HIROSHIMA MATH. J.
41 (2011), 137–152

Regularly varying solutions of second order nonlinear functional differential equations with retarded argument

KUSANO Takaŝi and V. MARIĆ

(Received February 11, 2010)

ABSTRACT. The existence of slowly and regularly varying solutions in the sense of Karamata implying nonoscillation is proved for a class of second order nonlinear retarded functional differential equations of Thomas-Fermi type. A motivation for such study is the extensively developed theory offering a number of properties of regularly and slowly varying functions ([2])—consequently of such solutions of differential equations. As an illustration, the precise asymptotic behaviour for $t \to \infty$ of the slowly varying solutions for a subclass of considered equations is presented.

1. Introduction

Theory of regular variation in the sense of Karamata has proved to be a powerful tool for the asymptotic analysis (e.g. nonoscillation, precise asymptotic behaviour), of solutions of second order linear and nonlinear ordinary differential equations, see [7]. For the reader's convenience we recall that a measurable function $L : [0, \infty) \to (0, \infty)$ is said to be *slowly varying* if it satisfies

$$L(\lambda t)/L(t) \to 1, \qquad \text{as } t \to \infty \text{ for } \forall\lambda > 0.$$

Furthermore, the function

$$f(t) = t^{\rho} L(t), \qquad \text{for } \rho \in \mathbf{R}$$

is said to be *regularly varying of index* ρ. The totality of these functions is denoted by $RV(\rho)$, and in particular $SV(= RV(0))$ stands for the totality of slowly varying functions.

One of the most important properties of slowly varying functions is the following representation theorem (see e.g. [2, Ch. 1]).

PROPOSITION 1.1. *$L(t) \in SV$ if and only if $L(t)$ is expressed in the form*

$$L(t) = c(t) \exp\left\{\int_a^t \delta(s)ds/s\right\}, \qquad t \geq a, \tag{1.1}$$

2010 *Mathematics Subject Classifications.* 34K06, 26A12.
Key words and phrases. functional differential equations, retarded argument, slowly, regularly varying functions, Karamata, Thomas-Fermi model, Schauder-Tychonoff fixed point theorem.

for some $a > 0$ *and some measurable functions* $c(t)$ *and* $\delta(t)$ *such that*

$$c(t) \to c_0 \in (0, \infty) \quad \textit{and} \quad \delta(t) \to 0 \qquad \textit{as} \ t \to \infty.$$

If in particular $c(t) = c_0$, then $L(t)$ is called a *normalized* slowly varying function. The order of growth or decay of $L(t)$ is severely limited in the sense that, for any $\varepsilon > 0$,

$$t^{\varepsilon} L(t) \to \infty \quad \text{and} \quad t^{-\varepsilon} L(t) \to 0 \qquad \text{as} \ t \to \infty. \tag{1.2}$$

Throughout the paper write ds/s as shorthand for $s^{-1}\,ds$.

We quote here the following result which is, on one hand, a typical one on the subject and on the other one, along with the Schauder-Tychonoff fixed point theorem, the main means for proving the results of this paper, [7, Th. 1.1].

PROPOSITION 1.2. *Consider the linear ordinary differential equation*

$$x''(t) = q(t)x(t), \tag{A}$$

where $q : [a, \infty) \to (0, \infty)$ *is continuous and integrable in* $[a, \infty)$. *Then there hold:*

(a) *Equation* (A) *possesses a fundamental set of solutions consisting of a decreasing normalized slowly varying solution* $x_0(t) = L_0(t)$ *and an increasing regularly varying solution of index* 1, $x_1(t) = tL_1(t)$ *with* $L_1(t) \sim L_0^{-1}(t)$ *as* $t \to \infty$, *if and only if*

$$Q(t) := t \int_t^{\infty} q(s)ds \to 0, \qquad \textit{as} \ t \to \infty. \tag{1.3}$$

(b) *These solutions can be for each* $T \geq a$, *respectively represented as*

$$x_0(t) = \exp\left\{ \int_T^t (v(s) - Q(s))ds/s \right\}, \tag{1.4}$$

where $v(t) \to 0$ *as* $t \to \infty$ *and satisfies the integral equation*

$$v(t) = t \int_t^{\infty} ((v(s) - Q(s))/s)^2 ds, \tag{1.5}$$

and

$$x_1(t) = \exp\left\{ \int_T^t (1 - Q(s) + w(s))ds/s \right\}, \tag{1.6}$$

where $w(t) \to 0$ *as* $t \to \infty$ *and satisfies the integral equation*

$$w(t) = t^{-1} \int_T^t [2Q(s) - (w(s) - Q(s))^2] ds. \tag{1.7}$$

Here the symbol $\sim$ denotes the asymptotic equivalence: $f(t) \sim g(t) \Leftrightarrow f(t)/g(t) \to 1$, as $t \to \infty$.

The study of second order *functional* differential equations by means of regular variation has been attempted for the first time by the present authors in [5]. There and in its continuation [6] the above Proposition has been generalized as follows:

Proposition 1.3. *Consider the linear functional differential equation*

$$x''(t) = q(t)x(g(t)),$$

where q is as in Proposition 1.2 and $g : [a, \infty) \to (0, \infty)$ is continuous, increasing and satisfies $g(t) < t$, $g(t) \to \infty$ as $t \to \infty$ and $\limsup_{t\to\infty} t/g(t) < \infty$. Then, the considered equation possesses a slowly varying solution and a regularly varying solution of index 1 if and only if (1.3) is satisfied.

The aim of this paper is to obtain results similar to those of Proposition 1.3 for the nonlinear (retarded, i.e. for $g(t) < t$) equation

$$x''(t) = q(t)x(g(t))^{\gamma} \tag{B}$$

for both the superlinear ($\gamma > 1$) and the sublinear cases ($\gamma < 1$).

Note that for $g(t) = t$, equation (B) is reduced to the celebrated Thomas-Fermi atomic model

$$x''(t) = q(t)x(t)^{\gamma}. \tag{C}$$

The structure of positive solutions of the retarded differential equation (B) is radically different from that of the ordinary differential equation (C). E.g., the superlinear equation (C) ($\gamma > 1$) always possesses a solution $x(t)$ which is defined and positive on a finite interval $[t_0, t_1)$ and blows up at t_1 in the sense that $x(t)$ and $x'(t) \to \infty$ as $t \to t_1 - 0$. On the other hand equation (B) admits no such solutions. In fact, suppose that $x(t)$ is a solution of (B) ($\gamma > 1$) on $[t_0, t_1)$ which blows up at t_1. Integrating (B) from t_0 to t yields

$$x'(t) = x'(t_0) + \int_{t_0}^{t} q(s)x(g(s))^{\gamma} ds, \qquad t < t_1.$$

Letting $t \to t_1 - 0$ and using $g(t) < t$, we have for $t \to t_1 - 0$,

$$\infty > \int_{t_0}^{t_1} q(s)x(g(s))^{\gamma} ds > \int_{t_0}^{t} q(s)x(g(s))^{\gamma} ds \to \infty,$$

which is a contradiction. For some additional differences see e.g. [1].

To avoid repetition we state here the following conditions on q and g valid throughout the text:

$q : [a, \infty) \to (0, \infty)$ is a continuous, integrable function,

$g : [a, \infty) \to (0, \infty)$ is a continuous function which is increasing and satisfies

$$g(t) < t \quad \text{and} \quad g(t) \to \infty, \qquad \text{as } t \to \infty. \tag{1.8}$$

Throughout the text all statements expressed by inequalities hold for $t \geq T$ and this adjective will be occasionally omitted. Also the number T need not be the same at each occurrence.

2. The superlinear case ($\gamma > 1$)

Let $x(t)$ be a positive solution of (B) on $[t_0, \infty)$. Since by (B), $x''(t) > 0$ for $t \geq t_1$, where t_1 is such that $g(t_1) = t_0$, $x'(t)$ is increasing for $t \geq t_1$. It follows that either $x'(t) < 0$ on $[t_1, \infty)$ or $x'(t) > 0$ on some $[t_2, \infty) \subset [t_1, \infty)$, which means that a positive solution of (B) is either decreasing or eventually increasing.

If $x'(t) < 0$ on $[t_1, \infty)$, then we must have $x'(t) \to 0$. In fact, if $x'(t) \to c < 0$, then $x'(t) < c$ for $t \geq t_1$, and integrating this inequality from t_1 to t gives $x(t) < x(t_1) + c(t - t_1) \to -\infty$, as $t \to \infty$ contradicting the positivity of $x(t)$. Hence, in this case $x(t)$ tends to a finite limit $x(\infty) \geq 0$ as $t \to \infty$.

If on the other hand $x'(t) > 0$ on $[t_2, \infty)$ then, due to $x''(t) > 0$, $x'(t)$ is eventually positive and increases to a finite or infinite limit $x'(\infty)$ as $t \to \infty$. In case $x'(\infty)$ is finite, then $x(t)$ satisfies $x(t)/t \to x'(\infty)$, that is, $x(t)$ is asymptotic to a constant multiple of t as $t \to \infty$. If $x'(\infty)$ is infinite, then for any M, one has for sufficiently large t, $x'(t) > M$ and so $x(t) > Mt$, whence $x(t) = t\varphi(t)$ with $\varphi(t) \to \infty$ as $t \to \infty$.

We first consider slowly varying solutions and prove:

THEOREM 2.1. *In addition to* (1.8) *suppose that*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} Q(s)ds/s < 1/e. \tag{2.1}$$

If

$$Q(t) = t\int_{t}^{\infty} q(s)ds \to 0, \qquad \text{as } t \to \infty, \tag{2.2}$$

then equation (B) *possesses a slowly varying solution.*

PROOF. First notice that SV solutions cannot increase, for being convex, they would violate (1.2). Let l be a positive constant less than $1/(4e)$. By

(2.1) and (2.2) one can choose $T > a$ so large that $g(T) > a$ and the inequalities $Q(t) \le l < 1/(4e)$ and $\int_{g(t)}^{t} Q(s)ds/s < 1/e$ hold for $t \ge T$.

Let Ξ denote the set of positive continuous functions $\xi(t)$ on $[g(T), \infty)$ which are nonincreasing and satisfy

$$\xi(t) = 1 \quad \text{for } g(T) \le t \le T, \qquad \xi(g(t))/\xi(t) \le e \quad \text{for } t \ge T. \tag{2.3}$$

It is clear that Ξ is a closed and convex subset of $C[g(T), \infty)$ which is a locally convex space equipped with the topology of uniform convergence on compact subintervals of $[g(T), \infty)$.

For each $\xi \in \Xi$ we define for $t \ge T$, $q_\xi(t) = q(t)\xi(g(t))^\gamma/\xi(t)$. Using (2.3) and the fact that $\gamma > 1$ and $\xi(t) \le 1$, we have

$$\xi(g(t))^\gamma/\xi(t) = \xi(g(t))^{\gamma-1}\xi(g(t))/\xi(t) \le e, \tag{2.4}$$

which implies that for $t \ge T$, $q_\xi(t) \le eq(t)$ and so due to (2.2) and the subsequent inequality,

$$Q_\xi(t) := t\int_t^\infty q_\xi(s)ds \to 0 \qquad \text{as } t \to \infty \tag{2.5}$$

and $Q_\xi(t) < 1/4$. Consequently, Proposition 1.2 applies to the family of linear ordinary differential equations

$$x''(t) = q_\xi(t)x(t), \qquad \xi \in \Xi \tag{2.6}$$

and ensures that for each $\xi \in \Xi$ equation (2.6) has a decreasing SV-solution $x_\xi(t)$ expressed in the form

$$x_\xi(t) = \exp\left\{\int_T^t (v_\xi(s) - Q_\xi(s))ds/s\right\}, \tag{2.7}$$

where $v_\xi(t)$ satisfies the integral equation (1.5) with v_ξ and Q_ξ replacing v and Q respectively.

We will show that there exists at least one $\xi \in \Xi$ for which the function $x_\xi(t)$ given by (2.7) exactly provides an SV-solution of equation (B). Use is made of the Schauder-Tychonoff fixed point theorem for this purpose. Let us define Φ to be the mapping which assigns to every $\xi \in \Xi$ the function $\Phi\xi$ given by

$$\Phi\xi(t) = 1 \quad \text{for } g(T) \le t \le T, \qquad \Phi\xi(t) = x_\xi(t) \quad \text{for } t \ge T.$$

Our task is to show that Φ is continuous and maps Ξ into a relatively compact subset of Ξ.

(i) Φ maps Ξ into itself: We divide $[T, \infty)$ into two subintervals $[T, T_1]$ and $[T_1, \infty)$ where $T_1 > T$ and such that $T = g(T_1)$. If $\xi \in \Xi$, then for $T \le t \le T_1$ we have by observing the choice of T_1,

$$\Phi\xi(g(t))/\Phi\xi(t) = 1/x_\xi(t) = \exp\left\{\int_T^t (Q_\xi(s) - v_\xi(s))ds/s\right\}$$
$$\leq \exp\left\{\int_{g(T_1)}^{T_1} eQ(s)ds/s\right\} \leq e,$$

and for $t \geq T_1$, arguing in the same way we get

$$\Phi\xi(g(t))/\Phi\xi(t) = \exp\left\{\int_{g(t)}^t (Q_\xi(s) - v_\xi(s))ds/s\right\} \leq e.$$

This implies that $\Phi\xi \in \Xi$, that is, $\Phi(\Xi) \subset \Xi$.

(ii) $\Phi(\Xi)$ is relatively compact in $C[g(T), \infty)$: The inclusion $\Phi(\Xi) \subset \Xi$ shows that $\Phi(\Xi)$ is locally uniformly bounded on $[g(T), \infty)$. If $\xi \in \Xi$, then for $t \geq T$,

$$(\Phi\xi)'(t) = x_\xi(t)(v_\xi(t) - Q_\xi(t))/t \geq x_\xi(t)(-Q_\xi(t)/t) \geq -eQ(t)/t,$$

which implies that $\Phi(\xi)$ is locally equicontinuous on $[g(T), \infty)$. The relative compactness of $\Phi(\Xi)$ in $C[g(T), \infty)$ follows from Arzela-Ascoli lemma.

(iii) Φ is a continuous mapping: Let $\{\xi_n\}$ be a sequence in Ξ converging to $\xi \in \Xi$, which means that the sequence $\{\xi_n(t)\}$ converges to $\xi(t)$ uniformly on compact subintervals of $[g(T), \infty)$. To show continuity of Φ we have to prove that $\{\Phi\xi_n(t)\}$ converges to $\Phi\xi(t)$ on any compact interval of $[g(T), \infty)$. Naturally it suffices to restrict our attention to the interval $[T, \infty)$. Noting that by using (2.7) and applying the mean value theorem, one obtains

$$|\Phi\xi_n(t) - \Phi\xi(t)| = |x_{\xi_n}(t) - x_\xi(t)| \leq \int_T^t (|v_{\xi_n}(s) - v_\xi(s)| + |Q_{\xi_n}(s) - Q_\xi(s)|)ds/s.$$

We need to verify that the two sequences $A_n = t^{-1}|v_{\xi_n}(t) - v_\xi(t)|$, $B_n = t^{-1}|Q_{\xi_n}(t) - Q_\xi(t)|$ converge to 0 uniformly on compact subintervals of $[T, \infty)$. The sequence B_n is easier to handle. In fact, we have the inequality

$$B_n \leq \int_t^\infty |q_{\xi_n}(s) - q_\xi(s)|ds \leq \int_t^\infty q(s)|\xi_n(g(s))^\gamma/\xi_n(s) - \xi(g(s))^\gamma/\xi(s)|ds.$$

Since the integrand of the last integral, denoted by $F_n(t)$, satisfies $F_n(t) \leq 2eq(t)$, by (2.4) and $F_n(t) \to 0$, $t \in [T, \infty)$ as $n \to \infty$, we conclude using the Lebesgue dominated convergence theorem that $B_n \to 0$ uniformly on $[T, \infty)$ as $n \to \infty$.

To deal with the sequence A_n we first note that in virtue of (1.5),

$$A_n \leq 4el\int_t^\infty |v_{\xi_n}(s) - v_\xi(s)|s^{-2}\,ds + 4el\int_t^\infty |Q_{\xi_n}(s) - Q_\xi(s)|s^{-2}\,ds \tag{2.8}$$

for $t \geq T$, where we have used the fact that $Q_{\xi_n}(t)$ and $Q_\xi(t)$ are less than or equal to $el < 1/4$ on $[T, \infty)$ which then holds also for $v_{\xi_n}(t)$ and $v_\xi(t)$ since $v_{\xi_n}(t) \leq Q_{\xi_n}(t)$, $v_\xi(t) \leq Q_\xi(t)$, $x_\xi(t)$ given by (2.7) being decreasing. Denoting the first integral in (2.8) by $w(t)$ we are able to transform (2.8) into the following differential inequality

$$(t^{4el}w(t))' \geq -4elt^{4el-1} \int_t^\infty |Q_{\xi_n}(s) - Q_\xi(s)| s^{-2}\, ds. \tag{2.9}$$

Integrating (2.9) from t to ∞ and noting that $t^{4el}w(t) \to 0$ as $t \to \infty$, since $v_\xi(t)$ does we obtain

$$w(t) \leq t^{-4el} \int_t^\infty |Q_{\xi_n}(s) - Q_\xi(s)| s^{4el-2}\, ds. \tag{2.10}$$

Combining (2.10) with (2.8) yields for $t \geq T$

$$t^{-1}|v_{\xi_n}(t) - v_\xi(t)| \leq 4elt^{-4el} \int_t^\infty |Q_{\xi_n}(s) - Q_\xi(s)| s^{4el-2}\, ds$$
$$+ 4el \int_t^\infty |Q_{\xi_n}(s) - Q_\xi(s)| s^{-2}\, ds,$$

which ensures that A_n also tends to zero uniformly on $[T, \infty)$ as $n \to \infty$, since B_n is such. We thus conclude that the sequence $\{\Phi\xi_n\}$ converges to $\Phi\xi$ in the topology of $C[g(T), \infty)$.

Therefore, applying the Schauder-Tychonoff fixed point theorem, we see that there exists $\xi \in \Xi$ such that $\xi = \Phi\xi$, which implies that $\xi(t) = x_\xi(t)$ for $t \geq T$, that is, $\xi(t)$ satisfies the differential equation $\xi''(t) = q_\xi(t)\xi(t)$ or equivalently $\xi''(t) = q(t)\xi(g(t))^\gamma$. Since $\xi(t)$ is slowly varying due to the definition of Φ, we have established the existence of an SV-solution for equation (B). This completes the proof of Theorem 2.1.

Remark 2.1. It is directly concluded that condition (2.1) is implied by

$$\limsup_{t\to\infty}\ t/g(t) < \infty \tag{2.11}$$

which is less general but simpler.

The preceding theorem gives only a sufficient condition for the existence of an SV solution which might tend to zero or to a positive constant i.e., the simplest ("trivial") SV solution. For the latter case there holds:

Theorem 2.2. *Let* (1.8) *and* (2.11) *hold, then equation* (B) *has a slowly varying solution* $x(t)$ *such that* $x(t) \to const. > 0$, *as* $t \to \infty$, *if and only if*

$$\int_a^\infty tq(t)dt < \infty. \tag{2.12}$$

Proof. "Only if". Let $x(t) \to c > 0$ as $t \to \infty$. Then, integrating equation (B) twice over (t, ∞) and integrating by part, one concludes that the integral $\int_t^\infty (s-t)q(s)ds$ converges. Since also $\int_t^\infty q(s)ds$ does by hypothesis, condition (2.12) follows.

"If". Condition (2.12) implies condition (2.2). Hence, by Theorem 2.1 and Remark 2.1, equation (B) has an SV solution which decreases and tends either to zero or to $c > 0$. Suppose $x(t) \to 0$. Then in virtue of the mean value theorem applied to the integral representation of $x(t)$, after dividing by $x(t)$, one has

$$1 \le x(g(t))^{\gamma-1} x(g(t))/x(t) \int_t^\infty (s-t)q(s)ds.$$

But, due to the representation (1.2) and (2.11), the quotient $x(g(t))/x(t)$ is bounded and so, by letting $t \to \infty$ in the preceding inequality, one has $1 \le 0$. Thus $x(t) \to c > 0$, qed.

This result generalizes P. K. Wong's Theorem 1.2 in [8] for the equation without deviating argument $x'' = xF(x,t)$ when $xF(x,t) = q(t)x^\gamma$.

Corollary 2.1. *Let* $q(t) \in RV(\alpha)$, $\alpha \le -2$. *Suppose that* (1.3) *holds and, instead of* (2.1) *that*

$$\int_{g(t)}^t Q(s)ds/s \to 0, \qquad \text{as } t \to \infty. \tag{2.13}$$

Then, the slowly varying solutions of equation (B) *obtained in the above theorems, have the following asymptotic behaviour*

$$x(t) \sim \left((\gamma-1)\int_T^t sq(s)ds\right)^{1/(1-\gamma)} \qquad \text{as } t \to \infty.$$

Proof. Since condition (2.13) implies (2.1), Theorem 2.1 ensures for some $\xi(t) \in \Xi$, the existence of a slowly varying solution $x(t)$ of equation (B) having the representation (2.7). Noting that $x(t)$ is decreasing and that $Q_\xi(t) \le eQ(t)$, by using (2.13) we obtain for $t \to \infty$

$$1 \le x(g(t))/x(t) \le \exp\left\{e\int_{g(t)}^t Q(s)ds/s\right\} \to 1,$$

whence $x(g(t)) \sim x(t)$, as $t \to \infty$.

Integrating (A) over $[t, \infty)$ and observing that $q(t) \in RV(\alpha)$, $\alpha \le -2$, and applying Karamata theorem ([7], Prop. 1) we get for $t \to \infty$

$$x'(t) \sim -\int_t^\infty q(s)(x(s))^\gamma ds \sim -tq(t)(x(t))^\gamma.$$

Another integration over (T, t) gives the desired result.

Remark 2.2. Note that condition (2.13) holds for any retarded argument $g(t)$ if $Q(t)/t$ is integrable on $[a, \infty)$, which is true for $\alpha < -2$. But then condition (2.13) implies (2.12) so that equation (B) has a trivial solution $x(t)$ i.e. tending to a positive constant. Hence only the case $\alpha = -2$ might lead to an SV solution tending to zero.

Example 2.1.

$$x''(t) = q(t)x(t/\log t)^{\gamma}, \qquad \gamma > 1,\ 0 < \delta \le 1$$

where $q(t) = r(t)/t^2(\log t)^{\delta}$ and $r(t)$ is a continuous positive function such that $r(t) \to \rho > 0$ as $t \to \infty$.

It is clear that $q(t) \in \mathrm{RV}(-2)$ and $Q(t) \sim \rho/(\log t)^{\delta}$ as $t \to \infty$, so that (1.3) holds.

One can, moreover, show that (2.13) is satisfied and an application of Corollary 2.1 gives for $0 < \delta < 1$

$$x(t) \sim ((1-\delta)/\rho(\gamma-1))^{1/(\gamma-1)}(\log t)^{(1-\delta)/(1-\gamma)}.$$

If in particular, $1 < \gamma < 2$, $\delta = 2 - \gamma$ and $\rho = 1$, then $x(t) \sim (\log t)^{-1}$ as $t \to \infty$. If in addition

$$r(t) = ((\log t - \log\log t)/\log t)^{\gamma}(1 + 2/\log t),$$

then the considered equation possesses an exact SV solution $x(t) = (\log t)^{-1}$.

For $\delta = 1$, Corollary 2.1 leads to

$$x(t) \sim (\rho(\gamma-1)\log\log t)^{1/(1-\gamma)} \qquad \text{as } t \to \infty.$$

It is easy to check that if $r(t)$ is given by

$$\begin{aligned} r(t) &= ((\log\log t - \log\log\log t)/\log\log t)^{\gamma/(\gamma-1)} \\ &\quad \times (1 + 1/\log t + \gamma/(\gamma-1)\log t \log\log t), \end{aligned}$$

then, the equation has an exact SV solution $x(t) = ((\gamma-1)\log\log t)^{1/(1-\gamma)}$.

Next we consider the existence of an RV(1) solution i.e. of the form $x(t) = t\ell(t)$ where $\ell(t)$ is some SV function and prove

Theorem 2.3. *Let* (1.8) *hold and let $L(t)$ be a normalized slowly varying function with $\delta(s)$ as in* (1.1) *and such that $L'(t) > 0$, $L(t) \to \infty$, as $t \to \infty$.*

Suppose that there exists a constant $K > 0$ such that for $t \ge T > a$,

$$t\int_t^{\infty} q(s)(g(s)L(g(s)))^{\gamma-1}ds \le K\delta(t). \tag{2.14}$$

Then, equation (B) *possesses a regularly varying solution of index* 1 *satisfying*

$$x(t) \le tL(t), \qquad \text{for } t \ge T. \tag{2.15}$$

Proof. First observe that, due to the assumptions on L, the function $\delta(t) = tL'(t)/L(t)$ is positive, tends to zero and the integral defining L is divergent.

Next put

$$q_L(t) = q(t)(g(t)L(g(t)))^{\gamma-1}, \qquad Q_L(t) = t\int_t^\infty q_L(s)ds. \tag{2.16}$$

Hence, due to (2.14) $Q_L(t) \to 0$, as $t \to \infty$ and so there exists $T > a$ such that $L(T) \geq 1$ and $K/L(T)^{\gamma-1} \leq 1$, and for $t \geq T$, $Q_L(t) \leq 1/8$.

Now the proof mimics the one of Theorem 2.1 with a different choice of the set Ξ and of the mapping Φ. Namely, this time we define Ξ to be the set of positive continuous functions $\xi(t) \in C[g(T), \infty)$ which are nondecreasing and satisfy

$$T \leq \xi(t) \leq tL(t)/L(T), \quad t \geq T, \quad \xi(t) = tL(t)/L(T), \quad g(T) \leq t \leq T. \tag{2.17}$$

It is clear that Ξ is a closed convex subset of the locally convex space $C[g(T), \infty)$. For each $\xi \in \Xi$ put

$$q_\xi(t) = q(t)\xi(g(t))^\gamma/\xi(t), \qquad Q_\xi(t) = t\int_t^\infty q_\xi(s)ds.$$

Using the increasing nature of $\xi(t)$, the inequality $g(t) < t$ and (2.17), we have

$$\xi(g(t))^\gamma/\xi(t) \leq \xi(g(t))^{\gamma-1} \leq (g(t)L(g(t))/L(T))^{\gamma-1}, \qquad t \geq T, \tag{2.18}$$

which implies that

$$Q_\xi(t) \leq Q_L(t)/L(T)^{\gamma-1} \leq 1/8, \qquad t \geq T, \text{ for all } \xi \in \Xi$$

and that by (2.14), $Q_\xi(t) \to 0$, as $t \to \infty$.

It then follows from Proposition 1.2 that every member of the family of linear differential equations

$$x''(t) = q_\xi(t)x(t), \qquad \xi \in \Xi, \tag{2.19}$$

possesses an RV(1)-solution $X_\xi(t)$ having the representation

$$X_\xi(t) = \exp\left\{\int_T^t (1 + w_\xi(s) - Q_\xi(s))ds/s\right\}, \tag{2.20}$$

where $w_\xi(t)$ is a solution of the integral equation (1.7) with $w_\xi(s)$ and $Q_\xi(s)$ replacing $w(s)$ and $Q(s)$ respectively, satisfying $w_\xi(t) \to 0$, $t \to \infty$ and so by $Q_\xi(t) \leq 1/8$, we have $|w_\xi(t)| \leq 1/4$.

We shall estimate $X_\xi(t)$ given by (2.20) from below and above. First we have $X_\xi(T) = 1$ and so $X_\xi(t) \geq 1$ for $t \geq T$. Next, using the inequality

$$\int_T^t w_\xi(s)ds/s \leq \int_T^t s^{-2}\int_T^s 2Q_\xi(r)drds \leq 2\int_T^t Q_\xi(s)ds/s,$$

following from the representation of $w_\xi(t)$, we find that

$$\int_T^t (w_\xi(s) - Q_\xi(s))ds/s \leq \int_T^t Q_\xi(s)ds/s \leq KL(T)^{1-\gamma}\int_T^t (L'(s)/L(s))ds$$
$$\leq \log(L(t)/L(T)),$$

where (2.14), the expression for $\delta(t)$, the inequality $KL(T)^{1-\gamma} \leq 1$ and the preceding one have been used. Thus we obtain

$$X_\xi(t) \leq tL(t)/TL(T). \tag{2.21}$$

Let us now define the mapping $\Phi : \Xi \to C[g(T), \infty)$ by

$$\Phi\xi(t) = TX_\xi(t), \quad t \geq T, \qquad \text{and} \qquad \Phi\xi(t) = tL(t)/L(T), \quad g(T) \leq t \leq T.$$

In view of the definition Φ and (2.21) we see that Φ maps Ξ into itself.

The inclusion $\Phi(\Xi) \subset \Xi$ shows that the set $\Phi(\Xi)$ is locally uniformly bounded on $[g(T), \infty)$. Formulas (2.20), (2.21) and $|w_\xi(t)| \leq 1/4$ lead to the inequality $(\Phi\xi)'(t) = TX_\xi'(t) \leq 5L(t)/4L(T)$ for $t \geq T$, holding for all $\xi \in \Xi$ which implies that $\Phi(\Xi)$ is locally equicontinuous on $[g(T), \infty)$. From these facts it follows via the Arzela-Ascoli lemma that $\Phi(\Xi)$ is relatively compact in $C[g(T), \infty)$.

Finally it can be verified that Φ is a continuous mapping in the topology of $C[T, \infty)$. Let $\{\xi_n(t)\}$ be any sequence in Ξ converging to $\xi(t) \in \Xi$ uniformly on compact subintervals of $[T, \infty)$. We shall prove that $\Phi\xi_n(t) \to \Phi\xi(t)$ uniformly on any compact subinterval of $[T, \infty)$.

To do so, we note that

$$|\Phi\xi_n(t) - \Phi\xi(t)| \leq T|X_{\xi_n}(t) - X_\xi(t)| \leq tL(t)/L(T)\int_T^t (|w_{\xi_n}(s) - w_\xi(s)|$$
$$+ |Q_{\xi_n}(s) - Q_\xi(s)|)ds/s.$$

Therefore, to establish the above mentioned convergence, it suffices to show that for $n \to \infty$

$$A_n = t^{-1}|w_{\xi_n}(t) - w_\xi(t)| \to 0 \qquad \text{and} \qquad B_n = t^{-1}|Q_{\xi_n}(t) - Q_\xi(t)| \to 0$$

uniformly on compact subintervals of $[T, \infty)$. The convergence of B_n follows by applying the Lebesgue dominated convergence theorem as in the proof of Theorem 2.1, using inequality (2.18) instead of (2.4).

To obtain the convergence of A_n one repeats the argument leading to the convergence of the sequence A_n in the proof of Theorem 2.1, using this time the representation for $w_\xi(t)$ instead of the representation for $v_\xi(t)$ and the inequality $Q_\xi(t) \le 1/8$ instead of $Q_\xi(t) \le 1/4$.

Thus all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled, so that there exists $\xi(t) \in \Xi$ such that $\xi(t) = \Phi\xi(t)$. By the definition of Φ this function $\xi(t)$ satisfies $\xi''(t) = q_\xi(t)\xi(t) = q(t)\xi(g(t))^\gamma$, $t \ge T$ and hence is a solution of equation (B). In view of (2.20) it is clear that $\xi(t) \in RV(1)$. The estimate (2.15) follows from (2.21) due to the definition of Φ.

REMARK 2.3. If condition (2.11) is assumed, then Theorem 2.3 holds when in (2.14) $g(s)$ is replaced by s.

Condition (2.14) requires that $Q_L(t)$ tends to zero at a particular rate. One can remove that request at the cost of restricting the coefficient $q(t)$ in equation (B).

To wit, there holds

THEOREM 2.4. *If, in addition to* (1.8), *the condition*

$$I_\varepsilon(t) := t\int_t^\infty q(s)g(s)^{\gamma-1+\varepsilon}ds \to 0, \qquad \text{as } t \to \infty$$

is satisfied for some $\varepsilon > 0$, *then equation* (B) *has a regularly varying solution of index* 1.

The proof is much the same as the one of Theorem 2.3. The only difference is the choice of the function $\xi(t)$ which here satisfies for some $m > 0$, $\xi(t) = \xi(T)$ for $g(t) \le t \le T$, $t \le \xi(t) \le t^{1+m}$ for $t \ge T$ and of the mapping $\Phi : \Xi \to C[g(T), \infty)$ being here $\Phi\xi(t) = T^{1+m}$ for $g(T) \le t \le T$ and $\Phi\xi(t) = X_\xi(t)$ for $t \ge T$. Also, property (1.2) is used to get an inequality analogous to $Q_\xi(t) \le 1/8$.

REMARK 2.4. If condition (2.11) is assumed, then Theorem 2.4 holds when in the condition $I_\varepsilon(t) \to 0$ as $t \to \infty$, $g(s)$ is replaced by s.

As in the SV case we want to obtain a necessary and sufficient condition for the existence of the simplest case of RV(1) solutions of equation (B) i.e. of the form $x(t) = tL(t)$ with $L(t) \to c > 0$ as $t \to \infty$. We prove

THEOREM 2.5. *Let* (1.8) *and* (2.11) *hold. Then equation* (B) *has a regularly varying solution* $x(t)$ *of index* 1 *such that*

$$x(t)/t \to c > 0, \qquad \text{as } t \to \infty \tag{2.22}$$

if and only if

$$I(a) := \int_a^\infty q(t)g(t)^\gamma dt < \infty. \tag{2.23}$$

Proof. The "only if" part. This follows directly from the integral representation for $x'(t)$ obtained by integrating equation (B) over $[t, \infty)$ since $x'(t) \to x'(\infty) = c$ as $t \to \infty$, and $x(g(s)) \sim cg(s)$ due to (2.22).

The "if" part. First notice that (2.23) implies $I(t) \to 0$ as $t \to \infty$, whence due to (2.11),

$$Q_g(t) := t \int_t^{\infty} q(s)g(s)^{\gamma-1} ds \to 0 \qquad \text{as } t \to \infty. \tag{2.24}$$

Moreover

$$\int_a^x Q_g(s)ds/s = \int_a^x \int_t^{\infty} q(s)g(s)^{\gamma-1} dsdt = Q_g(x) - Q_g(a) + \int_a^x q(s)sg(s)^{\gamma-1} ds.$$

Due to (2.23) and (2.11) the right-hand integral and so the left hand one converges to a constant $A > 0$.

Now we form the set Ξ of positive functions $\xi(t) \in C[g(T), \infty)$ which are nondecreasing and satisfy

$$\xi(t) = T \quad \text{for } g(T) \le t \le T \qquad \text{and} \qquad T \le \xi(t) \le Bt \quad \text{for } t > T, \tag{2.25}$$

where $B > 1$ is an arbitrary fixed constant, and define the mapping $\Phi : \Xi \to C[g(T), \infty)$ by

$$\Phi\xi(t) = T \quad \text{for } g(T) \le t \le T \qquad \text{and} \qquad \Phi\xi(t) = TX_{\xi}(t) \quad \text{for } t \ge T,$$

where $X_{\xi}(t)$ is the solution (2.20) of the equation (2.19) with

$$q_{\xi}(t) = q(t)\xi(g(t))^{\gamma}/\xi(t) \le q(t)\xi(g(t))^{\gamma-1} \le Mq(t)g(t)^{\gamma-1}, \qquad M = B^{\gamma-1}.$$

Hence

$$Q_{\xi}(t) = t \int_t^{\infty} q_{\xi}(s)ds \le t \int_t^{\infty} Mq(s)g(s)^{\gamma-1} ds = MQ_g(t).$$

Thus by (2.24), $Q_{\xi}(t) \to 0$ as $t \to \infty$ and so $Q_{\xi}(t) \le 1/8$ for $t \ge T$ and for each ξ which is needed here as in the previous proof.

We follow the same line of proof as before.

The mapping Φ maps Ξ into itself. The left-hand side inequality in (2.25) for $\Phi\xi(t)$ is proved as before and the right-hand one is obtained as follows:

First we construct the normalized slowly varying function $L(t)$ with $\delta(s) = MQ_g(s)/s$, so that $MQ_g(s)/s = L'(s)/L(s)$.

Also due to the convergence of the relevant integral, $L(t) \to C > 0$.

Arguing as in obtaining (2.21) we obtain for $t \ge T$, $X_{\xi}(t) \le tL(t)/TL(T)$. The right-hand inequality in (2.25) for $\Phi\xi(t)$ follows by $L(t)/L(T) \le B$ for $t \ge T$, provided $T > a$ is chosen sufficiently large. Hence $\Phi(\Xi) \subset \Xi$.

Now, from (2.25) one concludes that the set $\Phi(\Xi)$ is locally uniformly bounded on $[g(T), \infty)$.

Also, it follows from $(\Phi\xi)'(t) = TX_\xi'(t) \le 5B/4$, $t \ge T$, that $\Phi(\Xi)$ is locally equicontinuous on $[g(T), \infty)$.

The proof of the continuity of mapping Φ is the same as in Theorem 2.1. An application of the Schauder-Tychonoff fixed point theorem leads to the desired result as before.

This result generalizes P. K. Wong's Theorem 2.3 in [8] for the corresponding equation without deviating argument.

Note that if (2.11) holds, then in (2.23) $g(t)$ may be replaced by t.

3. The sublinear case ($\gamma < 1$)

The proofs of results in this section simply re-use the main idea of the previous ones in this paper. Namely, to combine Proposition 1.2 pertinent to the equation (A) (i.e. without the deviating argument) with the Schauder-Tychonoff fixed point theorem. The differences between these and the previous proofs in the text, consist in the construction of the set Ξ and of the operator Φ. It is our aim to stress only these facts and to neglect the calculations.

As in Section 2 we begin with consideration of SV solutions:

THEOREM 3.1. *Let* (1.8) *hold and let $M(t)$ denote a normalized slowly varying function which decreases to zero, with $\delta(t) = tM'(t)/M(t) < 0$. Assume in addition that there exists a constant $k > 1$ such that $M(g(t))/M(t) \le k$.*

If there exist a constant $K > 0$ and $T \ge a$ such that for $t \ge T$,

$$t\int_t^\infty q(s)M(g(s))^{\gamma-1}ds \le -K\delta(t),$$

then, equation (B) *possesses a slowly varying solution $x(t)$ such that $x(t) \ge M(t)$ for $t \ge T$.*

Here, for the proof we form the set Ξ of continuous nonincreasing functions $\xi(t)$ on $[g(T), \infty)$ which satisfy $\xi(t) = 1$, for $g(T) \le t \le T$, and $\xi(t) \ge M(t)/M(T)$, and $\xi(g(t))/\xi(t) \le k$, for $t \ge T$.

The mapping $\Phi : \Xi \to C[g(T), \infty)$ is in this case defined exactly as in the proof of Theorem 2.1 and the proof runs in the line of proof of Theorem 2.3.

EXAMPLE 3.1. It is left to the reader to show that Theorem 3.1 applies to the sublinear differential equation

$$x''(t) = q(t)x(t^\theta)^\gamma, \qquad 0 < \gamma < 1,\ 0 < \theta < 1,$$

where

$$q(t) = (\log(\theta \log t))^\gamma (t^2 \log t(\log\log t)^2)^{-1}(1 + 1/\log t + 2/\log t \log\log t).$$

One chooses here $M(t) = 1/\log\log t$. (One exact SV solution is $x(t) = 1/\log\log t$).

Next, we consider RV(1) solutions:

Theorem 3.2. *Let* (1.8) *hold. If*

$$t\int_t^\infty q(s)s^{\gamma-1}\,ds \to 0 \qquad \text{as } t \to \infty,$$

then equation (B) *possesses a regularly varying solution of index* 1.

Here, choose $T \ge a$ so that $(1-m)T^m \ge 1$ for some $0 < m < 1$, define the set Ξ of positive continuous functions $\xi(t)$ to be exactly the same as in the proof of Theorem 2.4. Also, define the mapping $\Phi : \Xi \to C[g(T), \infty)$ by $\Phi\xi(t) = T^{1+m}$, for $g(T) \le t \le T$, $\Phi\xi(t) = T^{1+m}X_\xi(t)$ for $t \ge T$, where $X_\xi(t)$ is given by (2.20).

Example 3.2. The sublinear retarded differential equation

$$x''(t) = q(t)x(\sigma t)^\gamma, \qquad 0 < \gamma < 1,\ 0 < \sigma < 1,$$

where $q(t) \sim k/t^{1+\gamma}(\log t)^\delta$ as $t \to \infty$, serves as an example when Theorem 3.2 can be applied.

If $k = 1/\sigma^\gamma$, $\delta = \gamma$ and $q(t) = 1/\sigma^\gamma t^{1+\gamma}(\log \sigma t)^\gamma$, it is easily checked that $x(t) = t\log t$ is one relevant solution.

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Kusano Takaŝi
Department of Applied Mathematics
Faculty of Science
Fukuoka University
Fukuoka, 841-0180, *Japan*
E-mail: kusano@zj8.so-net.ne.jp

V. Marić
Serbian Academy of Sciences
and Arts, Kneza Mihaila 35
11000 *Beograd, Serbia*
E-mail: vojam@uns.ac.rs