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Classification of 3-bridge arborescent links

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Abstract. In this paper, we give a complete classification of 3-bridge arborescent links.

1. Introduction

An *n-bridge sphere* of a link L in S^3 is a 2-sphere which meets L in $2n$ points and cuts (S^3, L) into n-string trivial tangles (B_1, t_1) and (B_2, t_2). Here, an *n-string trivial tangle* is a pair (B^3, t) of the 3-ball B^3 and n arcs properly embedded in B^3 parallel to the boundary of B^3. We call a link L an *n-bridge link* if L admits an n-bridge sphere and does not admit an $(n-1)$-bridge sphere. Two n-bridge spheres S_1 and S_2 of L are said to be *pairwise isotopic* (*isotopic*, in brief) if there exists a homeomorphism $f : (S^3, L) \to (S^3, L)$ such that $f(S_1) = S_2$ and f is *pairwise isotopic* to the identity, i.e., there is a continuous family of homeomorphisms $f_t : (S^3, L) \to (S^3, L)$ $(0 \le t \le 1)$ such that $f_0 = f$ and $f_1 = \mathrm{id}$. Two n-bridge spheres S_1 and S_2 are said to be *homeomorphic* if there exists an *orientation-preserving* homeomorphism $f : (S^3, L) \to (S^3, L)$ such that $f(S_1) = S_2$.

The only 1-bridge link is the unknot, and the 2-bridge links are completely classified by Schubert [27], by showing the uniqueness of 2-bridge spheres of 2-bridge links up to isotopy. Moreover, it is proved by Otal ([21] and [22]) that the unknot (resp. any 2-bridge link) admits a unique n-bridge sphere up to isotopy for $n \ge 1$ (resp. $n \ge 2$). These results were recently refined by Scharlemann and Tomova [26]. The bridge indices of Montesinos links are completely determined by Boileau and Zieschang [5]. In [14], the author constructed a family of links each of which admits infinitely many 3-bridge spheres up to isotopy. However, not much is known about 3-bridge links and 3-bridge spheres in general.

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Bridge presentations of links are intimately related with Heegaard splittings of closed orientable 3-manifolds (see, for example, [2]). Boileau, Collins and Zieschang [3] classified genus-2 Heegaard splittings of small Seifert fibered spaces. Kobayashi [17] characterized non-simple 3-manifolds of genus 2, and Morimoto [19] gave a list of all isotopy classes of genus-2 Heegaard splittings for certain graph manifolds containing essential tori.

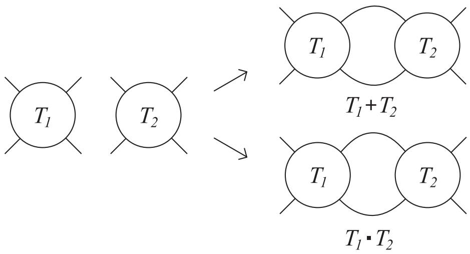


Fig. 1

In this paper, we classify 3-bridge arborescent links by using the results of [17] and [19]. Moreover, in the sequel of this paper, we classify their 3-bridge spheres up to isotopy. We first recall the definition of arborescent links. An *arborescent tangle* is a tangle obtained from rational tangles by repeatedly applying the operations in Figure 1. By an *arborescent link*, we mean a link obtained by closing an arborescent tangle with a trivial tangle (see [8]). Arborescent links are also defined by a plumbing construction from a weighted tree (see [10]). Arborescent links form an important family of links which contains 2-bridge links and Montesinos links, and the double branched covering of the 3-sphere S^3 branched over an arborescent link is a graph manifold. Bonahon and Siebenmann [6] gave a complete classification of arborescent links (cf. [9]).

We now state our main results. The following theorem gives the complete list of 3-bridge arborescent links, where two links are *equivalent* if there exists an orientation-preserving homeomorphism of S^3 which carries one of the two links to the other.

THEOREM 1. *A link L in S^3 is a* 3*-bridge arborescent link if and only if L is equivalent to one of the following links.*

(1) *The link* $L_1((\beta_1/\alpha_1, \beta_1'/\alpha_1'), (\beta_2/\alpha_2, \beta_2'/\alpha_2'))$ *in Figure* 2 (1).

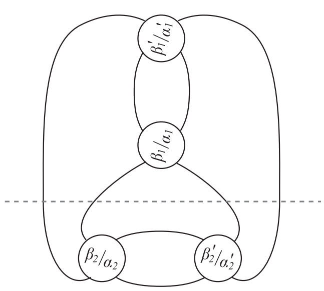
(2) *The link* $L_2((\beta_1/\alpha_1, \beta_1'/\alpha_1'), (1/\alpha_0), (\beta_2/\alpha_2, \beta_2'/\alpha_2'))$ *in Figure* 2 (2).

(3) *The link* $L_3((\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3), (1/2, -n/(2n+1)))$ *in Figure* 2 (3).

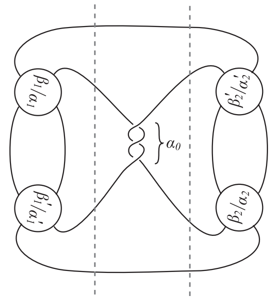
(4) *The Montesinos link* $L(-b; \beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ *(see Figure* 6*).*

Here, α_i, α_i', β_i, β_i' *are integers such that* $\alpha_i, \alpha_i' > 1$ *and* $g.c.d.(\alpha_i, \beta_i) = g.c.d.(\alpha_i', \beta_i') = 1$ $(i = 1, 2, 3)$, *and* α_0 *and* n *are integers such that* $|\alpha_0| > 1$ *and*

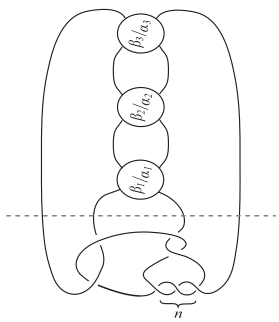
$|2n+1| > 1$. *In Figure 2, the circle encircling a rational number β/α represents the rational tangle of slope β/α.*



(1) $L_1((\beta_1/\alpha_1, \beta_1'/\alpha_1'), (\beta_2/\alpha_2, \beta_2'/\alpha_2'))$



(2) $L_2((\beta_1/\alpha_1, \beta_1'/\alpha_1'), (1/\alpha_0), (\beta_2/\alpha_2, \beta_2'/\alpha_2'))$



(3) $L_3((\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3), (1/2, -n/(2n+1)))$

Fig. 2

For each $i = 1, 2, 3$, we denote by $\mathscr{L}_i$ the family of links as in (i) in Theorem 1. In order to state a classification theorem of the links in $\mathscr{L}_1 \cup \mathscr{L}_2 \cup \mathscr{L}_3$, we prepare a notation.

Notation 1. Let $s_1, \dots, s_r$ and $s_1', \dots, s_r'$ be rational numbers whose denominators are greater than 1. We use the following notation.

- $(s_1, \dots, s_r) \approx (s_1', \dots, s_r')$ when $(s_1, \dots, s_r) = (s_1', \dots, s_r')$ in $(\mathbf{Q}/\mathbf{Z})^r$ and $\sum_{i=1}^{r} s_i = \sum_{i=1}^{r} s_i'$.
- $(s_1, \dots, s_r) \sim (s_1', \dots, s_r')$ when $(s_1, \dots, s_r) \approx (s_1', \dots, s_r')$ or $(s_r', \dots, s_1')$.

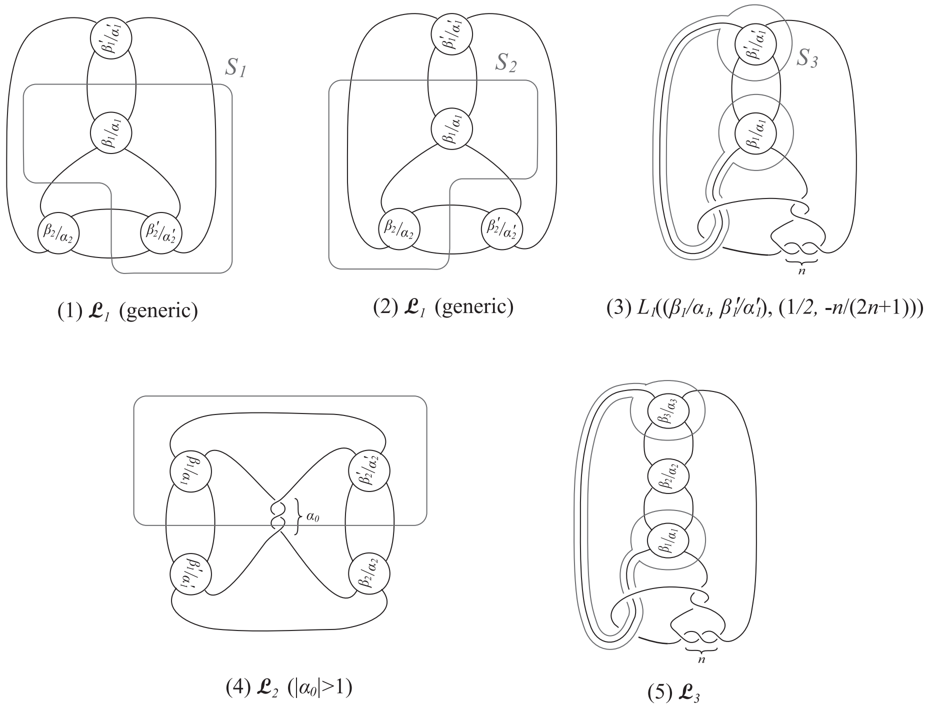
The following theorem gives the complete classification of the links in $\mathscr{L}_1 \cup \mathscr{L}_2 \cup \mathscr{L}_3$.

THEOREM 2. *Any link in* $\mathscr{L}_1 \cup \mathscr{L}_2 \cup \mathscr{L}_3$ *is not equivalent to a Montesinos link, and two links in distinct families of* $\mathscr{L}_1$, $\mathscr{L}_2$ *and* $\mathscr{L}_3$ *are not equivalent. Moreover, the links in each of the families are classified as follows.*

(1) $L_1((s_1, s_2), (s_3, s_4))$ *and* $L_1((s_1', s_2'), (s_3', s_4'))$ *are equivalent if and only if*
  (1-i) $(s_1, s_2) \sim (s_1', s_2')$ *and* $(s_3, s_4) \sim (s_3', s_4')$, *or*
  (1-ii) $(s_1, s_2) \sim (s_3', s_4')$ *and* $(s_3, s_4) \sim (s_1', s_2')$.

(2) $L_2((s_1, s_2), (1/\alpha_0), (s_3, s_4))$ *and* $L_2((s_1', s_2'), (1/\alpha_0'), (s_3', s_4'))$ *are equivalent if and only if* $\alpha_0 = \alpha_0'$ *and one of the following holds.*
  (2-i) $(s_1, s_2) \approx (s_1', s_2')$ *and* $(s_3, s_4) \approx (s_3', s_4')$,
  (2-ii) $(s_1, s_2) \approx (s_2', s_1')$ *and* $(s_3, s_4) \approx (s_4', s_3')$,
  (2-iii) $(s_1, s_2) \approx (s_3', s_4')$ *and* $(s_3, s_4) \approx (s_1', s_2')$, *or*
  (2-iv) $(s_1, s_2) \approx (s_4', s_3')$ *and* $(s_3, s_4) \approx (s_2', s_1')$.

(3) $L_3((s_1, s_2, s_3), (1/2, -n/(2n+1)))$ *and* $L_3((s_1', s_2', s_3'), (1/2, -n'/(2n'+1)))$ are equivalent if and only if $n = n'$ and $(s_1, s_2, s_3) \sim (s_1', s_2', s_3')$.



(1) $\mathcal{L}_1$ (generic) (2) $\mathcal{L}_1$ (generic) (3) $L_1((\beta_1/\alpha_1, \beta_1'/\alpha_1'), (1/2, -n/(2n+1)))$

(4) $\mathcal{L}_2$ ($|\alpha_0|>1$) (5) $\mathcal{L}_3$

Fig. 3

REMARK 1. (1) The classification of the links in $\mathscr{L}_1$ is already obtained by [11, Lemma 2.2]. Though the classification of the links in $\mathscr{L}_2$ and $\mathscr{L}_3$ may be also obtained by using the theory of Bonahon and Siebenmann [6], we give a direct proof in this paper.

(2) The Kinoshita-Terasaka knot and the Conway's 11 crossing knot (cf. [16, Example 3.8.4 and Fig. 3.8.1]) are equivalent to $L_2((-1/3, 1/2), (1/2), (-1/2, 1/3))$ and $L_2((1/2, -1/3), (1/2), (-1/2, 1/3))$, respectively. Theorem 2 (2) gives alternative proof of the inequivalence of these knots.

(3) Except for some special case, the dotted lines in Figure 2 give the characteristic decomposition of each link by essential Conway spheres (see [6] and Theorem 4 for the definition of the characteristic decomposition, and see Proposition 4 and Figure 12 for the exceptional cases).

In the proof of Theorem 1, we also obtain the 3-bridge spheres for the links as illustrated in Figure 3. In the sequel of this paper, we show that these 3-bridge spheres form a complete list of 3-bridge spheres for the links in Theorem 1 up to isotopy, and moreover, we give a necessary and sufficient condition for any two of these 3-bridge spheres to be isotopic.

This paper is organized as follows. In Section 2, we recall some basic properties of arborescent links. In Section 3, we recall a relation between 3-bridge spheres of links and genus-2 Heegaard surfaces of 3-manifolds. In Section 4, we recall the characterization of genus-2 graph manifolds given by Kobayashi [17]. In Section 5, we calculate the mapping class groups of some of the graph manifolds, which will be used in the rest of this paper. Finally, in Sections 7 and 8, we prove Theorems 1 and 2, respectively.

2. Seifert fibered spaces, graph manifolds and arborescent links

In this section, we recall (i) basic facts concerning Seifert fibered spaces, (ii) description of Seifert fibered spaces as double branched coverings due to Montesinos [18], and (iii) the characteristic decomposition theory of links established by Bonahon and Siebenmann [6].

For a given compact surface F with boundary, we denote by $F(\beta_1/\alpha_1, \beta_2/\alpha_2, \dots, \beta_r/\alpha_r)$ the orientable Seifert fibered space over F with Seifert indices $\beta_1/\alpha_1, \beta_2/\alpha_2, \dots, \beta_r/\alpha_r$. To be precise, consider the surface $F_0 := F \setminus \mathrm{Int}(D_1 \cup \cdots \cup D_r)$, where $\{D_i\}_{1 \le i \le r}$ is a set of r disjoint disks in $\mathrm{Int}(F)$, and let M_0 be the trivial S^1-bundle or the orientable twisted S^1-bundle according as F_0 is orientable or non-orientable. Then $F(\beta_1/\alpha_1, \beta_2/\alpha_2, \dots, \beta_r/\alpha_r)$ is obtained by gluing M_0, and r solid tori $V_1, \dots, V_r$, where the gluing homeomorphism is given as follows. Let F_0' be the image of a cross section of the bundle $M_0 \to F_0$, and let T_i be the component of ∂M_0 projecting to ∂D_i $(i = 1, 2, \dots, r)$. Let s_i be the intersection of T_i and F_0' and h_i be a fiber of M_0 on T_i. We orient s_i and h_i so that the ordered pair (s_i, h_i) gives the orientation of T_i induced by that of M_0. Then the gluing homeomorphism maps the boundary of the meridian disk of V_i to the loop on T_i representing the homology class $\alpha_i s_i + \beta_i h_i$.

We call $c := \partial F_0' \backslash (\bigcup_{i=1}^r s_i)$, i.e., the union of the components of $\partial F_0'$ projecting to ∂F, the *horizontal loop(s)*. We occasionally call a component of c a horizontal loop.

For a closed surface F, we denote by $F(b; \beta_1/\alpha_1, \beta_2/\alpha_2, \ldots, \beta_r/\alpha_r)$ the Seifert fibered space obtained as follows. Set $F_0 = F \backslash$(an open disk) and consider the Seifert fibered space $M_0 = F_0(\beta_1/\alpha_1, \beta_2/\alpha_2, \ldots, \beta_r/\alpha_r)$. Let c and h, respectively, be the horizontal loop and a regular fiber of the Seifert fibered space lying on the boundary torus. Then $F(b; \beta_1/\alpha_1, \beta_2/\alpha_2, \ldots, \beta_r/\alpha_r)$ is the Seifert fibered space obtained by gluing $F_0(\beta_1/\alpha_1, \beta_2/\alpha_2, \ldots, \beta_r/\alpha_r)$ and a solid torus so that the meridian loop of the solid torus is identified with the loop representing $c + bh$.

PROPOSITION 1 (cf. [23]). (1) *Let F be a compact surface with boundary, and consider two Seifert fibered spaces $M := F(\beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ and $M' := F(\beta_1'/\alpha_1', \ldots, \beta_r'/\alpha_r')$. Then there is an orientation-preserving homeomorphism $\varphi : M \to M'$ which preserves the Seifert fibration and maps the horizontal loop c of M to the horizontal loop c' of M', if and only if the following hold.*

(i) *After a permutation of indices,*

$$\left(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_r}{\alpha_r}\right) = \left(\frac{\beta_1'}{\alpha_1'}, \ldots, \frac{\beta_r'}{\alpha_r'}\right) \in (\mathbf{Q}/\mathbf{Z})^r.$$

(ii) $\displaystyle\sum_{i=1}^r \frac{\beta_i}{\alpha_i} = \sum_{i=1}^r \frac{\beta_i'}{\alpha_i'} \in \mathbf{Q}.$

(2) *Let F be a closed surface, and consider two Seifert fibered spaces $M := F(b; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ and $M' := F(b'; \beta_1'/\alpha_1', \ldots, \beta_r'/\alpha_r')$. Then there is an orientation-preserving homeomorphism $\varphi : M \to M'$ which preserves the Seifert fibration if and only if the following hold.*

(i) *After a permutation of indices,*

$$\left(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_r}{\alpha_r}\right) = \left(\frac{\beta_1'}{\alpha_1'}, \ldots, \frac{\beta_r'}{\alpha_r'}\right) \in (\mathbf{Q}/\mathbf{Z})^r.$$

(ii) $\displaystyle b + \sum_{i=1}^r \frac{\beta_i}{\alpha_i} = b' + \sum_{i=1}^r \frac{\beta_i'}{\alpha_i'} \in \mathbf{Q}.$

NOTATION 2. Let h and c, respectively, be a regular fiber and a horizontal loop of a Seifert fibered space $M = F(\beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$. Then we say that $M = F(\beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ w.r.t. h and c. The above proposition implies that $(\beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is uniquely defined up to the equivalence relation described in the proposition.

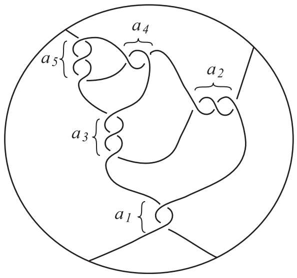


Fig. 4. $n = 5, a_0 = 0, a_1 = 2, a_2 = 3, a_3 = 3, a_4 = 2, a_5 = 3$ and $\beta/\alpha = 31/50$.

A *graph manifold* is a 3-manifold obtained by gluing Seifert fibered spaces along their boundaries. Graph manifolds are introduced and classified by Waldhausen [29].

A $(3,1)$-*manifold pair* is a pair (M,L) of a compact oriented 3-manifold M and a proper 1-submanifold L of M. By a *surface F in (M,L)*, we mean a surface F in M intersecting L transversely. Two surfaces F and F' in (M,L) are said to be *pairwise isotopic* (*isotopic*, in brief,) if there is a homeomorphism $f : (M,L) \to (M,L)$ such that $f(F) = F'$ and f is pairwise isotopic to the identity. We call a $(3,1)$-manifold pair a *tangle* if M is homeomorphic to B^3. A *trivial tangle* is a $(3,1)$-manifold pair (B^3, L), where L is the union of two arcs embedded in the 3-ball B^3 which bounds disjoint disks with arcs on the boundary of B^3. A *rational tangle* is a trivial tangle with its boundary fixed. A well-known fact is that rational tangles correspond to rational numbers, called the *slopes* of the rational tangles. For example, the rational tangle of slope β/α can be illustrated as in Figure 4, where α, β are defined by the continued fraction

$$\begin{aligned}\frac{\beta}{\alpha} &= -a_0 + [a_1, -a_2, \dots, \pm a_m] \\ &:= -a_0 + \cfrac{1}{a_1 + \cfrac{1}{-a_2 + \cfrac{1}{\cdots + \cfrac{1}{\pm a_m}}}}\end{aligned}$$

together with the condition that α and β are relatively prime and $\alpha \geq 0$. Here, the numbers a_i denote the numbers of right-hand half twists.

It is known that any 2-bridge link is obtained by closing a rational tangle with the trivial arcs. We denote by $S(\alpha, \beta)$ the 2-bridge link obtained by closing a rational tangle of slope β/α with the rational tangle of slope $1/0$.

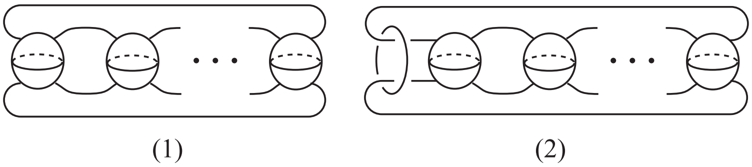


Fig. 5

A *Montesinos pair* is a $(3,1)$-manifold pair which is built from the pair in Figure 5 (1) or (2) by plugging some of the holes with rational tangles of finite slopes. We say that a Montesinos pair is *trivial* if it is homeomorphic to a rational tangle or $(S,P)\times I$, where S is a 2-sphere, P is the union of four distinct points on S and I is a closed interval. A *Montesinos link* is a link obtained by plugging the remaining holes of a Montesinos pair in Figure 5 (1) with rational tangles of finite slopes, as shown in Figure 6. Unless otherwise stated, we assume that β/α is not an integer, that is, $\alpha > 1$. The above Montesinos link is denoted by $L(-b;\beta_1/\alpha_1,\dots,\beta_r/\alpha_r)$. A Montesinos link is said to be *elliptic* if it is a nontrivial 2-bridge link or if $r=3$ and $\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}>1$. An *arborescent link* is a link in S^3 obtained by gluing some Montesinos pairs in their boundaries as in Figure 7.

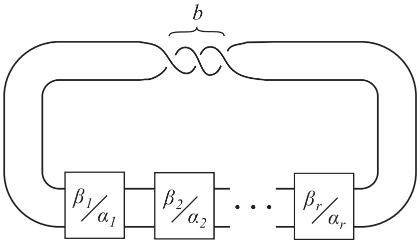


Fig. 6. $b=3$.

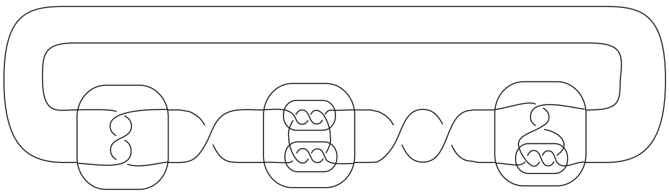


Fig. 7

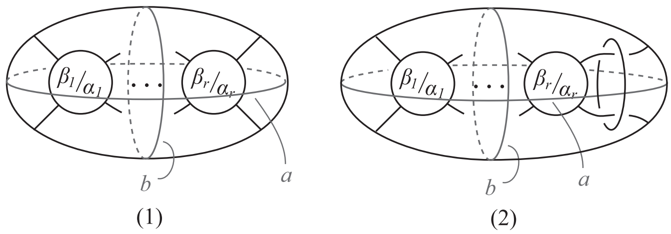


Fig. 8

The following proposition is a classical result due to Montesinos [18].

Proposition 2. (1) *Let* (N, L) *be the Montesinos pair in Figure* 8 (1). *Then the double branched covering of* N *branched over* L *is a Seifert fibered space* $D(\beta_1/\alpha_1, \beta_2/\alpha_2, \dots, \beta_r/\alpha_r)$ *over a disk. Moreover, the pre-images of the loops* a *and* b *in the figure, respectively, are the union of two parallel horizontal loops and the union of two regular fibers.*

(2) *Let* L *be a Montesinos link* $L(-b; \beta_1/\alpha_1, \dots, \beta_r/\alpha_r)$. *Then the double branched covering of* S^3 *branched over* L *is a Seifert fibered space* $S^2(-b; \beta_1/\alpha_1, \beta_2/\alpha_2, \dots, \beta_r/\alpha_r)$ *over the* 2*-sphere.*

(3) *Let* (N, L) *be the Montesinos pair in Figure* 8 (2). *Then the double branched covering of* N *branched over* L *is a Seifert fibered space* $M\ddot{o}(\beta_1/\alpha_1, \beta_2/\alpha_2, \dots, \beta_r/\alpha_r)$ *over a Möbius band. Moreover, the pre-images of the loops* a *and* b *in the figure, respectively, are the union of two parallel horizontal loops and the union of two regular fibers.*

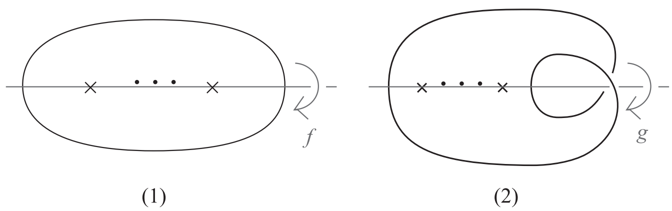


Fig. 9

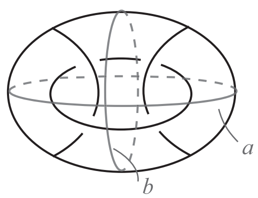


Fig. 10

REMARK 2. We denote the covering transformations of the double branched coverings in (1) and (3) in Proposition 2 by f and g, respectively. Then f and g are fiber-preserving involutions of $D(\beta_1/\alpha_1, \beta_2/\alpha_2, \dots, \beta_r/\alpha_r)$ and $M\ddot{o}(\beta_1/\alpha_1, \beta_2/\alpha_2, \dots, \beta_r/\alpha_r)$, respectively, and induce the involutions on the base orbifolds as illustrated in Figure 9.

If a Montesinos pair (N, L) is nontrivial, then the double branched covering of N branched over L is not homeomorphic to a solid torus nor $S^1 \times S^1 \times I$ (cf. [13, Examples VI.5]). The following remark, which is used to prove Proposition 4, is a direct consequence of [13, Examples VI.5 and Theorem VI.18].

REMARK 3. Let (N, L) be a nontrivial Montesinos pair with nonempty boundary, and let M be the double branched covering of N branched along L.

(1) If (N, L) is the *ring tangle* illustrated in Figure 10, then M admits two Seifert fibrations. Namely, M can be regarded as $D(-1/2, 1/2)$ w.r.t. h and c or a S^1-bundle over a Möbius band w.r.t. c and h, where c and h are simple loops in ∂M which project to the loops a and b, respectively, as in Figure 10 (see Notation 2).

(2) If (N, L) is not the ring tangle, then M admits a unique Seifert fibration up to isotopy.

Montesinos links are classified by the following theorem (see [31], [7, Chapter 12]).

THEOREM 3. *Let L be a Montesinos link $L(-b; \beta_1/\alpha_1, \dots, \beta_r/\alpha_r)$.*

(1) *If $r \leq 2$, then L is a 2-bridge link. To be precise,*
   (i) *if $r = 0$, then L is a torus link,*
   (ii) *if $r = 1$, then $L = L(-b; \beta_1/\alpha_1)$ is a 2-bridge link $S(b\alpha_1 - \beta_1, \alpha_1)$,*
   (iii) *if $r = 2$, then $L = L(-b; \beta_1/\alpha_1, \beta_2/\alpha_2)$ is a 2-bridge link $S(p, q)$, where $p = b\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2$ and $q = p\beta_1(b\alpha_2 + \beta_2)/|p|$.*

(2) *If $r > 2$, then L is not a 2-bridge link, and such links are classified by the ordered set of fractions $(\beta_1/\alpha_1, \dots, \beta_r/\alpha_r) \in (\mathbf{Q}/\mathbf{Z})^r$, up to cyclic permutations and reversal of order, together with the rational number $b_0 = b + \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j}$.*

Bonahon and Siebenmann [6] established a theory to decompose a link into simpler pieces in a canonical way. This decomposition consists of two steps. The first step is just the torus decomposition of the knot exterior. The second step is a decomposition by "Conway spheres". To state the result, we need to introduce some notation. A *Conway sphere* in (M, L) is a 2-sphere in $\mathrm{Int}(M)$ or in ∂M which meets L transversally in 4 points. A Conway sphere F is said to be *pairwise-compressible* if there is a disk D in $M \backslash L$ such that

$D \cap F = \partial D$ and ∂D does not bound a disk in $F \backslash L$. Otherwise, F is said to be *pairwise-incompressible*. A Conway sphere F is said to be *∂-parallel* if F splits M into two parts M_1 and M_2 such that for one of which, say M_1, we have a homeomorphism $(M_1, M_1 \cap L) \cong (F, F \cap L) \times [0,1]$. We say that a $(3,1)$-manifold pair (M, L) is *Conway-simple* if there does not exist a pairwise-incompressible, non-∂-parallel Conway sphere in $\mathrm{Int}(M)$ for (M, L). We sometimes call the pair $(F, F \cap L)$ a *Conway sphere* and denote it by (S^2, P).

A link L in S^3 is said to be *simple* if $S^3 \backslash L$ do not contain an essential torus. Bonahon and Siebenmann established the *characteristic decomposition theorem* for simple links ([6, Theorem 3.4]). The following theorem is a corollary of the characteristic decomposition theorem for simple arborescent links.

Theorem 4. *Let L be an arborescent link in S^3, which is simple. Then there is a 2-manifold $F \subset S^3$ which is unique up to pairwise isotopy of (S^3, L) and has the following properties.*

(1) *The components of F are pairwise-incompressible Conway spheres, no two of which are pairwise isotopic in (S^3, L).*

(2) *Each component N of the 3-manifold obtained from S^3 by splitting along F gives a Montesinos pair $(N, L \cap N)$.*

(3) *When any component is omitted from F, property (2) fails.*

Moreover, arborescent links with essential tori in their complements can be characterized by the following proposition (see [6] and [9]).

Proposition 3. *The following three families form a complete list of non-hyperbolic arborescent links.*

I. *L is the boundary of a single unknotted band, i.e., a torus knot or link of type $(2, n)$ for some $n \in \mathbf{Z}$.*

II. *L has two parallel components, each of which bounds a twice-punctured disk properly embedded in $S^3 \backslash L$.*

III. *L or its reflection is the pretzel link $P(p, q, r, -1) := M(-1; 1/p, 1/q, 1/r)$, where $p, q, r \geq 2$ and $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} \geq 1$.*

Remark 4. For families I and II, Figure 11 reveals an obvious annulus or Möbius band that forms an obstruction to the existence of a hyperbolic structure. Meanwhile, the pretzel links in family III contain incompressible tori when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ (by Oertel's work [20]) and are Seifert fibered when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ (see [25, Theorem 3.4]. In fact, such links are torus links unless (p, q, r) is a permutations of $(2, 2, n)$).

Using Theorem 4 and Proposition 3, we obtain the following proposition.

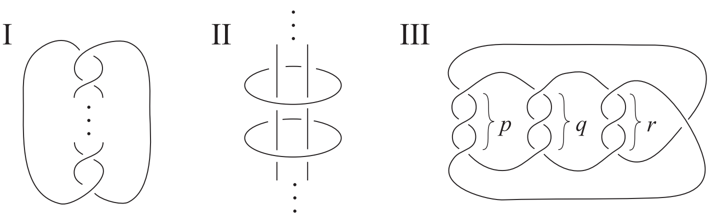
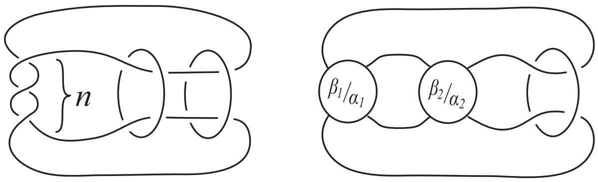


Fig. 11



(1) $n \neq 0$ (2) $\alpha_i > 1$ (i=1,2)

Fig. 12

PROPOSITION 4. *Let L be a 3-bridge arborescent link which is not a Montesinos link.*

(1) *If L is non-simple (i.e., $S^3 \backslash L$ contains an essential torus), then L is equivalent to the link in Figure 12 (1) for some $n \neq 0$. Thus, L is equivalent to $L_2((-1/2, 1/2), (1/n), (-1/2, 1/2)) \in \mathscr{L}_2$ or $L_1((-1/2, 1/2 - n), (-1/2, 1/2 - n)) \in \mathscr{L}_1$ according as $|n| > 1$ or $|n| = 1$.*

(2) *If L is simple and has a trivial characteristic decomposition, then L is equivalent to the link $L_1((\beta_1/\alpha_1, \beta_2/\alpha_2), (-1/2, 1/2)) \in \mathscr{L}_1$ (see Figure 12 (2)). In this case, the double branched covering $M_2(L)$ of S^3 branched over L is a Seifert fibered space $P^2(0; \beta_1/\alpha_1, \beta_2/\alpha_2)$, which contains a separating essential torus.*

(3) *If L is simple and has a nontrivial characteristic decomposition, then the pre-image of the family of Conway spheres in $M_2(L)$ is a family of separating tori and gives the (nontrivial) torus decomposition of $M_2(L)$.*

PROOF. Let L be a 3-bridge arborescent link and suppose that L is not a Montesinos link.

(1) Suppose that L is non-simple. Since the links in the family I in Proposition 3 are 2-bridge torus links and the links in the family III in Proposition 3 are Montesinos links, L has two parallel components each of which bounds a twice-punctured disk properly embedded in $S^3 \backslash L$ by Proposition 3. Since L is a 3-bridge link, L consists of 3 trivial components, and L is obtained by adding a parallel circle to one component of a 2-bridge link, say

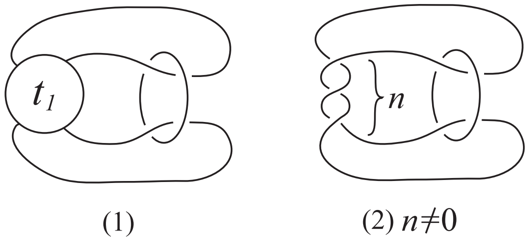


Fig. 13

L'. We show that L' is equivalent to the link in Figure 13 (2) for some $n \neq 0$. To this end, note that (S^3, L') is a union of some tangle, (B^3, t_1), and a ring tangle, (B^3, t_2), as shown in Figure 13 (1). Let $M_2(L')$ (resp. $M_2(t_i)$) be the double branched covering of S^3 (resp. B^3) branched over L' (resp. t_i), and set $T := M_2(t_1) \cap M_2(t_2)$. Since $M_2(L')$ is a lens space, the torus T bounds a solid torus V in $M_2(L')$. Since $M_2(t_2) \cong D(-1/2, 1/2)$ is not a solid torus, we see $V = M_2(t_1)$. Suppose that the meridian of $V = M_2(t_1)$ is identified with the loop representing $c^\alpha h^\beta$ for some integers α and β, where h and c are a regular fiber and a horizontal loop of $M_2(t_2) \cong D(-1/2, 1/2)$. If $|\alpha| > 1$, then $M_2(L')$ is a Seifert fibered space over a disk with three exceptional fibers, and hence, it is not a lens space, a contradiction. If $\alpha = 0$, then β must be ± 1, and $M_2(L')$ is the connected sum of two 3-dimensional projective space, a contradiction. Hence, $|\alpha| = 1$ and $M_2(L')$ is a Seifert fibered space $S^2(\pm\beta; -1/2, 1/2)$. This implies that L' is equivalent to the 2-bridge link in Figure 13 (2) for some nonzero integer $n(= \pm\beta)$ by [12, Corollary 4.12]. Hence L is equivalent to the link in Figure 12 (1). The remaining assertion is easily observed.

(2) Suppose that L is simple and that the characteristic decomposition of L is trivial. By the definition of the characteristic decomposition, (S^3, L) is a Montesinos pair with no boundary. Since L is not a Montesinos link by the assumption, (S^3, L) is obtained from the $(3, 1)$-manifold pair in Figure 5 (2) by plugging the holes with rational tangles of finite slopes. Note that L is a generalized Montesinos link in the sense of [5]. Since L is a 3-bridge link, it follows from [5, Theorem 2.1 and Figure 9] that L is equivalent to the link in Figure 12 (2).

(3) Suppose that L is simple and that L admits a nontrivial characteristic decomposition. Let F be a family of Conway spheres in (S^3, L) which gives the characteristic decomposition of L, and let $\{(N_i, N_i \cap L)\}_{i=1}^m$ be the Montesinos pairs in the decomposition. Since the double branched coverings of N_i branched over $N_i \cap L$ are Seifert fibered spaces, the double branched covering $M_2(L)$ of S^3 branched over L is a graph manifold.

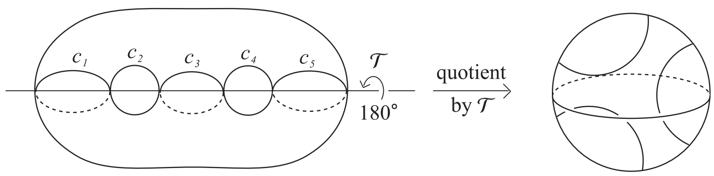


Fig. 14

Let p be the covering projection $M_2(L) \to S^3$, and assume, on the contrary, that $p^{-1}(F)$ does not give a (nontrivial) torus decomposition of $M_2(L)$. Then $p^{-1}(N_i) \cup p^{-1}(N_j)$ is a (connected) Seifert fibered space for some i, j $(i \neq j) \in \{1, \dots, m\}$.

If neither $(N_i, N_i \cap L)$ nor $(N_j, N_j \cap L)$ is the ring tangle in Figure 10, then each of $p^{-1}(N_i)$ and $p^{-1}(N_j)$ admits a unique Seifert fibration by Remark 3. The two Montesinos pairs $(N_i, N_i \cap L)$ and $(N_j, N_j \cap L)$ are glued so that the images of regular fibers of $p^{-1}(N_i)$ and $p^{-1}(N_j)$ are identified. Then, either $(N_i \cup N_j, (N_i \cup N_j) \cap L)$ is a Montesinos pair, or it contains a mutually parallel components contributing ring tangles and hence L is non-simple. This contradicts the assumption.

If $(N_i, N_i \cap L)$ or $(N_j, N_j \cap L)$ is a ring tangle, then its pre-image can be regarded as $D(-1/2, 1/2)$ or an S^1-bundle over Möbius band by Remark 3 (1). By an argument similar to that in the previous case, it can be seen that $(N_i, N_i \cap L) \cup (N_j, N_j \cap L)$ forms a Montesinos pair or L is non-simple. This again contradicts the assumption.

Hence, the pre-image of F gives the torus decomposition of $M_2(L)$. Moreover, each component of the pre-image of F is a separating torus, because each component of F separates S^3 and its pre-image in $M_2(L)$ is connected. $\square$

3. 3-bridge spheres and genus-2 Heegaard surfaces

Let M be a closed orientable 3-manifold of Heegaard genus 2, and let $(V_1, V_2; F)$ be a genus-2 *Heegaard splitting* of M, i.e., V_1 and V_2 are genus-2 handlebodies in M such that $M = V_1 \cup V_2$ and $F = \partial V_1 = \partial V_2 = V_1 \cap V_2$. By [2, Proof of Theorem 5], there is an involution τ on M which satisfies the following condition.

(∗) $\tau(V_i) = V_i$ $(i = 1, 2)$ and $\tau|_{V_i}$ is *equivalent* to the standard involution $\mathcal{T}$ on a standard genus-2 handlebody V as illustrated in Figure 14. To be precise, there is a homeomorphism $\psi_i : V_i \to V$ such that $\mathcal{T} = \psi_i(\tau|_{V_i})\psi_i^{-1}$ $(i = 1, 2)$.

Two involutions τ and τ' are said to be *strongly equivalent* if there exists a homeomorphism h on M such that $h\tau h^{-1} = \tau'$ and that h is isotopic to the identity map id_M.

Proposition 5. *Let M be a closed orientable 3-manifold and let $(V_1, V_2; F)$ be a genus-2 Heegaard splitting of M. Let τ and τ' be involutions of M satisfying the condition $(*)$. Then τ and τ' are strongly equivalent.*

Although the above proposition seems to be well-known, we could not find a proof in literature. For completeness, we present a proof which is obtained by refining the proof of [2, Theorem 8].

Proof of Proposition 5. By the assumption, there exist homeomorphisms $\psi_i : V_i \to V$ and $\psi_i' : V_i \to V$ such that $\psi_i(\tau|_{V_i})\psi_i^{-1} = \mathscr{T}$ and $\psi_i'(\tau'|_{V_i})\psi_i'^{-1} = \mathscr{T}$ $(i = 1, 2)$. Put $\varphi_1 := \psi_1'\psi_1^{-1} : V \to V$. By [2, Theorem 5], $\varphi_1|_{\partial V}$ can be isotoped to a homeomorphism $\varphi_1' : \partial V \to \partial V$ which commutes with $\mathscr{T}|_{\partial V}$. Since $\varphi_1|_{\partial V}$ extends to a homeomorphism $\varphi_1 : V \to V$, φ_1' also extends to a self-homeomorphism of V, which is denoted by φ_1' again. Since $\varphi_1|_{\partial V}$ and $\varphi_1'|_{\partial V}$ are isotopic, φ_1 and φ_1' are isotopic. By [2, Theorem 7], there exists a homeomorphism $\varphi_1'' : V \to V$ such that $\varphi_1''|_{\partial V} = \varphi_1'|_{\partial V}$ and φ_1'' commutes with the involution $\mathscr{T}$ on V. Since $\varphi_1''|_{\partial V} = \varphi_1'|_{\partial V}$, φ_1'' is isotopic to φ_1'. Put $\eta := \psi_1'^{-1}\varphi_1''\psi_1 : V_1 \to V_1$. Then

$$\begin{aligned}\eta(\tau|_{V_1})\eta^{-1} &= \psi_1'^{-1}\varphi_1''\psi_1(\tau|_{V_1})\psi_1^{-1}\varphi_1''^{-1}\psi_1' = \psi_1'^{-1}\varphi_1''\mathscr{T}\varphi_1''-1\psi_1' \\ &= \psi_1'^{-1}\mathscr{T}\psi_1' = \tau'|_{V_1}.\end{aligned}$$

Moreover, since φ_1'' is isotopic to φ_1, η is isotopic to $\psi_1'^{-1}\varphi_1\psi_1 = \mathrm{id}_{V_1}$.

Since $\eta|_F$ is isotopic to the identity map on $\partial V_2 = \partial V_1 = F$, it extends to a self-homeomorphism of M, which is isotopic to id_M. We use the same symbol, η, to denote the above homeomorphism on M. Note that $\eta\tau\eta^{-1} = \tau'$ on V_1, especially on $\partial V_2 = \partial V_1$. By applying the previous argument to $\psi_2'(\eta|_{V_2})\psi_2^{-1} : V \to V$, we can find a $\mathscr{T}$-equivariant homeomorphism $\varphi_2 : V \to V$ such that $\varphi_2|_{\partial V} = (\psi_2'\eta\psi_2^{-1})|_{\partial V}$. Set $\eta' := \psi_2'^{-1}\varphi_2\psi_2 : V_2 \to V_2$. Then we have $\eta'|_{\partial V_2} = \eta|_{\partial V_2}$, which implies that η' is isotopic to id_{V_2}, and $\eta'(\tau|_{V_2})\eta' = \tau'|_{V_2}$.

By gluing the (τ, τ')-equivariant homeomorphisms $\eta|_{V_1}$ and $\eta' : V_2 \to V_2$, we obtain a homeomorphism $h : M \to M$ such that $h\tau h^{-1} = \tau'$ and that h is isotopic to id_M. Hence we obtain the required result. $\square$

Two Heegaard splittings $(V_1, V_2; F)$ and $(W_1, W_2; G)$ of a 3-manifold M are said to be *isotopic* if there exists a self-homeomorphism f of M such that $f(F) = G$ and f is isotopic to the identity map id_M on M. Thus we regard

$(V_1, V_2; F)$ and $(V_2, V_1; F)$ as the same Heegaard splittings. We say that two Heegaard splittings $(V_1, V_2; F)$ and $(W_1, W_2; G)$ of a 3-manifold M are *homeomorphic* if there exists an *orientation-preserving* homeomorphism f of M such that $f(F) = G$.

For each genus-2 Heegaard splitting $(V_1, V_2; F)$, we call an involution of M satisfying the condition $(*)$ the *hyper-elliptic involution* associated with $(V_1, V_2; F)$ (or associated with F, in brief) and denote it by τ_F. By Proposition 5, the strong equivalence class of τ_F is uniquely determined by the isotopy class of $(V_1, V_2; F)$.

Let L be a 3-bridge link and let M be the double branched covering of S^3 branched over L. Let τ_L be the covering transformation on M. If S is a 3-bridge sphere of L, its pre-image in M is a genus-2 Heegaard surface F such that $\tau_F = \tau_L$. Moreover, the isotopy class of F is uniquely determined by that of S because a pairwise isotopy on (S^3, L) lifts to an isotopy on M. Thus we obtain the following map Φ_L from the set of 3-bridge spheres of L, up to isotopy, to the set of genus-2 Heegaard surfaces of M, up to isotopy, whose hyper-elliptic involutions are τ_L.

$$\begin{aligned}\Phi_L : &\{\text{3-bridge spheres of } L\}/\sim \\ &\to \{\text{genus-2 Heegaard surfaces } F \text{ of } M \text{ s.t. } \tau_F = \tau_L\}/\sim.\end{aligned}$$

It is obvious that Φ_L is surjective. We will discuss the injectivity of Φ_L in the sequel of this paper. Note that we also obtain the following map Φ.

$$\begin{aligned}\Phi : &\{(L, S) \mid L : \text{3-bridge link, } S : \text{3-bridge sphere of } L\}/\cong \\ &\to \{(M, F) \mid M : \text{genus-2 3-manifold,} \\ &\qquad F : \text{genus-2 Heegaard surface of } M\}/\cong.\end{aligned}$$

Here, $(L, S) \cong (L', S')$ means that there exists an orientation-preserving self-homeomorphism of S^3 which sends L to L' and S to S', whereas $(M, F) \cong (M', F')$ means that there exists an orientation-preserving homeomorphism from M to M' which sends F to F'. Then it is proved by Birman and Hilden [2, Theorem 8] that this map Φ is bijective.

4. Heegaard splittings of genus-2 graph manifolds

In [17], Kobayashi gave a classification of genus-2 closed Haken manifolds which admit nontrivial torus decompositions, by studying the intersection of Heegaard surfaces and essential tori. In this section, we recall the result and characterize genus-2 Heegaard splittings of genus-2 graph manifolds which admit nontrivial torus decompositions by separating essential tori.

We use the following notation.

$D[r]$ (resp. $M\ddot{o}[r]$, $A[r]$): the set of all orientable Seifert fibered spaces over a disk D (resp. a Möbius band $M\ddot{o}$, an annulus A) with r exceptional fibers.

SM_K: the set of the exteriors of the nontrivial 2-bridge knots which admit Seifert fibrations.

SM_L: the set of the exteriors of the nontrivial 2-bridge links different from the Hopf link which admit Seifert fibrations.

SL_K: the set of the exteriors of the 1-bridge knots in lens spaces each of which admits a Seifert fibration whose horizontal loop is a meridian loop.

KI: the twisted I-bundle on the Klein bottle.

Remark 5. The family SM_K (resp. SM_L, SL_K) consists of Seifert fibered spaces contained in the family M_K (resp. M_L, L_K) introduced in [17].

In the above, we regard S^3 and $S^2 \times S^1$ as lens spaces, and a knot in a lens space L_N is called a *1-bridge knot* if there is a Heegaard splitting $(V_1, V_2; F)$ of L_N of genus one such that $V_i \cap K$ $(i = 1, 2)$ is an arc trivially embedded in V_i. Here, an arc a in a solid torus V is said to be *trivially embedded* in V if there is a disk D in V such that $D \cap \partial V = b$ is an arc and $cl(\partial D - b) = a$.

The following lemmas characterize the families SM_K, SM_L and SL_K.

Lemma 1 ([17, Lemmas 4.2 and 4.4]). (1) *For a nontrivial 2-bridge knot $S(\alpha, \beta)$ with $|\beta| < |\alpha|$, its exterior belongs to SM_K if and only if $\beta/\alpha = 1/(2n+1)$ for some integer n with $|2n+1| \geq 3$. Moreover, the exterior $E(S(2n+1, 1))$ is homeomorphic to the Seifert fibered space $D(1/2, -n/(2n+1)) \in D[2]$ (w.r.t. a regular fiber and the meridian).*

(2) *For a 2-bridge link $S(\alpha, \beta)$ with $|\beta| < |\alpha|$, its exterior belongs to SM_L if and only if $\beta/\alpha = 1/(2n)$ for some integer n with $|n| \geq 2$. Moreover, the exterior $E(S(2n, 1))$ with $|n| \geq 2$ is homeomorphic to the Seifert fibered space $A(1/n) \in A[1]$ (w.r.t. a regular fiber and the meridians).*

Lemma 2 ([30, Lemma 1]). *Let K be a 1-bridge knot in a lens space such that its exterior $E(K)$ belongs to SL_K. Then $E(K)$ is homeomorphic to the Seifert fibered space $D(\beta_1/\alpha_1, \beta_2/\alpha_2) \in D[2]$ or $M\ddot{o}(1/\alpha) \in M\ddot{o}[0] \cup M\ddot{o}[1]$, where the meridian loop of K is a horizontal loop of the Seifert fibered space.*

Conversely, the Seifert fibered space $D(\beta_1/\alpha_1, \beta_2/\alpha_2) \in D[2]$ or $M\ddot{o}(1/\alpha) \in M\ddot{o}[0] \cup M\ddot{o}[1]$ is the exterior of a 1-bridge knot K in a lens space where the meridian of K is a horizontal loop.

Remark 6. When $E(K) \cong M\ddot{o}(1/\alpha)$ in the above lemma, the lens space containing K is homeomorphic to $P^2(0; 1/\alpha) \cong S^2(\alpha; -1/2, 1/2)$. Moreover, K is a regular fiber of $P^2(0; 1/\alpha)$, and the meridian of K is a horizontal loop of $E(K) \cong M\ddot{o}(1/\alpha)$ (see [30, Proof of Lemma 1]).

From the main theorem of Kobayashi [17] together with the above lemmas, we have the following characterization of genus-2 Heegaard splittings of genus-2 graph manifolds which admit nontrivial torus decompositions by separating essential tori. (Though the genus-2 manifolds admitting non-separating essential tori are also studied by Kobayashi [17], we do not need to study such manifolds in this paper.)

Theorem 5. *Let M be a closed, orientable, connected graph manifold with a Heegaard splitting of genus two. Assume that M admits a nontrivial torus decomposition by T such that each component of T is separating. Let $(V_1, V_2; F)$ be a genus-*2 *Heegaard splitting of M. Then F is ambient isotopic to a Heegaard surface, denoted by the same symbol F, which satisfies one of the following four conditions* (F1), (F2), (F3) *and* (F4) *(see Figure* 15*). Moreover, M is obtained by gluing Seifert fibered spaces as in* (M1-a), ..., (M4) *under each condition* (F1), (F2), (F3) *or* F(4) *as follows.*

(F1) *For $i = 1, 2$, $V_i \cap T$ consists of a single separating essential annulus.*

In this case, M is obtained from M_1 and M_2 by identifying their boundaries, where

(M1-a) $M_1 \in D[2]$ *and* $M_2 \in SL_K \cap D[2]$, *or*

(M1-b) $M_1 \in D[2]$ *and* $M_2 \in SL_K \cap M\ddot{o}[1]$,

where the regular fiber and a horizontal loop of M_1 are identified with the meridian loop and the regular fiber of M_2, respectively.

Moreover,

- *$M_1 \cap F$ is an essential annulus saturated in the Seifert fibration of M_1, and*
- *$M_2 \cap F$ is a* 2*-holed torus which gives a* 1*-bridge decomposition of the* 1*-bridge knot K such that $M_2 = E(K)$.*

(F2) *By exchanging V_1 and V_2 if necessary,*

(i) *$V_1 \cap T$ consists of two disjoint non-separating essential annuli satisfying the following condition: there exists a complete meridian disk system (D_1, D_2) of V_1 such that $D_1 \cap (V_1 \cap T) = \emptyset$ and $D_2 \cap (V_1 \cap T)$ consists of essential arcs properly embedded in each annulus of $V_1 \cap T$, and*

(ii) *$V_2 \cap T$ consists of disjoint non-parallel separating essential annuli.*

In this case, M is obtained from M_1 and M_2 by identifying their boundaries, where

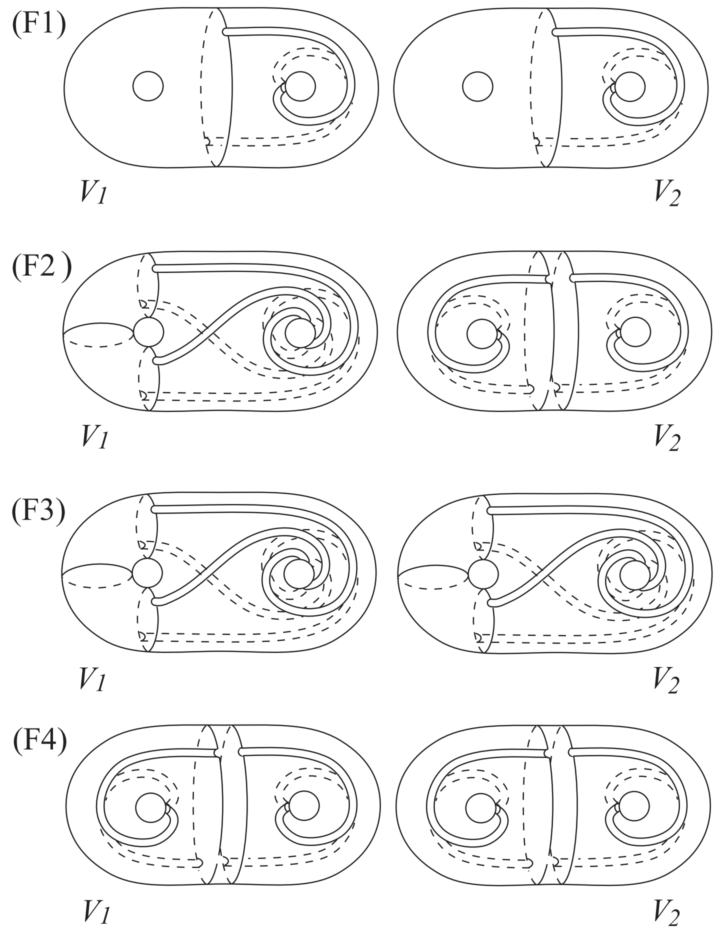


Fig. 15

(M2-a) $M_1 \in D[2]$ *and* $M_2 \in SM_K \subset D[2]$, *or*
(M2-b) $M_1 \in D[3]$ *and* $M_2 \in SM_K \subset D[2]$,
where the regular fiber and a horizontal loop of M_1 *are identified with the meridian loop and the regular fiber of* M_2, *respectively.*

Moreover,

- $M_1 \cap F$ *consists of two disjoint essential saturated annuli in* M_1 *which divide* M_1 *into three solid tori, and*
- *the* 2-*bridge knot corresponding to* M_2 *is* $S(2n+1, 1)$, *and* $M_2 \cap F$ *is a* 2-*bridge sphere.*

(F3) *For* $i = 1, 2$, $V_i \cap T$ *consists of two disjoint non-separating essential annuli satisfying the condition* (i) *in* (F2).

In this case,

(M3) *M is obtained from $M_1 \in M\ddot{o}[r]$ $(r = 1, 2)$ and $M_2 \in SM_K \subset D[2]$ by identifying their boundaries, where the regular fiber and a horizontal loop of M_1 are identified with the meridian loop and the regular fiber of M_2, respectively.*

Moreover,

- *$M_1 \cap F$ consists of two disjoint essential saturated annuli in M_1 which divide M_1 into two solid tori, and*
- *the 2-bridge knot corresponding to M_2 is $S(2n + 1, 1)$, and $M_2 \cap F$ is a 2-bridge sphere.*

(F4) *For $i = 1, 2$, $V_i \cap T$ consists of two disjoint non-parallel separating essential annuli satisfying the condition* (ii) *in* (F2).

In this case,

(M4) *M is obtained from $M_1, M_2 \in D[2]$ and $M_3 \in SM_L \subset A[1]$ by identifying their boundaries where the regular fibers and horizontal loops of M_i $(i = 1, 2)$ are identified with the meridian loops and regular fibers of M_3, respectively.*

Moreover,

- *$M_i \cap F$ is an essential saturated annulus in M_i $(i = 1, 2)$, and*
- *the 2-bridge link corresponding to M_3 is $S(2n, 1)$ $(|n| \geq 2)$, and $M_3 \cap F$ is a 2-bridge sphere.*

In the above theorem, a surface in a Seifert fibered space is said to be *saturated* in the Seifert fibration if it is a union of fibers.

PROOF OF THEOREM 5. The desired results follow from the main theorem of [17] and Lemmas 1 and 2. Here we note that we do not have the case with $M_1 \in D[2]$ and $M_2 \in SL_K \cap M\ddot{o}[0]$ (see Lemma 2), because Lemma 2 implies that in this case the Seifert fibration on M_1 extends to a Seifert fibration on M. The numbers of the conditions in this theorem correspond to the numbers of the conditions in the main theorem of [17] as follows.

Theorem 5	(M1-a), (M1-b)	(M2-a), (M2-b)	(M3)	(M4)
main theorem of [17]	(i)	(iii)	(ii)	(iv)

□

DEFINITION 1. We define M(1-a), M(1-b), M(2-a), M(2-b), M(3) and M(4) to be the families of 3-manifolds which satisfy the conditions (M1-a), (M1-b), (M2-a), (M2-b), (M3) and (M4) of Theorem 5, respectively. We set $\mathrm{M}(1) = \mathrm{M}(1\text{-a}) \cup \mathrm{M}(1\text{-b})$ and $\mathrm{M}(2) = \mathrm{M}(2\text{-a}) \cup \mathrm{M}(2\text{-b})$.

5. Mapping class groups

In this section, we calculate a certain subgroup of the mapping class groups of the Seifert fibered spaces and the graph manifolds which arose in Theorem 5. The results in this section are used in Sections 7, 8 and in the sequel of this paper.

Let M be a compact orientable 3-manifold obtained by gluing two 3-manifolds M_1 and M_2 along a torus T. We identify the universal cover of T with $\mathbf{R}^2$ and $\pi_1(T)$ with the action of $\mathbf{Z}^2$ on $\mathbf{R}^2$. Then T is identified with $\mathbf{R}^2/\mathbf{Z}^2$. By considering the regular neighborhood of T in M, we identify M with the union $M_1 \cup (T \times [1,2]) \cup M_2$, where $M_i \cap (T \times [1,2]) = T \times \{i\}$ for $i = 1,2$. For a rational number r and an oriented essential simple loop γ on T, an *r-Dehn twist*, D_γ^r, along T in the direction of γ is a self-homeomorphism of $T \times [1,2]$ defined as follows.

$$D_\gamma^r([\vec{x}], t) := ([\vec{x} + r\phi(t)\vec{\gamma}], t),$$

where $\vec{\gamma}$ is the element of $\mathbf{Z}^2 \subset \mathbf{R}^2$ corresponding to γ, $[\vec{x}]$ denotes the point of $\mathbf{R}^2/\mathbf{Z}^2$ determined by $\vec{x} \in \mathbf{R}^2$ and ϕ is a smooth function on $\mathbf{R}$ such that $\phi(\infty, 1] = 0$, $\phi[2, \infty) = 1$ and $\phi|_{[1,2]}$ is increasing.

If r is an integer, then $D_\gamma^r|_{T\times\{i\}} = id_{T\times\{i\}}$ for $i = 1,2$. Hence, D_γ^r extends to a self-homeomorphism $id_{M_1} \cup D_\gamma^r \cup id_{M_2}$, which we denote by D_γ^r again. We denote D_γ^1 by D_γ, and call it the *Dehn twist* along T in the direction of γ.

For a Seifert fibered space N, let $\mathscr{M}(N)$ be the subgroup of the (orientation-preserving) mapping class group of N which consists of the elements preserving the Seifert fibration of N and each of exceptional fibers.

Let M be a 3-manifold which belongs to $\mathrm{M}(1) \cup \mathrm{M}(2) \cup \mathrm{M}(3) \cup \mathrm{M}(4)$, where $\mathrm{M}(1)$, $\mathrm{M}(2)$, $\mathrm{M}(3)$ and $\mathrm{M}(4)$ are as in Definition 1. Then, by the definition, M is obtained by gluing two or three Seifert fibered spaces. Let $\mathscr{M}(M)$ denote the subgroup of the (orientation-preserving) mapping class group of M which consists of the elements preserving each piece of the torus decomposition, the Seifert fibration of each Seifert piece and each exceptional fiber. We remark that the group $\mathscr{M}(M)$ depends on the Seifert fibration of (Seifert pieces of) M. When we treat $\mathscr{M}(M)$, we consider the Seifert fibration of (Seifert pieces of) M which can be seen in context.

The reason why we are interested in the subgroup $\mathscr{M}(M)$ is that the hyper-elliptic involution τ_F associated with a genus-2 Heegaard surface F determines an element in $\mathscr{M}(M)$ (cf. [4, Proposition 20]). In this section, we calculate $\mathscr{M}(M)$ for some manifolds in Theorem 5. Throughout this paper, we do not distinguish between a self-homeomorphism and its isotopy class: we denote them by the same symbol.

Let M be a manifold in $\mathrm{M}(1) \cup \mathrm{M}(2) \cup \mathrm{M}(3) \cup \mathrm{M}(4)$, and let T be the union of tori which give the torus decomposition of M. Let $\mathscr{D}$ be the subgroup of $\mathscr{M}(M)$ generated by the all possible Dehn twists along T. Then we obtain the following.

LEMMA 3. (1) *$\mathscr{D} \cong \mathbf{Z}$ if M belongs to* $\mathrm{M}(1\text{-b})$, $\mathrm{M}(3)$ *or* $\mathrm{M}(4)$. *Moreover, $\mathscr{D}$ is generated by D_γ, where γ is a regular fiber of M_2, a regular fiber of M_1 or a regular fiber of M_3 according as M belongs to* $\mathrm{M}(1\text{-b})$, $\mathrm{M}(3)$ *or* $\mathrm{M}(4)$.

(2) *$\mathscr{D} = 1$ if M belongs to* $\mathrm{M}(1\text{-a})$ *or* $\mathrm{M}(2)$.

PROOF. Choose a normal orientation for T. For each closed-up component M_i of $M \backslash T$ which is *totally orientable* (i.e., M_i itself and its base space are orientable), pick out an (oriented) fiber $C_{i,j}$ on each component $T_{i,j}$ of ∂M_i. Let $\varepsilon_{i,j} = +1$ or -1 according as the orientations of T and ∂M_i coincide on $T_{i,j}$ or not. Let V be the subgroup of $H_1(T)$ generated by the elements $\sum_j \varepsilon_{i,j}[C_{i,j}]$, where M_i ranges over all totally orientable Seifert fibered closed-up components of $M \backslash T$. Then we have $\mathscr{D} \cong H_1(T)/V$ by [6, Proposition 15.2].

We prove the assertion for the case $M \in \mathrm{M}(4)$. (The remaining cases can be treated similarly.) Note that $H_1(T) = \langle c_1, h_1, c_2, h_2 \rangle \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$, where c_i and h_i are a horizontal loop and the regular fiber of $M_i \in D[2]$, respectively ($i = 1, 2$). By the definition of $\mathrm{M}(4)$, we may choose c_1 and c_2 so that they are identified with the regular fibers of $M_3 \in A[1]$. We assume that c_1 is homologous to $-c_2$ in M_3. Since M_1, M_2 and M_3 are totally orientable, $V = \langle h_1, h_2, c_1 - c_2 \rangle$. Hence, we have $\mathscr{D} \cong H_1(T)/V \cong \langle c_1 \rangle \cong \mathbf{Z}$. $\square$

When M is obtained by gluing two Seifert fibered spaces M_1 and M_2, let Δ be the subgroup of $\mathscr{M}(M_1) \times \mathscr{M}(M_2)$ consisting of all elements (f_1, f_2) such that $f_1|_T$ is isotopic to $f_2|_T$. When M is obtained by gluing three Seifert fibered spaces M_1, M_2 and M_3 along $T = T_1 \cup T_2$, where $T_i = M_i \cap M_3$ ($i = 1, 2$), let Δ be the subgroup of $\mathscr{M}(M_1) \times \mathscr{M}(M_2) \times \mathscr{M}(M_3)$ consisting of all elements (f_1, f_2, f_3) such that $f_1|_{T_1}$ and $f_2|_{T_2}$ are isotopic to $f_3|_{T_1}$ and $f_3|_{T_2}$, respectively. Then we have the following exact sequence (see [6, Theorem 15.1] and [24]).

$$1 \to \mathscr{D} \to \mathscr{M}(M) \to \Delta \to 1. \tag{1}$$

Next, we calculate $\mathscr{M}(N)$ for $N \in D[2] \cup M\ddot{o}[1] \cup A[1]$.

LEMMA 4. (1) *If N is a Seifert fibered space $D(\beta_1/\alpha_1, \beta_2/\alpha_2) \in D[2]$, then $\mathscr{M}(N) = \langle f \rangle \cong \mathbf{Z}_2$, where f is the involution of N in Remark* 2 *(see Figure* 16 (1)*)*.

(2) *If N is a Seifert fibered space $M\ddot{o}(\beta/\alpha) \in M\ddot{o}[1]$, then $\mathscr{M}(N) = \langle g_1, g_2, b \rangle \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$, where g_1 and g_2 are the involutions as illustrated in Figure* 16 (2) *and b is the Dehn twist along a saturated annulus A_b in Figure* 16 (2).

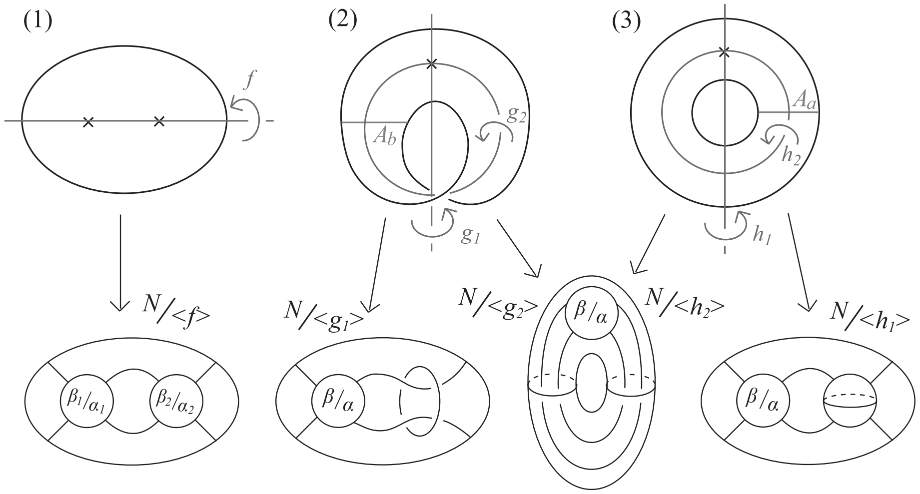


Fig. 16

(3) *If N is a Seifert fibered space $A(\beta/\alpha) \in A[1]$, then $\mathscr{M}(N)$ has a presentation*

$$\mathscr{M}(N) = \langle h_1, h_2, a \mid h_1^2, h_2^2, [h_1, h_2], [h_1, a], h_2 a h_2^{-1} a \rangle,$$

where h_1 and h_2 are the involutions as illustrated in Figure 16 (3) *and a is the Dehn twist along a saturated annulus A_a in Figure* 16 (3).

Remark 7. The Dehn twist along an annulus is defined as for the Dehn twist along a torus. We note that the Dehn twist b in (2) is isotopic to the π-rotation along fibers of $N \in M\ddot{o}[1]$ (see [15, Lemma 25.1]). From now on, b denotes the π-rotation along fibers of $N \in M\ddot{o}[1]$.

Proof of Lemma 4. By [15, Proposition 25.3], the full mapping class group of the Seifert fibered space N is generated by fiber-preserving homeomorphisms. Let F be the base space of N. Let $\mathscr{M}^0(N)$ be the subgroup of $\mathscr{M}(N)$ generated by those (fiber-preserving) elements which induce the identity map on F, and let $\mathscr{M}^*(F)$ denote the subgroup of the mapping class group of the pair (F, exceptional points), generated by those homeomorphisms which fix each of the exceptional points. By [15, Proposition 25.3], we have a split exact sequence

$$1 \to \mathscr{M}^0(N) \to \mathscr{M}(N) \to \mathscr{M}^*(F) \to 1.$$

Moreover, it is proved in [15, Lemma 25.2] that $\mathscr{M}^0(N)$ is isomorphic to the first relative homology group $H_1(F, \partial F)$ of the base space.

(1) Let N be a Seifert fibered space over a disk with two exceptional fibers. It is easy to see that $\mathcal{M}^*(D) \cong \mathbf{Z}_2$ for a disk D with one exceptional point. We note that the essential arcs properly embedded in a disk with two exceptional points are unique up to ambient isotopy. These imply that $\mathcal{M}^*(F) \cong \mathbf{Z}_2$ and the generator lifts to the self-homeomorphism f of N in Figure 16 (1). Since $\mathcal{M}^0(N) = 1$, we have the desired result.

(2) Let N be a Seifert fibered space over a Möbius band with one exceptional fiber. Then we can see that $\mathcal{M}^0(N) = \langle b \rangle \cong \mathbf{Z}_2$ (see [15, Lemma 25.2]) and $\mathcal{M}^*(F) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ whose generators lift to the self-homeomorphisms g_1 and g_2 of N. Hence, we obtain the desired result.

(3) Let N be a Seifert fibered space over an annulus with one exceptional fiber. Then we can see that $\mathcal{M}^0(N) = \langle a \rangle \cong \mathbf{Z}$ and $\mathcal{M}^*(F) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ whose generators lift to the self-homeomorphisms h_1 and h_2 of N. By using these results, we obtain the desired result. $\square$

In the remainder of this section, we describe $\mathcal{M}(M)$ for $M \in \mathrm{M}(1) \cup \mathrm{M}(4)$. To this end, we define a family of self-homeomorphisms of M.

Definition 2. (1) Let M be a manifold in M(1-a), i.e., $M = M_1 \cup (T \times [1,2]) \cup M_2$, where $M_1, M_2 \in D[2]$. We define a self-homeomorphism G_0 of M as follows.

$$G_0|_{M_1} = f_1, \qquad G_0|_{M_2} = f_2, \qquad G_0|_{T\times[1,2]} = R,$$

where f_i $(i = 1,2)$ is the involution f on $M_i \in D[2]$ in Remark 2 (see Figure 16 (1)), and R is the self-homeomorphism of $T \times [1,2]$ defined by $R([\vec{x}], t) = ([-\vec{x}], t)$.

(2) Let M be a manifold in M(1-b), i.e., $M = M_1 \cup (T \times [1,2]) \cup M_2$, where $M_1 \in D[2]$ and $M_2 \in M\ddot{o}[1]$. Let h_i be a regular fiber of M_i $(i = 1,2)$. We define self-homeomorphisms G_1, G_2, H and μ of M as follows. (Recall that D_γ^r denotes the r-Dehn twist along T in the direction of γ.)

$$\begin{aligned}
&G_1|_{M_1} = f, && G_1|_{M_2} = g_1, && G_1|_{T\times[1,2]} = R,\\
&G_2|_{M_1} = id, && G_2|_{M_2} = g_2, && G_2|_{T\times[1,2]} = D_{h_1}^{1/2},\\
&H|_{M_1} = id, && H|_{M_2} = b, && H|_{T\times[1,2]} = D_{h_2}^{1/2},\\
&\mu = D_{h_2}.
\end{aligned}$$

Here, f, g_1 and g_2 are involutions of M_1 or M_2 as described in Lemma 4, b is an involution of M_2 as described in Remark 7, and R is the involution of $T \times [1,2]$ defined in (1).

(3) Let M be a manifold in M(4), i.e., $M = M_1 \cup M_2 \cup ((T_1 \cup T_2) \times [1,2]) \cup M_3$, where $M_1, M_2 \in D[2]$, $M_3 \in A[1]$, $T_i \times \{1\} \subset M_i$ and $T_i \times \{2\} \subset M_3$ $(i = 1,2)$. Let $h_3^i (\subset T_i)$ be a regular fiber of M_3. We define self-homeomorphisms G_3 and λ of M as follows.

$$G_3|_{M_i} = f_i, \qquad G_3|_{M_3} = h_1, \qquad G_3|_{T_i \times [1,2]} = R \quad (i = 1,2),$$

$$\lambda_i = D_{h_3^i}.$$

Here, f_i $(i = 1,2)$ and R are the involutions of M_i and $T_i \times [1,2]$, respectively, as in (1), and h_1 is the involution of M_3 described in Lemma 4 (3). Since λ_1 and λ_2 are isotopic by Lemma 3, we denote them by λ.

Proposition 6. (1) *If* $M \in$ M(1-a) *and the decomposition of* M *into* M_1 *and* $M_2 \in D[2]$ *is the torus decomposition, then* $\mathcal{M}(M) = \langle G_0 \mid G_0^2 \rangle \cong \mathbf{Z}_2$.

(2) *If* $M \in$ M(1-b) *and the decomposition of* M *into* $M_1 \in D[2]$ *and* $M_2 \in M\ddot{o}[1]$ *is the torus decomposition, then* $\mathcal{M}(M)$ *has a group presentation*

$$\mathcal{M}(M) = \langle G_1, G_2, H, \mu \mid G_i^2 \ (i = 0,1), H^2\mu^{-1}, [G_1, G_2], [G_1, H]\mu, [G_2, H],$$
$$G_1 \mu G_1^{-1} \mu, [G_2, \mu], [H, \mu] \rangle$$

(3) *If* $M \in$ M(4) *and the decomposition of* M *into* $M_1, M_2 \in D[2]$ *and* $M_3 \in A[1]$ *is the torus decomposition, then* $\mathcal{M}(M)$ *has a group presentation*

$$\mathcal{M}(M) = \langle G_3, \lambda \mid G_3^2, (G_3\lambda)^2 \rangle$$

Proof. (2) Assume that $M \in$ M(1-b) and the decomposition of M into $M_1 \in D[2]$ and $M_2 \in M\ddot{o}[1]$ is the torus decomposition. We identify M with $M_1 \cup (T \times [1,2]) \cup M_2$ by considering the regular neighborhood of $T = \partial M_1 = \partial M_2$.

Let h_1 and $h_2 (\subset T)$ be regular fibers of M_1 and M_2, respectively. Then $\{h_1, h_2\}$ generates $\pi_1(T)$. We denote the Dehn twists along T in the direction of h_1 and h_2 by λ and μ, respectively. Then, by Lemma 3 (1), $\mathcal{D}$ is the infinite cyclic group generated by μ, where $\lambda = 1$.

On the other hand, we have $\mathcal{M}(M_1) = \langle f \rangle \cong \mathbf{Z}_2$ and $\mathcal{M}(M_2) = \langle g_1, g_2, b \rangle \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ by Lemma 4 (1) and (2). Then by considering matching conditions on T, we see that

$$\Delta = \langle (f, g_1), (id, g_2), (id, b) \rangle \cong (\mathbf{Z}_2)^3.$$

Let G_1, G_2 and H be the self-homeomorphisms of M as described in Definition 2. Then we have

$$\begin{gathered} G_i^2 = [G_1, G_2] = [G_2, H] = 1 \qquad (i = 1,2), \\ H^2 = \mu, \qquad G_1 H G_1^{-1} = H^{-1} \text{ and hence } [G_1, H] = \mu^{-1} \end{gathered} \tag{2}$$

in $\mathscr{M}(M)$. Moreover,

$$\iota_{G_1} = -id, \qquad \iota_{G_2} = id, \qquad \iota_H = id, \tag{3}$$

where ι_Y denotes the restriction of the inner automorphism $X \to YXY^{-1}$ induced by Y $(= G_1, G_2$ or $H)$ to $\mathscr{D} = \langle \mu \rangle$. Hence the above equalities (2) and (3), together with the exact sequence (1), implies the desired presentation of $\mathscr{M}(M)$.

(1), (3) can be proved similarly using the exact sequence (1) and Lemmas 3 and 4. $\square$

6. Principles of gluing

In this section, we describe useful facts which are used in the proof of Theorems 1, 2 and in the sequel of this paper.

Let M be a closed orientable 3-manifold obtained by gluing M_1 and M_2 along a torus T, and let F be a surface in M intersecting T transversely in n parallel essential loops, C, for some positive integer n. Then we may identify M with $M = M_1 \cup (T \times [1,2]) \cup M_2$ as in Section 5, and we may assume that $A := F \cap (T \times [1,2])$ consists of n parallel product annuli $C \times [1,2]$. Put $F_i := F \cap M_i$ for $i = 1,2$. Let γ be an oriented loop on T which meets each component of A once, and let $D_\gamma^{k/n}$ be the k/n-Dehn twist along T in the direction of γ defined in the previous section. We define $D_\gamma^{k/n}(F)$ by $F_1 \cup D_\gamma^{k/n}(A) \cup F_2$, where we assume that the components of A are "evenly spaced" on $T \times [1,2]$. We call it the *surface obtained from F by applying k/n-Dehn twist along T in the direction of γ*. Note that if F is a separating surface (e.g., Heegaard surface), then n is even, which implies $D_\gamma^{1/2}(F)$ is defined.

LEMMA 5. *Let M and F be as in the above, and assume that F is a genus-2 Heegaard surface of M. Assume also that the involution τ_F preserves the above decomposition, and set $\tau_i := \tau_F|_{M_i}$ $(i = 1,2)$ and $\tau_A := \tau_F|_{T\times[1,2]}$. Assume further that τ_A is hyper-elliptic, namely, $\tau_A([\vec{x}], t) = ([-\vec{x}], t)$.*

If $F' := D_\gamma^{1/2}(F)$ is also a genus-2 Heegaard surface of M, then $\tau_{F'} = D_\gamma \tau_F$.

PROOF. Consider the involution $\tau_A' = D_\gamma^{1/2} \tau_A D_\gamma^{-1/2}$ on $T \times [1,2]$. Then $\tau_A' = D_\gamma \tau_A$ on $T \times [1,2]$, because

$$\begin{aligned} D_\gamma^{1/2}\tau_A D_\gamma^{-1/2}([\vec{x}], t) &= D_\gamma^{1/2}\tau_A\big(\big[\vec{x} - \tfrac{1}{2}\phi(t)\vec{\gamma}\big], t\big) \\ &= D_\gamma^{1/2}\big(\big[-\vec{x} + \tfrac{1}{2}\phi(t)\vec{\gamma}\big], t\big) \\ &= D_\gamma([-\vec{x}], t) \\ &= D_\gamma \tau_A([\vec{x}], t). \end{aligned}$$

In particular, $\tau'_A|_{T\times\{i\}} = \tau_A|_{T\times\{i\}}$ for $i = 1,2$, and hence, we obtain an involution $\tau' := \tau_1 \cup \tau'_A \cup \tau_2$ on M.

Since

$$D_\gamma^{1/2}\tau_A D_\gamma^{-1/2}(D_\gamma^{1/2}(A)) = D_\gamma^{1/2}\tau_A(A) = D_\gamma^{1/2}(A),$$

we see that the involution τ' preserves $F' = F_1 \cup D_\gamma^{1/2}(A) \cup F_2$. Since τ_F preserves F_i $(i = 1,2)$, $\tau_F|_F : F \to F$ is orientation-preserving and τ_A is hyper-elliptic, we see that $\mathrm{Fix}(\tau_F) \cap A = \emptyset$. Hence

$$\mathrm{Fix}(\tau_F|_F) = \mathrm{Fix}(\tau_1|_{F\cap M_1}) \cup \mathrm{Fix}(\tau_2|_{F\cap M_2}) = \mathrm{Fix}(\tau'|_{F'}).$$

This implies $|\mathrm{Fix}(\tau'|_{F'})| = 6$, and hence τ' is the hyper-elliptic involution $\tau_{F'}$ associated with the Heegaard surface F'.

On the other hand, since $D_\gamma|_{M_i} = id$ for $i = 1,2$, we see from the above facts that

$$\begin{aligned}\tau_{F'} &= \tau' \\ &= \tau_1 \cup \tau'_A \cup \tau_2 \\ &= \tau_1 \cup D_\gamma \tau_A \cup \tau_2 \\ &= D_\gamma \tau_1 \cup D_\gamma \tau_A \cup D_\gamma \tau_2 \\ &= D_\gamma \tau_F. \end{aligned}$$

□

Lemma 6. *Let $M = M_1 \cup M_2$ be a closed orientable 3-manifold obtained by gluing two 3-manifolds M_1 and M_2 along a torus $T = \partial M_1 = \partial M_2$. Let F and F' be two surfaces in M, and put $F_i = F \cap M_i$ and $F'_i = F' \cap M_i$ for $i = 1,2$.*

(1) *Suppose that*
  (i) *$F \cap T = F' \cap T$ consists of n parallel essential loops on T for some positive integer n, and*
  (ii) *F_i is isotopic to F'_i in M_i $(i = 1,2)$.*
  Then F' is isotopic to $D_\gamma^{k/n}(F)$ for some integer k and for some loop γ on T which meets each component of $F \cap T = F' \cap T$ in one point.

(2) *Suppose that*
  (i) *F_1 is isotopic to F'_1 in M_1 by an isotopy fixing ∂M_1, and*
  (ii) *F_2 is isotopic to F'_2 in M_2.*
  Then F and F' are isotopic in M.

Proof. (1) Suppose that (1-i) and (1-ii) hold. Identify M with $M_1 \cup (T \times [1,2]) \cup M_2$. We may assume that each of $F \cap (T \times [1,2])$ and $F' \cap (T \times [1,2])$ consists of n parallel essential annuli in $T \times [1,2]$. By the condition (1-ii), we may assume that $F' \cap M_i = F \cap M_i$ $(i = 1,2)$ after an isotopic deformation of F' in $M = M_1 \cup (T \times [1,2]) \cup M_2$. Then $F' \cap$

$(T\times[1,2])$ consists of n parallel annuli such that $\partial(F'\cap(T\times[1,2]))=\partial(F\cap(T\times[1,2]))$. Thus it follows that $F'\cap(T\times[1,2])$ is isotopic to $D_\gamma^{k/n}(F\cap(T\times[1,2]))$ by an isotopy fixing $\partial(T\times[1,2])$ for some $k\in\mathbf{Z}$. Then F' is isotopic to $D_\gamma^{k/n}(F)$.

(2) Suppose that (2-i) and (2-ii) hold. By the condition (2-ii), we may assume that $F'\cap M_2=F\cap M_2$ after an isotopic deformation of F' in $M=M_1\cup M_2$. By the condition (2-i), $F'\cap M_1$ is isotopic to $F\cap M_1$ by an isotopy fixing ∂M_1. Hence, F' is isotopic to F in M. $\square$

LEMMA 7. *Let L be a link in S^3, and suppose that the $(3,1)$-manifold pair (S^3,L) is decomposed into (B_1^3,L_1) and (B_2^3,L_2) by a Conway sphere (S^2,P). Let S and S' be two surfaces in (S^3,L) such that*

(i) *$S\cap(S^2,P)=S'\cap(S^2,P)$ consists of n parallel loops on $S^2\backslash P$ each of which separates P into two families of two points, and*

(ii) *$S\cap B_i^3$ is isotopic to $S'\cap B_i^3$ in (B_i^3,L_i) $(i=1,2)$.*

Then S and S' are isotopic in (S^3,L).

PROOF. We identify (S^3,L) with $(B_1^3,L_1)\cup((S^2,P)\times[1,2])\cup(B_2^3,L_2)$, and we may assume that each of $S\cap(S^2\times[1,2])$ and $S'\cap(S^2\times[1,2])$ consists of n parallel essential annuli in $(S^2\backslash P)\times[1,2]$. Since $S\cap B_i^3$ is isotopic to $S'\cap B_i^3$ in (B_i^3,L_i) $(i=1,2)$, we may assume that $S\cap B_i^3=S'\cap B_i^3$ $(i=1,2)$ after an isotopic deformation of S' in $(S^3,L)=(B_1^3,L_1)\cup((S^2,P)\times[1,2])\cup(B_2^3,L_2)$. Then $S'\cap(S^2\times[1,2])$ consists of n parallel essential annuli in $(S^2\backslash P)\times[1,2]$ sharing the same boundary with $S\cap(S^2\times[1,2])$. It can be easily seen that $S\cap((S^2,P)\times[1,2])$ and $S'\cap((S^2,P)\times[1,2])$ are isotopic in $(S^2,P)\times[1,2]$ with their boundaries fixed (cf. [6, Theorem 8.3 (3)]). Hence, S and S' are isotopic in (S^3,L). $\square$

Let L be a link in S^3. Let (B_1,L_1) be a tangle in (S^3,L), and put $S_1=\partial B_1$. Let h be a homeomorphism from B_1 to the standard 3-ball $B^3\subset\mathbf{R}^3$ such that $h(S_1\cap L)=P\subset S^2=\partial B^3$, where $P=\left\{\left(0,\frac{\varepsilon_1}{\sqrt{2}},\frac{\varepsilon_2}{\sqrt{2}}\right)\mid\varepsilon_1,\varepsilon_2=\pm1\right\}$. Set $(B_2,L_2)=(S^3,L)\backslash\mathrm{Int}(B_1,L_1)$. Then we regard (S^3,L) as $(B_1,L_1)\cup(B_2,L_2)$, where (B_1,L_1) is glued to (B_2,L_2) by the identity map on S_1. Let g be the rotation through π about the third coordinate axis in $\mathbf{R}^3$. Let $\mu=h^{-1}gh:S_1\to S_1$ and let L' be a link in S^3 such that $(S^3,L')=(B_1,L_1)\cup_\mu(B_2,L_2)$ with μ as gluing map. The operation of replacing (S^3,L) by (S^3,L') is called a *mutation* of (S^3,L) along S_1, by the *mutation involution* μ (cf. [11]). Let c_g be the the circle $\{(x_1,x_2,0)\mid x_1^2+x_2^2=1\}$, and put $c_\mu:=h^{-1}(c_g)$. We call c_μ a *mutation loop* of the mutation.

LEMMA 8. *Let L, L', S_1, μ, c_μ be as above. Let γ be a component of the pre-image of c_μ in $M=M_2(L)(\cong M_2(L'))$. Then there is a homeomorphism $f:M_2(L)\to M_2(L')$ such that $f^{-1}\tau_{L'}f=D_\gamma\tau_L$.*

Proof. Let T be the pre-image of S_1 in M. Then T cuts $M = M_2(L)$ into two manifolds, M_1 and $M_2 \subset M$. Namely, M is obtained by gluing M_1 and M_2 by the identity map on T. Let $\tilde{\mu} \in \mathrm{Diff}(T)$ be a lift of μ, and observe that $\tilde{\mu}([\vec{x}]) = \left[\vec{x} + \frac{1}{2}\vec{\gamma}\right]$. Then $M_2(L')$ is obtained by gluing M_1 and M_2 along T by $\tilde{\mu}$. Identify $M = M_2(L)$ with $M_1 \cup (T \times [1,2]) \cup M_2$, where ∂M_i and $T \times \{i\}$ are identified by the identity map $(i = 1,2)$. Then $M_2(L')$ can be regarded as the manifold obtained from M_1, $T \times [1,2]$ and M_2 by gluing ∂M_1 and $T \times \{1\}$ by the identity map and gluing ∂M_2 and $T \times \{2\}$ by $\tilde{\mu}$. Thus there is a homeomorphism $f : M_2(L) \to M_2(L')$ defined by $f|_{M_i} = id$ $(i = 1,2)$ and $f|_{T\times[1,2]} = D_\gamma^{-1/2}$. On the other hand, we may assume that the involution τ_L on $M = M_1 \cup (T \times [1,2]) \cup M_2$ preserves the decomposition and $\tau_L|_{T\times[1,2]}$ is equal to the involution, $\tau_{[1,2]}$, defined by

$$\tau_{[1,2]}([\vec{x}], t) = ([-\vec{x}], t).$$

Set $\tau_i := \tau_L|_{M_i}$ $(i = 1,2)$. Then the involution $\tau_{L'}$ on $M_2(L') = M_1 \cup_{id} (T \times [1,2]) \cup_{\tilde{\mu}} M_2$ is given by

$$\tau_{L'}|_{M_i} = \tau_i \qquad (i = 1,2),$$

$$\tau_{L'}|_{T\times[1,2]} = \tau_{[1,2]}.$$

Hence, after identifying $M_2(L')$ with $M = M_2(L)$ by the homeomorphism f, the involution $\tau_{L'}$ is identified with

$$\begin{aligned} f^{-1}\tau_{L'}f &= \tau_1 \cup D_\gamma^{1/2}\tau_{[1,2]}D_\gamma^{-1/2} \cup \tau_2 \\ &= \tau_1 \cup D_\gamma \tau_{[1,2]} \cup \tau_2 \\ &= D_\gamma(\tau_1 \cup \tau_{[1,2]} \cup \tau_2) \\ &= D_\gamma \tau_L. \end{aligned}$$

□

7. Proof of Theorem 1

In this section, we prove Theorem 1, namely, we determine all 3-bridge arborescent links. We also describe all 3-bridge spheres of the links up to homeomorphism except for Montesinos links and for some special links (Proposition 7). The 3-bridge spheres of the exceptional links will be studied in the sequel of this paper.

Let L be a 3-bridge arborescent link. If L is a Montesinos link, then L satisfies the condition (4) in Theorem 1 by the result of Boileau and Zieschang [5]. So, in order to prove Theorem 1, we may assume that L is not a Montesinos link. Then, by Proposition 4, one of the following holds.

(i) L is equivalent to the link $L_2((-1/2,1/2),(1/\alpha),(-1/2,1/2)) \in \mathscr{L}_2$.
(ii) L is equivalent to the link $L_1((\beta_1/\alpha_1,\beta_2/\alpha_2),(-1/2,1/2)) \in \mathscr{L}_1$.
(iii) The double branched cover $M = M_2(L)$ is a graph manifold which admits a nontrivial torus decomposition by separating tori.

Hence we may assume L satisfies the condition (iii), in order to prove Theorem 1. Then M satisfies the assumption of Theorem 5 and the covering transformation τ_L is equal to the hyper-elliptic involution τ_F associated with a genus-2 Heegaard surface F of M, which is obtained as the pre-image of a 3-bridge sphere of L. Note, however, that every pair (M,F) in Theorem 5 does not necessarily yield an arborescent link, i.e., $(M,\mathrm{Fix}\,\tau_F)/\tau_F$ is not necessarily an arborescent link. In Proposition 7 below, we describe the link $(M,\mathrm{Fix}\,\tau_F)/\tau_F$ for each manifold M and genus-2 Heegaard surface F in Theorem 5, determine if it is an arborescent link, and identify the link with a link in Theorem 1 if it is an arborescent link. We also describe the 3-bridge sphere S of L obtained as the image of F.

PROPOSITION 7. *Let M be a graph manifold which satisfies the condition of Theorem* 5, *and F a genus-*2 *Heegaard surface. Let L be the link in S^3 obtained as the quotient $(M,\mathrm{Fix}\,\tau_F)/\tau_F$, and S the* 3-*bridge sphere of L obtained as the image of F. Then the following hold.*

(1) *Suppose the condition* (F1) *is satisfied.*

(1-a) *If the condition* (M1-a) *is satisfied, then L belongs to $\mathscr{L}_1$ and S is homeomorphic to the* 3-*bridge sphere S_1 or S_2 in Figure* 3 (1) *or* (2).

(1-b) *If the condition* (M1-b) *is satisfied, then L belongs to the set*

$$\{L_2((\beta_1/\alpha_1,\beta_1'/\alpha_1'),(1/\alpha_0),(\beta_2/\alpha_2,\beta_2'/\alpha_2')) \in \mathscr{L}_2 \mid (\beta_2/\alpha_2,\beta_2'/\alpha_2') \sim (-1/2,1/2)\}$$

and S is homeomorphic to the 3-*bridge sphere in Figure* 3 (4).

(2) *Suppose the condition* (F2) *is satisfied.*

(2-a) *If the condition* (M2-a) *is satisfied, then L belongs to the set*

$$\{L_1((\beta_1/\alpha_1,\beta_1'/\alpha_1'),(\beta_2/\alpha_2,\beta_2'/\alpha_2')) \in \mathscr{L}_1 \mid (\beta_2/\alpha_2,\beta_2'/\alpha_2') \sim (1/2,-n/(2n+1)) \text{ for some } n \text{ with } |2n+1|>1\}$$

and S is homeomorphic to the 3-*bridge sphere S_3 in Figure* 3.

(2-b) *If the condition* (M2-b) *is satisfied, then L belongs to $\mathscr{L}_3$ and S is homeomorphic to the* 3-*bridge sphere in Figure* 3 (5).

(3) *Suppose the condition* (F3) *is satisfied. Then L is not an arborescent link. Actually, L is equivalent to the link in Figure* 25.
(4) *Suppose the condition* (F4) *is satisfied. Then L belongs to $\mathscr{L}_2$ and S is homeomorphic to the* 3*-bridge sphere in Figure* 3 (4).

Remark 8. The proof of the uniquness of S up to homeomorphism in the case (1-b) is postponed to the sequel of this paper, because we need to prove additional facts.

Theorem 1 immediately follows from the above proposition. Thus the remainder of this section is devoted to the proof of Proposition 7.

Case 1. *F satisfies the condition* (F1).

Then M is obtained by gluing $M_1 \in D[2]$ and $M_2 \in SL_K$, where the regular fiber, h_1, of M_1 is identified with the meridian loop of M_2, which is a horizontal loop, c_2, of the Seifert fibered space M_2 by Lemma 2. Since the regular fiber, h_2, of M_2 intersects $h_1 = c_2$ in a single point, h_2 is regarded as a horizontal loop, c_1, of M_1. Let α_i, β_i, α_i', β_i' be integers such that

(i) $M_i = D(\beta_i/\alpha_i, \beta_i'/\alpha_i')$ w.r.t. h_i and c_i $(i = 1, 2)$ when M belongs to M(1-a), or
(ii) $M_1 = D(\beta_1/\alpha_1, \beta_1'/\alpha_1')$ w.r.t. h_1 and c_1 and $M_2 = M\ddot{o}(1/\alpha_2)$ w.r.t. h_2 and c_2 when M belongs to M(1-b).

Recall from Theorem 5 that

- $F \cap M_1$ is an essential saturated annulus in M_1, and
- $F \cap M_2$ is a 2-holed torus which gives a 1-bridge decomposition of the 1-bridge knot K such that $M_2 = E(K)$.

Let τ_1 be the involution of M_1 preserving the annulus $M_1 \cap F$, which is equivalent to the involution f in Remark 2 (see Figure 18 (1)). Let τ_2 be the "hyper-elliptic involution" of M_2 associated with the 1-bridge decomposition determined by $M_2 \cap F$ (see Figure 17). We may assume $\tau_1|_{\partial M_1} = \tau_2|_{\partial M_2}$ and hence obtain an involution $\tau = \tau_1 \cup \tau_2$ of M which preserves the Heegaard surface F. Moreover, we can easily see that $F/\langle\tau\rangle$ is a 2-sphere with 6 cone-points. Hence τ is the hyper-elliptic involution associated with F.

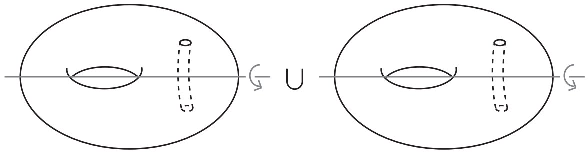


Fig. 17

Case 1.1. *M belongs to* M(1-a), *i.e.,* $M_1 \in D[2]$ *and* $M_2 \in SL_K \cap D[2]$.

By [19, Theorem 3 and Corollary 4.1], $F \cap M_2$ is isotopic to one of the two 2-holed tori in Figure 18 (2) by an isotopy fixing ∂M_2. To be precise, $F \cap M_2 = \mathrm{cl}(\partial V \setminus \partial M_2)$, where V is the regular neighborhood in M_2 of the graph obtained by connecting a horizontal loop and an exceptional fiber of M_2 by an arc as illustrated in Figure 18 (2). We can observe as in Figure 18 (2) that the hyper-elliptic involution τ_2, of M_2, associated with each of the 1-bridge decomposition is equivalent to the involution f in Remark 2. Each of the 2-holed torus in M_2 together with an essential annulus in M_1 uniquely determines a genus-2 Heegaard surface of M, and the Heegaard surface F is isotopic to one of these two Heegaard surfaces (see Lemma 6 (2) and [19, Proposition 5.2]). These Heegaard surfaces determine the same hyper-elliptic involution $\tau = \tau_1 \cup \tau_2$.

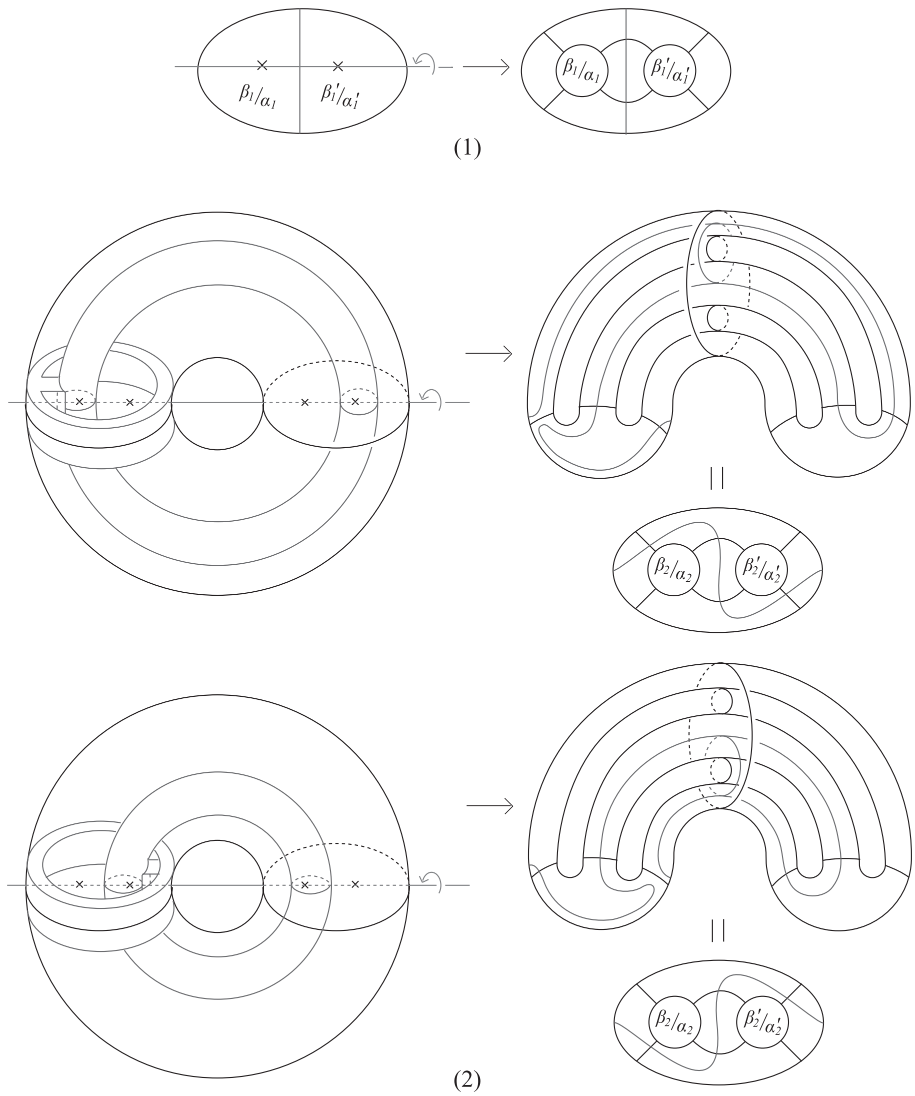
Moreover, since h_1 and c_1 are identified with c_2 and h_2, respectively, we see by Figure 8 (i) (note that a is the quotient of a horizontal loop and b is the quotient of a regular fiber) that L is equivalent to the link $L_1((\beta_1/\alpha_1, \beta_1'/\alpha_1'), (\beta_2/\alpha_2, \beta_2'/\alpha_2')) \in \mathscr{L}_1$. We note that, though the pairs $(\beta_i/\alpha_i, \beta_i'/\alpha_i')$ are defined only up to the equivalence relation in Proposition 1 (cf. Notation 2), the link type of the resulting link is not affected by the choice of the representative $(\beta_i/\alpha_i, \beta_i'/\alpha_i')$. (This corresponds to the easy if part of Theorem 2 (1).) For each genus-2 Heegaard surface F, the image of $F \cap M_i$ in $(M_i, \mathrm{Fix}\, \tau_i)/\tau_i$ is as illustrated in Figure 18, and hence, we see that F projects to one of the 3-bridge spheres S_1 and S_2 in Figure 3. Thus S is homeomorphic to S_1 or S_2 by virtue of [2, Theorem 8] (see Section 3). This completes the proof of the assertion (1-a) in Proposition 7.

Case 1.2. *M belongs to* M(1-b), *i.e.,* $M_1 \in D[2]$ *and* $M_2 \in SL_K \cap M\ddot{o}[1]$.

Note that $M_2 = M\ddot{o}(1/\alpha_2)$ for some $\alpha_2 \in \mathbf{Z}$ with $|\alpha_2| \geq 2$ by Lemma 2. We first describe the involution τ_2 of M_2.

Lemma 9. *τ_2 is equivalent to the involution g_1 in Lemma* 4.

Proof. Since the base orbifold is non-orientable, we see that $\pi_1(M_2)$ has trivial center. Hence by Tollefson [28, Corollary 7.2] (where readers should be careful about typos) we see that the strong equivalence class of the involution τ_2 is determined by its image in the mapping class group $\mathscr{M}(M_2) = \langle g_1, g_2, b \rangle \cong (\mathbf{Z}_2)^3$ (see Lemma 4 (2)). Note that $\mathscr{M}(M_2) \cong (\mathbf{Z}_2)^3$ is realized by the group action on M_2 as illustrated in Figure 19. To be precise, the natural projection $\mathrm{Diff}(M_2) \to \mathscr{M}(M_2)$ has a section $\mathfrak{s} : \mathscr{M}(M_2) \to \mathrm{Diff}(M_2)$ such that the image G of $\mathfrak{s}$ is described by Figure 19. In the figure, $M_2/\langle g_1 \rangle$ is equal to $N/\langle g_1 \rangle$ in Figure 16 with $\beta/\alpha = 1/\alpha$. The orbifold $M_2/\langle g_1, g_2 \rangle$ is



(2)

Fig. 18

obtained as the quotient of $M_2/\langle g_1 \rangle$ by the involution induced by g_2, and the orbifold M_2/G is obtained as the quotient of $M_2/\langle g_1, g_2 \rangle$ by the involution induced by b. Thus M_2/G has B^3 as the underlying space and the singular set forms the graph Γ in B^3 as illustrated in Figure 19. The group G is identified with the covering transformation group of the branched covering $M_2 \to M_2/G$, where the monodromy of the associated unbranched covering is given by

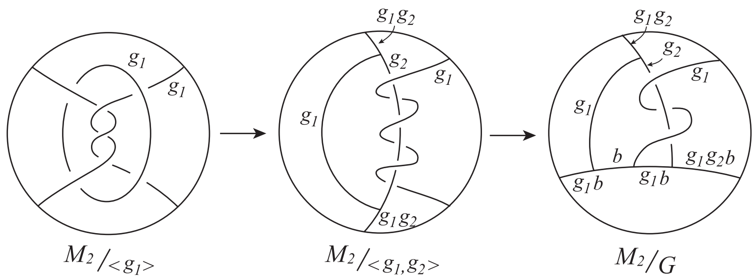


Fig. 19

$\varphi : \pi_1(B^3 \backslash \Gamma) \to G$ as illustrated in Figure 19. Here, each symbol near to an edge represent the image of the meridian of the edge by φ.

Since τ_2 has order 2, it is strongly equivalent to g_1, g_2, b, g_1g_2, g_1b, g_2b or g_1g_2b. We show that g_1 is the only possibility for τ_2. To this end, note that $M_2/\langle\tau_2\rangle$ must be a 3-ball because τ_2 is a hyper-elliptic involution associated with a 1-bridge decomposition of M_2. Since $\tau_2|_{\partial M_2}$ is a hyper-elliptic involution of the torus ∂M_2, τ_2 cannot be g_2, b nor g_2b. Thus τ_2 is strongly equivalent to g_1, g_1g_2, g_1b or g_1g_2b. However, we can see that $M_2/\langle g_1g_2\rangle$, $M_2/\langle g_1b\rangle$ and $M_2/\langle g_1g_2b\rangle$ are not homeomorphic to a 3-ball, as follows.

We first show this for g_1g_2. Note that $M_2/\langle g_1g_2\rangle \to (B^3, \Gamma)$ is the branched covering associated with the monodromy $\psi : \pi_1(B^3 \backslash \Gamma) \to G/\langle g_1g_2\rangle \cong (\mathbf{Z}_2)^2$. Let s and t be the elements of $G/\langle g_1g_2\rangle$ obtained as the images of $g_1(= g_2)$ and b, respectively. Then ψ is as illustrated in the left figure of Figure 20 (1). Then, by passing the intermediate covering space corresponding to $\langle s, t\rangle/\langle t\rangle$, we see that $M_2/\langle g_1g_2\rangle$ is a lens space of type $(2\alpha, 1)$ with an open 3-ball removed, which is not homeomorphic to B^3. By using Figure 20 (2) and (3), we can also see that neither $M_2/\langle g_1b\rangle$ nor $M_2/\langle g_1g_2b\rangle$ is a 3-ball. Hence, the only possibility for τ_2 is the involution g_1. This completes the proof of Lemma 9. $\square$

By Lemma 9, $(M_2, \mathrm{Fix}\,\tau_2)/\langle\tau_2\rangle = (M_2, \mathrm{Fix}\,g_1)/\langle g_1\rangle$ and it is the Montesinos pair as illustrated in the left figure of Figure 19. Since c_1 and h_1 are identified with h_2 and c_2, respectively, we see, by using Proposition 2 as in Case 1.1, that L is equivalent to the arborescent link in Figure 21, which in turn is equivalent to the link $L_2((\beta_1/\alpha_1, \beta_1'/\alpha_1'), (1/\alpha_2), (-1/2, 1/2)) \in \mathscr{L}_2$. As in Case 1.1, though $(\beta_1/\alpha_1, \beta_1'/\alpha_1')$ is defined only up to certain equivalence, this link is determined without ambiguity (see the if part of Theorem 2 (2)). This completes the proof of the assertion (1-b) in Proposition 7. We shall show in the sequel of this paper that the 3-bridge sphere of L obtained as the image of F is as illustrated in Figure 21.

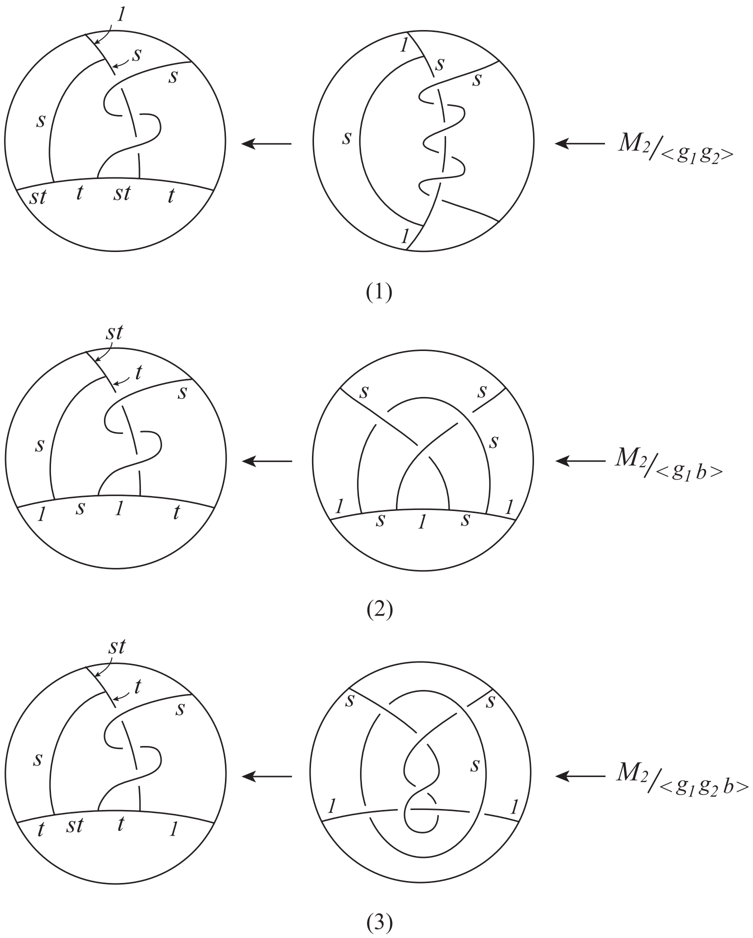


Fig. 20

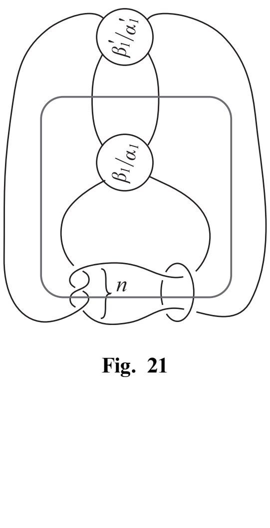


Fig. 21

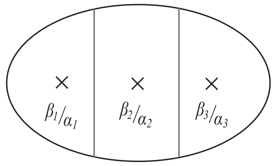


Fig. 22

CASE 2. *F satisfies the condition* (F2).

Then M is obtained by gluing $M_1 \in D[2] \cup D[3]$ and $M_2 = E(S(2n+1,1)) \in SM_K$, where the regular fiber, h_1, of M_1 is identified with the meridian loop of M_2, which is a horizontal loop, c_2, of the Seifert fibered space $M_2 = D(1/2, -n/(2n+1))$ (cf. Lemma 1). Since the regular fiber, h_2, of M_2 intersects $h_1 = c_2$ in a single point, h_2 is regarded as a horizontal loop, c_1, of M_1. Let α_i, β_i $(i = 1,2,3)$ be integers such that $M_1 = D(\beta_1/\alpha_1, \beta_2/\alpha_2) \in D[2]$ or $M_1 = D(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3) \in D[3]$ w.r.t. h_1 and c_1. Recall from Theorem 5 that

- $F \cap M_1$ consists of two essential saturated annuli in M_1 which divide M_1 into three solid tori, and
- $F \cap M_2$ is a 2-bridge sphere of the nontrivial 2-bridge knot $S(2n+1,1)$ such that $M_2 = E(S(2n+1,1))$.

If $M_1 = D(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3) \in D[3]$, we assume the following convention.

CONVENTION 1. *The singular fibers of indices β_1/α_1, β_2/α_2, β_3/α_3 are located in M_1 in this order with respect to the annuli $F \cap M_1$, as illustrated in Figure 22. Namely, the singular fiber of index β_2/α_2 is located in the "central component" of $M_1 \backslash F$*

This convention determines the ordered triple $(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ up to the equivalence relation $\sim$ in Notation 1.

Note that the isotopy type of $F \cap M_1$ in M_1 is uniquely determined under this convention. Moreover, the isotopy type of $F \cap M_2$ in M_2 is unique up to isotopy fixing ∂M_2 by [19, Theorem 4]. Hence, by Lemma 6 (2), there is a unique possibility for F up to isotopy under Convention 1.

Let τ_1 be the involution of M_1 preserving the annuli $F \cap M_1$ which is equivalent to the involution f in Remark 2. Let τ_2 be the involution of M_2 as illustrated in Figure 23. Then we may assume $\tau_1|_{\partial M_1} = \tau_2|_{\partial M_2}$, and hence we obtain an involution $\tau = \tau_1 \cup \tau_2$ of M which preserves the Heegaard surface F. Moreover, we can see that τ is the hyper-elliptic involution associated with F.

Since h_1 and c_1 are identified with c_2 and h_2, respectively, we see by Figure 23 and Figure 8 that L is equivalent to the link $L_1((\beta_1/\alpha_1, \beta_2/\alpha_2), (1/2, -n/(2n+1))) \in \mathscr{L}_1$ or $L_3((\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3), (1/2, -n/(2n+1))) \in \mathscr{L}_3$

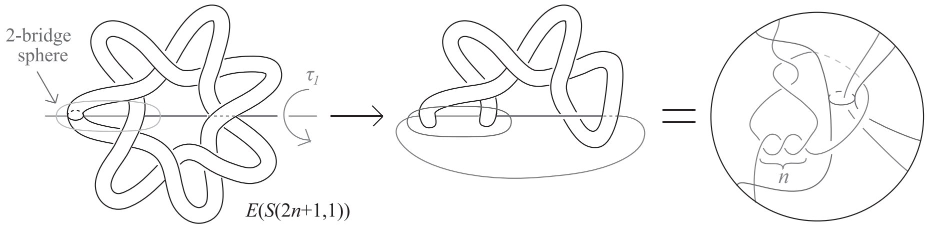


Fig. 23

according as M belongs to M(2-a) or M(2-b). As in Case 1, though $(\beta_1/\alpha_1, \beta_2/\alpha_2)$ or $(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ is defined only up to the equivalence, the link is determined without ambiguity under Convention 1, since the Montesinos pair $(M_2, \mathrm{Fix}\,\tau_2)/\tau_2$ admits a "horizontal" symmetry. Moreover, we see that F projects to the 3-bridge spheres in Figure 3 (3) or (5) according as M belongs to M(2-a) or M(2-b). Thus S is homeomorphic to the 3-bridge spheres in Figure 3 (3) or (5) according as M belongs to M(2-a) or M(2-b) by virtue of [2, Theorem 8]. This completes the proof of the assertion (2) in Proposition 7.

Case 3. *F satisfies the condition* (F3).

Then M is obtained by gluing $M_1 \in M\ddot{o}[r]$ $(r = 1, 2)$ and $M_2 = E(S(2n+1, 1)) \in SM_K$, where the regular fiber, h_1, of M_1 is identified with the meridian loop of M_2, which is a horizontal loop, c_2, of the Seifert fibered space $M_2 = D(1/2, -n/(2n+1))$ by Lemma 1. Since the regular fiber, h_2, of M_2 intersects $h_1 = c_2$ in a single point, h_2 is regarded as a horizontal loop, c_1, of M_1. Let α_i, β_i $(i = 1, 2)$ be integers such that $M_1 = M\ddot{o}(\beta_1/\alpha_1, \beta_2/\alpha_2) \in M\ddot{o}[r]$ $(r = 1, 2)$ w.r.t. h_1 and c_1. Here,

(i) $|\alpha_1| \geq 2$ and $\dfrac{\beta_2}{\alpha_2} = \dfrac{0}{1}$ if $r = 1$, and

(ii) $|\alpha_1|, |\alpha_2| \geq 2$ if $r = 2$.

Recall from Theorem 5 that

- $F \cap M_1$ consists of two saturated essential annuli in M_1 which divide M_1 into two solid tori, and
- $F \cap M_2$ is a 2-bridge sphere of the nontrivial 2-bridge knot $S(2n+1, 1)$ such that $M_2 = E(S(2n+1, 1))$.

Let τ_1 and τ_2, respectively, be the involutions of M_1 and M_2 as illustrated in Figure 24 (1) and (2). Then we may assume $\tau_1|_{\partial M_1} = \tau_2|_{\partial M_2}$, and hence, we obtain an involution $\tau = \tau_1 \cup \tau_2$ of M which preserves the Heegaard surface F. Moreover, we can see that τ is the hyper-elliptic involution associated with F. Hence $(M_i, \mathrm{Fix}\,\tau_i)/\tau_i$ $(i = 1, 2)$ is the (3,1)-manifold pair as illustrated in Figure 24, and L is equivalent to the link in Figure 25 (cf. [14]).

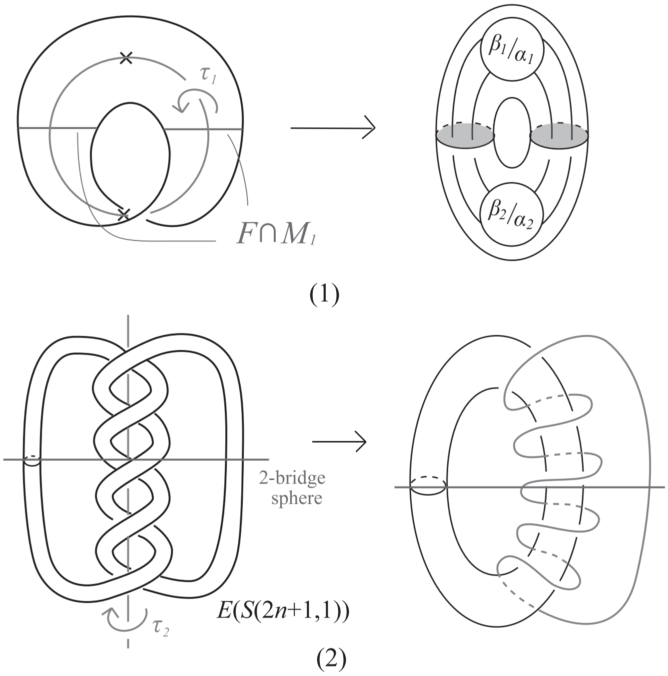


Fig. 24

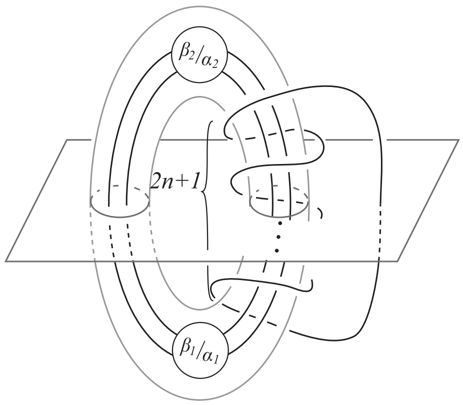


Fig. 25

In the following we show that L is not an arborescent link. Assume, on the contrary, that L is an arborescent link. Then, since L is non-simple (this fact is directly observed from Figure 25), L must be equivalent to the link in

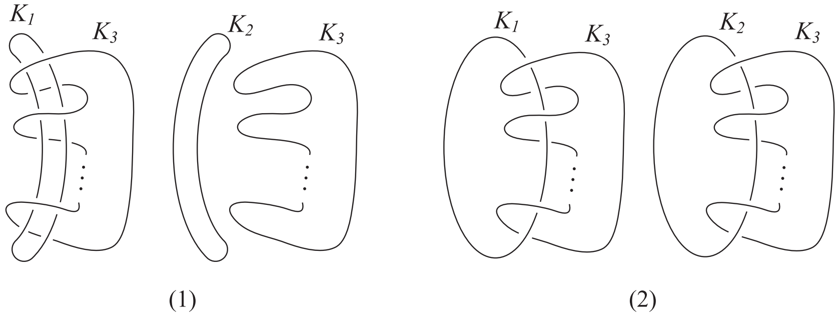


Fig. 26

Figure 12 (1) by Proposition 4 (1). Note that L must consist of 3 components. So, one of the following holds.

(i) $(\alpha_1, \beta_1) \equiv (\alpha_2, \beta_2) \equiv (0,1) \pmod 2$.

(ii) $(\alpha_1, \beta_1) \equiv (\alpha_2, \beta_2) \equiv (1,0)$ or $(1,1) \pmod 2$.

Let K_1 and K_2 be the two components of L "inside" the solid torus bounded by the torus in Figure 25, and let K_3 be the remaining component of L. Then both $K_1 \cup K_3$ and $K_2 \cup K_3$ are equivalent to the 2-component trivial link or a 2-bridge link $S(2(2n+1),1)$ ($|2n+1| \geq 3$) according as the condition (i) or (ii) holds (see Figure 26). On the other hand, the link in Figure 12 (1) is obtained by adding a loop parallel to a component of a 2-bridge link $S(4m, 2m+1)$ for some nonzero integer m, and hence, only one pair of its components forms a 2-component trivial link. This shows that $K_1 \cup K_3$ nor $K_2 \cup K_3$ cannot be the trivial link, and hence the case of Figure 26 (1) cannot occur. Thus both of $K_1 \cup K_3$ and $K_2 \cup K_3$ must be equivalent to $S(2(2n+1),1)$ ($|2n+1| \geq 3$). This implies that $S(4m, 2m+1)$ must be isotopic to $S(2(2n+1),1)$ and therefore $2m+1 \equiv \pm 1 \pmod{4m}$. Hence, m must be ± 1. However, since $|2n+1| \geq 3$, $S(2(2n+1),1)$ is not equivalent to $S(4,3) = S(4,-1)$ nor $S(-4,-1) = S(4,1)$, a contradiction. Hence, L is not an arborescent link. This completes the proof of the assertion (3) in Proposition 7.

Case 4. *F satisfies the condition* (F4).

Then M is obtained by gluing three Seifert fibered spaces $M_1, M_2 \in D[2]$ and $M_3 = E(S(2n,1)) \in A[1]$, where the regular fiber, h_i, of M_i $(i = 1,2)$ is identified with the meridian loop of M_3, which is a horizontal loop, c_3^i, of the Seifert fibered space $M_2 = A(1/n)$ (see Lemma 1). Since the regular fiber, $h_3^i \subset \partial M_i \cap \partial M_3$, of M_3 intersects $h_i = c_3^i$ in a single point, h_3^i is identified with a horizontal loop, c_i, of M_i $(i = 1,2)$. Let $\alpha_i, \beta_i, \alpha_i', \beta_i'$ $(i = 1,2)$ be integers such that $M_i = D(\beta_i/\alpha_i, \beta_i'/\alpha_i') \in D[2]$ w.r.t. h_i and c_i $(i = 1,2)$. Recall from Theorem 5 that

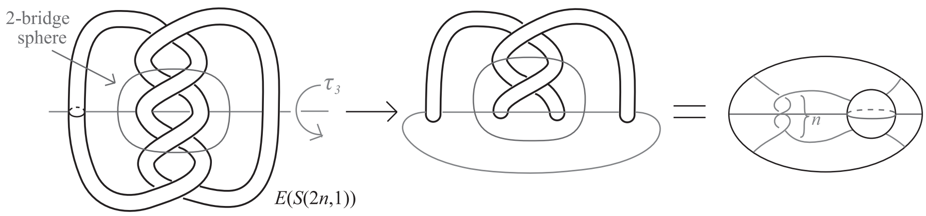


Fig. 27

- $F \cap M_i$ $(i = 1, 2)$ is an essential saturated annulus in M_i, and
- $F \cap M_3$ is a 2-bridge sphere of the 2-bridge link $S(2n, 1)$ $(|n| \geq 2)$ such that $M_3 = E(S(2n, 1))$.

Note that the isotopy type of $F \cap M_i$ in M_i $(i = 1, 2)$ is uniquely determined by the assumption. Moreover, the isotopy type of $F \cap M_3$ in M_3 is also uniquely determined by the assumption, because each 2-bridge link admits a unique 2-bridge sphere up to isotopy by [27]. Hence, the isotopy type of F is uniquely determined by the assumption modulo powers of $1/2$-Dehn twists along $M_i \cap M_3$ in the direction of the fiber of M_i $(i = 1, 2)$. Now, we assume the following convention.

CONVENTION 2. *The singular fibers of M_2 are located so that the singular fiber of M_1 of index β_1/α_1 and the singular fiber of M_2 of index β_2'/α_2' are contained in the same component of $M \backslash F$.*

By the above observation, we see that there are at most two possibilities for the homeomorphism type of F under Convention 2. Namely, if F^* is a Heegaard surface satisfying the condition (F4) and Convention 2, then the surface, F^{**}, obtained from F^* by applying $1/2$-Dehn twist along $M_i \cap M_3$ in the direction of the fiber of M_3 $(i = 1, 2)$, together with F^* forms a complete set of representatives of the Heegaard surfaces of M satisfying the condition (F4) and Convention 2. However, we can see by using the S^1-action on M_3 that F^{**} is isotopic to F^*. Hence, M admits a unique genus-2 Heegaard surface satisfying Convention 2 up to isotopy.

Let τ_i $(i = 1, 2)$ be the involution of M_i preserving the annuli $F \cap M_i$, which is equivalent to the involution f in Remark 2. Let τ_3 be the involution of M_3 as illustrated in Figure 27. Then we may assume $\tau_i|_{\partial M_i} = \tau_3|_{M_3 \cap M_i}$ for both $i = 1, 2$, and hence we obtain an involution $\tau = \tau_1 \cup \tau_2 \cup \tau_3$ of M preserving the Heegaard surface F. Moreover, we can see that τ is the hyper-elliptic involution associated with F.

Hence, by an argument similar to that in the previous cases, we see that L is equivalent to the 3-bridge link $L_2((\beta_1/\alpha_1, \beta_1'/\alpha_1'), (1/n), (\beta_2/\alpha_2, \beta_2'/\alpha_2')) \in \mathscr{L}_2$. Though the pairs $(\beta_i/\alpha_i, \beta_i'/\alpha_i')$ are defined only up to equivalence, the Convention 2 guarantees that the link is determined without ambiguity. Moreover, S is homeomorphic to the 3-bridge sphere as illustrated in Figure 3 (4) with $\alpha_0 = n$ by virtue of [2, Theorem 8].

This completes the proof of Proposition 7 and hence that of Theorem 1.

8. Proof of Theorem 2

In this section, we prove Theorem 2. Though this would follow from the classification of arborescent links by Bonahon and Siebenmann [6], we give an alternative proof by studying the double branched coverings. We remark that Theorem 2 (1) is already proved by Gordon and Luecke [11] by essentially the same method. However, we include its proof as a warm-up exercise for the more complicated proof of the remaining assertions, where we need to study not only the homeomorphism types of the double branched coverings but also the covering transformations.

Proposition 8. (1) *For a link* $L = L_1((s_1, s_2), (s_3, s_4))$, $M = M_2(L)$ *is decomposed into two Seifert fibered spaces* $M_1 = D(s_1, s_2)$ *and* $M_2 = D(s_3, s_4)$.

(i) *If* $(s_1, s_2) \sim (-1/2, 1/2)$ *(resp.* $(s_3, s_4) \sim (-1/2, 1/2)$*), then* M *is a Seifert fibered space* $P^2(0; s_3, s_4)$ *(resp.* $P^2(0; s_1, s_2)$*) over a projective plane* P^2.

(ii) *If* $(s_1, s_2) \not\sim (-1/2, 1/2)$ *and* $(s_3, s_4) \not\sim (-1/2, 1/2)$*, then the decomposition of* M *into* M_1 *and* M_2 *gives the torus decomposition of* M.

(2) *For a link* $L = L_2((s_1, s_2), (1/\alpha), (s_3, s_4))$, $M = M_2(L)$ *is decomposed into three Seifert fibered spaces* $M_1 = D(s_1, s_2)$, $M_2 = D(s_3, s_4)$ *and* $M_3 = A(1/\alpha)$.

(i) *If* $(s_1, s_2) \sim (s_3, s_4) \sim (-1/2, 1/2)$*, then* M *is a Seifert fibered space* $Kl(0; 1/\alpha)$ *over a Klein bottle* Kl.

(ii) *If* $(s_1, s_2) \sim (-1/2, 1/2)$ *and* $(s_3, s_4) \not\sim (-1/2, 1/2)$ *(resp.* $(s_1, s_2) \not\sim (-1/2, 1/2)$ *and* $(s_3, s_4) \sim (-1/2, 1/2)$*), then* M *has the torus decomposition into* $M_1 \cup M_3 = M\ddot{o}(1/\alpha)$ *and* $M_2 = D(s_3, s_4)$ *(resp.* $M_2 \cup M_3 = M\ddot{o}(1/\alpha)$ *and* $M_1 = D(s_1, s_2)$*).*

(iii) *If* $(s_1, s_2) \not\sim (-1/2, 1/2)$ *and* $(s_3, s_4) \not\sim (-1/2, 1/2)$*, then the decomposition of* M *into* M_1, M_2 *and* M_3 *gives the torus decomposition of* M.

(3) *For a link* $L = L_3((s_1, s_2, s_3), (1/2, -n/(2n+1)))$, $M = M_2(L)$ *admits the torus decomposition into two Seifert fibered spaces* $M_1 = D(s_1, s_2, s_3)$ *and* $M_2 = D(1/2, -n/(2n+1))$.

Proof. (1) Put $L = L_1((s_1, s_2), (s_3, s_4))$. Then we see, by using Proposition 2, that M is obtained from $M_1 = D(s_1, s_2)$ and $M_2 = D(s_3, s_4)$ by identifying their boundaries, where (h_1, c_1) is identified with (c_2, h_2). Here, h_i and c_i are a regular fiber and a horizontal loop of M_i, respectively, such that $M_i = D(s_1, s_2)$ or $D(s_3, s_4)$ according as $i = 1$ or 2 with respect to h_i and c_i.

(i) Assume that $(s_1, s_2) \sim (-1/2, 1/2)$. Then M_1 is identified with an S^1-bundle over a Möbius band, so that c_1 is a fiber and h_1 is a horizontal loop (Remark 3 (1)). Since (h_1, c_1) is identified with (c_2, h_2), M is homeomorphic to $P^2(0; s_3, s_4)$. Similarly, if $(s_3, s_4) \sim (-1/2, 1/2)$, then we have $M \cong P^2(0; s_3, s_4)$.

(ii) Assume that $(s_1, s_2) \not\sim (-1/2, 1/2)$ and $(s_3, s_4) \not\sim (-1/2, 1/2)$. Then the Seifert fibration of M_i is unique $(i = 1, 2)$ (see Remark 3 (2)), and hence the decomposition of M into M_1 and M_2 gives the nontrivial torus decomposition.

(2) and (3) can be proved similarly. $\square$

Proof of Theorem 2. By Proposition 8, the double branched coverings of S^3 branched along two links which belong to distinct families of $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_3$ are not homeomorphic. This implies that no two links in distinct families of $\mathscr{L}_1$, $\mathscr{L}_2$ or $\mathscr{L}_3$ are equivalent.

(1) Since the "if" part of the statement can be seen easily, we prove the "only if" part (cf. [11, Lemma 2.2]).

Assume that $L = L_1((s_1, s_2), (s_3, s_4))$ and $L' = L_1((s_1', s_2'), (s_3', s_4'))$ are equivalent. Then the oriented manifolds $M_2(L)$ and $M_2(L')$ are homeomorphic.

Assume that $(s_1, s_2) \sim (-1/2, 1/2)$. Then, by Proposition 8 (1), we have $(s_1', s_2') \sim (-1/2, 1/2)$ or $(s_3', s_4') \sim (-1/2, 1/2)$. Moreover, $M_2(L) \cong P^2(0; s_3, s_4)$, and $M_2(L') \cong P^2(0; s_3', s_4')$ or $P^2(0; s_1', s_2')$ according as $(s_1', s_2') \sim (-1/2, 1/2)$ or $(s_3', s_4') \sim (-1/2, 1/2)$. By the classification of Seifert fibered spaces, we have $(s_3, s_4) \sim (s_3', s_4')$ or $(s_3, s_4) \sim (s_1', s_2')$ according as $(s_1', s_2') \sim (-1/2, 1/2)$ or $(s_3', s_4') \sim (-1/2, 1/2)$. Hence, the statement (1) of Theorem 2 holds in this case. Similarly, we can see that the statement (1) of Theorem 2 holds when $(s_3, s_4) \sim (-1/2, 1/2)$.

Assume that $(s_1, s_2) \not\sim (-1/2, 1/2)$ and $(s_3, s_4) \not\sim (-1/2, 1/2)$. Then, by Proposition 8 (1), $M_2(L)$ has the nontrivial torus decomposition into $M_1 = D(s_1, s_2)$ and $M_2 = D(s_3, s_4)$. Then $M_2(L')$ also has the nontrivial torus decompositions into $M_1' = D(s_1', s_2')$ and $M_2' = D(s_3', s_4')$. Moreover, $M_2(L)$ is obtained from M_1 and M_2 by identifying (c_1, h_1) and (h_2, c_2), and $M_2(L')$ is obtained from M_1' and M_2' by identifying (c_1', h_1') and (h_2', c_2'). Let $f : M_2(L) \to M_2(L')$ be an orientation-preserving homeomorphism obtained as a lift of the homeomorphism from (S^3, L) to (S^3, L'). Then, by the uniqueness of torus decomposition, we may assume that $f(M_1) = M_1'$ or M_2'. Suppose $f(M_1) = M_1'$, and hence $f(M_2) = M_2'$. By the uniqueness of Seifert fibration of M_1, we may assume that $f(h_1) = \pm h_1'$. Since the covering transformation maps h_1 to h_1^{-1}, we can choose f so that $f(h_1) = h_1'$. Thus $f(c_2) = f(h_1) = h_1' = c_2'$. Since f is orientation-preserving, we see that $f(h_2) = h_2'$. This implies $f(c_1) = f(h_2) = h_2' = c_1'$. Thus $f|_{M_i} : M_i \to M_i'$ sends (c_i, h_i) to (c_i', h_i'). Hence, we have $(s_1, s_2) \sim (s_1', s_2')$ and $(s_3, s_4) \sim$

(s_3', s_4') by Proposition 1. Similarly, we can prove that $(s_1, s_2) \sim (s_3', s_4')$ and $(s_3, s_4) \sim (s_1', s_2')$ if $f(M_1) = M_2'$.

(2) Since the "if" part of the statement can be seen easily, we prove the "only if" part.

Assume that $L = L_2((s_1, s_2), (1/\alpha), (s_3, s_4))$ and $L' = L_2((s_1', s_2'), (1/\alpha'), (s_3', s_4'))$ are equivalent. Then $M_2(L)$ and $M_2(L')$ are homeomorphic.

Case 1. $(s_1, s_2) \sim (s_3, s_4) \sim (-1/2, 1/2)$.

By Proposition 8 (2) and the assumption, we see $M_2(L') \cong M_2(L) \cong Kl(0; 1/\alpha)$. By Proposition 8 (2) again, we see $(s_1', s_2') \sim (s_3', s_4') \sim (-1/2, 1/2)$ and $\alpha = \alpha'$.

Case 2. $(s_1, s_2) \sim (-1/2, 1/2)$ *and* $(s_3, s_4) \not\sim (-1/2, 1/2)$ *(or* $(s_1, s_2) \not\sim (-1/2, 1/2)$ *and* $(s_3, s_4) \sim (-1/2, 1/2)$*).*

By Proposition 8 (2), $M_2(L)$ has the torus decomposition into $D(s_1, s_2)$ and $M\ddot{o}(1/\alpha)$. Since $M_2(L') \cong M_2(L)$, we see, by Proposition 8 (2) again, that $(s_1', s_2') \sim (-1/2, 1/2)$ or $(s_3', s_4') \sim (-1/2, 1/2)$, and that $M_2(L')$ has the torus decomposition into $D(s_1', s_2')$ and $M\ddot{o}(1/\alpha')$, or $D(s_3', s_4')$ and $M\ddot{o}(1/\alpha')$. By using the gluing data and Proposition 1 as in the proof of (1), we obtain the desired conclusion.

Case 3. $(s_1, s_2) \not\sim (-1/2, 1/2)$ *and* $(s_3, s_4) \not\sim (-1/2, 1/2)$.

By Proposition 8 (2), $M_2(L)$ has the torus decomposition into $D(s_1, s_2)$, $A(1/\alpha)$ and $D(s_3, s_4)$. Since $M_2(L')$ is homeomorphic to $M_2(L)$, it has the torus decomposition into 3 pieces. Hence, we see by Proposition 8 (2) that $(s_1', s_2') \not\sim (-1/2, 1/2)$ and $(s_3', s_4') \not\sim (-1/2, 1/2)$ and that $M_2(L')$ has the torus decomposition into $D(s_1', s_2')$, $A(1/\alpha')$ and $D(s_3', s_4')$. By using the gluing data and Proposition 1 as in the proof of (1), we have $\alpha = \alpha'$, and

(i) $(s_1, s_2) \sim (s_1', s_2')$ and $(s_3, s_4) \sim (s_3', s_4')$, or

(ii) $(s_1, s_2) \sim (s_3', s_4')$ and $(s_3, s_4) \sim (s_1', s_2')$.

Suppose that (i) holds. Then one of the following holds.

(i-i) $(s_1, s_2) \approx (s_1', s_2')$ and $(s_3, s_4) \approx (s_3', s_4')$,

(i-ii) $(s_1, s_2) \approx (s_1', s_2')$ and $(s_3, s_4) \approx (s_4', s_3')$,

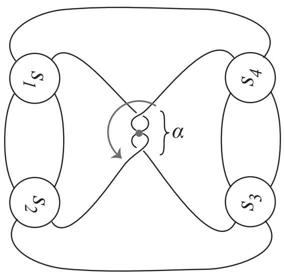
(i-iii) $(s_1, s_2) \approx (s_2', s_1')$ and $(s_3, s_4) \approx (s_3', s_4')$, or

(i-iv) $(s_1, s_2) \approx (s_2', s_1')$ and $(s_3, s_4) \approx (s_4', s_3')$.

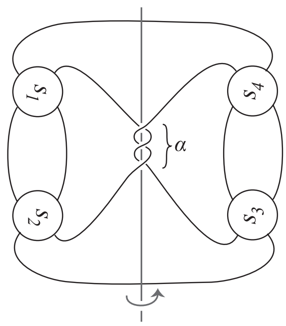
The conditions (i-i) and (i-iv), respectively, are nothing other than the conditions (2-i) and (2-ii). Moreover, if $(s_1, s_2) \approx (s_2, s_1)$ or $(s_3, s_4) \approx (s_4, s_3)$, i.e., $s_1 \equiv s_2$ or $s_3 \equiv s_4$ in $\mathbf{Q}/\mathbf{Z}$, then the condition (i-ii) and (i-iii) are also equivalent to (2-i) or (2-ii). So, we may assume that $s_1 \not\equiv s_2$ and $s_3 \not\equiv s_4$ in $\mathbf{Q}/\mathbf{Z}$. In the following, we show that (i-ii) or (i-iii) cannot happen under this assumption.

From now on, we identify $M_2(L) \cong M_2(L')$ with an oriented manifold M via an orientation-preserving homeomorphism, and we regard τ_L and $\tau_{L'}$ as involutions of M. Then τ_L and $\tau_{L'}$ are conjugate in the (orientation-preserving) mapping class group of M since L and L' are equivalent. Recall the assumption that $s_1 \not\equiv s_2$ and $s_3 \not\equiv s_4$ in $\mathbf{Q}/\mathbf{Z}$. If $(s_1, s_2) \not\sim (s_3, s_4)$, then $\mathcal{M}(M)$ is equal to the (orientation-preserving) mapping class group of M. If $(s_1, s_2) \sim (s_3, s_4)$, then $\mathcal{M}(M)$ is an index-2 subgroup of the (orientation-preserving) mapping class group of M.

We first assume that $(s_1, s_2) \not\sim (s_3, s_4)$, and hence $\mathcal{M}(M)$ is equal to the (orientation-preserving) mapping class group of M. Recall that $\mathcal{M}(M) = \langle G, \lambda \mid G^2, (G\lambda)^2 \rangle$, where we may assume that $\tau_L = G := G_3$ (see Definition 2 and Proposition 6 (3)). Note that L' is obtained from L by mutation along one of the essential Conway spheres which give the characteristic decomposition of L. Note also that the pre-image of the mutation loop in M is a regular fiber of $A(1/\alpha)$. Recall that $\lambda \in \mathcal{M}(M)$ is the Dehn twist along a component of the tori which give the torus decomposition of M in the direction of the regular fiber of $A(1/\alpha)$ (see Definition 2). Hence, by Lemma 8, we can see that $\tau_{L'}$ is conjugate to $\lambda\tau_L = \lambda G$ in $\mathcal{M}(M)$. Hence, λG must be conjugate to G in $\mathcal{M}(M)$, namely, there exists an element $\gamma \in \mathcal{M}(M)$ such that $\gamma^{-1} G \gamma = \lambda G$. Since $\gamma = G\lambda^m$ or λ^m for some integer m, we see that $\lambda^{-m} G \lambda^m = \lambda^{-2m} G$ is equal to λG, which implies $\lambda^{2m+1} = 1$. This is impossible since λ is of infinite order in $\mathcal{M}(M)$. Hence, (i-ii) or (i-iii) cannot be satisfied.



(1) $(s_1, s_2) \approx (s_3, s_4)$



(2) $(s_1, s_2) \approx (s_4, s_3)$

Fig. 28

Next, we assume that $(s_1, s_2) \sim (s_3, s_4)$, and hence $\mathcal{M}(M)$ is an index-2 subgroup of the (orientation-preserving) mapping class group of M. Let h be a symmetry of (S^3, L) illustrated in Figure 28 (1) or (2) according as $(s_1, s_2) \approx (s_3, s_4)$ or $(s_1, s_2) \approx (s_4, s_3)$, and let $\tilde{h}$ be a lift of h to M. Then

$\mathcal{M}(M) \sqcup \tilde{h}\mathcal{M}(M)$ gives a (left) coset decomposition of the (orientation-preserving) mapping class group of M. Since τ_L and $\tau_{L'}$ are conjugate in the (orientation-preserving) mapping class group of M, there exists an element $\gamma \in \mathcal{M}(M)$ such that $\gamma^{-1}\tau_L\gamma = \tau_{L'}$ or $(\tilde{h}\gamma)^{-1}\tau_L(\tilde{h}\gamma) = \tau_{L'}$. Since $\tilde{h}^{-1}\tau_L\tilde{h} = \tau_L$, we have $\gamma^{-1}\tau_L\gamma = \tau_{L'}$, and hence, τ_L and $\tau_{L'}$ are conjugate in $\mathcal{M}(M)$. Hence, as in the previous case, we can see that (i-ii) or (i-iii) cannot be satisfied.

The case when the condition (ii) holds can be treated similarly.

(3) Since the "if" part of the statement can be seen easily, we prove the "only if" part.

Assume that $L = L_3((s_1, s_2, s_3), (1/2, -n/(2n+1)))$ and $L' = L_3((s_1', s_2', s_3'), (1/2, -n'/(2n'+1)))$ are equivalent. Then by using the fact that $M_2(L) \cong M_2(L')$ and Proposition 8 (3), we see that $n = n'$, and $(s_1, s_2, s_3) \approx (s_1', s_2', s_3')$, (s_1', s_3', s_2'), (s_2', s_1', s_3'), (s_2', s_3', s_1'), (s_3', s_1', s_2') or (s_3', s_2', s_1'). We treat the case when s_1, s_2 and s_3 are mutually distinct in $\mathbf{Q}/\mathbf{Z}$. (The remaining case can be treated similarly.)

From now on, M denotes $M_2(L) \cong M_2(L')$, and note that $\mathcal{M}(M)$ is equal to the (orientation-preserving) mapping class group of M. We regard that τ_L and $\tau_{L'}$ as elements of $\mathcal{M}(M)$ which are conjugate in $\mathcal{M}(M)$. Recall that M is obtained from $M_1 \in D[3]$ and $M_2 \in SM_L$ by gluing them along their boundaries.

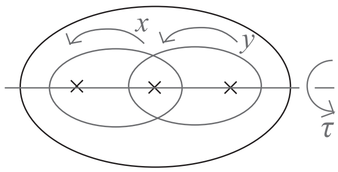


Fig. 29

To describe the group $\mathcal{M}(M)$, recall that the mapping class group G of a 3-punctured disk is the extension of the quotient of the 3-braid group B_3,

$$B_3/\langle (xy)^3 \rangle = \langle x, y \mid xyx = yxy, (xy)^3 \rangle,$$

by the order-2 cyclic group generated by the involution τ in Figure 29 (see, for example, [1, p. 35]). Here, x and y are elements corresponding to the standard generators of B_3 (see Figure 29). Hence,

$$\begin{aligned} G &= \langle x, y, \tau \mid xyx = yxy, (xy)^3, \tau^2, \tau x \tau = x^{-1}, \tau y \tau = y^{-1} \rangle \\ &\cong \langle a, b, \tau \mid a^3, b^2, \tau^2, \tau a \tau = ba^{-1}b, \tau b \tau = b \rangle, \end{aligned}$$

where $a = xy$ and $b = yxy$. Then, by [15, Proposition 25.3], we see that $\mathcal{M}(M_1)$ is isomorphic to the subgroup $P_3/\langle (xy)^3 \rangle \rtimes \langle \tau \rangle$ of G, generated by the images in G of the pure braid group $P_3 < B_3$ and τ. By using Lemma 3 and the exact sequence (1), we can see that $\mathscr{M}(M)$ is isomorphic to $\mathscr{M}(M_1)$. From now on, we identify $\mathscr{M}(M)$ with $\mathscr{M}(M_1)$, so $\mathscr{M}(M)$ is identified with the subgroup of G. Here, we may assume that $\tau_L = \tau$.

CLAIM 1. (1) *The centralizer $Z(\tau)$ of τ in G is equal to $\{1, yxy, \tau, yxy\tau\} \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$.*

(2) *The centralizer $Z(\tau)$ of τ in $\mathscr{M}(M)$ is equal to $\{1, \tau\} \cong \mathbf{Z}_2$.*

PROOF. (1) Let φ be the automorphism of G defined by $\varphi(\tau) = \tau$, $\varphi(a) = a^{-1}$ and $\varphi(b) = b$. Then the inner automorphism ι_τ of G induced by the involution τ is the composition of φ and the inner automorphism ι_b induced by the order-2 element b, because

$$\iota_\tau(a) = ba^{-1}b = \iota_b\varphi(a)$$

$$\iota_\tau(b) = b = \iota_b\varphi(b)$$

$$\iota_\tau(\tau) = \tau = \iota_b\varphi(\tau).$$

Note that any element g of G is represented uniquely by

$$\tau^{n_0}a^{n_1}b^{n_2}a^{n_3}b^{n_4}\dots a^{n_{2m-1}}b^{n_{2m}}$$

for some $n_0 \in \mathbf{Z}_2$, $n_{2i-1} \in \mathbf{Z}_3$ and $n_{2i} \in \mathbf{Z}_2$ $(i = 1, 2, \dots, m)$ such that $n_i \neq 0$ for any i but 0, 1 and $2m$. Then

$$\begin{aligned}\iota_\tau(g) &= \iota_b\varphi(\tau^{n_0}a^{n_1}b^{n_2}a^{n_3}b^{n_4}\dots a^{n_{2m-1}}b^{n_{2m}})\\ &= \iota_b(\tau^{n_0}a^{-n_1}b^{n_2}a^{-n_3}b^{n_4}\dots a^{-n_{2m-1}}b^{n_{2m}}).\end{aligned}$$

Suppose the word representing g contains the letter a, i.e., $n_1 \neq 0$, then the above word is equal to $\tau^{n_0}ba^{-n_1}b^{n_2}a^{-n_3}b^{n_4}\dots a^{-n_{2m-1}}b^{n_{2m}+1}$, and hence it is not equal to g. Thus, if g belongs to the centralizer of τ, then g does not contain a, and hence g is contained in the subgroup of G generated by τ and $b = xyx$, which is equal to $\{1, yxy, \tau, yxy\tau\}$. It is obvious that this group is contained in the centralizer of τ. Hence, we obtain the desired result.

(2) is a direct consequence of (1). $\square$

Note that $\tau_{L'}$ is conjugate to τ, $y^{-1}\tau y$, $x^{-1}\tau x$, $(xy)^{-1}\tau xy$, $(yx)^{-1}\tau yx$ or $(xyx)^{-1}\tau xyx (= \tau(xyx)^2 = \tau)$ in $\mathscr{M}(M)$ according as $(s_1, s_2, s_3) \approx (s_1', s_2', s_3')$, (s_1', s_3', s_2'), (s_2', s_1', s_3'), (s_2', s_3', s_1'), (s_3', s_1', s_2') or (s_3', s_2', s_1') (see Lemma 8).

Assume first that $(s_1, s_2, s_3) \approx (s_1', s_3', s_2')$. Since $\tau_{L'} = y^{-1}\tau y$ is conjugate to $\tau_L = \tau$ in $\mathscr{M}(M)$, there exists $\gamma \in \mathscr{M}(M)$ such that $y^{-1}\tau y = \gamma^{-1}\tau\gamma$. Then $\gamma y^{-1} \in Z(\tau) = \{1, \tau\}$ by Claim 1 (2), and hence $\gamma = y$ or τy. However, y and τy do not belong to $\mathscr{M}(M)$, a contradiction. Hence, $(s_1, s_2, s_3) \approx (s_1', s_3', s_2')$ cannot be satisfied.

Similarly, we can see that $(s_1, s_2, s_3) \approx (s_2', s_1', s_3')$, (s_2', s_3', s_1') or (s_3', s_1', s_2') cannot be satisfied. Hence, we have $(s_1, s_2, s_3) \sim (s_1', s_2', s_3')$. $\square$

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