \prod_i \tau_{\Omega_1}^{n_i} \tau_{\Omega_{2,k}}^{m_i} \tag{6.6}$$

determines a hyperbolic mapping class $[\lambda_k]^*$ in Mod_S^x whose associated Teichmüller disk Δ_k is not a member of $\mathscr{F}(\tilde{\Delta})$ (as defined in (2.7)), while it still holds that $I([\lambda_k]^*) = \tilde{\theta}$.

As $(\tilde{a},\tilde{b})$ fills $\tilde{S}$, we can choose a simple closed geodesic $\tilde{c}$ so that (i) $\tilde{c}$ is different from $\tilde{a}$ and $\tilde{b}$; and (ii) $\tilde{c}$ intersects both $\tilde{a}$ and $\tilde{b}$. There are many ways to acquire such a geodesic $\tilde{c}$. The easiest way is to choose $\tilde{c}$ to be the geodesic representative $t_{\tilde{a}}(\tilde{b})$ in the homotopy class of the image curve $\tilde{b}(t_{\tilde{a}})$. Clearly, the geodesic $\tilde{c}$ obtained in this way satisfies (i) and (ii) above.

Choose $\tilde{\Omega}$ so that $\tilde{c} \cap \tilde{\Omega} \neq \varnothing$. Let $\hat{c} \subset \mathbf{H}$ be a geodesic such that $\varrho(\hat{c}) = \tilde{c}$ and $\hat{c} \cap (\Omega_1 \cap \Omega_2) \neq \varnothing$. It is evident that $\hat{c}$ goes across maximal elements $U \in \mathscr{U}_{\Omega_1}$ and $V \in \mathscr{U}_{\Omega_2}$. Let $h \in G$ be the primitive simple hyperbolic element whose axis is $\hat{c}$ and whose repelling fixed point Y is covered by V. See Figure 4 (a).

Since X is the attracting fixed point of h and ∂V stays away from the repelling fixed point Y of h, $h^k(\partial V)$ shrinks to the point X for large positive

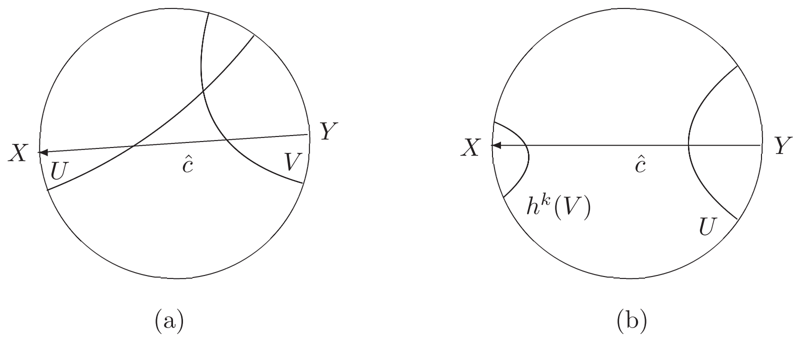

Fig. 4

integers k. As a consequence, we can choose a sufficiently large integer k_0 so that

$$h^k(V) \cup U = \mathbf{H} \qquad \text{for } k \geq k_0.$$

Figure 4 (b) shows the situation when k is large, not only $h^k(V)$ and U have an overlap, but also $h^k(V) \cup U$ covers the entire $\mathbf{H}$. Let

$$\tau_{\Omega_{2,k}} = h^k \tau_{\Omega_2} h^{-k}. \tag{6.7}$$

By a simple calculation, the maps λ_k as defined in (6.6) are lifts of $\tilde{\lambda}$. Further, if we write $\gamma_k = [\lambda_k]^*$, then obviously, $I(\gamma_k) = \tilde{\theta}$ for all $k \geq k_0$. By the same argument of Theorem 1.1 of [16], $\gamma_k \in \mathrm{Mod}_S^x$ are all hyperbolic mapping classes. By Bers [4], γ_k keeps a unique Teichmüller disk Δ_k invariant.

Theorem 1.1 (2) then follows from the following result.

THEOREM 6.1. *Some Teichmüller disks Δ_k are not members of $\mathscr{F}(\tilde{\Delta})$.*

7. Proof of Theorem 6.1 and Corollary 1.1

PROOF OF THEOREM 6.1. If all zeros of $\phi_{\tilde{\omega}}$ are punctures of $\tilde{S}$, then the set A (as defined in (3.15)) is empty. Hence by Lemma 3.3, all Δ_k obtained at the end of Section 6 are not members of $\mathscr{F}(\tilde{\Delta})$. Thus we may assume that $\{\tilde{z}_1, \ldots, \tilde{z}_m\}$, $m \geq 1$ is the set of non-puncture zeros of $\phi_{\tilde{\omega}}$. By taking a suitable power if necessary, we also assume that $\tilde{\omega}$ fixes these zeros.

Suppose that for all $k \geq k_0$, $\Delta_k \in \mathscr{F}(\tilde{\Delta})$. By Lemma 3.3 again, $\Delta_k \in A$ and thus $\Delta_k \in A_i$ for some $i \in \{1, 2, \ldots, m\}$, which tells us that there are infinitely many Teichmüller disks Δ_k that lie in the same set, say, A_1. Hence we may further assume without loss of generality that all Δ_k, $k \geq k_0$, lie in A_1. For any $k, l \geq k_0$, by Lemma 3.2, there are elements $h_k, h_l \in G$ such that $\gamma_k = h_k^* \circ \theta_1 \circ (h_k^*)^{-1}$ and $\gamma_l = h_l^* \circ \theta_1 \circ (h_l^*)^{-1}$. Let l be fixed and let $k \to +\infty$. We have

$$\theta_1 = (h_k^*)^{-1} \circ \gamma_k \circ h_k^* = (h_l^*)^{-1} \circ \gamma_l \circ h_l^*.$$

We conclude that γ_k and γ_l are conjugate by an element $g_k^* \in \mathrm{Mod}_S^x$ for some $g_k \in G$. That is,

$$\gamma_k = g_k^* \circ \gamma_l \circ (g_k^*)^{-1} \qquad \text{for an element } g_k \in G. \tag{7.1}$$

It follows immediately that

$$\log \lambda(\gamma_k) = \log \lambda(\gamma_l), \qquad \text{for all } k \geq k_0. \tag{7.2}$$

On the other hand, it was shown in [16] that

$$[\tau_{\Omega_1}]^* = t_a \qquad \text{and} \qquad [\tau_{\Omega_{2,k}}]^* = t_{b_k}, \tag{7.3}$$

where a has already been given as in (6.5) and $b_k \in \mathscr{S}$. Obviously, a and b_k are homotopic to $\tilde{a}$ and $\tilde{b}$, respectively, if a, b_k are viewed as curves on $\tilde{S}$.

From (6.6) and (7.3), we see that $\gamma_k = [\lambda_k]^*$ is determined by the pair (a, b_k) of simple closed geodesics on S. More precisely, for all $k \geq k_0$ we have

$$\gamma_k = [\lambda_k]^* = \left[\prod_i \tau_{\Omega_1}^{n_i} \tau_{\Omega_{2,k}}^{m_i}\right]^* = \prod_i t_a^{n_i} \circ t_{b_k}^{m_i} \in \langle t_a, t_{b_k} \rangle. \tag{7.4}$$

(where $\langle \eta, \xi \rangle$ denotes the group generated by ξ and η). For simplicity, in the rest of the article we write $\sigma_k = i(a, b_k)$. By Corollary 6.7 of Leininger [11], the dilatation $\lambda(\gamma_k)$ is greater or equal to the larger root of the quadratic equation

$$z^2 + (2 - \sigma_k^2)z + 1 = 0,$$

and the equality holds if and only if the pseudo-Anosov representative f_k of γ_k is of form $(t_a \circ t_{b_k})^{\pm 1}$. This implies that

$$\lambda(\gamma_k) \geq \frac{1}{2}(\sigma_k^2 - 2 + \sigma_k \sqrt{\sigma_k^2 - 4}). \tag{7.5}$$

An estimate shows that if $\sigma_k > 2$,

$$\frac{1}{2}(\sigma_k^2 - 2 + \sigma_k \sqrt{\sigma_k^2 - 4}) > \sigma_k^2 - \sigma_k - 1. \tag{7.6}$$

To complete the proof of Theorem 6.1, we need the following lemma.

Lemma 7.1. *As $k \to +\infty$, $\sigma_k^2 - \sigma_k - 1 \to +\infty$.*

Proof. By the definition, $\sigma_k = i(a, b_k)$. To see what the curves b_k are, we use (6.5), (6.7) and (7.3) to calculate as follows.

$$t_{b_k} = [\tau_{\Omega_{2,k}}]^* = (h^*)^k \circ [\tau_{\Omega_2}]^* \circ (h^*)^{-k} = (h^*)^k \circ t_b \circ (h^*)^{-k} = t_{(h^*)^k(b)}. \quad (7.7)$$

A basic fact about Dehn twists is that $t_\delta = t_\gamma$ if and only if the two curves δ and γ are homotopic to each other. From this fact along with (7.7), we see that

$$(h^*)^k(b) = b_k. \quad (7.8)$$

Recall that $h \in G$ (constructed in Section 6) is a primitive simple hyperbolic element. By Theorem 2 of [12] or Theorem 2 of [10], h^* is represented by a spin map $t_{c_0}^{-1} \circ t_{c_1}$, where $\{c_0, c_1\}$ bounds an x-punctured cylinder P. Since $\tilde{c}$ intersects both $\tilde{a}$ and $\tilde{b}$, it is not difficult to verify that the central curve c intersects both a and b (otherwise, $\tilde{c}$ world be disjoint from $\tilde{a}$ or $\tilde{b}$, which contradicts the definition of $\tilde{c}$). Now from Lemma 4.2, we deduce that

$$i(a, (h^*)^k(b)) \to +\infty,$$

as $k \to +\infty$. It follows from (7.8) that

$$\sigma_k = i(a, b_k) = i(a, (h^*)^k(b)) \to +\infty, \text{ as } k \to +\infty.$$

Hence $\sigma_k^2 - \sigma_k - 1 = \sigma_k(\sigma_k - 1) - 1 \to +\infty$ as $k \to +\infty$. This completes the proof of Lemma 7.1. □

Let us now return to the proof of Theorem 6.1. From Lemma 7.1, (7.5) and (7.6), we conclude that $\log \lambda(\gamma_k) \to +\infty$ as $k \to +\infty$. This contradicts (7.2), and hence the proof of Theorem 6.1 is complete. □

Proof of Corollary 1.1. By the argument of Theorem 1.1, we know that $\mathscr{T}$ is unbounded. The discreteness of $\mathrm{Spec}(\mathrm{Mod}_S)$ in $\mathbf{R}$ can be deduced from a theorem of [1] and [9]. Since $\mathscr{T} \subset \mathrm{Spec}(\mathrm{Mod}_S)$, $\mathscr{T}$ is also a discrete subset of $\mathbf{R}$. □

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