Infinitesimal isometries on tangent sphere bundles over three-dimensional manifolds

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Abstract. In this article, we study the infinitesimal isometries on tangent sphere bundles over orientable three-dimensional Riemannian manifolds. Focusing on the vector fields which do not preserve fibers, we show the existence of lifts to the bundles of orthonormal frames. These lifts enable us to analyze the infinitesimal isometries by the symmetry of principal fiber bundles. We prove that the tangent sphere bundle admits a non-fiber-preserving infinitesimal isometry if and only if the base manifold has the same constant sectional curvatures as the fibers have. As an application, we classify the infinitesimal isometries on tangent sphere bundles for the three dimensional case.

1. Introduction

For any fixed positive number λ, the tangent sphere bundle $T^\lambda M$ of radius λ over a Riemannian manifold (M, g) is defined to be the set of all tangent vectors of length λ, which is a hypersurface in the tangent bundle with the induced Sasaki metric g^S. For $\lambda = 1$, it was proved in [9] and [10] that the geodesic flow on a tangent sphere bundle is an infinitesimal isometry if and only if the base manifold is a space of constant curvature one. The generalization of this fact is a key to classifying the infinitesimal isometries on a tangent sphere bundle. This generalization for orientable two-dimensional Riemannian manifolds was studied and the infinitesimal isometries on the tangent sphere bundles were classified in [3]. In this article, we study the three-dimensional case, and see the differences coming from the dimensions.

In general, a vector field on a fiber space is called fiber preserving if the local one-parameter group of local transformations generated by the vector field maps each fiber into another one. For example, the lifts X^L and ϕ^L to $T^\lambda M$, which are to be described below, are fiber preserving, whereas the geodesic flow is not fiber preserving. If there exists an infinitesimal isometry which does not preserve fibers, it seems that the base space and the fibers have specific

common properties such as sectional curvatures. In Section 3, we study non-fiber-preserving infinitesimal isometries on $T^\lambda M$.

To analyze the infinitesimal isometries on $T^\lambda M$, we define lifts to the bundle of orthonormal frames, which enable us to make use of the symmetry of principal fiber bundles. Let $SO(M)$ be the bundle of oriented orthonormal frames over M. The bundle $SO(M)$ has a natural Riemannian metric G defined by

$$G(X, Y) = {}^t\theta(X) \cdot \theta(Y) + \frac{\lambda^2}{2} \operatorname{trace}({}^t\omega(X) \cdot \omega(Y))$$

$$\text{for } X, Y \in T_u SO(M), u \in SO(M),$$

where θ and ω denote the canonical form and the Riemannian connection form on $SO(M)$, respectively. In their papers [8], Takagi and Yawata classified the infinitesimal isometries of $(SO(M), G)$ when $\lambda = \sqrt{2}$ and $\dim M \neq 2, 3, 4$ nor 8. In the three-dimensional case, the principal $SO(3)$ bundle $SO(M)$ can be also regarded as a circle bundle over the tangent sphere bundle. Nagy used this bundle to study the geodesics of $T^1 M$ (cf. [5]). On this circle bundle, we define lifts of infinitesimal isometries on $T^\lambda M$ to those on $SO(M)$ in Section 3.

For a Riemannian manifold M with metric g, let $\mathfrak{i}(M, g)$ denote the Lie algebra of infinitesimal isometries on (M, g), $\mathfrak{D}^2(M)$ the Lie algebra of two-forms on M, and $\mathfrak{D}^2(M)_0$ the parallel two-forms in $\mathfrak{D}^2(M)$ with respect to the Riemannian connection ∇. Given $X \in \mathfrak{i}(M, g)$, the differentials of local transformations generated by X induce a vector field X^L on $T^\lambda M$. We call X^L the lift of X. On the other hand, we identify $\mathfrak{D}^2(M)$ with the set of all skew-symmetric tensor fields of type $(1, 1)$ on M in the usual manner. For $\phi \in \mathfrak{D}^2(M)$, let $\tilde{\phi}$ denote the corresponding skew-symmetric tensor field of type $(1, 1)$. Then, the one-parameter group of transformations $\exp t\tilde{\phi}$, $t \in \mathbf{R}$, induces a vector field ϕ^L on $T^\lambda M$. We call ϕ^L the lift of ϕ. The following theorems are the main results of this paper.

THEOREM 1. *Let (M, g) be a connected, orientable three-dimensional Riemannian manifold of class C^∞ and λ a positive number. Then, we have the following for $\mathfrak{i}(T^\lambda M, g^S)$.*

(i) *There exists a homomorphism Ψ of $\mathfrak{i}(T^\lambda M, g^S)$ into $\mathfrak{i}(SO(M), G)$.*

(ii) *The bundle $(T^\lambda M, g^S)$ admits a non-fiber-preserving infinitesimal isometry if and only if (M, g) is a space of constant curvature $1/\lambda^2$.*

In the three-dimensional case, the above result (ii) is the generalization of the fact stated at the beginning of this article. Since the fiber preserving case was studied in [2], the result (ii) completes classifying the infinitesimal isometries on the tangent sphere bundles.

THEOREM 2. *Let (M, g) be a connected, orientable three-dimensional Riemannian manifold of class C^∞ and λ a positive number. The Lie algebras of infinitesimal isometries on the tangent sphere bundles are classified as follows:*
  (i) *When (M, g) is not a space of constant curvature $1/\lambda^2$,*

$$\mathfrak{i}(T^\lambda M, g^S) = \operatorname*{span}_{\mathbf{R}}\{X^L, \phi^L; X \in \mathfrak{i}(M, g), \phi \in \mathfrak{D}^2(M)_0\}.$$

The brackets relations are given by

$$[X^L, Y^L] = [X, Y]^L, \qquad [\phi^L, \psi^L] = -[\phi, \psi]^L, \qquad [X^L, \phi^L] = -[\nabla X, \phi]^L,$$

where $X, Y \in \mathfrak{i}(M, g)$ and $\phi, \psi \in \mathfrak{D}^2(M)_0$.
  (ii) *When (M, g) is a space of constant curvature $1/\lambda^2$,*

$$\mathfrak{i}(T^\lambda M, g^S) = \operatorname*{span}_{\mathbf{R}}\{X^L, \mathscr{G}; X \in \mathfrak{i}(M, g)\},$$

where $\mathscr{G}$ denotes the geodesic flow on $T^\lambda M$. The brackets relations are given by

$$[X^L, Y^L] = [X, Y]^L, \qquad [X^L, \mathscr{G}] = 0.$$

Compared with the two-dimensional case (cf. [3]), non-fiber-preserving infinitesimal isometries in the three-dimensional case are, if they exist, essentially equivalent to the geodesic flow ignoring the infinitesimal isometries of the base space. Here, we review the two-dimensional case: *Let (N, h) be a connected, orientable two-dimensional Riemannian manifold of class C^∞.*
  (i) *When (N, h) is not a space of constant curvature $1/\lambda^2$, we have*

$$\mathfrak{i}(T^\lambda N, h^S) = \operatorname*{span}_{\mathbf{R}}\{X^L, \omega^L; X \in \mathfrak{i}(N, h)\},$$

$$[X^L, Y^L] = [X, Y]^L, \qquad [X^L, \omega^L] = 0,$$

where $X, Y \in \mathfrak{i}(N, h)$ and ω denotes the volume element of (N, h).
  (ii) *When (N, h) is a space of constant curvature $1/\lambda^2$, we have*

$$\mathfrak{i}(T^\lambda N, h^S) = \operatorname*{span}_{\mathbf{R}}\{X^L, \omega^L, \mathscr{G}, [\omega^L, \mathscr{G}]; X \in \mathfrak{i}(N, h)\},$$

$$[X^L, Y^L] = [X, Y]^L, \qquad [X^L, \omega^L] = 0, \qquad [X^L, \mathscr{G}] = 0,$$

$$[X^L, [\omega^L, \mathscr{G}]] = 0, \qquad [\omega^L, [\omega^L, \mathscr{G}]] = -\mathscr{G}, \qquad [\mathscr{G}, [\omega^L, \mathscr{G}]] = \omega^L,$$

where $\mathscr{G}$ denotes the geodesic flow on $T^\lambda N$.
  If the dimension of the base manifold is greater than two, the tangent sphere bundle is not isometric to the bundle of oriented orthonormal frames. For the three-dimensional case, we note that a fixed point theorem is applied

to the fiber to determine the lifts in Section 4. Differences coming from the dimensions can be also seen in the results gained by Blair in [1].

2. Orthonormal frames over a tangent sphere bundle

In this section, definitions and basic formulas used in this paper are summarized. More details are stated in [3]. In the course of this paper, we consistently assume that (M, g) is a connected, orientable three-dimensional Riemannian manifold of class C^∞.

Let TM be the tangent bundle of a Riemannian manifold (M, g), and $\pi_{TM} : TM \to M$ be the bundle projection. Recall that the connection map $K : TTM \to TM$ corresponding to the Riemannian connection ∇ is defined to be

$$K(Z) = \lim_{t \to 0} \frac{\tau_0^t(X(t)) - X}{t} \qquad \text{for } Z \in T_X TM, \, X \in TM,$$

where $X(t)$, $-\varepsilon < t < \varepsilon$ for some $\varepsilon > 0$, is a differentiable curve on TM satisfying $X(0) = X$, $\dot{X}(0) = Z$. Also $\tau_0^t(X(t))$ denotes the parallel displacement of $X(t)$ from $\pi_{TM}(X(t))$ to $\pi_{TM}(X)$ along the geodesic arc joining $\pi_{TM}(X(t))$ and $\pi_{TM}(X)$ in a normal neighborhood of $\pi_{TM}(X)$. Then the Sasaki metric g^S on TM is defined by the formula

$$g^S(Z, W) = g((\pi_{TM})_*(Z), (\pi_{TM})_*(W)) + g(K(Z), K(W))$$

$$\text{for } Z, W \in T_X TM, \, X \in TM.$$

For a fixed positive number λ, the total space of tangent sphere bundle $T^\lambda M$ over M is defined to be the set $\{X \in TM; g(X, X) = \lambda^2\}$. We also denote the induced metric on $T^\lambda M$ by g^S.

The bundle of oriented orthonormal frames $SO(M)$ is a principal fiber bundle over the base manifold M with structure group $SO(3)$. We denote this bundle simply by P. On the other hand, the set of all oriented orthonormal frames $SO(M)$ can be regarded as the total space of a circle bundle over the base manifold $T^\lambda M$. Denoting this bundle by Q, the bundle projection $\pi_Q : Q \to T^\lambda M$ is defined by

$$\pi_Q(u) = \lambda X_3 \qquad \text{for } u = (X_1, X_2, X_3) \in SO(M),$$

and the structure group $SO(2)$ acts on the bundle Q on the right as follows:

$$ua = \left(\sum_{k=1}^2 a_1^k X_k, \sum_{l=1}^2 a_2^l X_l, X_3 \right) \qquad \text{for } a = (a_j^i) \in SO(2),$$

where we identify $SO(2)$ with the subgroup of $SO(3)$ given by

$$\left\{ \begin{pmatrix} a_1^1 & a_2^1 & 0 \\ a_1^2 & a_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; (a_j^i) \in SO(2) \right\}.$$

Let $\mathfrak{o}(3)$ be the Lie algebra of $SO(3)$. An inner product $\langle \cdot, \cdot \rangle$ on the vector space $\mathfrak{o}(3)$ is defined by $\langle A, C \rangle = \text{trace}({}^t A \cdot C)$ for $A, C \in \mathfrak{o}(3)$. Let $\mathfrak{o}(2)^\perp$ denote the orthogonal complement of $\mathfrak{o}(2)$ in $\mathfrak{o}(3)$, and $p : \mathfrak{o}(3) \to \mathfrak{o}(2)$ be the orthogonal projection. Then, the composition $\omega_Q := p\omega$ defines a connection form on Q. The relation between the Sasaki metric g^S on $T^\lambda M$ and the metric G on $SO(M)$ is given by

$$G(X, Y) = g^S((\pi_Q)_* X, (\pi_Q)_* Y) + \frac{\lambda^2}{2} \langle \omega_Q(X), \omega_Q(Y) \rangle$$

$$\text{for } X, Y \in T_u SO(M), u \in SO(M).$$

Let N be a Riemannian manifold with metric h. Let $\mathfrak{F}(N)$ denote the set of all differentiable functions on N, and $\mathfrak{X}(N)$ the set of all differentiable vector fields on N, respectively. Suppose further that (N, h) has the structure of a fiber space as bundles $(T^\lambda M, g^S)$, (P, G) or (Q, G). The bundle projection is denoted by π_N. A vector field X on N is called fiber preserving if the local one-parameter group $\{\varphi_t\}$, $-\varepsilon < t < \varepsilon$, of local transformations generated by X maps each fiber of N into another one, where ε is a positive function on N. More precisely, the condition is that

$$\forall x \in N, \quad \forall y \in \pi_N^{-1}(\pi_N(x)), \quad \forall t \ (|t| < \varepsilon), \quad \pi_N(\varphi_t(x)) = \pi_N(\varphi_t(y)).$$

We call Y in $\mathfrak{X}(N)$ vertical if it is tangent to the fiber at each point of N. The vector field X on N is fiber preserving if and only if the commutator product $[X, Y]$ is vertical for any vertical vector field Y. At each point x in N, the horizontal subspace $(\mathscr{H}_N)_x$ of the tangent space $T_x N$ is expressed as the orthogonal complement of the vertical subspace $(\mathscr{V}_N)_x$ that is tangent to the fiber of N. Then the tangent space $T_x N$ is decomposed into a direct sum $T_x N = (\mathscr{H}_N)_x \oplus (\mathscr{V}_N)_x$.

$$T_u P = (\mathscr{H}_P)_u \oplus (\mathscr{V}_P)_u = \{B(\xi)_u; \xi \in \mathbf{R}^3\} \oplus \{A_u^*; A \in \mathfrak{o}(3)\},$$

$$T_u Q = (\mathscr{H}_Q)_u \oplus (\mathscr{V}_Q)_u = \{B(\xi)_u + A_u^*; \xi \in \mathbf{R}^3, A \in \mathfrak{o}(2)^\perp\} \oplus \{A_u^*; A \in \mathfrak{o}(2)\},$$

$$T_{X_3} T^\lambda M = (\mathscr{H}_{T^\lambda M})_{X_3} \oplus (\mathscr{V}_{T^\lambda M})_{X_3}$$

$$= \{(\pi_Q)_*(B(\xi)_u); \xi \in \mathbf{R}^3\} \oplus \{(\pi_Q)_*(A_u^*); A \in \mathfrak{o}(2)^\perp\},$$

where $B(\xi)$ denotes the standard horizontal vector field corresponding to $\xi \in \mathbf{R}^3$, and A^* the fundamental vector field corresponding to $A \in \mathfrak{o}(3)$. Aspects of these decompositions are studied in [4] and [5]. Given a vector field X on the base space of the bundle N, there exists a unique N-horizontal vector field X^{H_N} on N such that $(\pi_N)_*(X^{H_N}) = X$, which is called the horizontal lift of X to N. One then has

$$[A^*, X^{H_P}] = 0 \qquad \text{for any } A \in \mathfrak{o}(3), \tag{2.1}$$

likewise,

$$[A^*, X^{H_Q}] = 0 \qquad \text{for any } A \in \mathfrak{o}(2). \tag{2.2}$$

Each projection π_N is a Riemannian submersion. Especially, we shall use the following formula. Let ∇^S and D denote the Riemannian connections of $(T^\lambda M, g^S)$ and $(SO(M), G)$, respectively. Then we have

$$G(D_{X^{H_Q}} Y^{H_Q}, Z^{H_Q}) = g^S(\nabla^S_X Y, Z) \qquad \text{for } X, Y, Z \in \mathfrak{X}(T^\lambda M). \tag{2.3}$$

The following formulas are also frequently used throughout this paper. Let Ω denote the curvature form of V, then

$$\Omega(X^{H_P}, Y^{H_P}) = -\frac{1}{2}\omega([X^{H_P}, Y^{H_P}]) \qquad \text{for } X, Y \in \mathfrak{X}(M).$$

For any $A, C \in \mathfrak{o}(3)$ and $\xi, \eta, \zeta \in \mathbf{R}^3$, we have

$$G([B(\xi), B(\eta)], A^*) = -\lambda^2 \langle \Omega(B(\xi), B(\eta)), A \rangle, \tag{2.4}$$

$$[A^*, B(\xi)] = B(A\xi), \tag{2.5}$$

$$[A^*, C^*] = [A, C]^*, \tag{2.6}$$

$$G(D_{B(\xi)}B(\eta), B(\zeta)) = 0, \tag{2.7}$$

$$G(D_{B(\xi)}A^*, C^*) = 0, \tag{2.8}$$

$$G(D_{B(\xi)}A^*, B(\eta)) = \frac{\lambda^2}{2} \langle \Omega(B(\xi), B(\eta)), A \rangle, \tag{2.9}$$

$$G(D_{A^*}B(\xi), C^*) = 0, \tag{2.10}$$

$$D_{A^*}C^* = \frac{1}{2}[A, C]^*. \tag{2.11}$$

From formulas (2.6), (2.8), (2.9), and (2.11), we can easily confirm the following fact.

FACT 1. *The fundamental vector fields are infinitesimal isometries on* $(SO(M), G)$.

These formulas and Fact 1 are based on [6] and [8].

3.   The lifts of infinitesimal isometries

In this section, we define the lift $Z^{L_\varrho} \in \mathfrak{X}(Q)$ for $Z \in \mathfrak{i}(T^\lambda M, g^S)$ and prove that Z^{L_ϱ} is in $\mathfrak{i}(Q, G)$. This lift gives the homomorphism of $\mathfrak{i}(T^\lambda M, g^S)$ into $\mathfrak{i}(SO(M), G)$ stated in Theorem 1. We first define $e_i \in \mathbf{R}^3$ and $A_i \in \mathfrak{o}(3)$ for $i = 1, 2, 3$.

$$e_1 = {}^t(1,0,0), \qquad e_2 = {}^t(0,1,0), \qquad e_3 = {}^t(0,0,1),$$

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The system $\left\{ \frac{1}{\sqrt{2}} A_1, \frac{1}{\sqrt{2}} A_2 \right\}$ is an orthonormal basis of $\mathfrak{o}(2)^\perp$ and $\frac{1}{\sqrt{2}} A_3$ is a normal vector of $\mathfrak{o}(2)$. For the e_i and A_i, the following formulas will be frequently used in the argument below.

$$[A_1, A_2] = A_3, \quad [A_2, A_3] = A_1, \quad [A_3, A_1] = A_2; \quad A_1 e_1 = -e_3, \quad A_1 e_2 = 0,$$

$$A_1 e_3 = e_1, \quad A_2 e_1 = 0, \quad A_2 e_2 = -e_3, \quad A_2 e_3 = e_2,$$

$$A_3 e_1 = e_2, \quad A_3 e_2 = -e_1, \quad A_3 e_3 = 0.$$

Here, we note that the system $\left\{ B(e_1), B(e_2), B(e_3), \frac{1}{\lambda} A_1^*, \frac{1}{\lambda} A_2^*, \frac{1}{\lambda} A_3^* \right\}$ defines an orthonormal basis for each tangent space of $SO(M)$.

Given an infinitesimal isometry Z of $(T^\lambda M, g^S)$, we define the lift $Z^{L_\varrho} \in \mathfrak{X}(Q)$ by

$$Z^{L_\varrho} = Z^{H_\varrho} + \frac{1}{\lambda^2} G(D_{A_1^*} Z^{H_\varrho}, A_2^*) A_3^*. \tag{3.1}$$

From Theorem 1.1 in [3] and its proof, we know the following facts for the lift Z^{L_ϱ}.

FACT 2. *For an infinitesimal isometry X on $(T^\lambda M, g^S)$, we have that*
(i)   *The values of the Lie derivatives*

$$(L_{X^{L_\varrho}} G)(B(e_i), B(e_j)), \qquad (L_{X^{L_\varrho}} G)(A_i^*, A_j^*), \qquad (L_{X^{L_\varrho}} G)(B(e_i), A_k^*)$$

all vanish for $1 \le i, j \le 3$ and $1 \le k \le 2$,

(ii) *We denote the lift X^{L_Q} by $\Psi(X)$. Then the mapping $\Psi : \mathfrak{i}(T^\lambda M, g^S)$ $\to \mathfrak{X}(SO(M))$ gives a homomorphism from the Lie algebra of fiber preserving infinitesimal isometries of $(T^\lambda M, g^S)$ to that of $(SO(M), G)$.*

If Z is fiber preserving, we know that Z^{L_Q} is in $\mathfrak{i}(SO(M), G)$ from (ii) of the above facts. Throughout this section and Section 4, we assume that Z does not preserve a fiber of $T^\lambda M$. By (i) of the above facts, to prove Z^{L_Q} is in $\mathfrak{i}(SO(M), G)$, it suffices to show that the equality

$$(L_{Z^{L_Q}} G)(W, A_3^*) = 0 \tag{3.2}$$

holds for any P-horizontal vector field W.

PROPOSITION 1. *The lift Z^{L_Q} preserves the fibers of Q, that is, $[Z^{L_Q}, A_3^*] = 0$.*

PROOF. Since A_3^* is in $\mathfrak{i}(SO(M), G)$, and $\nabla^S Z$ is skew-symmetric with respect to g^S, we have

$$[A_3^*, Z^{L_Q}] = \frac{1}{\lambda^2} \{ A_3^* G(D_{A_1^*} Z^{H_Q}, A_2^*) \} A_3^*$$

$$= \frac{1}{\lambda^2} \{ G(D_{[A_3^*, A_1^*]} Z^{H_Q} + D_{A_1^*}[A_3^*, Z^{H_Q}], A_2^*) + G(D_{A_1^*} Z^{H_Q}, [A_3^*, A_2^*]) \} A_3^*$$

$$= \frac{1}{\lambda^2} \{ G(D_{A_2^*} Z^{H_Q}, A_2^*) - G(D_{A_1^*} Z^{H_Q}, A_1^*) \} A_3^* = 0,$$

where the formulas (3.1), (2.2), (2.6), and (2.3) were used in turn. □

Set $W_1 = [A_1^*, Z^{L_Q}]$ and $W_2 = [A_2^*, Z^{L_Q}]$. The pair of vector fields $\{W_1, W_2\}$ plays a vital role in the proofs of the theorems. From the Jacobi identity and Proposition 1, we have

$$[A_1^*, W_2] = [A_2^*, W_1]. \tag{3.3}$$

LEMMA 1. *The vector fields W_1 and W_2 are P-horizontal vector fields.*

PROOF. Let i, j be 1 or 2. Since A_i^* is in $\mathfrak{i}(SO(M), G)$, we have

$$G(W_i, A_j^*) = A_i^* G(Z^{L_Q}, A_j^*) - G(Z^{L_Q}, [A_i^*, A_j^*])$$

$$= G(D_{A_i^*} Z^{H_Q}, A_j^*) + G(Z^{H_Q}, D_{A_i^*} A_j^*) - G(D_{A_i^*} Z^{H_Q}, A_j^*) = 0$$

by using (3.1), (2.6), and (2.11) in turn. On the other hand, we have

$$G(W_i, A_3^*) = -Z^{L_Q} G(A_i^*, A_3^*) - G(A_i^*, [A_3^*, Z^{L_Q}]) = 0$$

by (i) in Fact 2 and Proposition 1. Hence, W_i is P-horizontal. □

If the equalities $W_1 = W_2 = 0$ hold on $SO(M)$, then the vector fields

$$[A_i^*, Z^{H_\varrho}] = W_i - \frac{1}{\lambda^2}\{A_i^* G(D_{A_1^*} Z^{H_\varrho}, A_2^*)\}A_3^* - \frac{1}{\lambda^2}G(D_{A_1^*} Z^{H_\varrho}, A_2^*)[A_i^*, A_3^*]$$

are vertical for $i = 1, 2$ on P. For any vertical vector field V on $T^\lambda M$, there exist $f^1, f^2 \in \mathfrak{F}(SO(M))$ such that $V^{H_\varrho} = f^1 A_1^* + f^2 A_2^*$. Then, the vector field

$$[Z, V] = [(\pi_Q)_* Z^{H_\varrho}, (\pi_Q)_* V^{H_\varrho}] = (\pi_Q)_* \sum_{i=1}^{2}\{(Z^{H_\varrho}f^i)A_i^* + f^i[Z^{H_\varrho}, A_i^*]\} \quad (3.4)$$

is also a vertical vector field on $T^\lambda M$. This conclusion contradicts the assumption that Z is not fiber preserving on $T^\lambda M$. Hence there exists an open set of $SO(M)$ where W_i does not vanish for either $i = 1$ or $i = 2$. In fact, we can assume that both W_1 and W_2 do not vanish on the open set, because we have

$$[A_3^*, W_1] = W_2 \qquad \text{and} \qquad [A_3^*, W_2] = -W_1 \qquad (3.5)$$

from the Jacobi identity and Proposition 1. Furthermore, we have the following lemma.

LEMMA 2. *Let U be the non-empty open set of $SO(M)$ defined by*

$$U = \{u \in SO(M); (W_1)_u \neq 0 \text{ and } (W_2)_u \neq 0\}.$$

Then, U is an open dense subset of the fibers $\pi_P^{-1}(\pi_P(U))$.

PROOF. Let i be 1 or 2, and u in U. Using a local expression $A_i^* = f_i^1 V_1^{H_\varrho} + f_i^2 V_2^{H_\varrho}$ for some functions $f_i^1, f_i^2 \in \mathfrak{F}(SO(M))$ and vertical vector fields $V_1, V_2 \in \mathfrak{X}(T^\lambda M)$, the nonzero P-horizontal component of $(W_i)_u$ is equal to that of $\{f_i^1[V_1, Z]^{H_\varrho} + f_i^2[V_2, Z]^{H_\varrho}\}_u$. Therefore, the horizontal component of either $[V_1, Z]_{\pi_Q(u)}$ or $[V_2, Z]_{\pi_Q(u)}$ does not vanish. This means that the local one-parameter group $\{\varphi_t\}$, $-\varepsilon < t < \varepsilon$, of local transformations generated by Z does not preserve the fiber $\pi_{T^\lambda M}^{-1}(\pi_P(u))$, that is,

$$\exists X, \exists Y \in \pi_{T^\lambda M}^{-1}(\pi_P(u)), \quad 0 < \exists\varepsilon' < \varepsilon, \quad -\varepsilon' < \forall t < \varepsilon',$$

$$\pi_{T^\lambda M}(\varphi_t(X)) \neq \pi_{T^\lambda M}(\varphi_t(Y)). \qquad (3.6)$$

Suppose that $W_i = 0$ holds on an open set O of $\pi_P^{-1}(\pi_P(U))$ and u is in $U \cap \pi_P^{-1}(\pi_P(O))$, where we can assume that $i = 1, 2$ from (3.5). Then, by the formula (3.4), we know that $[Z, V]$ is vertical on $\pi_Q(O)$ for any vertical vector field V on $T^\lambda M$. This implies that the local transformations $\{\varphi_t\}$ $(0 < \exists\varepsilon'' < \varepsilon', |t| < \varepsilon'')$ preserve the fibers of $\pi_Q(O)$. However, since each fiber of $T^\lambda M$ is a totally geodesic submanifold of $SO(M)$, each isometry φ_t maps the

whole fiber $\pi_{T^\lambda M}^{-1}(\pi_P(u))$ to a fiber, hence we have that

$$\{\varphi_t(X), \varphi_t(Y)\} \subset \varphi_t(\pi_{T^\lambda M}^{-1}(\pi_P(u))) = \pi_{T^\lambda M}^{-1}(\pi_{T^\lambda M}(\varphi_t(\pi_Q(u)))).$$

This contradicts (3.6), which comes from the open set O defined beneath the formula (3.6). $\qquad\square$

LEMMA 3. (i) $D_{A^*}[A^*, Z^{L\varrho}] + D_{[A^*, Z^{L\varrho}]}A^* = 0$ for all $A \in \mathfrak{o}(3)$.
(ii) $D_{A_1^*}W_2 + D_{W_1}A_2^* = 0$, $D_{A_2^*}W_1 + D_{W_2}A_1^* = 0$.

PROOF. (i) When A is in $\mathfrak{o}(2)$, the formula of (i) is trivial from Proposition 1. Let A be in $\mathfrak{o}(2)^\perp$ and $Y \in \mathfrak{X}(SO(M))$ an arbitrary P-horizontal lift. Then we have

$$
\begin{aligned}
G(D_{A^*}[A^*, Z^{L\varrho}], Y) &= A^* G(D_{A^*}Z^{L\varrho}, Y) && \text{(by Fact 1 and (2.1))} \\
&= -A^* G(D_Y Z^{L\varrho}, A^*) && \text{(by (i) in Fact 2)} \\
&= -G(D_Y[A^*, Z^{L\varrho}], A^*) && \text{(by Fact 1 and (2.1))} \\
&= G([A^*, Z^{L\varrho}], D_Y A^*) && \text{(by Lemma 1)} \\
&= -G(Y, D_{[A^*, Z^{L\varrho}]}A^*) && \text{(by Fact 1).}
\end{aligned}
$$

From this formula, we obtain the formula (i), because both $D_{A^*}[A^*, Z^{L\varrho}]$ and $D_{[A^*, Z^{L\varrho}]}A^*$ are P-horizontal by virtue of (2.8), (2.10), and Lemma 1.

(ii) Substituting $A^* = A_1^* + A_2^*$ in the formula of (i), we have

$$D_{A_1^*}W_2 + D_{A_2^*}W_1 + D_{W_1}A_2^* + D_{W_2}A_1^* = 0. \tag{3.7}$$

From (3.3) we also have

$$D_{A_1^*}W_2 - D_{W_2}A_1^* - D_{A_2^*}W_1 + D_{W_1}A_2^* = 0. \tag{3.8}$$

Adding each side of the formulas (3.7) and (3.8), we obtain

$$2(D_{A_1^*}W_2 + D_{W_1}A_2^*) = 0,$$

which implies the first formula of (ii). Then the second formula of (ii) is immediate. $\qquad\square$

LEMMA 4. Set $G_{ij} = G(W_i, W_j)$ for $1 \le i, j \le 2$. Then, the functions G_{ij} on $SO(M)$ satisfy the following equations. (i) $A_1^* G_{11} = 0$, (ii) $A_2^* G_{11} = -2A_1^* G_{12}$, (iii) $A_3^* G_{11} = 2G_{12}$, (iv) $A_1^* G_{22} = -2A_2^* G_{12}$, (v) $A_2^* G_{22} = 0$, (vi) $A_3^* G_{22} = -2G_{12}$, (vii) $A_3^* G_{12} = G_{22} - G_{11}$.

PROOF. We prove this lemma in the order (i), (v), (iii), (vii), (ii). The formulas of (iv) and (vi) are obtained by the same way as (ii) and (iii), respectively, so we omit the proof for these two formulas.

(i), (v)   Let i be 1 or 2.   From (i) of Lemma 3, we have

$$A_i^* G_{ii} = 2G(D_{A_i^*} W_i, W_i) = -2G(D_{W_i} A_i^*, W_i).$$

The right hand side of the above formula equals zero because DA_i^* is skew-symmetric with respect to the metric G by Fact 1.

(iii)   Since A_3^* is in $\mathfrak{i}(SO(M), G)$, by using the Jacobi identity and Proposition 1, we have

$$A_3^* G_{11} = 2G([A_3^*, W_1], W_1) = 2G([[A_3^*, A_1^*], Z^{L_Q}] + [A_1^*, [A_3^*, Z^{L_Q}]], W_1)$$

$$= 2G([A_2^*, Z^{L_Q}], W_1) = 2G_{21}.$$

(vii)   By a similar calculation to the above, we have

$$A_3^* G_{12} = G([A_3^*, W_1], W_2) + G(W_1, [A_3^*, W_2]) = G_{22} - G_{11}.$$

(ii)   Using the above results (i) and (iii), we can derive the left hand side of the formula (ii) from the right hand side as follows:

$$2A_1^* G_{12} = A_1^* A_3^* G_{11} = [A_1^*, A_3^*] G_{11} + A_3^* A_1^* G_{11} = -A_2^* G_{11}. \qquad \square$$

LEMMA 5.   *Set $W_3 = -[A_1^*, W_1]$, then the system $\{(W_1)_u, (W_2)_u, (W_3)_u\}$ forms an orthogonal basis of the P-horizontal space $(\mathscr{H}_P)_u$ at each point u in an open dense subset U' of the set U defined in Lemma 2.*

PROOF.   This lemma is proved by solving the equations given in Lemma 4 on each fixed fiber of the bundle P.   Let z be an arbitrary point in M, and set $S = \pi_Q(\pi_P^{-1}(z))$, which is a sphere of radius λ in the tangent space $T_z M$. To solve the equations on $\pi_P^{-1}(z)$, we choose arbitrary u_0 in $\pi_P^{-1}(z)$.   Let (x^1, x^2, x^3) be the canonical coordinate system with respect to the frame $u_0 \in SO(M)$.   Then, S is expressed as

$$\{(x^1, x^2, x^3) \in \mathbf{R}^3; (x^1)^2 + (x^2)^2 + (x^3)^2 = \lambda^2\}. \tag{3.9}$$

In this setting, we define a local coordinate system (x, y), $0 < x < \pi$, $0 < y < 2\pi$ for S satisfying

$$(x^1, x^2, x^3) = (\lambda \sin x \cos y, \lambda \sin x \sin y, \lambda \cos x). \tag{3.10}$$

There exists a local section $\sigma \subset \pi_Q^{-1}(S)$ of Q such that

$$(\pi_Q)_*((A_1^*)_u) = \left(\frac{\partial}{\partial x}\right)_{\pi_Q(u)} \quad \text{and} \quad (\pi_Q)_*((A_2^*)_u) = \left(\frac{1}{\sin x} \frac{\partial}{\partial y}\right)_{\pi_Q(u)} \quad \text{for } u \in \sigma.$$

Let θ denote a coordinate for the integral curves of A_3^* such that $d/d\theta = A_3^*$. Then, (x, y, θ) provides a local coordinate system for $\pi_Q^{-1}(S)$ on which the section σ is given by $(x, y, 0)$.   Then, the equations in Lemma 4 are written as

the following system of partial differential equations.

$$\begin{cases} \dfrac{\partial G_{11}}{\partial \theta} = 2G_{12}, \quad \dfrac{\partial G_{22}}{\partial \theta} = -2G_{12}, \quad \dfrac{\partial G_{12}}{\partial \theta} = G_{22} - G_{11}. \quad (3.11) \\[2mm] \dfrac{\partial G_{11}}{\partial x} = 0, \quad \dfrac{1}{\sin x}\dfrac{\partial G_{22}}{\partial y} = 0, \quad \dfrac{1}{\sin x}\dfrac{\partial G_{11}}{\partial y} + 2\dfrac{\partial G_{12}}{\partial x} = 0, \\[2mm] \dfrac{\partial G_{22}}{\partial x} + \dfrac{2}{\sin x}\dfrac{\partial G_{12}}{\partial y} = 0. \quad (3.12) \end{cases}$$

By the equations (3.11), the functions G_{11}, G_{22}, and G_{12} are of the forms

$$G_{11} = a\sin(2\theta + \gamma) + b, \qquad G_{22} = -a\sin(2\theta + \gamma) + b,$$

$$G_{12} = a\cos(2\theta + \gamma), \quad (3.13)$$

where a, b, and γ are functions on S. By substituting (3.13) in (3.12), we obtain that

$$\sin(2\theta + \gamma)\frac{\partial a}{\partial x} + a\cos(2\theta + \gamma)\frac{\partial \gamma}{\partial x} + \frac{\partial b}{\partial x} = 0, \quad (3.14)$$

$$-\frac{\sin(2\theta + \gamma)}{\sin x}\frac{\partial a}{\partial y} - \frac{a\cos(2\theta + \gamma)}{\sin x}\frac{\partial \gamma}{\partial y} + \frac{1}{\sin x}\frac{\partial b}{\partial y} = 0, \quad (3.15)$$

$$\frac{\sin(2\theta + \gamma)}{\sin x}\frac{\partial a}{\partial y} + \frac{a\cos(2\theta + \gamma)}{\sin x}\frac{\partial \gamma}{\partial y} + \frac{1}{\sin x}\frac{\partial b}{\partial y}$$

$$+ 2\cos(2\theta + \gamma)\frac{\partial a}{\partial x} - 2a\sin(2\theta + \gamma)\frac{\partial \gamma}{\partial x} = 0,$$

$$-\sin(2\theta + \gamma)\frac{\partial a}{\partial x} - a\cos(2\theta + \gamma)\frac{\partial \gamma}{\partial x} + \frac{\partial b}{\partial x} + \frac{2\cos(2\theta + \gamma)}{\sin x}\frac{\partial a}{\partial y}$$

$$- \frac{2a\sin(2\theta + \gamma)}{\sin x}\frac{\partial \gamma}{\partial y} = 0.$$

Eliminating $\partial b/\partial x$ and $\partial b/\partial y$ from the above four equations, we have that

$$\frac{\partial a}{\partial x} = -\frac{a}{\sin x}\frac{\partial \gamma}{\partial y}, \qquad \frac{\partial a}{\partial y} = a\sin x\frac{\partial \gamma}{\partial x}. \quad (3.16)$$

Then, we obtain

$$\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} = -\frac{1}{\sin x}\frac{\partial a}{\partial x}\frac{\partial \gamma}{\partial y} + \sin x\frac{\partial a}{\partial y}\frac{\partial \gamma}{\partial x} + \frac{a\cos x}{\sin^2 x}\frac{\partial \gamma}{\partial y}$$

$$+ a\left(\sin x - \frac{1}{\sin x}\right)\frac{\partial^2 \gamma}{\partial x\partial y}. \quad (3.17)$$

By differentiating (3.14) and (3.15) with respect to θ, we have that

$$\cos(2\theta + \gamma)\frac{\partial a}{\partial x} - a\sin(2\theta + \gamma)\frac{\partial \gamma}{\partial x} = 0, \qquad -\cos(2\theta + \gamma)\frac{\partial a}{\partial y} + a\sin(2\theta + \gamma)\frac{\partial \gamma}{\partial y} = 0,$$

which implies that

$$a\left(\frac{\partial a}{\partial x}\frac{\partial \gamma}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial \gamma}{\partial x}\right) = 0. \tag{3.18}$$

On the intersection of the set $\{X \in S; a(X) \neq 0\}$ and the great circle $x = \pi/2$ of the sphere S, we have from (3.17) and (3.18) that $\Delta a = 0/\lambda^2$, where Δ denotes the Laplacian on S. Since the coordinate system (x, y) for S is an arbitrary one, we have that $\Delta a = 0$ on the open set $\{X \in S; a(X) \neq 0\}$ of S, which implies that $\Delta a = 0$ on S because of the continuity of the function Δa. Since any harmonic function on a connected compact Riemannian manifold is constant, the function a is constant on S. Then, from (3.14), (3.15), and (3.16), we know that the functions b and γ are also constant on S. Since $G_{11} + G_{22} = 2b$, we have by Lemma 4 that

$$A_2^* G_{11} = A_2^*(2b - G_{22}) = 0, \qquad A_1^* G_{22} = A_1^*(2b - G_{11}) = 0.$$

Together with (i) of Lemma 4, the above formula implies that the function G_{11} is a constant on each fiber of $SO(M)$. Therefore, we also have $2G_{12} = A_3^* G_{11} = 0$ and $G_{22} - G_{11} = A_3^* G_{12} = 0$ by Lemma 4. We rewrite these formulas below for the citation.

$$A_2^* G_{11} = 0, \qquad A_1^* G_{22} = 0, \tag{3.19}$$

$$G_{12} = 0, \tag{3.20}$$

$$G_{11} = G_{22}. \tag{3.21}$$

We know that $W_3 = D_{W_1}A_1^* - D_{A_1^*}W_1$ is P-horizontal from Lemma 1, (2.8), and (2.10). By the Jacobi identity and (3.3), we have

$$\begin{aligned}
[A_1^*, [A_3^*, W_3]] &= [A_1^*, -[A_2^*, W_1] - [A_1^*, W_2]] \\
&= -2[A_1^*, [A_2^*, W_1]] \\
&= -2W_2 + 2[A_2^*, W_3].
\end{aligned}$$

Therefore, we know that if $W_3 = 0$ holds on an open set O of $SO(M)$, it also holds on O that $W_2 = 0$ and $W_1 = 0$ from the above formula and (3.21). Hence, $(W_3)_u \neq 0$ at each point u in an open dense subset U' of U, where U is the open set of $SO(M)$ defined in Lemma 2. The orthogonality of the system $\{W_1, W_2, W_3\}$ at each point in U' is given by (3.20) and the following simple

calculations. We have

$$G(W_1, W_3) = -G(W_1, [A_1^*, W_1]) = -\frac{1}{2} A_1^* G(W_1, W_1) = 0$$

by Fact 1 and (i) in Lemma 4, and we also have

$$
\begin{aligned}
G(W_2, W_3) &= -G(W_2, [A_1^*, W_1]) \\
&= -A_1^* G(W_2, W_1) + G([A_1^*, W_2], W_1) \quad \text{(by Fact 1)} \\
&= G([A_1^*, W_2], W_1) \quad \text{(by (3.20))} \\
&= G([A_2^*, W_1], W_1) \quad \text{(by (3.3))} \\
&= \frac{1}{2} A_2^* G(W_1, W_1) \quad \text{(by Fact 1)} \\
&= 0 \quad \text{(by (3.19))}.
\end{aligned}
$$

These equalities show that W_3 is perpendicular to both W_1 and W_2. $\qquad\square$

The following proposition gives the main part of (i) in Theorem 1. After we classify the infinitesimal isometries in the final section, we know that the lifts are endowed with the homomorphism between $\mathfrak{i}(T^\lambda M, g^S)$ and $\mathfrak{i}(SO(M), G)$.

PROPOSITION 2. *Given an infinitesimal isometry Z of a tangent sphere bundle $(T^\lambda M, g^S)$, the lift $Z^{L\varrho}$ defined by (3.1) is an infinitesimal isometry on the bundle of oriented orthonormal frames $(SO(M), G)$.*

PROOF. Let U be the open set of $SO(M)$ defined in Lemma 2. If the interior $(SO(M) \backslash U)^\circ$ is not the empty set, then we know that the restricted vector field $Z^{L\varrho} | (SO(M) \backslash U)^\circ$ is an infinitesimal isometry from (ii) in Fact 2. Therefore, according to Fact 2 and Lemma 5, it is sufficient to verify that the equality (3.2) holds on U for $W = W_1, W_2,$ and W_3. In fact, we have that

$$(L_{Z^{L\varrho}} G)(W_1, A_3^*)$$

$$= Z^{L\varrho} G(W_1, A_3^*) - G([Z^{L\varrho}, W_1], A_3^*) - G(W_1, [Z^{L\varrho}, A_3^*])$$

$$= -G([Z^{L\varrho}, [A_1^*, Z^{L\varrho}]], [A_1^*, A_2^*]) \quad \text{(by Proposition 1 and Lemma 1)}$$

$$= -A_1^* G([Z^{L\varrho}, [A_1^*, Z^{L\varrho}]], A_2^*) + G([A_1^*, [Z^{L\varrho}, [A_1^*, Z^{L\varrho}]]], A_2^*)$$

$$\quad \text{(by Fact 1)}$$

$$= A_1^* G([A_1^*, Z^{L\varrho}], [Z^{L\varrho}, A_2^*]) \quad \text{(by (i) in Fact 2 and Lemma 1)}$$

$$\quad + G([[A_1^*, Z^{L\varrho}], [A_1^*, Z^{L\varrho}]] + [Z^{L\varrho}, [A_1^*, [A_1^*, Z^{L\varrho}]]], A_2^*)$$

$$\quad \text{(by the Jacobi identity)}$$

$$= G([Z^{L\varrho}, [A_1^*, [A_1^*, Z^{L\varrho}]]], A_2^*) \qquad \text{(by (3.20))}$$

$$= -G([A_1^*, [A_1^*, Z^{L\varrho}]], [Z^{L\varrho}, A_2^*]) \qquad \text{(by (i) in Fact 2 and Lemma 1)}$$

$$= G([A_1^*, Z^{L\varrho}], [A_1^*, [Z^{L\varrho}, A_2^*]]) \qquad \text{(by Fact 1 and (3.20))}$$

$$= G([A_1^*, Z^{L\varrho}], [A_2^*, [Z^{L\varrho}, A_1^*]]) \qquad \text{(by (3.3))}$$

$$= -\frac{1}{2} A_2^* G([A_1^*, Z^{L\varrho}], [A_1^*, Z^{L\varrho}]) = 0 \qquad \text{(by Fact 1 and (3.19))}.$$

By the same calculation as the above, we also have $(L_{Z^{L\varrho}} G)(W_2, A_3^*) = 0$. Next, we have that

$$(L_{Z^{L\varrho}} G)(W_3, A_3^*)$$

$$= Z^{L\varrho} G(W_3, A_3^*) - G([Z^{L\varrho}, W_3], A_3^*) - G(W_3, [Z^{L\varrho}, A_3^*])$$

$$= G([Z^{L\varrho}, [A_1^*, W_1]], A_3^*) \qquad \text{(by Proposition 1 and Lemma 1)}$$

$$= G([[Z^{L\varrho}, A_1^*], W_1], A_3^*) + G([A_1^*, [Z^{L\varrho}, W_1]], A_3^*)$$

$$\text{(by the Jacobi identity)}$$

$$= A_1^* G([Z^{L\varrho}, W_1], A_3^*) - G([Z^{L\varrho}, W_1], [A_1^*, A_3^*]) \qquad \text{(by Fact 1)}$$

$$= G([Z^{L\varrho}, W_1], A_2^*)$$

$$\text{(by } (L_{Z^{L\varrho}} G)(W_1, A_3^*) = 0, \text{ Proposition 1, and (2.6))}$$

$$= -G(W_1, [Z^{L\varrho}, A_2^*]) \qquad \text{(by Lemma 1 and (i) in Fact 2)}$$

$$= G(W_1, W_2) = 0 \qquad \text{(by (3.20))}.$$

Thus, we verified the equality (3.2) on U. Since $L_{Z^{L\varrho}} G$ is a continuous tensor, it vanishes on $SO(M)$ entirely, which completes the proof of Proposition 2. $\square$

4. Curvatures of base spaces

Using the lifts studied in the previous section, we show in this section that (M, g) is a space of constant curvature $1/\lambda^2$, which is the necessary condition for (ii) in Theorem 1. For this purpose, we first study a relation between the lift $Z^{L\varrho}$ and the standard horizontal vector fields.

LEMMA 6. *The norms of the horizontal vector fields W_i and $[A_i^*, W_j]$, $1 \le i, j \le 2$, are constants on each fiber of P.*

PROOF. The functions G_{ij}, $1 \le i, j \le 2$, are constants on each fiber of U from Lemma 4 and the formulas (3.19), (3.20), and (3.21). Therefore we know

that $U = \pi_P^{-1}(\pi_P(U))$ by the continuity of the functions G_{ij}, and that the norms of W_i are constants on each fiber. On the other hand, since the lift Z^{L_Q} is an infinitesimal isometry from Proposition 2, the brackets $[A_i^*, W_j]$ are also infinitesimal isometries on $SO(M)$. For any $A \in \mathfrak{o}(3)$, we have

$$A^*G([A_i^*, W_j], [A_i^*, W_j])$$

$$= 2G([A^*, [A_i^*, W_j]], [A_i^*, W_j])$$

$$= -2[A_i^*, W_j]G(A^*, [A_i^*, W_j]) + 2G(A^*, [[A_i^*, W_j], [A_i^*, W_j]]) = 0.$$

Hence, the norms of $[A_i^*, W_j]$ are also constants on each fiber. $\square$

LEMMA 7. *For the brackets* $[A_i^*, W_j]$, $1 \leq i, j \leq 3$, *we have the following.* (i) $[A_1^*, W_2] = [A_2^*, W_1] = 0$, (ii) $[A_1^*, W_1] = [A_2^*, W_2] = -W_3$, (iii) $[A_1^*, W_3] = W_1$, (iv) $[A_2^*, W_3] = W_2$, (v) $[A_3^*, W_1] = W_2$, (vi) $[A_3^*, W_2] = -W_1$, (vii) $[A_3^*, W_3] = 0$.

PROOF. (i) The formula $[A_1^*, W_2] = [A_2^*, W_1]$ was given by (3.3). Let U be the open set defined in Lemma 2. If the interior of the complement of U is not empty, where W_1 vanishes identically at each point, then the equality $[A_1^*, W_2] = 0$ holds on the set $(SO(M) \setminus U)^\circ$ by virtue of

$$2[A_1^*, W_2] = [A_2^*, W_1] + [A_1^*, W_2]$$

$$= [[A_3^*, A_1^*], W_1] + [A_1^*, [A_3^*, W_1]]$$

$$= [A_3^*, [A_1^*, W_1]].$$

Therefore, it is sufficient to verify that the equality $[A_1^*, W_2] = 0$ holds on U. At each point of U, the vector field $[A_1^*, W_2]$ is perpendicular to both W_1 and W_2, because we have from (3.3) and (3.19) that

$$G([A_1^*, W_2], W_1) = G([A_2^*, W_1], W_1) = \frac{1}{2}A_2^*G(W_1, W_1) = 0,$$

$$G([A_1^*, W_2], W_2) = \frac{1}{2}A_1^*G(W_2, W_2) = 0.$$

By Lemma 5 and Lemma 6, we know that $W_3 \neq 0$ at each point of the open set U. Hence, we can write $[A_1^*, W_2] = \alpha[A_1^*, W_1]$ on U, where $\alpha \in \mathfrak{F}(U)$ is constant on each fiber of P by Lemma 6. Then, we have

$$[A_3^*, [A_1^*, W_2]] = \alpha[A_3^*, [A_1^*, W_1]]. \tag{4.1}$$

The left hand side of the above equality is calculated by the Jacobi identity and (3.5) as follows:

$$[A_3^*, [A_1^*, W_2]] = [[A_3^*, A_1^*], W_2] + [A_1^*, [A_3^*, W_2]] = [A_2^*, W_2] - [A_1^*, W_1]. \tag{4.2}$$

The right hand side of the equality (4.1) is calculated by the Jacobi identity, (3.3), and (3.5) as follows:

$$\alpha[A_3^*, [A_1^*, W_1]] = \alpha([[A_3^*, A_1^*], W_1] + [A_1^*, [A_3^*, W_1]]) = 2\alpha[A_1^*, W_2]. \quad (4.3)$$

We have by (4.2) and (4.3) that

$$[A_2^*, W_2] = 2\alpha[A_1^*, W_2] + [A_1^*, W_1]. \quad (4.4)$$

By (4.2), we further have

$$
\begin{aligned}
[A_3^*, [A_3^*, [A_1^*, W_2]]] &= [A_3^*, [A_2^*, W_2]] - [A_3^*, [A_1^*, W_1]] \\
&= [[A_3^*, A_2^*], W_2] + [A_2^*, [A_3^*, W_2]] \\
&\quad - [[A_3^*, A_1^*], W_1] - [A_1^*, [A_3^*, W_1]] \\
&= -[A_1^*, W_2] - [A_2^*, W_1] - [A_2^*, W_1] - [A_1^*, W_2] \\
&= -4[A_1^*, W_2].
\end{aligned}
$$

On the other hand, by (4.3) and (4.4), we also have

$$
\begin{aligned}
[A_3^*, \alpha[A_3^*, [A_1^*, W_1]]] &= 2\alpha[[A_3^*, A_1^*], W_2] + 2\alpha[A_1^*, [A_3^*, W_2]] \\
&= 2\alpha[A_2^*, W_2] - 2\alpha[A_1^*, W_1] \\
&= 2\alpha(2\alpha[A_1^*, W_2] + [A_1^*, W_1]) - 2\alpha[A_1^*, W_1] \\
&= 4\alpha^2[A_1^*, W_2].
\end{aligned}
$$

Consequently, we obtain the formula $-4[A_1^*, W_2] = 4\alpha^2[A_1^*, W_2]$ from the above formulas, which implies that $[A_1^*, W_2] = 0$.

(ii)   From the Jacobi identity and (i), we have

$$[A_2^*, W_2] - [A_1^*, W_1] = [[A_3^*, A_1^*], W_2] + [A_1^*, [A_3^*, W_2]] = [A_3^*, [A_1^*, W_2]] = 0.$$

Since the definition of W_3 is $W_3 = -[A_1^*, W_1]$, we obtain (ii) by the formula above.

(iii), (iv), (vii)   By using (ii), the Jacobi identity, (i), and (3.5) in turn, we have that

$$
\begin{aligned}
[A_1^*, W_3] &= [A_1^*, -[A_2^*, W_2]] \\
&= -[[A_1^*, A_2^*], W_2] - [A_2^*, [A_1^*, W_2]] = -[A_3^*, W_2] = W_1, \\
[A_2^*, W_3] &= [A_2^*, -[A_1^*, W_1]] \\
&= -[[A_2^*, A_1^*], W_1] - [A_1^*, [A_2^*, W_1]] = [A_3^*, W_1] = W_2,
\end{aligned}
$$

$$[A_3^*, W_3] = [A_3^*, -[A_1^*, W_1]]$$

$$= -[[A_3^*, A_1^*], W_1] - [A_1^*, [A_3^*, W_1]] = -2[A_2^*, W_1] = 0.$$

The formulas (v) and (vi) had already been given by (3.5).   □

LEMMA 8.   *The norms of the P-horizontal vector fields W_1, W_2, and W_3 are the same nonzero-constants on $SO(M)$.*

PROOF.   Since W_1, W_2, and W_3 are infinitesimal isometries on $SO(M)$, we have that

$$W_1 G(W_1, W_1) = 2G([W_1, W_1], W_1) = 0,$$

$$W_2 G(W_1, W_1) = 2G([W_2, W_1], W_1) = -2W_1 G(W_2, W_1) + 2G(W_2, [W_1, W_1]) = 0,$$

$$W_3 G(W_1, W_1) = 2G([W_3, W_1], W_1) = -2W_1 G(W_3, W_1) + 2G(W_3, [W_1, W_1]) = 0,$$

where Lemma 5 was applied.   Therefore, the norm of W_1 is a constant on the open set U defined in Lemma 2. Since the norm of W_1 is a continuous function on $SO(M)$, the domain U must equal $SO(M)$.   By (3.21), the norm of W_2 is the same as that of W_1.   As for the norm of W_3, it also coincides with that of W_1 from the following calculation.

$$G(W_3, W_3) = G([A_1^*, W_1], [A_2^*, W_2])$$

$$= A_1^* G(W_1, [A_2^*, W_2]) - G(W_1, [A_1^*, [A_2^*, W_2]])$$

$$= -A_1^* G(W_1, W_3) - G(W_1, [[A_1^*, A_2^*], W_2] + [A_2^*, [A_1^*, W_2]])$$

$$= -G(W_1, [A_3^*, W_2])$$

$$= G(W_1, W_1),$$

where Lemma 7, Fact 1, the Jacobi identity, and Lemma 5 were applied.
□

Dividing Z by the constant norm $\|W_1\|$, we assume $\|W_i\| = 1$ for $i = 1, 2, 3$ in the remainder of this section.   The following proposition shows that the vector fields W_1, W_2, and W_3 are equivalent to standard horizontal vector fields.

PROPOSITION 3.   *Let (M, g) be a connected, orientable three-dimensional Riemannian manifold of class C^∞ and λ a positive number. If the bundle $(T^\lambda M, g^S)$ admits a non-fiber-preserving infinitesimal isometry Z, then there exists a nonzero constant μ such that*

$$[A_i^*, Z^{L_\varrho}] = \mu B(e_i) \quad and \quad [A_i^*, [A_i^*, Z^{L_\varrho}]] = -\mu B(e_3) \qquad for \ i = 1, 2.$$

PROOF.   By normalizing W_1 appropriately, we shall show that

$$\sum_{i=1}^{3} G(W_i, B(e_i)) = 3$$

on $SO(M)$.   Set $K = \sum_{i=1}^{3} G(W_i, B(e_i))$.   We first show that the function K is constant on each fiber of the bundle P.   We have by (2.5) and Lemma 7 that

$$A_1^* K = G([A_1^*, W_1], B(e_1)) + G(W_1, [A_1^*, B(e_1)]) + G([A_1^*, W_2], B(e_2))$$
$$+ G(W_2, [A_1^*, B(e_2)]) + G([A_1^*, W_3], B(e_3)) + G(W_3, [A_1^*, B(e_3)])$$
$$= -G(W_3, B(e_1)) - G(W_1, B(e_3)) + 0 + 0 + G(W_1, B(e_3)) + G(W_3, B(e_1))$$
$$= 0,$$

$$A_2^* K = G([A_2^*, W_1], B(e_1)) + G(W_1, [A_2^*, B(e_1)]) + G([A_2^*, W_2], B(e_2))$$
$$+ G(W_2, [A_2^*, B(e_2)]) + G([A_2^*, W_3], B(e_3)) + G(W_3, [A_2^*, B(e_3)])$$
$$= 0 + 0 - G(W_3, B(e_2)) - G(W_2, B(e_3)) + G(W_2, B(e_3)) + G(W_3, B(e_2))$$
$$= 0.$$

By using these equalities, we also have that $A_3^* K = [A_1^*, A_2^*]K = 0$.   Hence, K is constant on each fiber of P.

We next fix an arbitrary fiber of $T^\lambda M$.   We denote this fixed fiber by S, which is a sphere of radius λ.   By the fixed point theorem, we can choose u_0 in $\pi_P^{-1}(\pi_{T^\lambda M}(S))$ such that $(W_3)_{u_0} = B(e_3)_{u_0}$ holds, where, if necessary, we adopt $-Z$ in $\mathfrak{i}(T^\lambda M, g^S)$ from the beginning instead of Z.   Let (x^1, x^2, x^3) be the canonical coordinate system with respect to the frame $u_0 \in SO(M)$.   In this setting, we define a local coordinate system (x, y) for S given by (3.9) and (3.10).   Since

$$A_3^* G(W_1, B(e_3)) = G(W_2, B(e_3)), \qquad A_3^* G(W_2, B(e_3)) = -G(W_1, B(e_3)),$$

the functions $G(W_1, B(e_3))$ and $G(W_2, B(e_3))$ can be written as follows using the same local coordinate system (x, y, θ) for the fiber of $SO(M)$ as in the proof of Lemma 5.

$$G(W_1, B(e_3)) = c \sin(\theta + \rho), \qquad G(W_2, B(e_3)) = c \cos(\theta + \rho), \qquad (4.5)$$

where c and ρ are functions on the sphere S.   We have by Fact 1, (2.5), and Lemma 7 that

$$A_1^* G(W_1, B(e_3)) + A_2^* G(W_2, B(e_3))$$
$$= G(W_1, B(e_1)) + G(W_2, B(e_2)) - 2G(W_3, B(e_3)). \qquad (4.6)$$

By substituting (4.5) in the left hand side of the equality (4.6), we have

$$A_1^* G(W_1, B(e_3)) + A_2^* G(W_2, B(e_3))$$

$$= \left(\frac{\partial c}{\partial x} - \frac{c}{\sin x}\frac{\partial \rho}{\partial y}\right)\sin(\theta + \rho) + \left(c\frac{\partial \rho}{\partial x} + \frac{1}{\sin x}\frac{\partial c}{\partial y}\right)\cos(\theta + \rho).$$

However, the right hand side of the equality (4.6) is independent of the value of θ owing to the calculation that

$$A_3^*\{G(W_1, B(e_1)) + G(W_2, B(e_2)) - 2G(W_3, B(e_3))\}$$

$$= G(W_2, B(e_1)) + G(W_1, B(e_2)) - G(W_1, B(e_2)) - G(W_2, B(e_1))$$

$$\quad - 2G(0, B(e_3)) - 2G(W_3, 0)$$

$$= 0,$$

where Fact 1, (2.5), and Lemma 7 have been used. Hence, from the orthogonality of the trigonometric functions, we know that both sides of the equality (4.6) vanish. Hence, we have $K = 3G(W_3, B(e_3))$. Since the restricted function $K|S$ is a constant function, we know the value at the point $\pi_Q(u_0)$, that is $K|S = 3G(W_3, B(e_3))_{u_0} = 3G(B(e_3), B(e_3))_{u_0} = 3$. Therefore $\sum_{i=1}^{3} G(W_i, B(e_i)) = 3$. This formula implies $W_i = B(e_i)$ for $i = 1, 2, 3$. $\square$

Now we prove the necessary condition for (ii) in Theorem 1 by calculating the curvature of the base space (M, g). Applying the Jacobi identity, Fact 1, Proposition 2, Proposition 1, and Proposition 3 in turn, we have

$$G([Z^{L_\varrho}, [A_2^*, W_1]], A_3^*)$$

$$= G([[Z^{L_\varrho}, A_2^*], W_1], A_3^*) + G([A_2^*, [Z^{L_\varrho}, W_1]], A_3^*)$$

$$= G([-W_2, W_1], A_3^*) + A_2^* G([Z^{L_\varrho}, W_1], A_3^*) - G([Z^{L_\varrho}, W_1], [A_2^*, A_3^*])$$

$$= G([W_1, W_2], A_3^*) + A_2^*\{Z^{L_\varrho} G(W_1, W_3) - G(W_1, [Z^{L_\varrho}, A_3^*])\}$$

$$\quad - G([Z^{L_\varrho}, W_1], A_1^*)$$

$$= G([W_1, W_2], A_3^*) + A_2^*\{Z^{L_\varrho}0 - G(W_1, 0)\} - Z^{L_\varrho} G(W_1, A_1^*)$$

$$\quad + G(W_1, [Z^{L_\varrho}, A_1^*])$$

$$= G([W_1, W_2], A_3^*) - Z^{L_\varrho}0 + G(W_1, -W_1)$$

$$= G([B(e_1), B(e_2)], A_3^*) - G(B(e_1), B(e_1))$$

$$= G([B(e_1), B(e_2)], A_3^*) - 1.$$

From the above formula, (i) of Lemma 7, and (2.4), we have

$$\langle \Omega(B(e_1), B(e_2)), A_3 \rangle = -\frac{1}{\lambda^2}. \tag{4.7}$$

Let X_1 and X_2 be any orthonormal pair of tangent vectors at a point of M, and $X_3 = X_1 \times X_2$ the vector product as the part of a frame $u \in SO(M)$ given by $u = (X_1, X_2, X_3)$. Let R denote the Riemannian curvature tensor of (M, g). The sectional curvature for the tangent plane spanned by X_1 and X_2 is calculated by (4.7):

$$g(R(X_1, X_2)X_2, X_1) = (2\Omega(B(e_1)_u, B(e_2)_u)e_2, e_1)$$

$$= -\langle \Omega(B(e_1), B(e_2)), A_3 \rangle = \frac{1}{\lambda^2},$$

where $(\cdot, \cdot)$ denotes the canonical inner product on the three-dimensional Euclidian space. Consequently, we assert that (M, g) is a space of constant curvature $1/\lambda^2$. We have proved the necessary condition for (ii) in Theorem 1.

The proof of the converse part of (ii) in Theorem 1 is based on a fact in [8], which gives a condition for the standard horizontal vector fields to be infinitesimal isometries of the frame bundle. In fact, when (M, g) is a space of constant curvature $1/\lambda^2$, we can see that the Lie derivative $L_{B(e_3)}G$ vanishes by (2.4), (2.5), and (2.7) as follows:

$$(L_{B(e_3)}G)(B(e_i), B(e_j)) = 0, \qquad (L_{B(e_3)}G)(A_i^*, A_j^*) = 0,$$

$$(L_{B(e_3)}G)(B(e_i), A_j^*) = 0,$$

for $1 \le i, j \le 3$. Hence, $B(e_3)$ is an element of $\mathfrak{i}(SO(M), G)$ in this case, and hence the geodesic flow $\mathscr{G} = (\pi_Q)_*(\lambda B(e_3))$ is in $\mathfrak{i}(T^\lambda M, g^S)$ because the mapping π_Q is a Riemannian submersion. As the geodesic flow is not fiber preserving, we have completed the proof of (ii) in Theorem 1.

5. The classification

The Lie algebra $\mathfrak{i}(T^\lambda M, g^S)$ is classified in this section; as a result we know that the lifts are endowed with the homomorphism between $\mathfrak{i}(T^\lambda M, g^S)$ and $\mathfrak{i}(SO(M), G)$. We first review a fact for parallel two-forms.

FACT 3. *Let (N, h) be a three-dimensional Riemannian manifold which has a point where no sectional curvature vanishes. Then, the Lie algebra of parallel two-forms $\mathfrak{D}^2(N)_0$ with respect to the Riemannian connection ∇^h of (N, h) is trivial.*

PROOF.   The above fact is proved for spaces of nonzero constant curvature of general dimension greater than two (cf. [8-II, Lemma 15]).   Here, we review this fact in a basic manner when dim $N = 3$.   We consider $\phi \in \mathfrak{D}^2(M)$ as a skew-symmetric tensor field $\tilde{\phi}$ of type $(1,1)$ by the formula $h(\tilde{\phi}(X), Y) = \phi(X, Y)$, $X, Y \in \mathfrak{X}(N)$.   Suppose that there exists nonzero $\phi \in \mathfrak{D}^2(M)_0$.   Then, there exists a parallel axis vector field A with respect to the rotation $\exp t\tilde{\phi}$, $t \in \mathbf{R}$, for the tangent spaces such that $\tilde{\phi}(X) = A \times X$.   Since $\nabla^h A$ is identically zero, the sectional curvature for any tangent plane that contains A at the point of N is equal to zero, which implies the contrapositive statement of Fact 3.

$\square$

Now we classify the Lie algebra $\mathfrak{i}(T^\lambda M, g^S)$ to prove Theorem 2.   Given an infinitesimal isometry Z of $(T^\lambda M, g^S)$, set $\mu = \|[A_1^*, Z^{L\varrho}]\|$.

If $\mu = 0$, then Z is fiber preserving.   From the study in [2], there exist $X \in \mathfrak{i}(M, g)$ and $\phi \in \mathfrak{D}^2(M)_0$ such that Z is uniquely decomposed as

$$Z = X^L + \phi^L. \tag{5.1}$$

The brackets relations

$$[X^L, Y^L] = [X, Y]^L, \qquad [\phi^L, \psi^L] = -[\phi, \psi]^L, \qquad [X^L, \phi^L] = -[\nabla X, \phi]^L \tag{5.2}$$

for $X, Y \in \mathfrak{i}(M, g)$ and $\phi, \psi \in \mathfrak{D}^2(M)$ are given in [3].

On the other hand, if $\mu \neq 0$, then Z is not fiber preserving.   From (ii) of Theorem 1, we know that the base space (M, g) is a space of constant curvature $1/\lambda^2$.   Set $Z_\pm = Z \pm \mu\lambda^{-1}\mathscr{G}$.   Since the geodesic flow $\mathscr{G}$ is an infinitesimal isometry in this case, the vector fields $Z_\pm$ are also in $\mathfrak{i}(T^\lambda M, g^S)$. Note that $\mathscr{G}^{L\varrho} = \lambda B(e_3)$.   Then, we have

$$[A_i^*, Z_\pm^{L\varrho}] = [A_i^*, Z^{L\varrho} \pm \mu B(e_3)] = [A_i^*, Z^{L\varrho}] \pm \mu B(e_i) \qquad \text{for } i = 1, 2.$$

Applying Proposition 3 to the formula above, we know that either $[A_i^*, Z_+^{L\varrho}]$ or $[A_i^*, Z_-^{L\varrho}]$ vanishes, which implies that either Z_+ or Z_- preserves the fibers of $T^\lambda M$ (cf. Lemma 2).   Hence, there exist $X \in \mathfrak{i}(M, g)$ and $\phi \in \mathfrak{D}^2(M)_0$ such that either Z_+ or Z_- is written as $X^L + \phi^L$, where we know $\phi = 0$ from Fact 3.   Consequently, we obtain the decomposition, either

$$Z = X^L - \mu\lambda^{-1}\mathscr{G} \qquad \text{or} \qquad Z = X^L + \mu\lambda^{-1}\mathscr{G}. \tag{5.3}$$

Since the local transformations generated by $X \in \mathfrak{i}(M, g)$ are local isometries, each of them maps a geodesic to a geodesic, so the local transformations generated by X and those of the geodesic flow commute.   Hence, we have

$$[X^L, \mathscr{G}] = 0. \tag{5.4}$$

Since $\Psi(X^L)$ is a fiber preserving infinitesimal isometry on P, the horizontal subspaces $\mathscr{H}_P$ and the vertical subspaces $\mathscr{V}_P$ are invariant by the differentials of the local isometries generated by $\Psi(X^L)$. Therefore, we know that $[\Psi(X^L), B(e_3)]$ is P-horizontal. At the same time we have

$$(\pi_Q)_*([\Psi(X^L), B(e_3)]) = (\pi_Q)_*([(X^L)^{H_Q}, (\lambda^{-1}\mathscr{G})^{H_Q}]) = \lambda^{-1}[X^L, \mathscr{G}] = 0,$$

which implies that $[\Psi(X^L), B(e_3)]$ is Q-vertical. Hence, $[\Psi(X^L), B(e_3)] = 0$ holds on $SO(M)$. Consequently, we have that $[\Psi(X^L), \Psi(\mathscr{G})] = [\Psi(X^L), \lambda B(e_3)] = 0$, and hence we obtain the brackets relation

$$[\Psi(X^L), \Psi(\mathscr{G})] = \Psi([X^L, \mathscr{G}]). \tag{5.5}$$

With the assertion (ii) in Theorem 1, formulas (5.1), (5.2), (5.3), and (5.4) give the classification for the Lie algebras of infinitesimal isometries on the tangent sphere bundles, which completes the proof of Theorem 2.

According to this classification, Proposition 2 and formula (5.5) imply that the mapping Ψ defined in Fact 2 gives a homomorphism of $\mathfrak{i}(T^\lambda M, g^S)$ into $\mathfrak{i}(SO(M), G)$. The proof of (i) in Theorem 1 is also completed.

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