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An application of capacitary functions to an inverse inclusion problem[1]

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Abstract. An efficient application of capacitary functions for compact subsets of the Royden harmonic boundary to an inverse inclusion problem concerning spaces of Dirichlet finite and mean bounded harmonic functions in the classification theory of Riemann surfaces is given.

1. Introduction

As usual we denote by $H(R)$ the linear space of harmonic functions u on an open Riemann surface R and by $HD(R)$ the subspace of $H(R)$ consisting of $u \in H(R)$ with finite Dirichlet integral $D(u; R)$ of u taken over R (cf. e.g. [1]):

$$D(u; R) := \int_R du \wedge *du < +\infty. \tag{1}$$

An end W of R is a subregion (i.e. open and connected subset) of R such that, firstly, $R \backslash \overline{W}$ is a regular subregion of R and, secondly, the relative boundary ∂W of W coincides with that $\partial(R \backslash \overline{W})$ of $R \backslash \overline{W}$. Then the relative class $H(W; \partial W)$ is the linear subspace of $H(W)$ consisting of $u \in H(W) \cap C(R)$ with $u \mid R \backslash W = 0$ and the relative class $HD(W; \partial W)$ is the linear subspace of $H(W; \partial W)$ consisting of $u \in H(W; \partial W)$ with finite Dirichlet integral $D(u; R) = D(u; W)$. Then the mutual Dirichlet integral $D(u, v; R)$ of u and v in $HD(W; \partial W)$ taken over R (and in reality over W) is given by

$$D(u, v; R) := \int_R du \wedge *dv \tag{2}$$

and $HD(W; \partial W)$ forms a Hilbert space with the mutual Dirichlet integral $D(\cdot, \cdot; W)$ as its inner product so that its norm is $D(\cdot; R)^{1/2}$. The normal derivative measure $*du$ of $u \in HD(W; \partial W)$, if it exists, is a Radon measure on the Royden harmonic boundary $\delta_{\mathscr{R}} = \delta_{\mathscr{R}} R$ of R (cf. e.g. [2], [12]) such that

$$D(v, u; R) = \int_{\delta_{\mathscr{R}} R} v * du \tag{3}$$

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for every $v \in HD(W;\partial W)$ (cf. e.g. [5]). A variational 2-capacity, or simply a capacity, $\mathrm{cap}(K)$ of a compact subset $K \subset \delta_{\mathscr{R}}R$ with respect to an end W of R is, by definition,

$$\mathrm{cap}(K) := \inf D(f;R), \tag{4}$$

where the infimum is taken with respect to Dirichlet finite continuous functions f on R belonging to the local Sobolev space $W^{1,2}_{\mathrm{loc}}(R)$ such that $f \mid K \geqq 1$ and $f \mid R \backslash W \leqq 0$. Recall that a property is said to hold quasieverywhere (abbreviated as q.e.) on $\delta_{\mathscr{R}}R$ if it holds on $\delta_{\mathscr{R}}R$ except for a subset of $\delta_{\mathscr{R}}R$ whose every compact subset is of capacity zero. The capacitary function c_K of K is the unique extremal function for the variation (4). We obtained the following characterization of the capacitary function c_K in [9].

Theorem A. *A function h on R is identical with the capacitary function c_K on R if and only if h satisfies the following four conditions: first of all, $h \in HD(W;\partial W)$; second, the normal derivative measure $*dh$ of h exists on $\delta_{\mathscr{R}}R$ and $*dh \geqq 0$ on $\delta_{\mathscr{R}}R$; third, $*dh = 0$ on $\delta_{\mathscr{R}}R \backslash K$; and, fourth and lastly, $h = 1$ q.e. on K.*

It is stated in [9] that there are many expected applications of this result, and actually one such was given there. The purpose of this paper is to exhibit another one of these applications by giving a simple and elementary proof based essentially upon the above characterization of capacitary functions to the following recent interesting result obtained by Masaoka [6]:

Theorem B. *If the harmonic Hardy space $HM_p(R)$ of exponent p in $(1,2) \cup (2,+\infty)$ on R coincides with the space $HD(R)$, then the linear dimension of $HM_p(R)$, denoted by* $\dim HM_p(R)$*, is finite:* $\dim HM_p(R) < \infty$.

Here the harmonic Hardy space $HM_p(R)$ of the exponent $p \in (1,+\infty)$ is the class of functions $u \in H(R)$ such that the subharmonic function $|u|^p$ is dominated by a harmonic function on R. It is remarkable that the above result is invalid for $p = 2$ (cf. [7]). At least if we stand on the view point of the inverse inclusion problem in the classification theory of Riemann surfaces, then the above result for the case $p \in (1,2)$ is really meaningful by virtue of the basic inclusion relation (cf. e.g. [10])

$$HM_2(R) \supset HD(R) \tag{5}$$

but any possible proof for this case is more or less almost trivial. Regardless what the intent of the above result for the case $p \in (2,+\infty)$ is, it seems likely that no proof of this part can be too easy. The original proof of Masaoka, the author of [6], itself is no exception: it heavily relies upon the Doob characterization ([3]) for a function $u \in H(R)$ to belong to $HD(R)$ in terms of fine boundary values of u on the Martin boundary, which is highly deep and tough and actually requires exquisitely elaborate cares to apply. On the contrary, our proof for the case $2 < p < \infty$ using capacitary functions is quite simple, elementary, and relatively easy. For completeness we will also append a proof, though almost trivial, for the case $1 < p < 2$. We add one more remark here. If R is parabolic, i.e. $R \in \mathcal{O}_G$ (the class of Riemann surfaces carrying no Green function), then both of $HM_p(R)$ $(1 < p < +\infty)$ and $HD(R)$ are reduced to the real number field $\mathbf{R}$. Thus to avoid the trivial situation we may assume that R is hyperbolic, i.e. $R \notin \mathcal{O}_G$, hereafter in this paper.

2. Proof

Theorem B will be proven for the case $p \in (2, +\infty)$ first and then, just for the sake of completeness, for the case $p \in (1, 2)$ as a supplement. Before starting the proof, we make a preparatory consideration valid for any $p \in (1, +\infty)$. Hereafter throughout this paper we always assume that

$$HM_p(R) = HD(R) \tag{6}$$

for some $p \in (1, +\infty)$. We need to consider two more subspaces of $H(R)$: the linear subspace $HB(R)$ consisting of bounded harmonic functions on R and the subspace $HBD(R) := HB(R) \cap HD(R)$. As a direct consequence of (6), we have $HD(R) = HM_p(R) \supset HB(R)$ so that

$$HB(R) = HBD(R), \tag{7}$$

i.e. every bounded harmonic function on R is automatically of finite Dirichlet integral over R. Let $\delta_{\mathscr{W}} = \delta_{\mathscr{W}} R$ ($\delta_{\mathscr{R}} = \delta_{\mathscr{R}} R$, resp.) be the Wiener (Royden, resp.) harmonic boundary of R (cf. e.g. [2], [12]). In general, $HB(R) \,|\, \delta_{\mathscr{W}} = C(\delta_{\mathscr{W}})$ but $HBD(R) \,|\, \delta_{\mathscr{R}}$ is only uniformly dense in $C(\delta_{\mathscr{R}})$. In view of (7), however, $HBD(R) \,|\, \delta_{\mathscr{R}}$ is uniformly closed in $C(\delta_{\mathscr{R}})$ so that $HBD(R) \,|\, \delta_{\mathscr{R}} = C(\delta_{\mathscr{R}})$. Therefore $C(\delta_{\mathscr{W}}) = C(\delta_{\mathscr{R}})$ as Banach lattices and a fortiori

$$\delta := \delta_{\mathscr{W}} \equiv \delta_{\mathscr{R}} \tag{8}$$

(cf. e.g. [13]). Since δ, as $\delta_{\mathscr{W}}$, is a Stonean space, the family of open subsets of δ contains the base consisting of clopen (i.e. closed and open) subsets of δ.

We fix an end W of $R \notin \mathcal{O}_G$. As we considered the relative class $H(W; \partial W)$ corresponding to the original class $H(R)$, we also consider the relative classes $HX(W; \partial W) := HX(W) \cap H(W; \partial W)$ corresponding to original classes $HX(R)$ for $X = M_p$, D, B, and BD. It is easy to see that there is the bijective mapping T of $H(R)$ onto $H(W; \partial W)$ such that $|Tu - u|$ is dominated by a potential p_u on R for every $u \in H(R)$ (cf. e.g. [11]); moreover, T gives a vector space isomorphism between $H(R)$ and $H(W;\partial W)$; T preserves the order, i.e. $Tu \geqq 0$ if and only if $u \geqq 0$ for $u \in H(R)$; T preserves each of the properties $X = M_p$, D, B, BD, i.e.

$$T(HX(R)) = HX(W;\partial W) \qquad (X = M_p, D, B, BD). \tag{9}$$

Hence assuming (6) is equivalent to assuming that

$$HM_p(W;\partial W) = HD(W;\partial W). \tag{10}$$

Then Theorem B is equivalent to that if (10) is valid for some $p \in (1,2) \cup (2,\infty)$, then $\dim HM_p(W;\partial W) < \infty$. We will prove Theorem B in this latter form.

Let $\omega = \omega_o$ be the harmonic measure on δ relative to W with a reference point $o \in W$, i.e. ω is the unique Borel measure on δ such that

$$H_\varphi^W(o) = \int_\delta \varphi \, d\omega \tag{11}$$

for every $\varphi \in C(\delta)$, where H_φ^W is the usual PWB solution on W to the Dirichlet problem on W with boundary data 0 on ∂W and φ on δ. Then $HM_p(W;\partial W)$ forms a Banach space equipped with the norm $\|\cdot; W\|_p$ given by

$$\|u; W\|_p := \left(\int_\delta |u|^p d\omega\right)^{1/p} \tag{12}$$

for every $u \in HM_p(W;\partial W)$. Hence we have

$$HM_p(W;\partial W) \,|\, \delta = L^p(\delta, \omega). \tag{13}$$

Similarly, $HD(W;\partial W)$ forms a Hilbert space equipped with the inner product $D(\cdot,\cdot; W)$ so that its norm is $D(\cdot; W)^{1/2}$. Observe that $S := HM_p(W;\partial W) \equiv HD(W;\partial W)$ forms a Banach space under the norm $\|\cdot; W\|_p + D(\cdot; W)^{1/2}$ since the convergence of any sequence in S in $\|\cdot; W\|_p$ ($D(\cdot; W)^{1/2}$, resp.) implies its local uniform convergence. The identity mapping of S onto $HM_p(W;\partial W)$ ($HD(W;\partial W)$, resp.) is clearly continuous and thus the Banach interior mapping principle (cf. e.g. [4], [13]) assures that S and $HM_p(W;\partial W)$ ($HD(W;\partial W)$, resp.) are bicontinuously linear isomorphic as Banach spaces. Hence the Banach spaces $(HM_p(W;\partial W), \|\cdot; W\|_p)$ and $(HD(W;\partial W), D(\cdot; W)^{1/2})$ are bicontinuously linear isomorphic by the identity mapping so that there exists a constant $C = C_p \in [1, +\infty)$ such that

$$C^{-1}\|u; W\|_p \leqq D(u; W)^{1/2} \leqq C\|u; W\|_p \tag{14}$$

for every $u \in HM_p(W;\partial W) \equiv HD(W;\partial W)$. We are now ready to proceed to the proof of Theorem B mainly for the case $p \in (2, +\infty)$.

2.1. Proof for the case $2 < p < \infty$

The standing assumption throughout this part of the proof is (10):

$$HM_p(W;\partial W) = HD(W;\partial W).$$

We are to show that $\dim HM_p(W;\partial W) < +\infty$, or, since we have (13):

$$HM_p(W;\partial W)\,|\,\delta = L^p(\delta,\omega),$$

we have to show that $\dim L^p(\delta,\omega) < \infty$. Recall that δ is a Stonean compact Hausdorff space, ω is a regular Borel measure on δ with $0 < \omega(\delta) < 1$, and $\omega(U) > 0$ for any nonempty open subset U of δ. Because of these, we see in general that $\dim L^p(\delta,\omega) = \#\delta$ (the number of points in δ) under the convention that $\dim\cdot$ and $\#\cdot$ are either a finite number in $\mathbf{N}$ (the set of positive integers) or $+\infty$ without distinguishing among infinite cardinalities. Thus we are to show that $\#\delta < +\infty$. We prove this by contradiction. Thus we assume contrariwise that $\#\delta = +\infty$. Then we can maintain that there exists at least one accumulation point in δ. Otherwise every point of δ is isolated in δ. Then a single point set $\{d\}$ is a clopen neighborhood of each point $d \in \delta$ in δ and $\bigcup_{d\in\delta}\{d\}$ is an open covering of the compact set δ so that there exists a finite subset $\{d_1, d_2, \ldots, d_n\}$ such that $\bigcup_{1\leqq i\leqq n}\{d_i\} = \delta$ or $\delta = \{d_1, d_2, \ldots, d_n\}$, contradicting our, though erroneous, assumption $\#\delta = +\infty$. Hence there is an accumulation point d in δ. Since δ is a clopen neighborhood of d, we can find a point $d_1 \in \delta$ and a clopen neighborhood $N_1 \subset \delta$ of d_1 such that $d \in \delta\backslash N_1$. Since $\delta\backslash N_1$ is a clopen neighborhood of d, we can find a point $d_2 \in \delta\backslash N_1$ and a clopen neighborhood $N_2 \subset \delta\backslash N_1$ of d_2 such that $d \in \delta\backslash(N_1 \cup N_2)$. Since $\delta\backslash(N_1 \cup N_2)$ is a clopen neighborhood of d, we can find a point $d_3 \in \delta\backslash(N_1 \cup N_2)$ and a clopen neighborhood $N_3 \subset \delta\backslash(N_1 \cup N_2)$ of d_3 such that $d \in \delta\backslash(N_1 \cup N_2 \cup N_3)$. Repeating this procedure we can choose an infinite sequence $(d_i)_{i\in\mathbf{N}}$ of mutually distinct points $d_i \in \delta$ $(i \in \mathbf{N})$ and a sequence $(N_i)_{i\in\mathbf{N}}$ of mutually disjoint clopen neighborhoods N_i of d_i $(i \in \mathbf{N})$ in δ. Then the relation

$$\sum_{i\in\mathbf{N}} \omega(\{d_i\}) = \omega(\{d_i : i \in \mathbf{N}\}) \leqq \omega(\delta) < 1$$

implies that $\omega(\{d_i\}) \to 0$ $(i \to \infty)$. Choose a sequence $(U_i)_{i\in\mathbf{N}}$ of mutually disjoint clopen neighborhood U_i of d_i in δ such that $U_i \subset N_i$ and $0 < \omega(U_i) \leqq 2\omega(\{d_i\}) + 1/i$ for each $i \in \mathbf{N}$. Finally by choosing a suitable subsequence $(V_n)_{n\in\mathbf{N}}$ of $(U_i)_{i\in\mathbf{N}}$ we can assume that

$$\left(\frac{\omega(V_{n+1})}{\omega(V_n)}\right)^{1/p} < \varepsilon^n \qquad (n \in \mathbf{N}), \tag{15}$$

where ε is an arbitrarily chosen but then fixed in $(0, 1/2)$. Note that each V_n is clopen in δ $(n \in \mathbf{N})$ and $V_n \cap V_m = \emptyset$ $(n \neq m)$.

Let e_n be the harmonic measure function of the clopen subset $V_n \subset \delta$ on W so that $e_n \in HB(W; \partial W)$ $(\subset HM_p(W; \partial W) = HD(W; \partial W))$ with the boundary condition $e_n \,|\, \partial W = 0$, $e_n \,|\, V_n = 1$, and $e_n \,|\, \delta \backslash V_n = 0$ for each $n \in \mathbf{N}$. As an auxiliary function to e_n, we consider the capacitary function $c_n := c_{V_n}$ of the compact (and clopen in reality) subset V_n of δ so that $c_n \in HBD(W; \partial W)$ $(\subset HM_p(W; \partial W) = HD(W; \partial W))$ with the boundary condition $c_n \,|\, \partial W = 0$, $c_n \,|\, V_n = 1$, $*dc_n \,|\, \delta \geqq 0$, and $*dc_n \,|\, \delta \backslash V_n = 0$ for each $n \in \mathbf{N}$. Here, originally $c_n = 1$ not necessarily on the whole V_n but only on V_n except for a subset of capacity zero. As a consequence of (7) the vanishingness of capacities and harmonic measures for subsets of δ are identical (cf. [8]). This with the fact that V_n is not merely compact but also open (i.e. clopen) assures that $c_n \equiv 1$ on V_n. Observe that $D(c_n; W) = \operatorname{cap}(V_n) \leqq D(e_n; W)$ by the definition of $\operatorname{cap}(V_n)$ and thus by (14) we have

$$C^{-1} \|c_n; W\|_p \leqq D(c_n; W)^{1/2} \leqq D(e_n; W)^{1/2} \leqq C \|e_n; W\|_p \tag{16}$$

for every $n \in \mathbf{N}$. In view of the fact that $0 \leqq c_n \leqq 1$ on $\partial W \cup W \cup \delta$ and $c_n = 1$ on V_n, we have

$$\|c_n; W\|_p = \left(\int_\delta |c_n|^p d\omega \right)^{1/p} \geqq \left(\int_{V_n} d\omega \right)^{1/p} = \omega(V_n)^{1/p}.$$

Similarly, but more trivially, we have

$$\|e_n; W\|_p = \left(\int_\delta |e_n|^p d\omega \right)^{1/p} = \left(\int_{V_n} d\omega \right)^{1/p} = \omega(V_n)^{1/p}.$$

Incorporating the above two displayed relations with (16) we deduce that

$$C^{-2} \omega(V_n)^{2/p} \leqq D(c_n; W) \leqq D(e_n; W) \leqq C^2 \omega(V_n)^{2/p} \qquad (n \in \mathbf{N}). \tag{17}$$

We fix a sequence $(a_n)_{n \in \mathbf{N}}$ of strictly positive numbers a_n given by

$$a_n^2 \omega(V_n)^{2/p} = \frac{1}{n} \qquad (n \in \mathbf{N}). \tag{18}$$

Using this sequence $(a_n)_{n \in \mathbf{N}}$ and the sequence $(e_n)_{n \in \mathbf{N}}$ of harmonic measure functions e_n, we consider the main sequence $(u_n)_{n \in \mathbf{N}}$ of functions u_n on $\partial W \cup W \cup \delta$ defined by

$$u_n := \sum_{1 \leqq i \leqq n} a_i e_i \qquad (n \in \mathbf{N}) \tag{19}$$

on $\partial W \cup W \cup \delta$ belonging to $HBD(W; \partial W)$, and similarly by using the sequence $(a_n)_{n \in \mathbf{N}}$ again and the sequence $(c_n)_{n \in \mathbf{N}}$ of capacitary functions c_n, we also consider the auxiliary sequence $(v_n)_{n \in \mathbf{N}}$ to the main sequence $(u_n)_{n \in \mathbf{N}}$ given by

$$v_n := \sum_{1 \leqq i \leqq n} a_i c_i \qquad (n \in \mathbf{N}) \tag{20}$$

also on $\partial W \cup W \cup \delta$ belonging to $HBD(W; \partial W)$.

We will evaluate $\|u_n; W\|_p$ and $D(u_n; W)^{1/2}$ for every $n \in \mathbf{N}$. We work on the former first. Since $|u_n|^p = 0$ on $\delta \backslash \bigcup_{1 \leqq i \leqq n} V_i$ and $|u_n|^p = |a_i e_i|^p = a_i^p e_i = a_i^p$ on each V_i $(1 \leqq i \leqq n)$, we see that

$$\|u_n; W\|_p^p = \int_\delta |u_n|^p d\omega = \int_{\bigcup_{1 \leqq i \leqq n} V_i} |u_n|^p d\omega = \sum_{1 \leqq i \leqq n} \int_{V_i} a_i^p \, d\omega = \sum_{1 \leqq i \leqq n} a_i^p \omega(V_i).$$

In view of (18), $a_i^p \omega(V_i) = 1/i^{p/2}$ $(1 \leqq i \leqq n)$ and thus

$$\|u_n; W\|_p^p = \sum_{1 \leqq i \leqq n} \frac{1}{i^{p/2}} \leqq \sum_{i \in \mathbf{N}} \frac{1}{i^{p/2}} =: K < +\infty$$

because $p/2 > 1$. Hence we see that

$$\|u_n; W\|_p \leqq K^{1/p} < +\infty \qquad (n \in \mathbf{N}). \tag{21}$$

As the core of the present proof, the essential task is the estimation of the Dirichlet norm $D(u_n; W)^{1/2}$ for every $n \in \mathbf{N}$ and for the purpose the auxiliary functions v_n $(n \in \mathbf{N})$ will play a decisive role. To save the notations we write $D(\cdot)$ and $D(\cdot, \cdot)$ for the Dirichlet integral $D(\cdot; W)$ and the mutual Dirichlet integral $D(\cdot, \cdot; W)$ omitting the integrating domain W. We observe the inequalities

$$D(u_n) + D(v_n) - 2D(u_n, v_n) \geqq 0 \qquad (n \in \mathbf{N}) \tag{22}$$

since the lefthand side of the above is $D(u_n - v_n) \geqq 0$. We next compute $D(u_n, v_n)$ and $D(v_n)$. First we have

$$D(u_n, v_n) = D\left(\sum_{1 \leqq i \leqq n} a_i e_i, \sum_{1 \leqq j \leqq n} a_j c_j\right) = \sum_{1 \leqq i \leqq n} a_i^2 D(e_i, c_i) + 2 \sum_{i < j}^{1, \ldots, n} a_i a_j D(e_i, c_j).$$

Since $e_i * dc_i = *dc_i = c_i * dc_i$ on δ, we see that

$$D(e_i, c_i) = \int_\delta e_i * dc_i = \int_\delta c_i * dc_i = D(c_i) \qquad (1 \leqq i \leqq n).$$

Similarly as above we have $e_i * dc_j = 0$ on δ for $i \neq j$ and thus

$$D(e_i, c_j) = \int_\delta e_i * dc_j = 0 \qquad (1 \leqq i < j \leqq n).$$

By the above three displayed relations, we obtain

$$D(u_n, v_n) = \sum_{1 \leqq i \leqq n} a_i^2 D(c_i) \qquad (n \in \mathbf{N}). \tag{23}$$

We turn to the computation of $D(v_n)$ $(n \in \mathbf{N})$. As before, viewing

$$D(v_n) = D\left(\sum_{1 \leqq i \leqq n} a_i c_i, \sum_{1 \leqq j \leqq n} a_j c_j\right),$$

we deduce that

$$D(v_n) = \sum_{1 \leqq i \leqq n} a_i^2 D(c_i) + 2 \sum_{i<j}^{1,\dots,n} a_i a_j D(c_i, c_j). \tag{24}$$

Since $0 \leqq c_i \leqq 1$ and $*dc_j \geqq 0$ on δ and further $*dc_j \,|\, \delta \setminus V_j = 0$ and $c_j \,|\, V_j = 1$ (for this, cf. the introduction of c_j in the second paragraph of the present section 2.1), we have $0 \leqq c_i * dc_j \leqq *dc_j = c_j * dc_j$ on δ and therefore

$$D(c_i, c_j) = \int_\delta c_i * dc_j \leqq \int_\delta c_j * dc_j = D(c_j) = \left(\frac{D(c_j)}{D(c_i)}\right)^{1/2} \cdot (D(c_i) D(c_j))^{1/2},$$

i.e. we have obtained that

$$D(c_i, c_j) \leqq \left(\frac{D(c_j)}{D(c_i)}\right)^{1/2} \cdot (D(c_i) D(c_j))^{1/2} \qquad (1 \leqq i < j \leqq n). \tag{25}$$

By (17), $C^{-1} \omega(V_k)^{1/p} \leqq D(c_k)^{1/2} \leqq C \omega(V_k)^{1/p}$ $(k = i, j)$ and these imply

$$\left(\frac{D(c_j)}{D(c_i)}\right)^{1/2} \leqq C^2 \left(\frac{\omega(V_j)}{\omega(V_i)}\right)^{1/p} = C^2 \left(\frac{\omega(V_{i+1})}{\omega(V_i)} \cdot \frac{\omega(V_{i+2})}{\omega(V_{i+1})} \cdot \dots \cdot \frac{\omega(V_j)}{\omega(V_{j-1})}\right)^{1/p}$$

$(1 \leqq i < j \leqq n)$. By the choice (15), we see that

$$\left(\frac{D(c_j)}{D(c_i)}\right)^{1/2} \leqq C^2 \varepsilon^i \cdot \varepsilon^{i+1} \cdot \dots \cdot \varepsilon^{j-1} = C^2 \varepsilon^{(j-i)(i+j-1)/2} \leqq C^2 \varepsilon^{i(j-i)}.$$

That is, we have obtained

$$\left(\frac{D(c_j)}{D(c_i)}\right)^{1/2} \leqq C^2 \varepsilon^{i(j-i)} \qquad (1 \leqq i < j \leqq n). \tag{26}$$

Again by (17) and (18), we see that

$$a_i a_j (D(c_i)D(c_j))^{1/2} \leqq C a_i \omega(V_i)^{1/p} \cdot C a_j \omega(V_j)^{1/p} = C^2(1/ij)^{1/2} \leqq C^2$$

so that we have deduced that

$$a_i a_j (D(c_i)D(c_j))^{1/2} \leqq C^2 \qquad (1 \leqq i < j \leqq n). \tag{27}$$

Putting (25), (26), and (27) together, we conclude that

$$a_i a_j D(c_i, c_j) \leqq C^4 \varepsilon^{i(j-i)} \qquad (1 \leqq i < j \leqq n)$$

and thus we see that

$$\begin{aligned}
\sum_{i<j}^{1,\dots,n} a_i a_j D(c_i, c_j) &\leqq C^4 \sum_{1 \leqq i \leqq n} \left(\sum_{i+1 \leqq j \leqq n} \varepsilon^{i(j-i)} \right) \\
&\leqq C^4 \sum_{i \in \mathbf{N}} \left(\sum_{k \in \mathbf{N}} \varepsilon^{ik} \right) = C^4 \sum_{i \in \mathbf{N}} \frac{\varepsilon^i}{1-\varepsilon^i} \\
&\leqq C^4 \sum_{i \in \mathbf{N}} \frac{\varepsilon^i}{1-\varepsilon} = C^4 \frac{\varepsilon}{(1-\varepsilon)^2} \leqq 2C^4,
\end{aligned}$$

i.e. we get the estimate of the second term on the righthand side of (24):

$$2 \sum_{i<j}^{1,\dots,n} a_i a_j D(c_i, c_j) \leqq 4C^4 \qquad (n \in \mathbf{N}). \tag{28}$$

Finally, concerning the estimate of the first term on the righthand side of (24), or the righthand side term of (23), we examine $a_i^2 D(c_i)$: by (17) and (18) we see that

$$\sum_{1 \leqq i \leqq n} a_i^2 D(c_i) \geqq C^{-2} \sum_{1 \leqq i \leqq n} a_i^2 \omega(V_i)^{2/p} = C^{-2} \sum_{1 \leqq i \leqq n} \frac{1}{i} \geqq C^{-2}(\gamma + \log n),$$

where γ is the Euler constant so that

$$\sum_{1 \leqq i \leqq n} a_i^2 D(c_i) \geqq C^{-2}(\gamma + \log n) \qquad (n \in \mathbf{N}). \tag{29}$$

Putting (23) and (24) to (22) we obtain

$$D(u_n) + \sum_{1 \leqq i \leqq n} a_i^2 D(c_i) + 2 \sum_{i<j}^{1,\dots,n} a_i a_j D(c_i, c_j) - 2 \sum_{1 \leqq i \leqq n} a_i^2 D(c_i) \geqq 0$$

so that

$$D(u_n) \geqq \sum_{1 \leqq i \leqq n} a_i^2 D(c_i) - 2 \sum_{i<j}^{1,\dots,n} a_i a_j D(c_i, c_j).$$

Applying the estimates (28) and (29) to the righthand side of the above, we get

$$D(u_n) \geqq C^{-2}(\gamma + \log n) - 4C^4 \qquad (n \in \mathbf{N}). \tag{30}$$

Putting (21) and the above (30) to the rightmost inequality of (14), we obtain

$$(C^{-2}(\gamma + \log n) - 4C^4)^{1/2} \leqq CK^{1/p}$$

for every $n \in \mathbf{N}$, which is clearly impossible and thus we are done. □

2.2. Proof for the case $1 < p < 2$ (Supplementary)

Supposing $HM_p(W; \partial W) = HD(W; \partial W)$, we are to show that $\dim HM_p(W; \partial W) < +\infty$ or equivalently $\dim L^p(\delta, \omega) < +\infty$, which is equivalent to $\#\delta < +\infty$. In view of

$$HD(W; \partial W) = HM_p(W; \partial W) \supset HM_2(W; \partial W) \supset HD(W; \partial W),$$

we see that $HM_p(W; \partial W) = HM_2(W; \partial W)$ so that

$$L^p(\delta, \omega) = L^2(\delta, \omega). \tag{31}$$

Contrary to the assertion, assume that $\#\delta = +\infty$ and we are to derive a contradiction. From $\#\delta = +\infty$ it follows that there is a sequence $(V_n)_{n \in \mathbf{N}}$ of mutually disjoint clopen subset V_n of δ such that

$$0 < \omega(V_n) < n^{-(4-p)/(2-p)} \qquad (n \in \mathbf{N}). \tag{32}$$

Let e_n be the harmonic measure function of V_n relative to W $(n \in \mathbf{N})$ and choose $a_n \in (0, +\infty)$ such that

$$a_n^2 \omega(V_n) = \frac{1}{n} \qquad (n \in \mathbf{N}). \tag{33}$$

Then consider

$$u := \sum_{n \in \mathbf{N}} a_n e_n.$$

Since, under the assumption (33), the requirement (32) is equivalent to the inequalities that $a_n^p \omega(V_n) < 1/n^2$, we can offhand see that

$$u \in L^p(\delta, \omega) \backslash L^2(\delta, \omega),$$

contradicting (31). □

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