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On the chromatic $\mathrm{Ext}^0(M^1_{n-1})$ on $\Gamma(m+1)$ for an odd prime

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Abstract. Ravenel introduced spectra $T(m)$ for $m \geq 0$ interpolating the Brown-Peterson spectrum BP and the sphere spectrum S in [5]. Since the homotopy groups of BP are well known, it is interesting to study differences among the homotopy groups of $T(m)$'s to study the homotopy groups of spheres. He also introduced the localization functor L_n on the stable homotopy category in [4]. To study the difference of $L_nT(m)$'s for a fixed integer n, we consider the corresponding chromatic E_1-term $\mathrm{Ext}^0(M^1_{n-1})$ on $\Gamma(m+1)$ for each m, and determine it for $m+1 \geq (n-2)(n-1)$ in this paper. The results show that the structures depend on a integer $\left[\dfrac{m+1}{n-1}\right]$. Here $[x]$ denotes the greatest integer that does not exceed x.

1. Introduction

Let p be an odd prime number, and S denote the p-localized sphere spectrum. Determination of the homotopy groups of S is one of main problems in the stable homotopy theory. Consider the Brown-Peterson spectrum BP at the prime p. The spectrum BP gives rise to the Adams-Novikov spectral sequence abutting to $\pi_*(X)$ for a spectrum X with E_2-term $E_2^*(X) = \mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*(X))$, where BP_*BP is the Hopf algebroid

$$(BP_*, BP_*BP) = (\mathbf{Z}_{(p)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots])$$

associated with BP. Then, the E_2-term $E_2^*(S)$ is an approximation of the homotopy groups $\pi_*(S)$. Ravenel [5] constructed ring spectra $T(m)$ for $m \geq 0$ characterized by

$$BP_*(T(m)) = BP_*[t_1, t_2, \dots, t_m] \subset BP_*BP$$

as BP_*BP-comodule algebras. These spectra interpolate between the sphere spectrum S and the Brown-Peterson spectrum BP. Since we know the homotopy groups $\pi_*(BP) = BP_*$ of BP, it is interesting to understand the difference between $E_2^*(T(m))$ and $E_2^*(T(m+1))$ in order to study the E_2-term

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$E_2^*(S)$. Miller, Ravenel and Wilson [3] introduced the chromatic spectral sequence to study the Adams-Novikov E_2-term for computing the homotopy groups $\pi_*(V(k))$ of the Smith-Toda spectrum $V(k)$ characterized by $BP_*(V(k)) = BP_*/I_{k+1}$ for $I_k = (p, v_1, \dots, v_{k-1})$. In order to set up the spectral sequence, they introduced BP_*BP-comodules N_k^t and M_k^t defined inductively by $N_k^0 = BP_*/I_k$, $M_k^t = v_{k+t}^{-1}N_k^t$ and the short exact sequence

$$0 \to N_k^t \hookrightarrow M_k^t \to N_k^{t+1} \to 0. \tag{1.1}$$

The spectral sequence is obtained from the exact couple given by applying the Ext group $H^*- = \mathrm{Ext}^*_{BP_*BP}(BP_*, -)$ to the short exact sequence (1.1). The spectral sequence is also applied for computing the E_2-term $E_2^*(T(m) \wedge V(k))$, which is isomorphic to $\mathrm{Ext}^*_{\Gamma(m+1)}(BP_*, BP_*/I_{k-1})$ by Ravenel [5], where $\Gamma(m+1) = BP_*BP/(t_1, \dots, t_m)$ is the induced Hopf algebroid. Indeed, use $H_m^*- = \mathrm{Ext}^*_{\Gamma(m+1)}(BP_*, -)$ instead of H^*-, and we have the chromatic spectral sequence

$$E_1^{s,t} = H_m^t M_k^s \Rightarrow H_m^{s+t} N_k^0.$$

The E_1-term $H_m^t M_n^0$ for $n < m+2$ is determined by Ravenel (*cf.* [5]) as follows:

$$H_m^* M_n^0 = K[w_0] \otimes E(h_{k,j} : m+1 \le k \le m+n, j \in \mathbf{Z}/n). \tag{1.2}$$

where

$$w_0 = v_{n+m}, \qquad K(n)_* = \mathbf{Z}/p[v_n, v_n^{-1}], \qquad K = K(n)_*[v_{n+1}, \dots, v_{n+m-1}],$$

and $h_{k,j} \in H_m^1 M_n^0$ is represented by the cocycle $t_k^{p^j}$ of the cobar complex $\Omega^*_{\Gamma(m+1)} M_n^0$. This shows that the modules $H_m^* M_n^0$ have a uniform structure for $m+1 > n-1$. In general, it seems that $H_m^t M_k^s$ gets harder to be determined as t and s get larger. So we consider $H_m^0 M_{n-1}^1$ in this paper. If $m+1 \ge n(n-1)$, the modules $H_m^0 M_{n-1}^1$ are determined in [2] and [6], and the results show that the structures are uniform as well in this range. In [1], Ichigi, Nakai and Ravenel determined the modules $H_m^0 M_2^1$ for $m \ge 2$, whose structures depend on m and are not uniform. We expand their result for $n \ge 3$ here. In order to explain the result, we state here our key lemma, which is a paraphrase of [3, Remark 3.11]:

LEMMA 1.3. *Let n and m be positive integers with $n-1 < m+1$. Suppose the following two conditions:*

1) *For each integer $k \ge 0$, there is an element $x_k \in v_n^{-1}BP_*$ such that $x_k \equiv w_0^{p^k} \mod I(1)$ and*

$$d(x_k) \equiv u^{a_k} v_n^{a'_k} w_0^{a''_k} g_k \mod I(a_k+1)$$

for nonnegative integers a_k, a'_k *and* a''_k, *and for* $g_k \in \{t_{m+i}^{p^j} : 0 < i \le n, j \in \mathbf{Z}/n\}$. *Here,* $d = \eta_R - \eta_L$, $u = v_{n-1}$ *and* $I(k) = I_{n-1} + (u^k)$.

2) *The elements* $w_0^{(s-1)p^k + a''_k} g_k$ *for nonnegative integers* s *and* k *with* $p \not| s$ *represent linearly independent generators over* K *in* $H^1_m M^0_n$.

Then, $H^0_m M^1_{n-1}$ *is isomorphic to*

$$K[u, u^{-1}]/K[u] \oplus \bigoplus_{k \ge 0, p \not| s > 0} K[u]/(u^{a_k})\langle x_k^s/u^{a_k}\rangle$$

as a $K[u]$*-module. Here,* $K[u]/(u^a)\langle x\rangle$ *denotes the* $K[u]$*-module generated by* x, *which is isomorphic to* $K[u]/(u^a)$.

In the known cases, there are elements x_k satisfying the conditions of the lemma, and the structure of $H^0_m M^1_{n-1}$ is described by the integers a_k. In particular, if $n = 3$, the structures depend on m as the following table of a_k ([1, p. 3802]):

	$m = 2$	$m = 3, 4$	$m \ge 5$
$0 \le k < 3$	p^k		
$3 \le k < 6$	$p^k + p^{k-1}$		
$6 \le k < 9$	$p^k + p^{k-1} + p^{k-3}$	$p^k + p^{k-1} + p^{k-2}$	
$k \ge 9$	$p^k + p^{k-1} + p^{k-3} + a_{k-6}$	$p^k + p^{k-1} + p^{k-2} + p^{k-9}Q_2 + a_{k-6}$	$p^k + p^{k-1} + p^{k-2} + a_{k-4}$

Here, $Q_2 = p^{m+1} - p^4$. In this paper, we construct elements satisfying the condition of Lemma 1.3 (see Lemma 3.15), and obtain our main result:

Theorem 1.4. *Suppose that* $n \ge 3$, $m + 1 > n - 1$ *and* $m + 1 \ge (n-2)(n-1)$. *Then the module* $H^0_m M^1_{n-1}$ *is isomorphic to the module of Lemma* 1.3 *with integers* a_k *given by the following table:*

	$m \in J_2$	$m \in J_1$	$m \in J_0$
$(j-1)n \le k < jn$ $(1 \le j < n)$	$p^{k-j+1}e(j)$		
$(n-1)n \le k < n^2$	$p^{k-n+1}e(n) + p^{k-(n-1)n}Q_{n-1}$	$p^{k-n+1}e(n)$	
$k \ge n^2$	$p^{k-n+1}e(n) + p^{k-(n-1)n}Q_{n-1} + a_{k-2n}$	$p^{k-n+1}e(n) + p^{k-n^2}Q_{n-1} + a_{k-2n}$	$p^{k-n+1}e(n) + a_{k-n-1}$

Here,

$$J_k = \begin{cases} \{m \in \mathbf{Z} : m+1 \geq n(n-1)\} & k = 0 \\ \{m \in \mathbf{Z} : (n-k)(n-1) \leq m+1 < (n-k+1)(n-1)\} & 0 < k < n-1, \\ \{m \in \mathbf{Z} : n-1 < m+1 < 2(n-1)\} & k = n-1 \end{cases}$$

$e(k) = (p^k - 1)/(p-1)$ *for* $k \geq 0$, *and*

$$Q_l = p^{m+1} - p^{l(n-1)}.$$

This shows that the smaller the integer m is, the more complex the structure of $H_m^0 M_n^1$ is, as we expected. Our computation also suggests that the integers a_k for smaller m fit in the following table (see Lemma 3.2):

	$m \in J_{n-1}$	$\cdots$	$m \in J_{n-l}$	$\cdots$	$m \in J_2$
$0 \leq k < n$	$p^k e(1) = p^k$				
$n \leq k < 2n$	$p^{k-1} e(2) = p^k + p^{k-1}$				
$2n \leq k < 3n$	$p^{k-2} e(3) + p^{k-2n} Q_2$	$p^{k-2} e(3)$			
$\cdots$		$\ddots$	$\cdots$		
$ln \leq k < (l+1)n$			$p^{k-l} e(l+1) + p^{k-ln} Q_l$	$\cdots$	$p^{k-l} e(l+1)$
$\cdots$				$\ddots$	$\cdots$
$(n-1)n \leq k < n^2$					$p^{k-n+1} e(n) + p^{k-(n-1)n} Q_{n-1}$
$k \geq n^2$					$p^{k-n+1} e(n) + p^{k-(n-1)n} Q_{n-1} + a_{k-2n}$

So far, we have some difficulty in computation to fill in the blanks of the table. Here the case $m \in J_n$ is excluded, since the result (1.2) of Ravenel's holds for $m+1 > n-1$.

In the same manner as [8] and [7], we obtain a $v_n^{-1} BP$-local spectrum W_{n-1}^1 such that $v_n^{-1} BP_*(W_{n-1}^1) = M_{n-1}^1$ if $n^2 + n \leq 2p$. The module $H_m^* M_{n-1}^1$ is the E_2-term of the Adams-Novikov spectral sequence converging to $\pi_*(W_{n-1}^1)$. For such a large prime number p, the Adams-Novikov spectral sequence converging to $\pi_*(W_{n-1}^1)$ collapses and poses no extension problem, and hence we obtain the following

Corollary 1.5. *The groups* $H_m^0 M_{n-1}^1$ *are subgroups of the homotopy groups* $\pi_*(W_n^1)$.

2. Preliminaries

In the following, we fix the positive integers m and $n \geq 3$ satisfying the condition

$$(n-2)(n-1) \leq m+1 < n(n-1), \qquad \text{or} \qquad m \in J_1 \cup J_2.$$

Let $\eta_R : BP_* \to \Gamma(m+1) = BP_*BP/(t_1, \dots, t_m)$ be the right unit of the Hopf algebroid $\Gamma(m+1)$. We have the formulas of Hazewinkel's and Quillen's:

$$v_i = p\ell_i - \sum_{j=1}^{i-1} \ell_j v_{i-j}^{p^j} \in BP_* \otimes \mathbf{Q} = \mathbf{Q}[\ell_1, \ell_2, \dots],$$

$$\eta_R(\ell_i) = \ell_i + \sum_{j=1}^{i-m} \ell_{i-m-j} t_{m+j}^{p^{i-m-j}} \in \Gamma(m+1) \otimes \mathbf{Q}.$$

We use the notations

$$u_k = v_{n-1+k}, \qquad u = u_0 = v_{n-1}, \qquad w_k = v_{n+m+k} = u_{m+k+1} \qquad \text{and} \qquad s_k = t_{m+k};$$

$$P = p^{n-1} \qquad \text{and} \qquad Q = p^{m+1}; \qquad \text{and}$$

$$I = I_{n-1} = (p, v_1, \dots, v_{n-2}) \qquad \text{and} \qquad I(k) = I + (u^k).$$

We now have the following lemma by routine computations using above formulas and the fact that the right unit η_R is an algebra map.

Lemma 2.1. *The differential* $d = \eta_R - \eta_L : BP_* \to \Gamma(m+1)$ *acts on generators* v_i *as follows:*

$$d(v_i) \equiv 0 \mod I \qquad \text{for } i \leq n+m,$$

$$d(w_k) \equiv \sum_{j=0}^{k} (u_j s_{k+1-j}^{p^j P} - u_j^{p^{k-j} Q} s_{k+1-j}) - \omega_k \mod I \quad \text{for } 0 \leq k < n, \qquad \text{and}$$

$$d(w_n) \equiv \sum_{j=1}^{n} (u_j s_{k+1-j}^{p^j P} - u_j^{p^{k-j} Q} s_{k+1-j}) \mod I(1) = I_n,$$

where

$$\omega_k = \begin{cases} 0 & k < n-1, \\ u w_{0,n-2} & k = n-1 \end{cases} \tag{2.2}$$

for the element $w_{0,i}$ *defined by*

$$d(w_0^{p^{i+1}}) \equiv u^{p^{i+1}} s_1^{p^{i+1}P} - u^{p^{i+1}Q} s_1^{p^{i+1}} + pw_{0,i} \tag{2.3}$$

$\mathrm{mod}(p^2, v_1, \dots, v_{n-2})$ *in* $\Gamma(m+1)$.

Note that the ideal $(p^2, v_1, \dots, v_{n-2})$ is invariant in $\Gamma(m+1)$.

3. The elements x_n

Since we are working on modules localized at $u_1 = v_n$, it is justified to put $u_1 = 1$. We furthermore consider integers $e_P(k)$ for $k \geq 0$, Q_k and $\tilde{Q}_k$ for $k \geq -1$, and P_k for $0 \leq k \leq n$:

$$e_P(k) = \frac{P^k - 1}{P - 1}, \qquad Q_k = Q - [P^k], \qquad \tilde{Q}_k = Q - e_P(k+1) \qquad \text{and}$$

$$P_k = \begin{cases} P^k e(k+1) & 0 \leq k < n \\ pPP_{n-1} + (pP)^{v-1} Q_{n-1} & k = n \text{ and } m \in J_v \end{cases}$$

Here $[r]$ for a rational number r denotes the greatest integer that does not exceed r.

Now we introduce elements X_k and X'_k of $u_1^{-1}BP_*$ for $0 \leq k \leq n$, which correspond to the elements x_{kn} and x_{kn+1} in Lemma 1.3.

$$\begin{aligned}
X_k &= \begin{cases} w_0 & k = 0 \\ \overline{X}_k + \overline{\overline{X}}_k & k > 0 \end{cases} \\
\overline{X}_k &= \begin{cases} (X'_{k-1})^P + (-1)^k u^{pPP_{k-1}} \tilde{X}_{k-1} & (k,m) \in S_1 \\ X_{n-1}^{pP} - (-1)^n u^{P_n - pP_{n-1}} W_n & (k,m) \in S_2 \end{cases} \\
\overline{\overline{X}}_k &= \begin{cases} (-1)^k u^{pPP_{k-1} + Q_{k-1}} \tilde{X}'_{k-1} & (k,m) \in S_1 \\ -u^{P_n - (P_{n-1} + Q_{n-1})} X_{n-1} & (k,m) \in S_2 \end{cases} \\
X'_k &= X_k^p - (-1)^k W_{k+1} \\
W_k &= u^{pP_{k-1}} w_k^{P^{k-1}} - \overline{W}_k \\
\overline{W}_k &= u^{pP_{k-1}} \sum_{j=2}^{k} (u_j^{P^{k-1}} \tilde{X}_{k-j}^{p^j P^{j-1}} - u_j^{p^{k-j} P^{k-1} Q} \tilde{X}_{k-j}^{P^{j-2}}),
\end{aligned} \tag{3.1}$$

where S_1 and S_2 are the sets $\{(k,m) \in \mathbf{Z}^2 : 0 < k < n, \text{ or } k = n \text{ and } m \in J_1\}$ and $\{(n,m) \in \mathbf{Z}^2 : m \in J_2\}$, respectively, and $\tilde{X}_j$ and $\tilde{X}'_j$ denote $(-1)^j u^{-P_j} X_j$ and $(-1)^j u^{-pP_j} X'_j$, respectively.

Lemma 3.2. *The differential* $d = \eta_R - \eta_L : u_1^{-1}BP_* \to u_1^{-1}\Gamma(m+1)$ *acts on* X_k *and* X_k' *for* $k \geq 0$ *as follows:*

$$d(X_k) \equiv (-1)^k u^{P_k}(s_{k+1}^{P^{k+1}} - u^{Q_k}s_{k+1}^{P^k}) \bmod I(P_k + Q_k + \tilde{Q}_{k-1}),$$

$$d(X_k') \equiv (-1)^k u^{pP_k}(s_{k+1}^{P^k} - u^{P^k}s_{k+2}^{P^{k+1}} + \omega_{k+1}^{P^k}) \bmod I(pP_k + Q_{k-1})$$

for $k \leq n-2$. *For* $k = n-1$ *and* n, *we have*

$$d(X_{n-1}) \equiv (-1)^{n-1}u^{P_{n-1}}((s_n^{P^n} - w_{0,n-2}^{P^{n-1}}) - u^{Q_{n-1}}(s_n^{P^{n-1}} - w_{0,n-2}^{P^{n-2}}))$$

$$\bmod I(P_{n-1} + Q_{n-1} + \tilde{Q}_{n-2}),$$

$$d(X_{n-1}') \equiv (-1)^{n-1}u^{pP_{n-1}}s_n^{P^{n-1}} \bmod I(pP_{n-1} + P^{n-1}) \quad \text{for } m \in J_1, \qquad \text{and}$$

$$d(X_n) \equiv (-1)^n u^{P_n}w_{0,n-2}^{P^{n-2}} \bmod I(P_n + P^{n-1} - \varepsilon(v-1)Q)$$

where v *is* 1 *or* 2 *with* $m \in J_v$.

Remark. In the case $m \in J_0$, $d(X_n) \equiv (-1)^{n+1}u^{pPP_{n-1}}w_{0,n-2}^{P^{n-1}}$ $\bmod I(pPP_{n-1} + Q_{n-1})$ in our notation (see [6]).

Proof. By Lemma 2.1, we have $d(X_0) = d(w_0) \equiv us_1^P - u^Q s_1 \bmod I$ and compute

$$d(X_0') \equiv d(X_0)^p - u^p d(w_1)$$
$$\equiv u^p s_1^{pP} - u^p(us_2^P + s_1^{pP} - s_1) \bmod I(pQ).$$

Since $pQ \geq pP_0 + Q_{-1}$, we have the first step of induction.

Suppose that we have the congruences on $d(X_i)$ and $d(X_i')$ for $i \leq k-1 < n-1$. We notice that $Q_i \geq 0$ and $(k,m) \in S_1$ in this case. Then, the congruence on $d(X_k)$ follows from computation

$$(3.3) \qquad d(\overline{X}_k) \equiv (-1)^{k-1}u^{pPP_{k-1}}(s_k^{P^k} - u^{P^k}s_{k+1}^{P^{k+1}} + \omega_k^{P^k})$$
$$+ (-1)^k u^{pPP_{k-1}}(s_k^{P^k} - u^{Q_{k-1}}s_k^{P^{k-1}})$$
$$\equiv (-1)^k u^{pPP_{k-1}}(u^{P^k}s_{k+1}^{P^{k+1}} - \omega_k^{P^k} - u^{Q_{k-1}}s_k^{P^{k-1}})$$

$\bmod I(pPP_{k-1} + Q_{k-1} + \tilde{Q}_{k-2})$,

$$(3.4) \qquad d(\overline{\overline{X}}_k) \equiv (-1)^k u^{pPP_{k-1}+Q_{k-1}}(s_k^{P^{k-1}} - u^{P^{k-1}}s_{k+1}^{P^k} + \omega_k^{P^{k-1}})$$

$\bmod I(pPP_{k-1} + Q_{k-1} + Q_{k-2})$, and

$$pPP_{k-1} + PQ_{k-2} > pPP_{k-1} + Q_{k-1} + Q_{k-2}$$
$$> pPP_{k-1} + Q_{k-1} + \tilde{Q}_{k-2} = P_k + Q_k + \tilde{Q}_{k-1}.$$

Since $Q_{k+1-j} < Q_{k-j}$ and

$$d(u^{P_{k+1-j}}\tilde{X}_{k+1-j}) \equiv u^{P_{k+1-j}} s_{k+2-j}^{P^{k+2-j}} \mod I(P_{k+1-j} + Q_{k+1-j})$$

for $j \geq 2$, we have

$$d(\overline{W}_{k+1}) \equiv u^{pP_k} \sum_{j=2}^{k+1} (u_j^{P^k} s_{k+2-j}^{p^j P^{k+1}} - u_j^{p^{k+1-j}P^k Q} s_{k+2-j}^{P^k})$$

mod $I(pP_k + Q_{k-1})$. By Lemma 2.1,

$$d(u^{pP_k} w_{k+1}^{P^k}) \equiv u^{pP_k} \Bigg(u^{P^k} s_{k+2}^{P^{k+1}} + s_{k+1}^{pP^{k+1}} - s_{k+1}^{P^k} + \sum_{j=2}^{k+1} (u_j s_{k+2-j}^{p^j P} - u_j^{p^{k+1-j}Q} s_{k+2-j})^{P^k} - \omega_{k+1}^{P^k} \Bigg)$$

mod $I(pP_k + p(pP)^k Q)$, and so

$$d(W_{k+1}) \equiv u^{pP_k}(u^{P^k} s_{k+2}^{P^{k+1}} + s_{k+1}^{pP^{k+1}} - s_{k+1}^{P^k} - \omega_{k+1}^{P^k})$$

mod $I(pP_k + Q_{k-1})$. If $k = n-1$ and $m \in J_1$, this is replaced by

$$d(W_n) \equiv u^{pP_{n-1}}(s_n^{pP^n} - s_n^{P^{n-1}}) \mod I(pP_{n-1} + P^{n-1}). \tag{3.5}$$

Noticing that $d(X_k^p)$ is congruent to $(-1)^k u^{pP_k} s_{k+1}^{pP^{k+1}}$ modulo $I(pP_k + pQ_k)$, we obtain the congruence on $d(X_k')$, and the induction completes.

Turn to the case for $k = n$. Observing (3.3), (3.4) and (3.5) under the congruences on $d(X_{n-1})$ and $d(X_{n-1}')$, we obtain the congruences

$$d(\overline{X}_n) \equiv \begin{cases} (-1)^{n-1} u^{P_n} \sigma \mod I(pPP_{n-1} + p^{n-2}P^{n-1}) & m \in J_1 \\ (-1)^n u^{P_n} \sigma' \mod I(P_n + P^{n-1}) & m \in J_2, \end{cases}$$

and

$$d(\overline{\overline{X}}_n) \equiv \begin{cases} (-1)^n u^{P_n} \sigma' \mod I(P_n + P^{n-1}) & m \in J_1 \\ (-1)^{n+1} u^{P_n} \sigma \mod I(P_n - Q_{n-1}) & m \in J_2. \end{cases}$$

for $\sigma = s_n^{P^{n-1}} - w_{0,n-2}^{P^{n-2}}$ and $\sigma' = s_n^{P^{n-1}}$. Note that $pPP_{n-1} + p^{n-2}P^{n-1} \geq P_n + P^{n-1}$ if $m \in J_1$, since $p^{n-2}P^{n-1} \geq Q$, and also $0 < -Q_{n-1} < P^{n-1}$ if $m \in J_2$. These show the congruence on $d(X_n)$. □

Lemma 3.6. *There exists an element Y such that $d(Y) \equiv w_{0,n-2}^{P^{n-2}} - X_{n-2}^{p-1} d(X_{n-2}) \mod I(P_{n-2} + P^{n-2}(p^2 - p - 1))$.*

PROOF. Put $Y_k = X_{k-1}^{pP} - X_k$. Then,

$$Y_k \equiv 0 \mod I(pP^k) \tag{3.7}$$

for $(k,m) \in S_1$, since W_k is a multiple of $u^{pP^{k-1}}$ by the definition (3.1). Note that $(pP)^{n-2-k}((p-1)pP^k + pPP_{k-1}) = P_{n-2} + P^{n-2}(p^{n-k} - e(n-k))$. Then,

$$Y_k^{(p-1)(pP)^{n-2-k}} d(Y_k)^{(pP)^{n-2-k}} \equiv Y_k^{(p-1)(pP)^{n-2-k}} (d(X_{k-1}^{pP}) - d(X_k))^{(pP)^{n-2-k}} \equiv 0$$

mod $I(P_{n-2} + P^{n-2}(p^{n-k} - e(n-k)))$. Now, we define the element Y by

$$pY \equiv X_0^{p(pP)^{n-2}} - X_{n-2}^p - \sum_{k=1}^{n-2} Y_k^{p(pP)^{n-2-k}} \mod (p^2, v_1, \dots, v_{n-2}),$$

which is verified to be the one of the lemma by (2.3). □

Consider a numerical sequence $\{\alpha_k\}_{k \geq 0}$ given by the recurrence formula

$$\alpha_k = (pP)^{k-2}\alpha + \alpha_{k-2} \qquad \text{for } k \geq 2. \tag{3.8}$$

Then,

$$\alpha_k = (pP)^{\varepsilon(k)} e_2(k - \varepsilon(k))\alpha + \alpha_{\varepsilon(k)} \tag{3.9}$$

for integers

$$e_2(2k) = \frac{(pP)^{2k} - 1}{(pP)^2 - 1} \qquad \text{and} \qquad \varepsilon(k) = \frac{1 - (-1)^k}{2}.$$

Note that

$$\alpha_k - pP\alpha_{k-1} = \begin{cases} \alpha + \alpha_0 - pP\alpha_1 & \varepsilon(k) = 0 \\ \alpha_1 - pP\alpha_0 & \varepsilon(k) = 1 \end{cases} \tag{3.10}$$

for $k \geq 1$. We introduce a notation $(\alpha, \alpha_0, \alpha_1)_k$, which denotes the number α_k:

$$(\alpha, \alpha_0, \alpha_1)_k = \alpha_k.$$

We introduce integers A_k and c_k for $k \geq 0$ by use of this:

$$A_k = \begin{cases} P_k & (k+1, m) \in S_1 \\ P_{n-1} + Q_{n-1} & (k+1, m) \in S_2 \\ (P_n, P_{n-2}, A_{n-1})_{k-n+2} & k \geq n. \end{cases}$$

$$c_k = \begin{cases} pP^k & (k, m) \in S_1 \\ P_n - p^2PP_{n-2} & (k, m) \in S_2 \\ (pPP_n, c_{n-1}, c_n)_{k-n+1} & k > n \end{cases} \tag{3.11}$$

We replace X_n by

$$X_n - (-1)^n u^{P_n} Y$$

for Y in Lemma 3.6, and define the elements X_k for $k > n$ inductively by

$$X_k = X_{k-1}^{pP} - (-1)^{k-n} u^{(pP)^{k-n} P_n} X_{k-2}^{p-1} Y_{k-2}. \tag{3.12}$$

for

$$Y_k = X_{k-1}^{pP} - X_k.$$

Lemma 3.13. *The element Y_k for $k \geq n-1$ is a multiple of u^{c_k}.*

Proof. The case for $(k,m) \in S_1$ is given by (3.7), and the case for $(k,m) \in S_2$ follows from the fact that $u^{P_n - pP_{n-1}} \overline{W}_n$ is divisible by $u^{P_n - p^2 P P_{n-2}}$. For $k > n$, we see that $c_k = (pP)^{k-n} P_n + c_{k-2}$ by (3.12). □

We further introduce integers A''_k and elements G_k defined by

$$A''_k = \begin{cases} 0 & k < n \\ ((p-1)(pP)^{n-2}, 0, 0)_{k-n+2} & k \geq n \end{cases}$$

$$G_k = \begin{cases} s_{k+1}^{p^{k+1}} & k < n \\ G_{k-2} & k \geq n \end{cases}$$

Then, we have

Lemma 3.14. *For $i \geq 0$,*

$$\begin{aligned} d(X_{n+i}) &\equiv (-1)^n u^{(pP)^i P_n} X_{n+i-2}^{p-1} d(X_{n+i-2}) \mod I(A_{n+i} + (p-1)P^{n+i-2}) \\ &\equiv (-1)^n u^{A_{n+i}} w_0^{A''_{n+i}} G_{n+i} \mod I(A_{n+i} + 1). \end{aligned}$$

Proof. The second congruence follows from the first one, which we show by induction. By Lemmas 3.2 and 3.6, we see that

$$d(X_n) \equiv (-1)^n u^{P_n} X_{n-2}^{p-1} d(X_{n-2}) \mod I(P_n + P_{n-2} + P^{n-2}(p^2 - p - 1)),$$

which shows the case for $i = 0$, since $A_n = P_n + P_{n-2}$.

Suppose that the congruence holds for i. We put $B_i = A_{n+i} + (p-1)P^{n+i-2}$ for $i \geq 0$. Since for $k \geq n$, $A_k - pPA_{k-1} = A_{n-2} + \varepsilon(v)Q_{n-1}$ if $\varepsilon(k-n) = 0$, and $= pP^{n-1} + \varepsilon(v-1)Q_{n-1}$ if $\varepsilon(k-n) = 1$ by (3.10) and (3.11), we see that neither of pPB_i nor $pPA_{n+i} + c_{n+i-1}$ are less than B_{i+1}. The case for $i+1$ is now given as the sum of the congruences

$$d(X_{n+i}^{pP}) \equiv (-1)^n u^{(pP)^{i+1}P_n} X_{n+i-2}^{(p-1)pP} d(X_{n+i-2}^{pP}) \bmod I(pPB_i),$$

$$d((-1)^{n+1} u^{(pP)^{i+1}P_n} X_{n+i-2}^{(p-1)pP} Y_{n+i-1})$$
$$\equiv (-1)^{n+1} u^{(pP)^{i+1}P_n} X_{n+i-2}^{(p-1)pP} (d(X_{n+i-2}^{pP}) - d(X_{n+i-1}))$$

$\bmod I((pP)^{i+1}P_n + pPA_{n+i-2} + c_{n+i-1}) = I(pPA_{n+i} + c_{n+i-1}).$ □

We define integers a_k and a''_k, and elements x_k and g_k as follows:

$$a_{in+j} = p^j A_i$$
$$a''_{in+j} = p^j A''_i$$
$$x_{in+j} = X_i^{p^j}$$
$$g_{in+j} = G_i^{p^j}$$

for $i \geq 0$ and $0 \leq j < n$. We notice that the integer a_k is the same as the one in the introduction.

LEMMA 3.15. *These elements and integers satisfy the assumption of Lemma 1.3.*

PROOF. Note that $t_k^{p^n}$ is homologous to $u_1^{p-1} t_k$ in $u_1^{-1}\Gamma(m+1)/I_n$. Then, the first part of the assumption follows from the definition of elements and Lemmas 3.2 and 3.14.

We prove the second part by showing the following assertion on $\ell \geq 0$.

$(3.16)_{(\ell)}$ The elements $w_0^{(s-1)p^k + a''_k}\gamma_k$ for non-negative integers s, k with $p \nmid s$ and $k < \ell$ are linearly independent over K.

Here, γ_k denotes the elements of $H^1_m M^0_n$ represented by the cocycle g_k. We notice that γ_k for $0 \leq k < n^2$ are the independent generators of the $K[w_0]$-module $H^1_m M^0_n$. For $\ell < n^2$, it is trivial, since $\gamma_k \neq \gamma_{k'}$ if $k \neq k'$ in this range. Suppose that $(3.16)_\ell$ holds for $\ell \geq n^2 - 1$, and that we have $w_0^{(s-1)p^\ell + a''_\ell}\gamma_\ell = \sum_{k=0}^{\ell-1} \lambda_k w_0^{(s_k-1)p^k + a''_k}\gamma_k$ for $\lambda_k \in K$ and $s_k \geq 0$ prime to p. Since $H^1 M^0_n$ is a free K-module over the generators γ_k, we may assume that $\lambda_k \in \mathbf{Z}/p$. We further suppose that there is an integer k such that $\lambda_k \neq 0$. For each integer a, we write $a = i_a n + j_a$ with $0 \leq j_a < n$. Then, we see that $j_k = j_\ell$, and $(s_k - 1)p^k + a''_k = (s-1)p^\ell + a''_\ell$. It follows that $a''_\ell \equiv (s_k - 1)p^k + a''_k \bmod p^\ell$, and so $A''_{i_\ell} - A''_{i_k} \equiv (s_k - 1)(pP)^{i_k} \bmod p^{\ell - j_k}$. On the other hand, by (3.9), we see that $A''_{i_\ell} - A''_{i_k} \equiv \pm(pP)^{n-2} \bmod(p(pP)^{n-2})$ if $i_k \not\equiv i_\ell \bmod 2$, and $\equiv -(pP)^{i_k} \bmod(p(pP)^{i_k})$ otherwise. These imply that $i_k \equiv i_\ell \bmod 2$ and that p divides s_k, which contradicts to the hypothesis of s_k. Hence, the coefficients λ_k's are all zero, and $(3.16)_{(\ell+1)}$ holds. Thus, the induction completes. □

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