

## Characterizations of BMO by $A_p$ weights and $p$ -convexity

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**ABSTRACT.** We show that the Lebesgue spaces for defining BMO can be replaced by  $p$ -convex rearrangement-invariant quasi-Banach function spaces associated with  $A_p$ -weighted measures.

### 1. Introduction

In this paper, we apply the notion of  $p$ -convexity to study the characterizations of BMO by rearrangement-invariant quasi-Banach function spaces (r.-i.q-B.f.s) on  $(\mathbf{R}^n, \omega)$  where  $\omega \in A_\infty$ .

The notion of  $p$ -convexity  $1 \leq p \leq \infty$  for Banach lattices was introduced in [2, 4, 10]. For the extension of the notion of  $p$ -convexity to quasi-Banach space, the reader is referred to [3] p. 156.

The notion of  $p$ -convexity was used to study the isomorphic properties of Banach lattices (see [11] Volume II, Section 1.d). In this paper, we find that BMO can be characterized by an r.-i.q-B.f.s. on  $(\mathbf{R}^n, \omega)$ ,  $\omega \in A_p$ , if it is  $p$ -convex.

On one hand, this paper shows that  $p$ -convexity is not only an abstract notion arising from the Banach space geometry, it also has applications on the study of some concrete function spaces. On the other hand, this paper generalizes the characterizations of BMO by replacing the Lebesgue spaces  $L^p$ ,  $1 \leq p < \infty$ , by rearrangement-invariant quasi-Banach function spaces.

We recall some results on the characterizations of BMO. Let  $\mathcal{B} = \{B(x_0, r) : x_0 \in \mathbf{R}^n, r > 0\}$  where  $B(x_0, r) = \{x \in \mathbf{R}^n : |x - x_0| < r\}$ . Let us denote the center and the radius of  $B \in \mathcal{B}$  by  $c_B$  and  $r_B$ , respectively. Let  $\mathcal{M}_0$  be the set of locally integrable function on  $\mathbf{R}^n$ .

Recall that a locally integrable function  $f$  belongs to BMO if

$$\|f\|_{\text{BMO}} = \sup_{B \in \mathcal{B}} \frac{\|(f - f_B)\chi_B\|_{L^1}}{\|\chi_B\|_{L^1}} < \infty$$

where  $f_B = \frac{1}{|B|} \int_B f(x) dx$ .

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More generally, BMO can be defined via the  $L^p$  norm. That is,

$$\text{BMO} = \left\{ f \in \mathcal{M}_0 : \sup_{B \in \mathcal{B}} \frac{\|(f - f_B)\chi_B\|_{L^p}}{\|\chi_B\|_{L^p}} < \infty \right\}.$$

Thus, the characterization of BMO can be generalized by examining whether we have the following identification

$$\text{BMO} = \left\{ f \in \mathcal{M}_0 : \sup_{B \in \mathcal{B}} \frac{\|(f - f_B)\chi_B\|_{L^p(\omega)}}{\|\chi_B\|_{L^p(\omega)}} < \infty \right\} \tag{1.1}$$

where  $\omega \in A_\infty$ .

As Lemma 2 shows, one inclusion is easily obtained. One of the main result of this paper is that the identity (1.1) is true provided that  $\omega \in A_p$ , see Theorem 3.1.

We further extend our result by considering the r.i.q-B.f.s.  $Y_\omega$  on  $(\mathbf{R}^n, \omega)$  where  $\omega \in A_\infty$ . That is, we investigate whether the characterization

$$\text{BMO} = \left\{ f \in \mathcal{M}_0 : \sup_{B \in \mathcal{B}} \frac{\|(f - f_B)\chi_B\|_{Y_\omega}}{\|\chi_B\|_{Y_\omega}} < \infty \right\} \tag{1.2}$$

is valid. This result is presented in Theorem 3.5.

For any r.i.q-B.f.s  $Y_\omega$  on  $(\mathbf{R}^n, \omega)$ , we introduce  $\text{BMO}_{Y_\omega}$ . It is defined by

$$\text{BMO}_{Y_\omega} = \left\{ f \in \mathcal{M}_0 : \sup_{B \in \mathcal{B}} \frac{\|(f - f_B)\chi_B\|_{Y_\omega}}{\|\chi_B\|_{Y_\omega}} < \infty \right\}.$$

Write  $\|f\|_{\text{BMO}_{Y_\omega}} = \sup_{B \in \mathcal{B}} \frac{\|(f - f_B)\chi_B\|_{Y_\omega}}{\|\chi_B\|_{Y_\omega}}$ .

To prove the embedding  $\text{BMO} \hookrightarrow \text{BMO}_{Y_\omega}$ , we establish the John-Nirenberg inequality for r.i.q-B.f.s on  $(\mathbf{R}^n, \omega)$ , see Proposition 3.2. In fact, with that proposition, the embedding

$$\text{BMO} \hookrightarrow \text{BMO}_{Y_\omega} \tag{1.3}$$

holds for r.i.q-B.f.s  $Y_\omega$  on  $(\mathbf{R}^n, \omega)$ ,  $\omega \in A_\infty$ .

The unweighted version of the characterization (1.2) is presented in [7]. For the unweighted case, an expected condition on the Boyd indices of the rearrangement-invariant Banach function space (r.i.B.f.s.)  $X$  on  $(\mathbf{R}^n, |\cdot|)$  where  $|\cdot|$  is the Lebesgue measure is enough to guarantee the characterization of BMO by  $X$ . However, when  $Y_\omega$  is an r.i.q-B.f.s. on  $(\mathbf{R}^n, \omega)$  and  $\omega \in A_p$ , we need the notion of  $p$ -convexity to establish the reverse inclusion of (1.3).

Moreover, for a general r.i.q-B.f.s.  $Y_\omega$  on  $(\mathbf{R}^n, \omega)$ , it is not necessarily a subset of  $\mathcal{M}_0$ . That is,  $f_B$  is not necessarily well defined for any  $B \in \mathcal{B}$  and  $f \in Y_\omega$ .

In Theorem 3.5, we figure out a condition imposed on an r.i.q-B.f.s. so that on one hand, it is a subset of  $\mathcal{M}_0$  and on the other hand, the characterization (1.2) is valid. More precisely, we find that if  $Y_\omega$  is  $p$ -convex and its Boyd's indices satisfying  $p \leq p_{Y_\omega} \leq q_{Y_\omega} < \infty$ , then  $Y_\omega \subset \mathcal{M}_0$  and we have the characterization (1.2).

**2. Background materials**

We present the notations and terminologies used in this paper.

Even though the  $A_p$  class is well-known, for completeness, we offer the definition of  $A_p$  weight functions.

DEFINITION 2.1. For  $1 < p < \infty$ , a locally integrable function  $\omega : \mathbf{R}^n \rightarrow [0, \infty)$  is said to be an  $A_p$  weight if

$$\sup_{B \in \mathcal{B}} \left( \frac{1}{|B|} \int_B \omega(x) dx \right) \left( \frac{1}{|B|} \int_B \omega(x)^{-p'/p} dx \right)^{p/p'} < \infty$$

where  $p' = \frac{p}{p-1}$ . A locally integrable function  $\omega : \mathbf{R}^n \rightarrow [0, \infty)$  is said to be an  $A_1$  weight if

$$\frac{1}{|B|} \int_B \omega(y) dy \leq C\omega(x), \quad a.e. \ x \in B$$

for some constant  $C > 0$ . We define  $A_\infty = \bigcup_{p \geq 1} A_p$ .

For any  $\omega \in A_\infty$  and any Lebesgue measurable set  $E$ , write  $\omega(E) = \int_E \omega(x) dx$ . We have the following characterization of  $A_\infty$  weight (see [6] Theorem 9.3.3 (d)).

THEOREM 2.1. A locally integrable function  $\omega : \mathbf{R}^n \rightarrow [0, \infty)$  belongs to  $A_\infty$  if and only if there exist an  $\varepsilon > 0$  and a constant  $C_0 > 0$  such that for any  $B \in \mathcal{B}$  and all measurable subsets  $E$  of  $B$ , we have

$$\frac{\omega(E)}{\omega(B)} \leq C_0 \left( \frac{|E|}{|B|} \right)^\varepsilon. \tag{2.1}$$

We recall the John-Nirenberg inequality in the next theorem (see [6], Theorem 7.1.6).

THEOREM 2.2. There exist constants  $C_1, C_2 > 0$  such that for any  $\gamma > 0$  and any  $B \in \mathcal{B}$ ,

$$|\{x \in B : |f(x) - f_B| > \gamma\}| \leq C_1 \exp \left[ -\frac{C_2 \gamma}{\|f\|_{BMO}} \right] |B|, \quad f \in BMO \setminus \mathcal{C}$$

where  $\mathcal{C}$  denotes the set of constant functions.

We state some background materials for rearrangement-invariant quasi-Banach function spaces.

Let  $\omega \in A_\infty$ . For any Lebesgue measurable function  $f$ , denote its decreasing rearrangement with respect to  $(\mathbf{R}^n, \omega)$  by  $f^{*, \omega}$ .

We recall the definition of rearrangement-invariant Banach function space from [1, Chapter 1, Definitions 1.1 and 1.3, and Chapter 2, Definition 4.1]. For any  $\omega \in A_\infty$ , let  $\mathcal{M}_\omega$  denote the class of  $\omega$ -measurable functions.

**DEFINITION 2.2.** Let  $\omega \in A_\infty$ . A mapping  $\rho : \mathcal{M}_\omega \rightarrow [0, \infty]$  is said to be a rearrangement-invariant Banach function norm if for all  $\omega$ -measurable functions  $f, g, \{f_n\}_{n=1}^\infty$  on  $\mathbf{R}^n$  and  $a > 0$ , we have

- (1)  $\rho(f) = 0 \Leftrightarrow f = 0$   $\omega$ -a.e.,  $\rho(af) = a\rho(f)$ ,  $\rho(f + g) \leq \rho(f) + \rho(g)$
- (2)  $0 \leq g \leq f$   $\omega$ -a.e.  $\Rightarrow \rho(g) \leq \rho(f)$
- (3)  $0 \leq f_n \uparrow f$   $\omega$ -a.e.  $\Rightarrow \rho(f_n) \uparrow \rho(f)$
- (4)  $\omega(E) < \infty \Rightarrow \rho(\chi_E) < \infty$
- (5)  $\omega(E) < \infty \Rightarrow \int_E f(x)\omega(x)dx \leq C_E\rho(f)$  for some  $C_E > 0$ .
- (6)  $\rho(f) = \rho(g)$  for every pair of equimeasurable functions  $f, g$ .

The collection  $Y_\omega$  of all functions  $f$  in  $\mathcal{M}_\omega$  for which  $\rho(|f|) < \infty$  is called a *rearrangement-invariant Banach function space* (r.-i.B.f.s.). The norm of  $Y_\omega$  is given by  $\|\cdot\|_{Y_\omega} = \rho(|\cdot|)$ .

For any r.-i.B.f.s.  $Y_\omega$ , according to the Luxemburg representation theorem (see [1] Chapter 2, Theorem 4.10), we have a norm  $\rho_{Y_\omega} : \mathcal{M}([0, \infty)) \rightarrow [0, \infty)$  where  $\mathcal{M}([0, \infty))$  is the set of Lebesgue measurable functions on  $[0, \infty)$  such that

$$\|f\|_{Y_\omega} = \rho_{Y_\omega}(f^{*, \omega}).$$

We find that this property is crucial on the definition of Boyd's indices. On the other hand, the validity of the Luxemburg representation theorem relies on the fact that the associated space of an r.-i.B.f.s. is not trivial. But for a general quasi-Banach function space, the associated space may be trivial. That is, item (5) of Definition 2.2 does not necessarily hold for a general quasi-Banach function space. For instance, when  $0 < p < 1$ , the associated space of  $L^p(\mathbf{R}^n)$  is equal to  $\{0\}$ .

Therefore, we use the subsequent definition for rearrangement-invariant quasi-Banach function spaces.

**DEFINITION 2.3.** Let  $\omega \in A_\infty$ . A quasi-Banach function space  $Y_\omega$  on  $(\mathbf{R}^n, \omega)$  is rearrangement-invariant (r.-i.) if

- (1)  $\|\cdot\|_{Y_\omega}$  is a quasi-norm;
- (2)  $\|\cdot\|_{Y_\omega}$  satisfy item (2)–(4) in Definition 2.2;

(3) there exists a quasi-norm  $\rho_{Y_\omega} : \mathcal{M}([0, \infty)) \rightarrow [0, \infty]$  such that

$$\|f\|_{Y_\omega} = \rho_{Y_\omega}(f^{*,\omega}).$$

We combine the definition of Boyd indices for r.-i.B.f.s. from [1], Chapter 3, Definition 5.10 and the definition of Boyd indices for r.-i.q-B.f.s. from [12] to give the following definition.

DEFINITION 2.4. For each  $t > 0$  and any Lebesgue measurable function  $f$  on  $[0, \infty)$ , let  $E_t$  denote the dilation operator defined by

$$(E_t f)(x) = f(tx), \quad x \geq 0.$$

The Boyd indices of an r.-i.q-B.f.s.  $Y_\omega$  are the numbers defined by

$$\begin{aligned} p_{Y_\omega} &= \sup\{p : \exists C > 0, \forall t < 1, \rho_{Y_\omega}(E_t f) \leq Ct^{-1/p} \rho_{Y_\omega}(f)\}, \\ q_{Y_\omega} &= \inf\{q : \exists C > 0, \forall t > 1, \rho_{Y_\omega}(E_t f) \leq Ct^{-1/q} \rho_{Y_\omega}(f)\}. \end{aligned}$$

We have  $0 \leq p_{Y_\omega} \leq q_{Y_\omega} \leq \infty$ . For any quasi-Banach function space  $Y_\omega$ , let  $Y'_\omega$  be the associated space (the Köthe dual) of  $Y_\omega$  (see [13] p. 35).

LEMMA 1. Let  $Y_\omega$  be an r.-i.B.f.s. on  $(\mathbf{R}^n, \omega)$ . For any Lebesgue measurable set  $E$  with  $\omega(E) < \infty$ , we have

$$\|\chi_E\|_{Y_\omega} \|\chi_E\|_{Y'_\omega} = \omega(E). \tag{2.2}$$

The above lemma is crucial to establish the main result in [7]. The proof of the above lemma is given in [1] Chapter 2, Theorem 5.2.

The identification  $\text{BMO}_{L^p(\omega)} = \text{BMO}$  is valid provided that  $\omega \in A_p$ . To apply this result to r.-i.q-B.f.s. on  $(\mathbf{R}^n, \omega)$ , we introduce the notion of  $p$ -th power ( $1/p$ -convexification). For any  $0 < p < \infty$  and any quasi-Banach function space  $Y_\omega$  on  $(\mathbf{R}^n, \omega)$ , define the  $p$ -th power of  $Y_\omega$ ,  $Y_\omega^p$  by

$$f \in Y_\omega^p \Leftrightarrow |f|^{1/p} \in Y_\omega,$$

and the quasi-norm of  $Y_\omega^p$  is defined by  $\|f\|_{Y_\omega^p} = \| |f|^{1/p} \|_{Y_\omega}^p$ . The reader is referred to [13] Section 2.2 for a complete discussion on the notion of  $p$ -th power of quasi-Banach function space. For  $0 < p < 1$ ,  $Y_\omega^p$  is a Banach space (see [5] Proposition 1.11) while for  $1 < p < \infty$ , it is a quasi-Banach space (see [13] Chapter 2, Proposition 2.22).

As claimed on the introduction, conditions on the Boyd indices are not sufficient to assert the characterization of BMO by  $A_p$  weights. We need another notion from the geometry of quasi-Banach space. Let  $0 < p < \infty$ . A

quasi-Banach function space  $X$  is said to be  $p$ -convex if there exists a constant  $C > 0$  such that

$$\left\| \left( \sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_X \leq C \left( \sum_{i=1}^n \|f_i\|_X^p \right)^{1/p}$$

for any  $\{f_i\}_{i=1}^n \subset X$ .

Any  $p$ -convex quasi-Banach function space is also  $r$ -convex provided that  $0 < r \leq p$  (see [3] Lemma 4).

The following proposition gives a procedure to obtain an equivalent norm for a  $p$ -convex quasi-Banach function space. That procedure was already presented in [4, 10] for Banach lattices.

**PROPOSITION 2.3.** *Let  $1 \leq p < \infty$ . If the quasi-Banach function space  $Y_\omega$  is  $p$ -convex, then*

$$\eta_{[p]}(f) = \inf \left\{ \sum_{i=1}^n \|f_i\|_{Y_\omega^p} : |f| \leq \sum_{i=1}^n |f_i|, f_i \in Y_\omega^p, i = 1, 2, \dots, n, n \in \mathbf{N} \right\} \quad (2.3)$$

is a lattice norm and is equivalent to  $\|\cdot\|_{Y_\omega^p}$ . Hence,  $Y_\omega$  is normable and admits

$$\eta(f) = (\eta_{[p]}(|f|^p))^{1/p}$$

as an equivalent lattice norm.

The proof of the above proposition is given by [13, Proposition 2.23].

### 3. The characterizations of BMO

We present several embedding and characterizations of BMO in this section. We first pay our attention to the weighted Lebesgue space  $L^p(\omega)$ ,  $1 \leq p < \infty$ ,  $\omega \in A_\infty$ .

**LEMMA 2.** *Let  $0 < p < \infty$  and  $\omega \in A_\infty$ . We have*

$$BMO \hookrightarrow BMO_{L^p(\omega)}. \quad (3.1)$$

**PROOF.** According to the John-Nirenberg inequality, we have

$$|\{x \in B : |f(x) - f_B| > \gamma\}| \leq C_1 \exp \left[ -\frac{C_2 \gamma}{\|f\|_{BMO}} \right] |B|.$$

Applying inequality (2.1), we find that

$$\omega(\{x \in B : |f(x) - f_B| > \gamma\}) \leq C_0 C_1^p \exp \left[ -\frac{C_2 \varepsilon \gamma}{\|f\|_{BMO}} \right] \omega(B) \quad (3.2)$$

for some  $C_0 > 0$ . Hence, there exists constant  $C_3 > 0$  so that for any  $B \in \mathcal{B}$ ,

$$\begin{aligned} & \frac{1}{\omega(B)} \|(f - f_B)\chi_B\|_{L^p(\omega)}^p \\ &= \frac{p}{\omega(B)} \int_0^\infty \gamma^{p-1} \omega(\{x \in B : |f(x) - f_B| > \gamma\}) d\gamma \\ &\leq C_0 C_1^\varepsilon p \int_0^\infty \gamma^{p-1} \exp\left[-\frac{C_2 \varepsilon \gamma}{\|f\|_{\text{BMO}}}\right] d\gamma \leq C_3 \|f\|_{\text{BMO}}^p. \end{aligned}$$

Therefore, the embedding (3.1) is valid. □

LEMMA 3. *If  $1 \leq p < \infty$  and  $\omega \in A_p$ , then*

$$BMO_{L^p(\omega)} \hookrightarrow BMO. \tag{3.3}$$

PROOF. When  $1 < p$ , by the Hölder inequality, we obtain

$$\int_B |f(x) - f_B| dx \leq \left( \int_B |f(x) - f_B|^p \omega(x) dx \right)^{1/p} \left( \int_B \omega(x)^{-p'/p} dx \right)^{1/p'}.$$

Therefore, the  $A_p$  condition concludes that

$$\int_B |f(x) - f_B| dx \leq C \|(f - f_B)\chi_B\|_{L^p(\omega)} \frac{|B|}{\omega(B)^{1/p}}.$$

Similarly, the proof of the embedding (3.3) for  $p = 1$  follows from the definition of  $A_1$  weight functions. □

The above lemmas offer a new characterization for BMO by  $L^p(\omega)$ ,  $\omega \in A_p$ . We will generalize this characterization of BMO to r.i.q-B.f.s. on the rest of this section. Even though the characterization of BMO by  $L^p(\omega)$  is a special case of the following results, the proof for the general case is, in fact, based on this special case.

THEOREM 3.1. *Let  $1 \leq p < \infty$  and  $\omega \in A_p$ . We have*

$$BMO = BMO_{L^p(\omega)}.$$

*The norms are mutually equivalent.*

We now turn our attention to the r.i.q-B.f.s. on  $(\mathbf{R}^n, \omega)$ ,  $\omega \in A_\infty$ .

We prove the embedding  $BMO \hookrightarrow BMO_{Y_\omega}$  by using the John-Nirenberg inequality. We have a supporting lemma for establishing the John-Nirenberg inequality on  $Y_\omega$ .

LEMMA 4. Let  $\omega \in A_\infty$ . If  $Y_\omega$  is an r.-i.-q.-B.f.s. on  $(\mathbf{R}^n, \omega)$  with  $q_{Y_\omega} < \infty$ , then for any  $q > q_{Y_\omega}$ , there exists a constant  $C > 0$  such that for any  $x_0 \in \mathbf{R}^n$  and  $R > r > 0$ , we have

$$\frac{\|\chi_{B(x_0, r)}\|_{Y_\omega}}{\|\chi_{B(x_0, R)}\|_{Y_\omega}} \leq C \left( \frac{\omega(B(x_0, r))}{\omega(B(x_0, R))} \right)^{1/q}. \quad (3.4)$$

The proof of Lemma 4 follows from the definition of Boyd's indices and the facts that

$$\frac{\|\chi_{B(x_0, r)}\|_{Y_\omega}}{\|\chi_{B(x_0, R)}\|_{Y_\omega}} = \frac{\rho_{Y_\omega}(\chi_{[0, \omega(B(x_0, r))])}}{\rho_{Y_\omega}(\chi_{[0, \omega(B(x_0, R))])}}$$

and  $E_t \chi_{[0, \omega(B(x_0, R))]} = \chi_{[0, \omega(B(x_0, r))]}$  where  $t = \frac{\omega(B(x_0, R))}{\omega(B(x_0, r))}$ .

We have the following John-Nirenberg inequality for r.-i.-q.-B.f.s..

PROPOSITION 3.2. Let  $\omega \in A_\infty$ . If  $Y_\omega$  is an r.-i.-q.-B.f.s. on  $(\mathbf{R}^n, \omega)$  with  $q_{Y_\omega} < \infty$ , then there exist  $K_1, K_2 > 0$  such that for any  $\gamma > 0$  and any  $B \in \mathcal{B}$ ,

$$\|\chi_{\{x \in B: |f(x) - f_B| > \gamma\}}\|_{Y_\omega} \leq K_1 \exp\left[-\frac{K_2 \gamma}{\|f\|_{BMO}}\right] \|\chi_B\|_{Y_\omega}, \quad \forall f \in BMO \setminus \mathcal{C}.$$

PROOF. As  $\omega \in A_\infty = \bigcup_{1 \leq p < \infty} A_p$ , we have  $C_4, \delta > 0$  (see [6] Corollary 9.3.4 and Proposition 9.1.5.(9)) such that for any  $\lambda > 1$ ,

$$\omega(B(x_0, \lambda r)) \leq C_4 \lambda^\delta \omega(B(x_0, r)), \quad \forall x_0 \in \mathbf{R}^n.$$

It suffices to consider the case when  $\gamma$  is large, so, without loss of generality, we assume that  $(C_4 C_0)^{1/\delta} C_1^{e/\delta} \exp\left[-\frac{C_2 e \gamma}{\delta \|f\|_{BMO}}\right] < 1$ .

According to the John-Nirenberg inequality for  $\omega$  (see (3.2)), we find that

$$\omega(\{x \in B: |f(x) - f_B| > \gamma\}) \leq \omega(\tilde{B})$$

where  $\tilde{B} \in \mathcal{B}$  with

$$c_{\tilde{B}} = c_B \quad \text{and} \quad r_{\tilde{B}} = (C_4 C_0)^{1/\delta} C_1^{e/\delta} \exp\left[-\frac{C_2 e \gamma}{\delta \|f\|_{BMO}}\right] r_B.$$

As  $Y_\omega$  is r.-i. with respect to  $\omega$  (see [1] Chapter 2, Definition 5.1 and Corollary 5.3), we assert that

$$\|\chi_{\{x \in B: |f(x) - f_B| > \gamma\}}\|_{Y_\omega} \leq \|\chi_{\tilde{B}}\|_{Y_\omega}.$$

In view of  $q_{Y_\omega} < \infty$ , (2.1) and Lemma 4 guarantee that for any  $q > q_{Y_\omega}$

$$\|\chi_{\{x \in B: |f(x) - f_B| > \gamma\}}\|_{Y_\omega} \leq K_1 \exp\left[-\frac{K_2 \gamma}{\|f\|_{BMO}}\right] \|\chi_B\|_{Y_\omega}$$

where  $K_1 = (C_4 C_0)^{ne/q\delta} C_1^{ne^2/q\delta}$  and  $K_2 = \frac{C_2 ne^2}{q\delta}$ .  $\square$

**THEOREM 3.3.** *Let  $\omega \in A_\infty$ . Suppose that  $Y_\omega$  is an  $r$ - $i$ - $q$ -B.f.s. on  $(\mathbf{R}^n, \omega)$  with  $q_{Y_\omega} < \infty$ . Then, we have the embedding*

$$BMO \hookrightarrow BMO_{Y_\omega}. \tag{3.5}$$

**PROOF.** Let  $\kappa$  be the Aoki-Rolewicz index for the quasi-Banach function space  $Y_\omega$  (see [9] Theorem 1.3). That is,  $\kappa$  is a number such that  $\|\cdot\|_{Y_\omega}^\kappa$  is sub-additive on  $Y_\omega$ . For any  $j \in \mathbf{N}$ , Proposition 3.2 gives

$$\|\chi_{\{x \in B: 2^j < |f(x) - f_B| \leq 2^{j+1}\}}\|_{Y_\omega}^\kappa \leq K_1^\kappa \exp\left[-\frac{\kappa K_2 2^j}{\|f\|_{BMO}}\right] \|\chi_B\|_{Y_\omega}^\kappa$$

because  $q_{Y_\omega} < \infty$ . Multiplying  $2^{(j+1)\kappa}$  on both sides and summing over  $j$ , we find that

$$\|(f - f_B)\chi_B\|_{Y_\omega}^\kappa \leq C \|f\|_{BMO}^\kappa \|\chi_B\|_{Y_\omega}^\kappa \tag{3.6}$$

for some constant  $C > 0$ . More precisely, we have the above inequality because

$$\sum_{j \in \mathbf{N}} 2^{jk} \exp\left[-\frac{\kappa K_2 2^j}{\|f\|_{BMO}}\right] \leq C \int_0^\infty s^{\kappa-1} \exp\left[-\frac{\kappa K_2 s}{\|f\|_{BMO}}\right] ds \leq C \|f\|_{BMO}^\kappa$$

for some constant  $C > 0$  independent of  $f \in BMO$  and  $B \in \mathcal{B}$ . The embedding (3.5) follows from inequality (3.6). □

The condition  $q_{Y_\omega} < \infty$  is the best condition for the embedding (3.5) in term of Boyd's indices. For instance, when  $Y_\omega = L^\infty$ , the upper Boyd indices of  $Y_\omega$  is infinity. We see that the embedding (3.5) does not hold because  $BMO_{L^\infty} = L^\infty$  (see [7]).

**COROLLARY 3.4.** *Let  $\omega \in A_\infty$ . If  $Y_\omega$  is an  $r$ - $i$ - $q$ -B.f.s. on  $(\mathbf{R}^n, \omega)$  with  $q_{Y_\omega} < \infty$ , then for any  $f \in BMO$  and for all  $\mu < \frac{K_2}{\|f\|_{BMO}}$  ( $K_2$  is given in Proposition 3.2), we have*

$$\|\exp[\mu|f - f_B|]\chi_B\|_{Y_\omega} \leq C(\mu, f) \|\chi_B\|_{Y_\omega},$$

for some constant  $C(\mu, f)$  independent of  $B \in \mathcal{B}$ .

To obtain the embedding of  $BMO_{Y_\omega} \hookrightarrow BMO$ , we encounter a technical obstacle. Any  $p$ -convex  $r$ - $i$ - $q$ -B.f.s.  $Y_\omega$  on  $(\mathbf{R}^n, \omega)$  possesses two quasi-norms. The first one  $\|\cdot\|_{Y_\omega}$  is not a norm but it is rearrangement-invariant. The second one  $\eta(\cdot)$  is a norm but it is not necessarily rearrangement-invariant.

On one hand, the merit of having an rearrangement-invariant quasi-norm is that the Boyd type interpolation theorem can be applied to  $Y_\omega$  (see [12]). On the other hand, the advantage of possessing a norm is that the associate

space (Köthe dual) of  $Y_\omega$  is non-trivial. Even though  $Y_\omega$  does not necessarily have an rearrangement-invariant norm, the following lemma shows how to incorporate these two separated properties to obtain a generalization of Lemma 1 for  $Y_\omega$ .

LEMMA 5. *Let  $\omega \in A_\infty$ . Suppose that  $Y_\omega$  is an r.-i.-q.-B.f.s. on  $(\mathbf{R}^n, \omega)$  with  $1 < p_{Y_\omega} \leq q_{Y_\omega} < \infty$  and  $Y_\omega$  is 1-convex. Then, the associate space of  $Y_\omega$  is nontrivial and we have two constants  $D_1, D_2 > 0$  such that for any Lebesgue measurable set  $E$  with  $\omega(E) < \infty$ ,*

$$D_1\omega(E) \leq \|\chi_E\|_{Y_\omega} \|\chi_E\|_{Y'_\omega} \leq D_2\omega(E). \tag{3.7}$$

PROOF. As  $Y_\omega$  is 1-convex, it possesses an equivalent lattice norm  $\eta$  and, hence,  $Y'_\omega \neq \{0\}$ . Denote the associate norm for  $\eta$  by  $\eta'$ . Thus,  $\eta'(\cdot)$  is an equivalent norm for  $\|\cdot\|_{Y'_\omega}$ . Furthermore, by [1], Chapter 1, Theorem 2.4, we have a constant  $C > 0$  so that

$$\int_{\mathbf{R}^n} |f(x)g(x)|\omega(x)dx \leq \eta(f)\eta'(g) \leq C\|f\|_{Y_\omega}\|g\|_{Y'_\omega}$$

for any  $f \in Y_\omega$  and  $g \in Y'_\omega$ . The first inequality in (3.7) follows by taking  $f = g = \chi_E$ .

To establish the second inequality of (3.7), we consider the linear operator

$$P_E(f) = \left( \frac{1}{\omega(E)} \int_E f(x)\omega(x)dx \right) \chi_E$$

where  $E$  is a Lebesgue measurable set with  $\omega(E) < \infty$ . For any  $1 \leq p \leq \infty$ ,  $P_E$  is uniformly bounded on  $L^p(\omega)$ . More precisely,  $\|P_E\|_{L^p(\omega) \rightarrow L^p(\omega)} = 1$ . Thus, for any  $1 \leq p, q \leq \infty$  and  $E$ ,  $P_E$  is of joint weak type  $(p, p; q, q)$  (see [1], Chapter 4, Theorem 4.11). According to Theorem 3 of [12] or Theorem 4.4 of [8],  $P_E$  is bounded on  $Y_\omega$  with operator norm independent of  $E$ . That is, there is a constant  $C > 0$  such that for any Lebesgue measurable set  $E$  and any  $f \in Y_\omega$ ,

$$\frac{1}{\omega(E)} \left| \int_E f(x)\omega(x)dx \right| \|\chi_E\|_{Y_\omega} = \|P_E(f)\|_{Y_\omega} \leq C\|f\|_{Y_\omega}.$$

Then,

$$\|\chi_E\|_{Y'_\omega} = \sup \left\{ \left| \int_E f(x)\omega(x)dx \right| : \|f\|_{Y_\omega} \leq 1 \right\} \leq C \frac{\omega(E)}{\|\chi_E\|_{Y_\omega}}. \quad \square$$

We now present the main results of this paper. Theorems 3.5 and 3.6 give the characterizations of BMO by  $A_p$  weighted r.-i.-q.-B.f.s. for  $p > 1$  and  $p = 1$ ,

respectively. To obtain Theorem 3.5, we use the openness property for  $A_p$  weight functions when  $p > 1$  (see [6] Corollary 9.2.6). That is,

$$A_p = \bigcup_{1 < r < p} A_r. \tag{3.8}$$

**THEOREM 3.5.** *Let  $1 < p < \infty$  and  $\omega \in A_p$ . Suppose that  $Y_\omega$  is an  $r$ -i.q-B.f.s. on  $(\mathbf{R}^n, \omega)$  with  $p \leq p_{Y_\omega} \leq q_{Y_\omega} < \infty$  and  $Y_\omega$  is  $p$ -convex. Then,  $Y_\omega \subseteq \mathcal{M}_0$ ,*

$$BMO_{Y_\omega} = BMO$$

and  $\|\cdot\|_{BMO_{Y_\omega}}$  is an equivalent norm of  $\|\cdot\|_{BMO}$ .

**PROOF.** We first show that  $Y_\omega \subseteq \mathcal{M}_0$ . Using property (3.8), we see that  $\omega \in A_r$  for some  $r$  slightly smaller than  $p$ . By using Lemma 4 of [3],  $Y_\omega$  is  $r$ -convex. Therefore,  $\eta_{[r]}$  is well-defined. Denote the associate norm of  $\eta_{[r]}$  by  $\eta'_{[r]}$ . From Lemma 5, for any  $B \in \mathcal{B}$ , we have  $\chi_B \in Y_\omega \cap Y'_\omega$ . For any  $f \in Y_\omega$ , the Hölder inequality ensures that

$$\begin{aligned} |f|_B &= \frac{1}{|B|} \int_B |f(x)| dx \\ &\leq \frac{1}{|B|} \left( \int_B |f(x)|^r \omega(x) dx \right)^{1/r} \left( \int_B \omega^{-r'/r}(x) dx \right)^{1/r'} \end{aligned}$$

where  $r'$  is the conjugate of  $r$ . Using the Hölder inequality for  $\eta_{[r]}$  and the definition of  $A_r$ , we obtain

$$|f|_B \leq C(\eta_{[r]}(|f|^r)\eta'_{[r]}(\chi_B))^{1/r} \frac{1}{\omega(B)^{1/r}} \leq C\eta(f)(\eta'_{[r]}(\chi_B))^{1/r} \frac{1}{\omega(B)^{1/r}}.$$

As  $Y_\omega^r$  is  $\frac{p}{r}$ -convex and the lower Boyd index of  $Y_\omega^r$  satisfies  $p_{Y_\omega^r} = \frac{pY_\omega}{r} \geq \frac{p}{r} > 1$ , we are allowed to apply Lemma 5 to  $\|\cdot\|_{Y_\omega^r}$ . Moreover,  $\eta$ ,  $\eta_{[r]}$  and  $\eta'_{[r]}$  are equivalent to  $\|\cdot\|_{Y_\omega}$ ,  $\|\cdot\|_{Y_\omega^r}$  and  $\|\cdot\|_{(Y_\omega^r)'}$ , respectively. Thus,

$$|f|_B \leq C\|f\|_{Y_\omega} \left( \frac{\omega(B)}{\|\chi_B\|_{Y_\omega^r}} \right)^{1/r} \frac{1}{\omega(B)^{1/r}} \leq C\|f\|_{Y_\omega} \frac{1}{\|\chi_B\|_{Y_\omega}} < \infty.$$

That is,  $f_B$  is well-defined and  $Y_\omega \subseteq \mathcal{M}_0$ .

It remains to prove the embedding  $BMO_{Y_\omega} \hookrightarrow BMO$ . Theorem 3.1 ensures that  $BMO_{L^r(\omega)} = BMO$ . Thus, for any  $f \in BMO_{Y_\omega}$ , we obtain

$$\int_B |f(x) - f_B|^r \omega(x) dx \leq \eta_{[r]}(|f - f_B|^r \chi_B) \eta'_{[r]}(\chi_B).$$

Similarly, as  $Y_\omega^r$  is  $\frac{p}{r}$ -convex and  $p_{Y_\omega^r} = \frac{pY_\omega}{r} \geq \frac{p}{r} > 1$ , applying Lemma 5 to  $\|\cdot\|_{Y_\omega^r}$  again and using the fact that  $\eta_{[r]}$  and  $\eta'_{[r]}$  are equivalent to  $\|\cdot\|_{Y_\omega^r}$  and  $\|\cdot\|_{(Y_\omega^r)'}$ , respectively, we obtain

$$\begin{aligned} \int_B |f(x) - f_B|^r \omega(x) dx &\leq C \frac{(\eta(|f - f_B| \chi_B))^r \omega(B)}{\|\chi_B\|_{Y_\omega^r}} \\ &= C \frac{\| |f - f_B| \chi_B \|_{Y_\omega^r}^r \omega(B)}{\|\chi_B\|_{Y_\omega^r}^r} \end{aligned}$$

where the constant  $C > 0$  is independent of  $B \in \mathcal{B}$  and  $f \in \text{BMO}_{Y_\omega}$ . Hence, the inequality

$$\frac{\|(f - f_B)\chi_B\|_{L^r(\omega)}}{\|\chi_B\|_{L^r(\omega)}} \leq C \frac{\| |f - f_B| \chi_B \|_{Y_\omega}}{\|\chi_B\|_{Y_\omega}}$$

is valid and the embedding  $\text{BMO}_{Y_\omega} \hookrightarrow \text{BMO}$  follows apparently.  $\square$

Using Lemma 1 instead of Lemma 5 when  $\omega \in A_1$ , we have the following result.

**THEOREM 3.6.** *Let  $\omega \in A_1$ . If  $Y_\omega$  is an r.-i.B.f.s. on  $(\mathbf{R}^n, \omega)$ , then*

$$\text{BMO}_{Y_\omega} = \text{BMO}$$

*and  $\|\cdot\|_{\text{BMO}_{Y_\omega}}$  is an equivalent norm of  $\|\cdot\|_{\text{BMO}}$ .*

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