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Bridge decompositions with distances at least two

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ABSTRACT. For n-bridge decompositions of links in S^3, we propose a practical method to ensure that the Hempel distance is at least two.

1. Introduction

Hempel distance is a measure of complexity originally defined for Heegaard splittings of 3-manifolds [7]. The definition can be extended to bridge decompositions of links and it has been successfully applied to knot theory. For example, extending Hartshorn's [6] study for Heegaard splittings, Bachman-Schleimer [1] showed that the distance of a bridge decomposition of a knot bounds from below the genus of any essential surface in the knot exterior. Extending Scharlemann-Tomova's [13] for Heegaard splittings, Tomova [14] showed that the distance of a bridge decomposition bounds from below the bridge number of the knot or the Heegaard genus of the knot exterior.

However, it is difficult to calculate the Hempel distance of a general Heegaard splitting or bridge decomposition. While estimating it from above is a simple task in principle, it is a hard problem to estimate the distance from below.

For a Heegaard splitting, Casson-Gordon [4] introduced the rectangle condition to ensure that the distance is at least two. Lee [8] gave a weak version of rectangle condition which guarantees the distance to be at least one. Berge [2] gave a criterion for a genus two Heegaard splitting which guarantees the distance to be at least three. Lustig-Moriah [9] also gave a criterion to estimate the distance of a Heegaard splitting from below.

On the other hand, we could not find corresponding results for bridge decompositions in literature. In this paper, we observe that a bridge decomposition of a link in S^3 can be described by a *bridge diagram*, and show that the *well-mixed condition* for a bridge diagram guarantees the distance to be at least two (see Section 3 for definitions). It may be regarded as a variation of the rectangle condition for Heegaard diagrams.

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THEOREM 1. *Suppose $(T_+, T_-; P)$ is an n-bridge decomposition of a link in S^3 for $n \geq 3$. If a bridge diagram of $(T_+, T_-; P)$ satisfies the well-mixed condition, the Hempel distance $d(T_+, T_-)$ is at least two.*

Recently, Masur-Schleimer [12] found an algorithm to calculate the Hempel distance of a Heegaard splitting with a bounded error term. The author imagine that their algorithm may also be appliable to bridge decompositions. However, the point of our result is its practicality: for any given bridge decomposition, we can easily obtain a bridge diagram and check whether it satisfies the well-mixed condition.

2. Bridge decompositions and the Hempel distance

Suppose L is a link in S^3 and P is a 2-sphere dividing S^3 into two 3-balls B_+ and B_-. Assume that L intersects P transversally and let τ_ε be the intersection of L with B_ε for each $\varepsilon = \pm$. That is to say, (S^3, L) is decomposed into $T_+ := (B_+, \tau_+)$ and $T_- := (B_-, \tau_-)$ by P. We call the triple $(T_+, T_-; P)$ an *n-bridge decomposition* of L if each T_ε is an n-string trivial tangle. Here, T_ε is called an *n-string trivial tangle* if τ_ε consists of n arcs parallel to the boundary of B_ε. Obviously 1-bridge decompositions are possible only for the trivial knot, so we assume $n \geq 2$ in this paper.

Consider a properly embedded disk D in B_ε. We call D an *essential disk* of T_ε if ∂D is essential in the surface $\partial B_\varepsilon \backslash \tau_\varepsilon$ and D is disjoint from τ_ε. Here, a simple closed curve on a surface is said to be *essential* if it neither bounds a disk nor is peripheral in the surface. Note that essential disks of T_+ and T_- are bounded by some essential simple closed curves on the $2n$-punctured sphere $P \backslash L$.

The essential simple closed curves on $P \backslash L$ form a 1-complex $\mathscr{C}(P \backslash L)$, called the *curve graph* of $P \backslash L$. The vertices of $\mathscr{C}(P \backslash L)$ are the isotopy classes of essential simple closed curves on $P \backslash L$ and a pair of vertices spans an edge of $\mathscr{C}(P \backslash L)$ if the corresponding isotopy classes can be realized as disjoint curves. In the case of $n = 2$, this definition makes the curve graph a discrete set of points and so a slightly different definition is used.

The *Hempel distance* (or just the *distance*) of $(T_+, T_-; P)$ is defined by

$$d(T_+, T_-) := \min\{d([\partial D_+], [\partial D_-]) \mid D_\varepsilon \text{ is an essential disk of } T_\varepsilon. \ (\varepsilon = \pm)\}$$

where $d([\partial D_+], [\partial D_-])$ is the minimal distance between $[\partial D_+]$ and $[\partial D_-]$ measured in $\mathscr{C}(P \backslash L)$ with the path metric. Because the curve graph is connected [10], the distance $d(T_+, T_-)$ is a finite non-negative integer.

For 2-bridge decompositions, there is a unique essential disk for each of the 2-string trivial tangles. Moreover, the curve graph of a 4-punctured sphere is well understood (see Sections 1.5 and 2.1 in [11] for example) and so we can calculate the exact distance.

Suppose $(T_+, T_-; P)$ is an n-bridge decomposition of a link L for $n \geq 3$. If $d(T_+, T_-) = 0$, there are essential disks D_+, D_- of T_+, T_-, respectively, such that $[\partial D_+] = [\partial D_-]$. We can assume $\partial D_+ = \partial D_-$ indeed and so $D_+ \cup D_-$ is a 2-sphere in S^3. Therefore, $(T_+, T_-; P)$ is separated by the sphere into an m-bridge decomposition and an $(n-m)$-bridge decomposition of sublinks of L. By the definition of essential disks, m is more than 0 and less than n. Conversely, we can conclude that the distance is at least one if $(T_+, T_-; P)$ is not a such one.

3. Bridge diagrams and the well-mixed condition

Suppose $(T_+, T_-; P)$ is an n-bridge decomposition of a link L in S^3 and $T_+ = (B_+, \tau_+)$, $T_- = (B_-, \tau_-)$. For each $\varepsilon = \pm$, the n arcs of τ_ε can be disjointly projected into P. Let $p : L \to P$ be such a projection. A *bridge diagram* of $(T_+, T_-; P)$ is a diagram of L obtained from $p(\tau_+)$ and $p(\tau_-)$. In the terminology of [5], τ_+, τ_- are the overpasses and the underpasses of L.

Note that the boundary of a regular neighborhood of each arc of $p(\tau_\varepsilon)$ in P bounds an essential disk of T_ε separating an arc of τ_ε. In this sense a bridge diagram represents a family of essential disks of T_+, T_-. So we can think of it as something like a Heegaard diagram for a Heegaard splitting.

It is well known that a bridge decomposition is displayed as a "plat" as in Figure 1 (See [3]). Now we describe how to convert a plat presentation to a bridge diagram. For example, consider a 3-bridge decomposition with a plat presentation as in the left of Figure 2. Here P can be isotoped onto any height, so start with P in the position P_s. The top in the right of Figure 2 illustrates a view of a canonical projection of the arcs t_+^1, t_+^2, t_+^3 on P from B_+ side. In our pictures, $p(t_+^1)$, $p(t_+^2)$, $p(t_+^3)$ are represented by a solid line, a dotted line, a broken line, respectively. Shifting P to the position P_1, the projections are as the second in the right of Figure 2. Shifting P further to

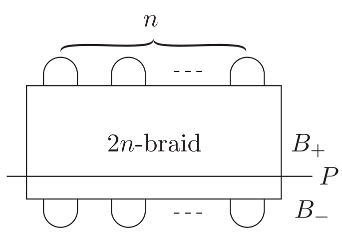

Fig. 1

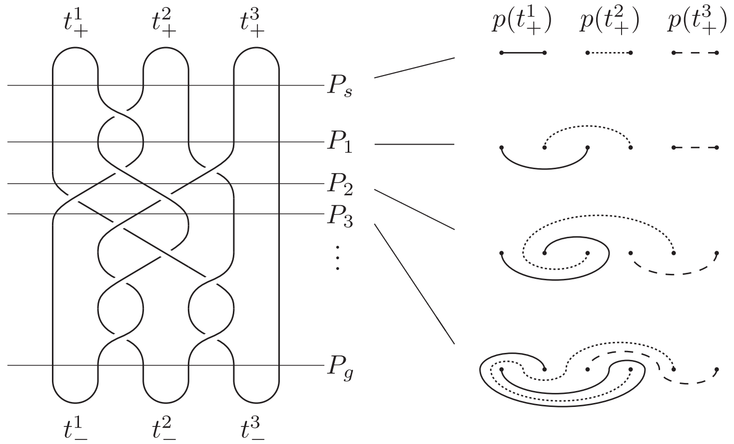

Fig. 2

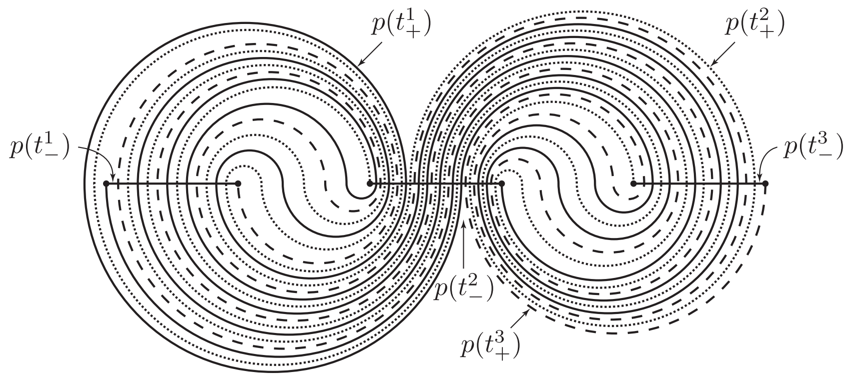

Fig. 3

the position P_2, the projections are as the third. By continuing this process, the projections are as in Figure 3 when P is in the position P_g. Then we can find a canonical projection of the arcs t_-^1, t_-^2, t_-^3 and obtain a bridge diagram.

Next we study the distance of this 3-bridge decomposition. Since the link L is connected, the bridge decomposition cannot be separated into smaller ones. It follows that the distance is at least one. Consider the simple closed curve c as in Figure 4. The curve c is essential in $P \backslash L$ and disjoint from both $p(t_+^1)$ and $p(t_-^1)$. Recall that the boundary of a small neighborhood of $p(t_+^1)$, $p(t_-^1)$ in P bounds an essential disk D_+^1 of T_+ and an essential disk D_-^1 of T_-, respectively. So there are an edge between $[\partial D_+^1]$, $[c]$ and an edge between $[c]$,

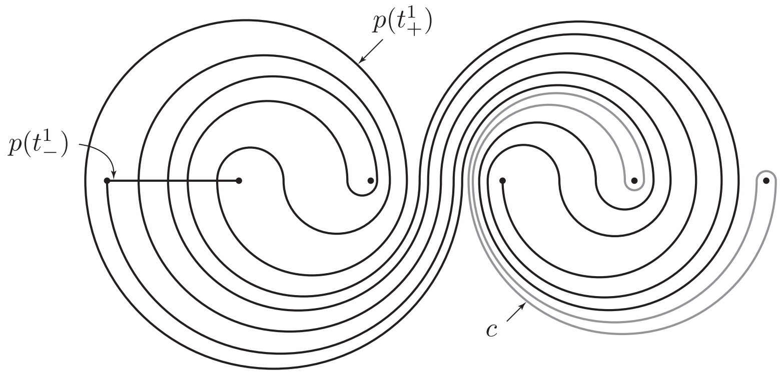

Fig. 4

$[\partial D_-^1]$ in the curve graph $\mathscr{C}(P\backslash L)$. By definition, the distance is at most two. It is true that there is no direct edge between $[\partial D_+^1]$ and $[\partial D_-^1]$. However, this is not enough to conclude that the distance is equal to two because there are infinitely many essential disks of T_+, T_- other than D_+^1, D_-^1.

As shown in [2], [4], [8] and [9], sufficiently complicated Heegarrd diagram implies a large distance of the Heegaard splitting. We can expect that sufficiently complicated bridge diagram also implies a large distance of the bridge decomposition. A bridge diagram should be pretty complicated if it satisfies the *well-mixed condition*, which we define in the following.

Denote the arcs of each τ_ε by $t_\varepsilon^1, t_\varepsilon^2, \dots, t_\varepsilon^n$. Let l be a loop on P containing $p(\tau_-)$ such that $p(t_-^1), p(t_-^2), \dots, p(t_-^n)$ are located in l in this order. We can assume that $p(\tau_+)$ has been isotoped in $P\backslash L$ to have minimal intersection with l. For the bridge diagram of Figure 3, it is natural to choose l to be the closure in $P \cong S^2$ of the horizontal line containing $p(t_-^1) \cup p(t_-^2) \cup p(t_-^3)$. Let $H_+, H_- \subset P$ be the hemi-spheres divided by l and let δ_i $(1 \le i \le n)$ be the component of $l\backslash p(\tau_-)$ which lies between $p(t_-^i)$ and $p(t_-^{i+1})$. (Here the indices are considered modulo n.) Let $\mathscr{A}_{i,j,\varepsilon}$ be the set of components of $p(\tau_+) \cap H_\varepsilon$ separating δ_i from δ_j in H_ε for a distinct pair $i, j \in \{1, 2, \dots, n\}$ and $\varepsilon \in \{+, -\}$. For example, Figure 5 displays $\mathscr{A}_{1,2,+}$ for the above bridge diagram. Note that $\mathscr{A}_{i,j,\varepsilon}$ consists of parallel arcs in H_ε.

Definition 1. (1) A bridge diagram satisfies the (i, j, ε)-*well-mixed condition* if in $\mathscr{A}_{i,j,\varepsilon} \subset H_\varepsilon$, a subarc of $p(t_+^r)$ is adjacent to a subarc of $p(t_+^s)$ for all distinct pair $r, s \in \{1, 2, \dots, n\}$.

(2) A bridge diagram satisfies the *well-mixed condition* if it satisfies the (i, j, ε)-well-mixed condition for all combinations of a distinct pair $i, j, \in \{1, 2, \dots, n\}$ and $\varepsilon \in \{+, -\}$.

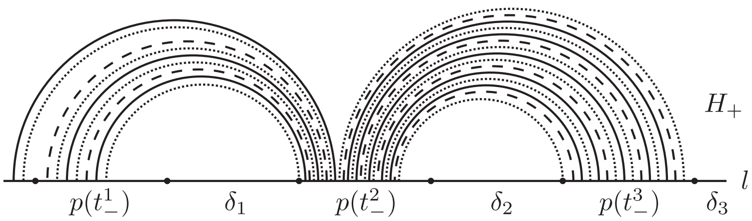

Fig. 5

As in Figure 5, the bridge diagram in Figure 3 amply satisfies the $(1, 2, +)$-well-mixed condition. One can also check the (i, j, ε)-well-mixed condition for all the other combinations $(i, j, \varepsilon) = (1, 2, -), (2, 3, +), (2, 3, -), (3, 1, +), (3, 1, -)$. Hence the bridge diagram in Figure 3 satisfies the well-mixed condition.

4. Proof of the theorem

Firstly, consider an essential disk D_- of T_-. Assume that D_- has been isotoped so that $|\partial D_- \cap l|$ is minimal. Here, $|\cdot|$ denotes the number of connected components of a topological space.

LEMMA 1. *There exist a distinct pair $i, j \in \{1, 2, \dots, n\}$ and $\varepsilon \in \{+, -\}$ such that ∂D_- includes a subarc connecting δ_i and δ_j in H_ε.*

PROOF. Since the arcs of τ_- are projected to subarcs of l, there exists a disk E_- in B_- such that $\partial E_- = l$ and $\tau_- \subset E_-$. The essential disk D_- must have non-empty intersection with E_-. The closed components of $D_- \cap E_-$ can be eliminated by an isotopy of $\operatorname{Int} D_-$. Then $D_- \cap E_-$ is a non-empty family of properly embedded arcs in D_-. Consider an outermost subdisk D_-^0 of D_- cut off by an arc of them. For the minimality of $|\partial D_- \cap l|$, we can see that $\partial D_-^0 \cap \partial D_-$ connects δ_i and δ_j in H_ε for a distinct pair $i, j \in \{1, 2, \dots, n\}$ and $\varepsilon \in \{+, -\}$. □

Secondly, consider an essential disk D_+ of T_+. Assume that D_+ has been isotoped so that $|\partial D_+ \cap p(\tau_+)|$ is minimal.

LEMMA 2. *Suppose c is an essential simple closed curve on $P \backslash L$ disjoint from ∂D_+. There exist a distinct pair $r, s \in \{1, 2, \dots, n\}$ such that no subarc of c connects $p(t_+^r)$ and $p(t_+^s)$ directly (i.e. its interior is disjoint from $p(\tau_+)$).*

PROOF. Let E_+^i be a disk of parallelism between t_+^i and $p(t_+^i)$ for each $i = 1, 2, \dots, n$ so that $E_+^1, E_+^2, \dots, E_+^n$ are pairwise disjoint. The closed components of $D_+ \cap (E_+^1 \cup E_+^2 \cup \cdots \cup E_+^n)$ can be eliminated by an isotopy of

Int D_+. If $D_+ \cap (E_+^1 \cup E_+^2 \cup \cdots \cup E_+^n)$ is empty, D_+ separates the n disks $E_+^1, E_+^2, \dots, E_+^n$ into two classes in B_+. Since D_+ is essential, both these classes are not empty. If $D_+ \cap (E_+^1 \cup E_+^2 \cup \cdots \cup E_+^n)$ is not empty, it consists of properly embedded arcs in D_+. Consider an outermost subdisk D_+^0 of D_+ cut off by an arc of them, say, an arc of $D_+ \cap E_+^k$. Then, $D_+^0 \cup E_+^k$ separates the $(n-1)$ disks $E_+^1, \dots, E_+^{k-1}, E_+^{k+1}, \dots, E_+^n$ into two classes in B_+. Since $|\partial D_+ \cap p(t_+^k)|$ is minimal, both these classes are not empty. Anyway, by choosing r and s from the indexes of the disks of separated classes, the lemma follows. □

Assume that the distance of $(T_+, T_-; P)$ is less than two. There are disjoint essential disks D_+, D_- of T_+, T_-, respectively. If ∂D_- contains a subarc connecting δ_i and δ_j in H_ε, it intersects all the arcs of $\mathscr{A}_{i,j,\varepsilon}$. In particular, if two arcs of $\mathscr{A}_{i,j,\varepsilon}$ are adjacent in H_ε, a subarc of ∂D_- connects them directly. The above observations and the well-mixed condition are almost enough to lead to a contradiction, but only the following should be checked:

Lemma 3. *The disks D_+ and D_- can be isotoped preserving the disjointness so that $|\partial D_+ \cap p(\tau_+)|$ and $|\partial D_- \cap l|$ are minimal.*

Proof. Note that any isotopy of ∂D_ε in $P \backslash L$ can be realized by an isotopy of D_ε in $B_\varepsilon \backslash \tau_\varepsilon$ for $\varepsilon = \pm$.

If $|\partial D_+ \cap p(\tau_+)|$ is not minimal, there are a subarc of ∂D_+ and a subarc α of $p(\tau_+)$ cobounding a disk Δ_+ in $P \backslash L$. Since D_+, D_- are disjoint, $\partial D_- \cap \Delta_+$ consists of arcs parallel into α. Let Δ_+^0 be an outermost disk of the parallelisms. By assumption, $p(\tau_+)$ has minimal intersection with l and so no component of $l \cap \Delta_+^0$ has both end points on α. By an isotopy of ∂D_- across Δ_+^0, we can reduce $|\partial D_- \cap \Delta_+|$ without increasing $|\partial D_- \cap l|$. After pushing out ∂D_- from Δ_+ in this way, we can reduce $|\partial D_+ \cap p(\tau_+)|$ by an isotopy of ∂D_+ across Δ_+.

If $|\partial D_- \cap l|$ is not minimal, there are a subarc of ∂D_- and a subarc β of l cobounding a disk Δ_- in $P \backslash L$. The intersection $\partial D_+ \cap \Delta_-$ consists of arcs parallel into β. Let Δ_-^0 be an outermost disk of the parallelisms. By the minimality of $|l \cap p(\tau_+)|$, no component of $p(\tau_+) \cap \Delta_-^0$ has both end points at β. By an isotopy of ∂D_+ across Δ_-^0, we can reduce $|\partial D_+ \cap \Delta_-|$ without increasing $|\partial D_+ \cap p(\tau_+)|$. After pushing out ∂D_+ from Δ_- in this way, we can reduce $|\partial D_- \cap l|$ by an isotopy of ∂D_- across Δ_-. □

Theorem 1 implies that the 3-bridge decomposition in Figure 2 has distance at least two. Since we have shown that it is at most two, the distance is exactly two. We can work out in this way fairly many n-bridge decompositions, especially for $n = 3$.

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