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A series associated to generating pairs of a once punctured torus group and a proof of McShane's identity

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ABSTRACT. We give a proof of McShane's identity in [5] based on the investigation on the arrangement of axes of simple hyperbolic elements in a once punctured torus group which are represented by palindromic words. Our argument includes a short proof of the fact that the linear measure of the infinitesimal Birman-Series set is zero.

1. Introduction

Let $\mathbf{T}$ be a once punctured torus equipped with a finite area hyperbolic metric. We denote by $|\gamma|$ the length of a closed geodesic γ on $\mathbf{T}$. G. McShane proved in [5] the identity

$$\sum_{\gamma} \arcsin\left(\frac{1}{\cosh(|\gamma|/2)}\right) = \frac{\pi}{2}, \tag{1.1}$$

where the sum is taken over all simple closed geodesics passing through a fixed pair of Weierstrass points. In this note we give an alternative proof of (1.1). Let G be a *once punctured torus group*, that is, a group of hyperbolic motions on the unit disk D with the factor surface $\mathbf{T} = D/G$. G is freely generated by a pair of neighbors $\{a, b\}$ (for definition, see Section 2.1). It acts also on the boundary ∂D of D in the complex plane $\mathbf{C}$. The axes of a and b meet at a single point O of D. Let E be the closure in ∂D of the set of fixed points of generators (simple and primitive hyperbolic elements) in G whose axes pass through O. In [5] E is called the *infinitesimal Birman-Series set*, and a component of $\partial D - E$ a *gap* of E. Our proof follows the usual steps: show that the left hand side of (1.1) is a quarter of the sum of angles subtended by gaps with respect to O and deduce (1.1) from the fact that the linear measure $|E|$ of E is zero (see [2]). However our technique is based on theorems in [3] and [4] which characterize generators whose axes pass through O in terms of the words of symbols in $\Gamma = \{a, a^{-1}, b, b^{-1}\}$. By this characterization we

2000 *Mathematics Subject Classification*. Primary 20H10; Secondary 57M50 establish a correspondence between positively oriented pairs of neighbors whose axes pass through O and the gaps of E. So the identity (4.2) we obtain first is expressed in the language of Fuchsian groups and does not involve the hyperbolic geometry in appearance. In Section 5 we show that (1.1) and (4.2) are identical. Our argument includes an elementary proof of that $|E| = 0$.

2. Once punctured torus group and pair of neighbors

2.1. We regard D as a model of the hyperbolic plane. For two distinct points p and q of ∂D, $L(p,q)$ will denote the directed hyperbolic line with initial point p and terminal point q.

The group $\mathscr{H}(D)$ of orientation-preserving motions on D is identified with $SU(1,1)/\{\pm I\}$. A hyperbolic element g of $\mathscr{H}(D)$ has two fixed points in ∂D, the repelling fixed point p_g and the attracting fixed point q_g. The *axis* $ax(g)$ of g is the directed line $L(p_g, q_g)$.

The once punctured torus group G is a Fuchsian subgroup of $\mathscr{H}(D)$. Let g be a hyperbolic element of G. Then its axis $ax(g)$ projects to a closed geodesic on $\mathbf{T}$ which will be denoted by γ_g. The element g is called *simple* if γ_g is a simple curve and *primitive* if $g = h^n$ for an $h \in G$ and an integer n, then $n = \pm 1$. A simple and primitive hyperbolic element in G is called a *generator*. Two generators g and h in G are called *neighbors* if they correspond to a pair of simple closed curves on $\mathbf{T}$ with intersection number 1 and the axes of g and h intersect in D.

An ordered pair of neighbors $\{a,b\}$ in G is said to be *positively oriented* when the axis of b cuts the axis of a from the right to the left. For two pairs of neighbors $\{a,b\}$ and $\{a',b'\}$, we write $\{a,b\} \sim \{a',b'\}$ if there exists $c \in G$ such that $\{a',b'\} = c^{-1}\{a,b\}c = \{c^{-1}ac, c^{-1}bc\}$. We write also $a \sim a'$ if a' is conjugate to a in G. If $\{a,b\}$ is a positively oriented pair of neighbors and $\{a',b'\} \sim \{a,b\}$, then $\{a',b'\}$ is positively oriented, since each $c \in G$ is an orientation-preserving homeomorphism of D.

We fix a pair of neighbors $\{a,b\}$. Each element of G is written as a word of the symbols a, a^{-1}, b and b^{-1}. If $W = e_1 e_2 \dots e_n$, where $e_i \in \Gamma = \{a, a^{-1}, b, b^{-1}\}$, then n is called the *length* of W and denoted by $\ell(W)$. For each $e \in \Gamma$ we let $n_e(W)$ denote the number of e_i's which equals e. Since G is free on a and b, each $g \in G$ is represented by a unique *reduced* word W_g, the shortest expression of g as a word.

2.2. Each simple closed curve on $\mathbf{T}$ is isotopic to a unique geodesic curve. Hence we can identify the set of the conjugacy classes of generators in G with the set of isotopy classes of oriented simple closed curves on $\mathbf{T}$. Then a characterization of generators by words in $\Gamma = \{a, a^{-1}, b, b^{-1}\}$ is

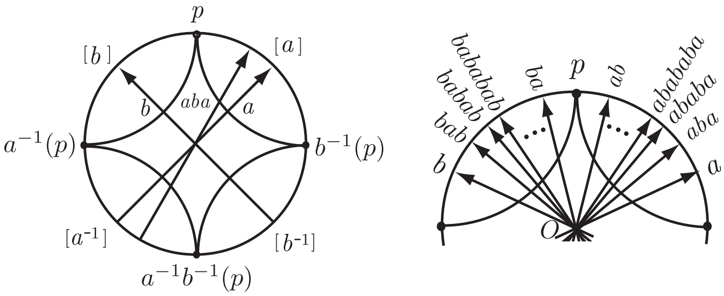

Fig. 1

Theorem 2.1 ([3], see also Theorem 5.1 in [1]). *Up to permutations of Γ which interchange a and b, a and a^{-1}, or b and b^{-1} the word W representing a generator g in G is up to cyclic permutations either a or of the form*

$$a^{n_1} b a^{n_2} b \dots a^{n_k} b$$

where $\{n_1, n_2, \dots, n_k\} \subset \{n, n+1\}$ *for some positive integer n.*

In [3] the theorem concerns the free group $F(a,b)$ of rank 2 and its statement is true for any element g of $F(a,b)$ which forms a basis with another element h. It is important to note that for a generator, its reduced word has at most two symbols in Γ.

2.3. We owe the following description to [1]. Let $\{a,b\}$ be positively oriented. We denote by $J_{\{a,b\}}$ the subarc of ∂D between q_a and q_b which does not contain p_a. Then the fixed point p of $aba^{-1}b^{-1}$ lies in $J_{\{a,b\}}$. (To see this, apply the proof of Proposition 33.23 in [9] by setting $T^{-1} = a$ and $U^{-1} = b$.) Likewise we see that $a^{-1}(p)$, $a^{-1}b^{-1}(p)$ and $b^{-1}(p)$ are situated in ∂D as in Figure 1. The four points p, $a^{-1}(p)$, $a^{-1}b^{-1}(p)$ and $b^{-1}(p)$ divide ∂D to four arcs. We label the arcs as follows: $[b]$ is the arc between p and $a^{-1}(p)$, $[a^{-1}]$ is the arc between $a^{-1}(p)$ and $a^{-1}b^{-1}(p)$, $[b^{-1}]$ is the arc between $a^{-1}b^{-1}(p)$ and $b^{-1}(p)$, and $[a]$ is the arc between $b^{-1}(p)$ and p. Let $W = e_1 \dots e_r e_{r+1}$ be a reduced word of the symbols in Γ, then we define $[W] = e_1 \dots e_r [e_{r+1}]$.

Let $W = e_1 \dots e_r a$ be a reduced word. Since a sends $[b^{-1}] \cup [a] \cup [b]$ into $[a]$ and a is orientation-preserving, the subarcs $[Wb^{-1}]$, $[Wa]$ and $[Wb]$ of $[W]$ are arranged in anticlockwise order. We can say the same thing when (a,b) is replaced by (b^{-1}, a), (a^{-1}, b^{-1}) and (b, a^{-1}). If $W \in G$ is a cyclically reduced word and hyperbolic, then $\{[W^n]\}_{n=1}^{\infty}$ is a decreasing sequence of arcs and the attracting fixed point q_W of W equals $\bigcap_{n=1}^{\infty} [W^n]$. This observation leads to the following lemma.

LEMMA 2.1. *Let W_1 and W_2 be cyclically reduced words for hyperbolic elements in G with distinct axes. Let m and n be positive integers such that $\ell(W_1^m) \geq \ell(W_2)$ and $\ell(W_2^n) \geq \ell(W_1)$. Let $W_1^m = e_1 e_2 \dots e_p$ and $W_2^n = f_1 f_2 \dots f_q$. Then q_{W_1} precedes q_{W_2} in anticlockwise order around ∂D starting from p if and only if*

(i) *(e_1, f_1) equals (b, a^{-1}), (b, b^{-1}), (b, a), (a^{-1}, b^{-1}), (a^{-1}, a) or (b^{-1}, a), or*

(ii) *There is an index number r such that $e_i = f_i$ for each $i = 1, 2, \dots, r$ and either*
  - (a) *$e_r = a$ and $(e_{r+1}, f_{r+1}) = (a, b)$, (b^{-1}, b) or (b^{-1}, a), or*
  - (b) *$e_r = b^{-1}$ and $(e_{r+1}, f_{r+1}) = (b^{-1}, a)$, (a^{-1}, a) or (a^{-1}, b^{-1}), or*
  - (c) *$e_r = a^{-1}$ and $(e_{r+1}, f_{r+1}) = (a^{-1}, b^{-1})$, (b, b^{-1}) or (b, a^{-1}), or*
  - (d) *$e_r = b$ and $(e_{r+1}, f_{r+1}) = (b, a^{-1})$, (a, a^{-1}) or (a, b).*

We define two transformations on the set of pairs of neighbors:

$$\omega_1\{g, h\} = \{g, hg\}, \qquad \omega_2\{g, h\} = \{gh, h\}.$$

Note that gh is a generator, because γ_{gh} is isotopic to a Dehn twist of γ_g along γ_h. For positive integers n we define also

$$\sigma_n\{g, h\} = \{(gh)^{n-1}g, (gh)^n g\}, \qquad \sigma_{-n}\{g, h\} = \{(hg)^n h, (hg)^{n-1}h\}.$$

These are pairs of neighbors, because

$$\begin{aligned} \{g, h\} &\xrightarrow{\omega_1} \{g, hg\} \xrightarrow{\omega_2^{n-1}} \{g(hg)^{n-1}, hg\} \\ &\xrightarrow{\omega_1} \{g(hg)^{n-1}, hgg(hg)^{n-1}\} \sim \{(gh)^{n-1}g, (gh)^n g\}, \\ \{g, h\} &\xrightarrow{\omega_2} \{gh, h\} \xrightarrow{\omega_1^{n-1}} \{gh, h(gh)^{n-1}\} \\ &\xrightarrow{\omega_2} \{ghh(gh)^{n-1}, h(gh)^{n-1}\} \sim \{(hg)^n h, (hg)^{n-1}h\}. \end{aligned}$$

Since $\{a, b\}$ is positively oriented, so are the pairs $\sigma_n\{a, b\}$. This can be seen from Lemma 2.1, but more easily from (5.1) below. Note that entries of all $\sigma_n\{a, b\}$ are palindromes in the symbols a and b. Let $\mathcal{G}$ denote the semigroup generated by $\{\sigma_n : n \in \mathbf{Z}\}$, where σ_0 is defined to be the identity.

2.4. For any positively oriented pair of neighbors $\{a, b\}$ in a once-punctured torus group, $x = |\mathrm{tr}\, a|$, $y = |\mathrm{tr}\, b|$ and $z = |\mathrm{tr}\, ab|$ satisfy

$$x^2 + y^2 + z^2 - xyz = 0. \tag{2.1}$$

On the other hand, a triple of numbers (x, y, z) satisfying $x > 2$, $y > 2$, $z > 2$ and (2.1) determines a unique conjugacy class of positively oriented pairs of neighbors in once-punctured torus groups. Let A, B be matrices in $SU(1, 1)$

such that $x = \mathrm{tr}\, A$ and $y = \mathrm{tr}\, B$, $z = |\mathrm{tr}\, AB|$ and $ABA^{-1}B^{-1}$ is parabolic. Assume that $ax(B)$ cuts $ax(A)$ from the right to the left. Then $z = \mathrm{tr}\, AB$ and $\mathrm{tr}\, ABA^{-1}B^{-1} = -2$ (see [9, Lemma 33.21]). If we normalize A and B so that $ABA^{-1}B^{-1}$ fixes 1 and that the axes of A and B meet at 0, then we have uniquely

$$A = \begin{pmatrix} \dfrac{x}{2} & \dfrac{xz - 2y - 2ix}{2z} \\ \dfrac{xz - 2y + 2ix}{2z} & \dfrac{x}{2} \end{pmatrix}, \quad B = \begin{pmatrix} \dfrac{y}{2} & \dfrac{yz - 2x + 2iy}{2z} \\ \dfrac{yz - 2x - 2iy}{2z} & \dfrac{y}{2} \end{pmatrix}. \tag{2.2}$$

We have also

$$AB = \begin{pmatrix} \dfrac{z - 2i}{2} & \dfrac{z}{2} \\ \dfrac{z}{2} & \dfrac{z + 2i}{2} \end{pmatrix}, \qquad BA = \begin{pmatrix} \dfrac{z + 2i}{2} & \dfrac{z}{2} \\ \dfrac{z}{2} & \dfrac{z - 2i}{2} \end{pmatrix}. \tag{2.3}$$

3. Palindrome pair of neighbors

3.1. Let $\{a, b\}$ be a pair of neighbors in G. Let O be the intersecting point of the axes $ax(a)$ and $ax(b)$. The axis $ax(g)$ of a hyperbolic element g of G passes through O if and only if the reduced word $W_g = e_1 e_2 \dots e_r$ for g of the symbols in $\{a, a^{-1}, b, b^{-1}\}$ is a palindrome, that is, $e_i = e_{r+1-i}$, $i = 1, 2, \dots, r$. To show this, we assume that O is the origin. A hyperbolic element $A \in SU(1,1)$ has its axis passing through O if and only if $A^* = A$, where

$$A^* = \begin{pmatrix} s & q \\ r & p \end{pmatrix} \qquad \text{for } A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Therefore $ax(g)$ passes through O if and only if $W_g = e_1 e_2 \dots e_r$ is a palindrome, because

$$(e_1 e_2 \dots e_r)^* = e_r^* \dots e_2^* e_1^* = e_r \dots e_2 e_1.$$

This fact has interesting applications. See, for example, [4] and [6]. By Theorem 2.1 we have

Lemma 3.1. *Let W be the reduced word for a generator g with axis passing through O. Then, after a suitable permutation of symbols, $W = a$ or*

$$W = a^{n_1} b a^{n_2} b \dots b a^{n_k} b a^{n_{k+1}}$$

satisfying (i) $n_i = n_j$ *if* $i + j = k + 2$, (ii) $\{2n_1, n_2, \dots, n_k\}$ *equals either* $\{2n_1\}$, $\{2n_1, 2n_1 + 1\}$ *or* $\{2n_1 - 1, 2n_1\}$ *and* (iii) $n_a(W) = n_1 + \dots + n_{k+1} > k = n_b(W)$.

3.2. Let p_1 and p_2 be distinct points of $\partial D - \{p\}$. We write $p_1 \prec p_2$ if p_1 precedes p_2 in anticlockwise order around ∂D starting from p. Let q_{-n} be the attracting fixed point of $(ba)^{n-1}b$ for $n = 1, 2, \dots$, and q_n the attracting fixed point of $(ab)^n a$ for $n = 0, 1, \dots$. Then by Lemma 2.1 $q_n \prec q_{n+1}$ for all integers n. See Figure 1, where an axis is labeled by the element of G which keeps it invariant. Since $q_n \in [(ab)^n]$ and $q_{-n} \in [(ba)^{n-1}]$ for all $n \geq 2$, we have $\lim_{n\to\infty} q_n = q_{ab}$ and $\lim_{n\to-\infty} q_n = q_{ba}$. We denote by $I_{\{a,b\}}$ the arc on ∂D between q_{ab} and q_{ba} which contains the fixed point p of $aba^{-1}b^{-1}$. We shall call $I_{\{a,b\}}$ the *gap* associated to the pair $\{a, b\}$. Its meaning is clarified by the theorem below. If $n \neq 0$, then $J_{\sigma_n\{a,b\}}$ is the interval in $J_{\{a,b\}}$ between q_{n-1} and q_n. Thus

$$J_{\{a,b\}} = I_{\{a,b\}} \cup \bigcup_{n\neq 0} J_{\sigma_n\{a,b\}}. \tag{3.1}$$

Two distinct intervals in the right hand side have disjoint interiors.

THEOREM 3.1. *In $I_{\{a,b\}}$ there are no terminal points of axes of generators which pass through O. Let $|I_{\{a,b\}}|$ denote the angle subtended by the arc $I_{\{a,b\}}$ with respect to the center O. Then*

$$|I_{\{a,b\}}| = 2 \arcsin\left(\frac{2}{|\mathrm{tr}\, ab|}\right). \tag{3.2}$$

Proof. Suppose that a generator g has its axis which passes through O and the terminal point q_g in $I_{\{a,b\}}$. Since G is discrete, q_g cannot be p. If q_g lies between p and q_{ab}, then by Lemmas 2.1 and 3.1 the reduced word for g has the form $W = a^{n_1}ba^{n_2}b \dots ba^{n_2}ba^{n_1}$, where n_1 is a positive integer. Thus WW is of the form $(ab)^n aaW_1$ for some non-negative integer n and some word W_1. By Lemma 2.1, $q_W = q_{WW} \prec q_{ab} = q_{(ab)^{n+1}}$. This is a contradiction. We can prove in the same way that g cannot have an axis which passes through O and ends between p and q_{ba}.

Now we prove the second statement of the theorem. If $\{a, b\}$ is a positively oriented pair of neighbors and if $x = \mathrm{tr}\, a$, $y = \mathrm{tr}\, b$ (taken to be positive) and $z = |\mathrm{tr}\, ab|$, then $\{a, b\}$ is simultaneously conjugate to $\{A, B\}$, where A and B are as in (2.2). Since the conjugation is done by a conformal automorphism of the unit disk, we need only to consider the pair $\{A, B\}$. The attracting fixed points of the matrices in (2.3) satisfiy

$$\mathrm{Im}(q_{AB}) = -\frac{2}{z}, \qquad \mathrm{Im}(q_{BA}) = \frac{2}{z}. \tag{3.3}$$

Thus $|I_{\{A,B\}}| = 2 \arcsin(2/z) = 2 \arcsin(2/\mathrm{tr}\, AB)$. Now we complete the proof. □

Recall that $J_{\{a,b\}}$ is the subarc of ∂D between q_a and q_b which contains $I_{\{a,b\}}$. Suppose that $J_{\{a,b\}}$ is seen from O with the angle $|J_{\{a,b\}}|$. For the matrices A and B as above we have

$$q_A = \frac{xz - 2y - 2ix}{z\sqrt{x^2 - 4}}, \qquad q_B = \frac{yz - 2x + 2iy}{z\sqrt{y^2 - 4}}. \tag{3.4}$$

Now (3.3) and (3.4) together with the conformality of Möbius transformations yield

$$\frac{|I_{\{a,b\}}|}{|J_{\{a,b\}}|} \geq \min\left\{ \arcsin\left(\frac{2}{z}\right) \Big/ \arcsin\left(\frac{2x}{z\sqrt{x^2-4}}\right), \arcsin\left(\frac{2}{z}\right) \Big/ \arcsin\left(\frac{2y}{z\sqrt{y^2-4}}\right) \right\}.$$

Since the function $\arcsin(t)/\arcsin(\theta t)$ is decreasing for $t \in (0, \theta^{-1})$ for each $\theta > 1$, we obtain

$$\frac{|I_{\{a,b\}}|}{|J_{\{a,b\}}|} \geq c = \min\left\{ \frac{2}{\pi} \arcsin\left(\frac{\sqrt{|\mathrm{tr}\, g|^2 - 4}}{|\mathrm{tr}\, g|} \right) : g \text{ is a hyperbolic element of } G \right\}. \tag{3.5}$$

Finally we remark that the ratio $|I_{\{a,b\}}|/|J_{\{a,b\}}|$ tends to 1 as $\min\{\mathrm{tr}\, a, \mathrm{tr}\, b\} \to \infty$.

4. Sequences of palindrome pairs of neighbors

We fix a positively oriented pair of neighbors $\{a,b\}$. Let O denote the intersecting point of the axes of a and b. Let $\mathscr{P}(a,b)$ be the $\mathscr{G}$-orbit of $\{a,b\}$. More precisely $\mathscr{P}(a,b)$ is the minimal set satisfying the following conditions:

(i) $\{a,b\} \in \mathscr{P}(a,b)$

(ii) If $\{g,h\} \in \mathscr{P}(a,b)$ then $\{(gh)^{n-1}g, (gh)^n g\}$, $\{(hg)^n h, (hg)^{n-1}h\} \in \mathscr{P}(a,b)$ for any positive integer n.

Likewise we define $\mathscr{P}(b,a^{-1})$, $\mathscr{P}(a^{-1},b^{-1})$ and $\mathscr{P}(b^{-1},a)$ by the $\mathscr{G}$-orbits of $\{b,a^{-1}\}$, $\{a^{-1},b^{-1}\}$ and $\{b^{-1},a\}$, respectively. Let $\mathscr{P} = \mathscr{P}(a,b) \cup \mathscr{P}(b^{-1},a) \cup \mathscr{P}(a^{-1},b^{-1}) \cup \mathscr{P}(b,a^{-1})$. For each generator which belongs to a pair in $\mathscr{P}$, the corresponding word in Γ is a palindrome. Hence its axis passes through O. Let $\Gamma = \{a, a^{-1}, b, b^{-1}\}$.

Proposition 4.1. *If the reduced word W_f in Γ for a generator f of G is a palindrome, then f belongs to a pair in $\mathscr{P}$.*

Proof. We introduce an algorithm to find a pair $\{a',b'\}$ in $\mathscr{P}$ such that $n_{a'}(W') + n_{b'}(W') = 1$, where W' is the reduced word for f in $\{a',b'\}$. Then the last equation means either $f = a'$ or $f = b'$.

Since the arguments for the proof are similar for other cases, we treat only the case where $W = W_f$ is a word in $\{a,b\}$ and $n_a(W) > n_b(W)$. So we assume that W is $a^{m_1}ba^{m_2}b\dots ba^{m_k}ba^{m_{k+1}}$ with $m_1,\dots,m_{k+1}$ positive and $m_i = m_j$ if $i+j=k+2$. We have $n_a(W) = m_1+\dots+m_{k+1}$, $n_b(W)=k$ and $l(W)=n_a(W)+n_b(W)$. There are three cases.

Case 1: $k=1$ or $\{m_2,\dots,m_k\}=\{2m_1\}$,

Case 2: $k\geq 2$ and $\{2m_1,m_2,\dots,m_k\}=\{2m_1,2m_1+1\}$,

Case 3: $k\geq 2$ and $\{2m_1,m_2,\dots,m_k\}=\{2m_1-1,2m_1\}$.

For Cases 1 and 2, let $a_1=a$ and $b_1=a^{m_1}ba^{m_1}$. Then $\{a_1,b_1\}=\sigma_1^{m_1}\{a,b\}=\{a,a^{m_1}ba^{m_1}\}\in\mathscr{P}$. Case 1 means $W=(a^{m_1}ba^{m_1})^k$. Since f is a generator, $k=1$ and hence $f=b_1$. For Case 2, f has the form $W_1=b_1^{n_1}a_1b_1^{n_2}a_1\dots a_1b_1^{n_l}a_1b_1^{n_{l+1}}$. Since

$$n_{b_1}(W_1)=n_b(W),\qquad n_{a_1}(W_1)=n_a(W)-2m_1n_b(W),\tag{4.1}$$

we have $l(W_1)<l(W)$. Next we consider Case 3. If $m_1>1$, then let $\{a_1,b_1\}=\{a,a^{m_1-1}ba^{m_1-1}\}=\sigma_1^{m_1-1}\{a,b\}\in\mathscr{P}$. Then $\{a_1,b_1\}$ belongs to $\mathscr{P}$ and f is written as

$$W_1=a_1^{n_1}b_1a_1^{n_2}b_1\dots b_1a_1^{n_l}b_1a_1^{n_{l+1}}.$$

Here $n_k\in\{1,2\}$, $k=2,\dots,l$, and $n_1=n_{l+1}=1$. Since

$$n_{a_1}(W_1)=n_a(W)-2(m_1-1)n_b(W),\qquad n_{b_1}(W_1)=n_b(W),$$

we have $l(W_1)<l(W)$. If $m_1=1$, then W is written as $(ab)^{n_1}a(ab)^{n_2}a\dots(ab)^{n_l}a$ with positive integers $n_1,\dots,n_l$. Let $n=\min\{n_1,\dots,n_l\}$. Since $\{a,ab\}$ is a generating pair of G, Theorem 2.1 yields subcases.

Case 3-1: $\{n_1,\dots,n_l\}=\{n\}$,

Case 3-2: $\{n_1,\dots,n_l\}=\{n,n+1\}$ and $n_1=n$,

Case 3-3: $\{n_1,\dots,n_l\}=\{n,n+1\}$ and $n_1=n+1$.

Let $\{a_1,b_1\}=\{(ab)^na,(ab)^{n+1}a\}=\sigma_{n+1}\{a,b\}\in\mathscr{P}$. Case 3-1 means that $W=((ab)^na)^l$. Since f is a generator, $l=1$ and $W=a_1$. We can write W as $W_1=a_1^{p_1}b_1a_1^{p_2}\dots a_1^{p_s}b_1a_1^{p_{s+1}}$ for Case 3-2 and $W_1=b_1^{p_1}a_1b_1^{p_2}\dots b_1^{p_s}a_1b_1^{p_{s+1}}$ for Case 3-3, with some positive integers $p_1,\dots,p_{s+1}$. For Case 3-2, the word W_1 in the pair $\{a_1,b_1\}=\{(ab)^{n_1}a,(ab)^{n_1+1}a\}$ satisfies

$$n_{a_1}(W_1)=(n_1+1)n_a(W)-(n_1+2)n_b(W),$$

$$n_{b_1}(W_1)=-n_1n_a(W)+(n_1+1)n_b(W).$$

Thus we have $l(W_1)<l(W)$. For Case 3-3, with the pair $\{a_1,b_1\}=\{(ab)^{n_1-1}a,(ab)^{n_1}a\}$, we have the equations

$$n_{a_1}(W_1)=n_1n_a(W)-(n_1+1)n_b(W),\qquad n_{b_1}(W_1)=-(n_1-1)n_a(W)+n_1n_b(W)$$

and hence $l(W_1)<l(W)$. For all the cases above, if $1<\ell(W_1)<\ell(W)$, we repeat this step with $\{a,b\}$ replaced by $\{a_1,b_1\}$. Then after finite steps we find a pair $\{a', b'\}$ in $\mathscr{P}$ such that $l(W') = 1$, where W' is the reduced word for f in $\{a', b'\}$. □

Let $\mathscr{P}_1$ be the collection of $\{a, b\}$, $\{b^{-1}, a\}$, $\{a^{-1}, b^{-1}\}$ and $\{b, a^{-1}\}$. We define $\mathscr{P}_n$, $n = 2, 3, \ldots$, inductively by the collection of all $\sigma_m\{c, d\}$ with $\{c, d\} \in \mathscr{P}_{n-1}$ and $m \in \mathbf{Z} - \{0\}$. Thus, a pair $\{g, h\}$ of $\mathscr{P}$ which belongs to $\mathscr{P}_n$ has the form $\{(cd)^{m-1}c, (cd)^m c\}$ or $\{(dc)^m d, (dc)^{m-1}d\}$ for some $\{c, d\} \in \mathscr{P}_{n-1}$ and for some positive integer m. By (3.1) applied to the pairs in $\mathscr{P}_1$, we see that ∂D is divided into four gaps $I_{\{a,b\}}$, $I_{\{b^{-1},a\}}$, $I_{\{a^{-1},b^{-1}\}}$, $I_{\{b,a^{-1}\}}$ and infinitely many subarcs $J_{\{g,h\}}$, $\{g, h\} \in \mathscr{P}_2$. Each $J_{\{g,h\}}$ is in turn divided into the gap $I_{\{g,h\}}$ and subarcs $J_{\sigma_m\{g,h\}}$ of $\mathscr{P}_3$ defined for all non-zero integers m. By continuing this observation we see that ∂D is divided into the union of gaps $I_{\{g,h\}}$ with $\{g, h\} \in \mathscr{P}$ and its complement E. Let us consider the sequence of sets $E_n = \partial D - \bigcup_{k=1}^{n} \bigcup_{\{g,h\} \in \mathscr{P}_k} I_{\{g,h\}}$. We apply (3.5) to all $\{g, h\} \in \mathscr{P}$ to have $|I_{\{g,h\}}| \ge c|J_{\{g,h\}}|$. Let $|\cdot|$ denote also the angular measure on ∂D with respect to O. Then $|E_{n+1}| \le (1 - c)|E_n| < (1 - c)^n |E_1|$ for all n. Thus we obtain that $|E| = 0$, a result due to Birman and Series [2]. By Proposition 4.1, E is the closure of the set of all fixed points of generators whose axes pass through O, or the infinitesimal Birman-Series set in [5]. By using (3.2) we obtain

$$\sum_{\{g,h\} \in \mathscr{P}} 2 \arcsin\left(\frac{2}{|\mathrm{tr}\, gh|}\right) = 2\pi. \tag{4.2}$$

Let $\mathscr{P}(a) = \mathscr{P}(a, b) \cup \mathscr{P}(b^{-1}, a)$. Since $I_{\{g,h\}}$ and $I_{\{g^{-1},h^{-1}\}}$ are antipodal with respect to O, $\{g, h\}$ and $\{g^{-1}, h^{-1}\}$ contribute the same angle to the sum. Hence we obtain

$$\sum_{\{g,h\} \in \mathscr{P}(a)} \arcsin\left(\frac{2}{|\mathrm{tr}\, gh|}\right) = \sum_{\{g,h\} \in \mathscr{P}(a)} \arcsin\left(\frac{1}{\cosh(|\gamma_{gh}|/2)}\right) = \frac{\pi}{2}, \tag{4.3}$$

where γ_{gh} is the simple closed geodesic which is the projection of the axis of gh.

5. Equivalence of the series constants

In this section we prove that the two identities (1.1) and (4.3) are identical when the pair of Weierstrass points P_2 and P_3 is chosen as described below.

5.1. For materials in this paragraph, see [7]. Let W_g be the reduced word for a generator g in $\Gamma = \{a, a^{-1}, b, b^{-1}\}$. Then in the homology group $H_1(\mathbf{T}) = G/[G, G]$, g is homologous to $n_a(W_g)a + n_b(W_g)b$, and $n_a(W_g)$ and $n_b(W_g)$ are coprime integers. For each pair of neighbors $\{g, h\}$, there exists a homeomorphism φ of $\mathbf{T}$ onto itself which sends γ_a and γ_b to γ_g and

γ_h, respectively. Obviously $\{g,h\}$ is positively oriented if and only if φ is orientation-preserving. Since both $\{a,b\}$ and $\{g,h\}$ give bases of $H_1(\mathbf{T})$,

$$\det\begin{pmatrix} n_a(W_g) & n_b(W_g) \\ n_a(W_h) & n_b(W_h) \end{pmatrix} = \pm 1, \tag{5.1}$$

and the determinant above equals 1 if and only if $\{g,h\}$ is positively oriented.

Let $\mathscr{S}$ denote the set of isotopy classes of unoriented simple closed curves in $\mathbf{T}$. We can identify $\mathscr{S}$ with the set of unoriented closed geodesics, because each isotopy class has a unique geodesic representative γ. If the axes of two generators g and g' project to γ, then g' is conjugate either to g or to g^{-1}, and hence $n_b(W_g)/n_a(W_g) = n_b(W_{g'})/n_a(W_{g'})$. Thus $n_b(W_g)/n_a(W_g)$ depends only on γ. We write $slope(\gamma) = n_b(W_g)/n_a(W_g)$ and define a mapping $slope : \mathscr{S} \to \hat{\mathbf{Q}} = \mathbf{Q} \cup \{\frac{1}{0}\}$.

There exists a complex number τ with $\mathrm{Im}(\tau) > 0$ such that $\mathbf{C}_\tau = \mathbf{C} - (\mathbf{Z} + \mathbf{Z}\tau)$ is a covering surface of $\mathbf{T}$ such that the lifts of a and b define the transformations $z \mapsto z+1$ and $z \mapsto z+\tau$, respectively, generating the group of covering transformations $\tilde{G} \cong H_1(\mathbf{T})$. We say that a straight line in $\mathbf{C}$ has *slope* q/p if it is parallel to the line passing through 0 and $p + q\tau$. Each pair of coprime integers (p,q) defines a simple closed curve c in $\mathbf{T}$, which is the projection of a line in $\mathbf{C}_\tau$ with slope q/p. Since the correspondence $q/p \mapsto [c]$ is the inverse of *slope*,

Lemma 5.1. *The mapping slope* $: \mathscr{S} \to \hat{\mathbf{Q}}$ *which sends* γ_g *to* $n_b(W_g)/n_a(W_g)$ *is bijective.*

By this lemma we identify $\mathscr{S}$ with $\{\gamma_{q/p} : p/q \in \hat{\mathbf{Q}}\}$, where $\gamma_{q/p}$ is the geodesic curve with $slope(\gamma_{q/p}) = q/p$. Let $\tilde{P}_1$, $\tilde{P}_2$ and $\tilde{P}_3$ denote the $\tilde{G}$-orbits of the points $\frac{1}{2} + \frac{1}{2}\tau$, $\frac{1}{2}\tau$ and $\frac{1}{2}$, respectively, and let P_1, P_2 and P_3 be their projections in $\mathbf{T}$. If the puncture is filled by a point P_4, then P_1, P_2, P_3 and P_4 are the Weierstrass points of the torus $\overline{\mathbf{T}} = \mathbf{T} \cup \{P_4\}$. We divide $\mathscr{S}$ into three subsets $\mathscr{S}_{12}$, $\mathscr{S}_{13}$ and $\mathscr{S}_{23}$ so that $\gamma_{q/p} \in \mathscr{S}_{jk}$ if $\gamma_{q/p}$ passes through P_j and P_k, or equivalently there exists a line with slope q/p which meets points of $\tilde{P}_j$ and $\tilde{P}_k$. Therefore, $\gamma_{q/p}$ belongs to $\mathscr{S}_{12}$, $\mathscr{S}_{13}$ or $\mathscr{S}_{23}$ in accordance with $(p,q) \equiv (1,0),\ (0,1)$ or $(1,1) \bmod 2$.

The projection of O is P_1, because it is the intersection of $\gamma_a = \gamma_{0/1}$ and $\gamma_b = \gamma_{1/0}$. If $\{g,h\} \in \mathscr{P}(a)$, then the axes of g and h pass through O. Hence γ_g and γ_h pass through P_1. By (5.1) either γ_g or γ_h belongs to $\mathscr{S}_{12}$ and the other belongs to $\mathscr{S}_{13}$. Then γ_{gh} belongs to $\mathscr{S}_{23}$. So we can define a mapping $\Phi : \mathscr{P}(a) \to \mathscr{S}_{23}$ by the correspondence $\Phi(\{g,h\}) = \gamma_{gh}$. For the rest of this section we will show that Φ is bijective.

Let $\{f, g\}$ be a pair in $\mathscr{P}(a)$. Let $W_f = w_1 \dots w_m$ be the reduced word for f and $W_g = w_{m+1} \dots w_n$ the one for g. Since fg is also a generator, by Lemma 3.1, either $\{w_1, \dots, w_n\} \subset \{a, b\}$ or $\{w_1, \dots, w_n\} \subset \{b^{-1}, a\}$. Thus their juxtaposition $W_{fg} = w_1 \dots w_m w_{m+1} \dots w_n$ is the reduced word for fg. Note that W_{fg} is cyclically reduced too. Suppose that $\gamma_{f_1g_1} = \gamma_{f_2g_2}$ for two pairs $\{f_1, g_1\}$ and $\{f_2, g_2\}$ in $\mathscr{P}(a)$. Since f_2g_2 is conjugate either to f_1g_1 or $(f_1g_1)^{-1}$, $W_{f_2g_2}$ is a cyclic permutation of $W_{f_1g_1}$ or $W_{f_1g_1}^{-1}$. This is possible only when exactly either $\mathscr{P}(a, b)$ or $\mathscr{P}(b^{-1}, a)$ contains both $\{f_1, g_1\}$ and $\{f_2, g_2\}$, and $W_{f_2g_2}$ is a cyclic permutation of $W_{f_1g_1}$. Since the proof for the other case can be modified easily, we consider the case where $\{f_1, g_1\} \in \mathscr{P}(a, b)$. Let $p_1 = n_a(W_{f_1})$, $q_1 = n_b(W_{f_1})$, $r_1 = n_a(W_{g_1})$, $s_1 = n_b(W_{g_1})$, $p_2 = n_a(W_{f_2})$, $q_2 = n_b(W_{f_2})$, $r_2 = n_a(W_{g_2})$ and $s_2 = n_b(W_{g_2})$. We show that $r_1 = r_2$ and $s_1 = s_2$. If this is not the case, we can assume without loss of generality that $r_1 \geq r_2$ and $s_1 > s_2$ if $r_1 = r_2$. Since $W_{f_2g_2}$ is a cyclic permutation of $W_{f_1g_1}$, $p_1 + r_1 = p_2 + r_2$ and $q_1 + s_1 = q_2 + s_2$. Since $\{f_1, g_1\}$ and $\{f_2, g_2\}$ are positively oriented,

$$0 = \det\begin{pmatrix} p_1 + r_1 & q_1 + s_1 \\ r_1 & s_1 \end{pmatrix} - \det\begin{pmatrix} p_1 + r_1 & q_1 + s_1 \\ r_2 & s_2 \end{pmatrix} = \det\begin{pmatrix} p_1 + r_1 & q_1 + s_1 \\ r_1 - r_2 & s_1 - s_2 \end{pmatrix}.$$

Thus $r_1 - r_2 > 0$ and $s_1 - s_2 > 0$, and there exist coprime positive integers m and n with

$$m(p_1 + r_1, q_1 + s_1) = n(r_1 - r_2, s_1 - s_2).$$

Since $p_1 + r_1$ and $q_1 + s_1$ are coprime too, n must be 1. But this contradicts that $p_1 + r_1 > r_1 - r_2$ or $q_1 + s_1 > s_1 - s_2$. Thus $r_1 = r_2$ and $s_1 = s_2$ and hence $p_1 = p_2$ and $q_1 = q_2$. By Lemma 5.1 f_1 and f_2 are conjugate and so are g_1 and g_2. Since they are simple and primitive, and their axes pass through O, $f_1 = f_2$ and $g_1 = g_2$. We conclude that the map Φ is injective.

5.2. In what follows all rational numbers q/p are such that p and q are coprime and $p > 0$. We identify $\hat{\mathbf{Q}}$ with the set of vertices of the Farey tessellation $\mathscr{T}$ of the upper half plane (see [7]): Two vertices q/p and s/r are connected by an edge in $\mathscr{T}$ if and only if

$$\det\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \pm 1. \tag{5.2}$$

If q/p, q_1/p_1 and q_2/p_2 are vertices of a triangle in $\mathscr{T}$ and the edge connecting q_1/p_1 and q_2/p_2 separates q/p from $-q/p$, then

$$\frac{q}{p} = \frac{q_1 + q_2}{p_1 + p_2}. \tag{5.3}$$

(If $q/p = -n/1$ is a negative integer, let $q_1/p_1 = -1/0$ and $q_2/p_2 = (-n + 1)/1$.)

Let $\gamma_{q/p} \in \mathscr{S}_{23}$ and choose q_1/p_1 and p_2/q_2 as above. Then by (5.2) and (5.3) either γ_{q_1/p_1} or γ_{q_2/p_2} belongs $\mathscr{S}_{12}$ and the other belongs to $\mathscr{S}_{13}$. The identity (5.2) means that γ_{q_1/p_1} and γ_{q_2/p_2} meet at a single point, and this point must be P_1. Therefore γ_{q_1/p_1} and γ_{q_2/p_2} define a pair of neighbors $\{f,h\}$ such that $\gamma_f = \gamma_{q_1/p_1}$, $\gamma_h = \gamma_{q_2/p_2}$ and $\gamma_{fh} = \gamma_{q/p}$ and such that the axes of f and h pass through O. Since $\gamma_{fh} = \gamma_{hf}$, by interchanging f and h, if necessary, we assume that $\{f,h\}$ is positively oriented. Moreover, by replacing $\{f,h\}$ by $\{f^{-1},h^{-1}\}$, if necessary, we assume that the reduced word for f has the symbols in $\{a,b\}$ or in $\{b^{-1},a\}$. We consider the case where the word is in $\{a,b\}$. The other case follows simply by replacing $\{a,b\}$ by $\{b^{-1},a\}$. By Proposition 4.1 there are a generator g and $\sigma \in \mathscr{G}$ such that $\{f,g\} = \sigma\{a,b\}$. Since both $\{f,g\}$ and $\{f,h\}$ are positively oriented,

$$\det\begin{pmatrix} 1 & 0 \\ n_f(W_h) & n_g(W_h) \end{pmatrix} = n_g(W_h) = 1,$$

where W_h is the reduced word for h in $\{f,f^{-1},g,g^{-1}\}$. Since the axes of f, g and h pass through O, W_h is a palindrome in $\{f^{\pm 1},g\}$. Therefore $h = f^n g f^n$ for some integer n. If $n \geq 0$, then $\{f,h\} = \sigma_1^n \sigma\{a,b\} \in \mathscr{P}(a)$ and $\gamma_{fh} = \Phi(\{f,h\})$. So in order to show that Φ is surjective, what is left for us is to prove

Lemma 5.2. *Let $\{f,g\} \in \mathscr{P}(a,b)$. If $h = f^{-n} g f^{-n}$ for a positive integer n, then there exists a pair $\{f_1,g_1\} \in \mathscr{P}(a)$ such that $f_1 g_1$ is conjugate to fh or to $(fh)^{-1}$ and hence $\Phi(\{f_1,g_1\}) = \gamma_{fh}$.*

Proof. Let $\{f,g\} = \sigma_{m_1}\sigma_{m_2}\ldots\sigma_{m_p}\{a,b\}$. Our proof is by induction on p. If $p=0$, that is, if $\{f,g\} = \{a,b\}$, then $\{f,h\} = \{a, a^{-n} b a^{-n}\}$. In this case let $\{f_1,g_1\} = \{a^{n-1} b^{-1} a^{n-1}, a\} = \sigma_{-1}^{n-1}\{b^{-1},a\}$. Then we have $f_1 g_1 \sim (fh)^{-1}$. If $p > 0$, let $\{c,d\} = \sigma_{m_2}\ldots\sigma_{m_p}\{a,b\} \in \mathscr{P}(a,b)$. Then $\{f,g\} = \{(cd)^{m-1}c, (cd)^m c\}$ if $m = m_1 > 0$ and $\{f,g\} = \{(dc)^m d, (dc)^{m-1} d\}$ if $m = -m_1 > 0$.

If $n = 1$, then $fh = gf^{-1} = cd$ if $m_1 > 0$ and $fh = (dc)^{-1}$ if $m_1 < 0$. In this case we can let $\{f_1,g_1\} = \{c,d\}$. Now we assume that $n \geq 2$. If $m = m_1 \geq 2$, then let $\{f_1,g_1\} = \sigma_{-1}^{n-2}\sigma_{m-1}\{c,d\}$. So $f_1 = ((cd)^{m-1}c)^{n-2}((cd)^{m-2}c)\cdot((cd)^{m-1}c)^{n-2}$ and $g_1 = (cd)^{m-1}c$. Since

$$\begin{aligned} h^{-1} &= ((cd)^{m-1}c)^n((cd)^m c)^{-1}((cd)^{m-1}c)^n \\ &= ((cd)^{m-1}c)^{n-1}((cd)^{m-2}c)((cd)^{m-1}c)^{n-1}, \end{aligned}$$

we have $f^{-1}h^{-1} = f_1 g_1$. If $m_1 = 1$, then $\{f,g\} = \{c, cdc\}$ and $\{f,h\} = \{c, c^{-n+1} d c^{-n+1}\}$. In this case, we replace $\{f,g\}$ by $\{c,d\} = \sigma_{m_2}\ldots\sigma_{m_p}\{a,b\}$. Then $\{f,h\} = \{f, f^{-n+1} g f^{-n+1}\}$. Since σ is replaced by $\sigma_{m_2}\ldots\sigma_{m_p}$, by hypothesis of induction there is a pair $\{f_1, g_1\} \in \mathscr{P}(a)$ with $\Phi\{f_1, g_1\} = \gamma_{fh}$. If $m = -m_1 > 0$, then let $\{f_1, g_1\} = \sigma_{-1}^{n-2}\sigma_{-(m+1)}\{c, d\}$. So $f_1 = ((dc)^m d)^{n-2} \cdot ((dc)^{m+1} d)((dc)^m d)^{n-2}$ and $g_1 = (dc)^m d$. Since

$$h^{-1} = ((dc)^m d)^n ((dc)^{m-1} d)^{-1} ((dc)^m d)^n = ((dc)^m d)^{n-1} ((dc)^{m+1} d)((dc)^m d)^{n-1},$$

we have $f^{-1}h^{-1} = f_1 g_1$. □

Now we complete the proof that Φ is bijective. Thus (4.3) and (1.1) are identical.

6. McShane's identity for torus with one hole

Let G be a Fuchsian group generated by a and b such that D/G is a torus with one boundary curve. We assume that $ax(b)$ cuts $ax(a)$ from the right to the left and that $x = \operatorname{tr} a$, $y = \operatorname{tr} b$ are positive. Then $z = \operatorname{tr} ab > 0$ and $t = \operatorname{tr}(aba^{-1}b^{-1}) < -2$, where $t = -xyz + x^2 + y^2 + z^2 - 2$, and the conjugacy class of G is determined by the quadraple (x, y, z, t) (see [9, 33.D]). It has a representative generated by

$$A = \begin{pmatrix} \dfrac{x}{2} & \dfrac{-2y + xz - ix\sqrt{2-t}}{2\sqrt{z^2 - t - 2}} \\ \dfrac{-2y + xz + ix\sqrt{2-t}}{2\sqrt{z^2 - t - 2}} & \dfrac{x}{2} \end{pmatrix},$$

$$B = \begin{pmatrix} \dfrac{y}{2} & \dfrac{-2x + yz + iy\sqrt{2-t}}{2\sqrt{z^2 - t - 2}} \\ \dfrac{-2x + yz - iy\sqrt{2-t}}{2\sqrt{z^2 - t - 2}} & \dfrac{y}{2} \end{pmatrix}.$$

Here A and $B \in SU(1,1)$ are chosen so that the axes of A and B intersect at the origin and that the real axis is perpendicular to the axis of $ABA^{-1}B^{-1}$. Let $p = 1$. A similar argument to the one in Section 3 shows that the subarc $I_{\{A,B\}}$ on ∂D between q_{AB} and q_{BA} which contains p is a gap for the group generated by A and B. Let $J_{\{A,B\}}$ denote the subarc between q_A and q_B which contains p. Since

$$\operatorname{Im} q_A = -\frac{x\sqrt{2-t}}{\sqrt{x^2-4}\sqrt{z^2-t-2}}, \qquad \operatorname{Im} q_B = \frac{y\sqrt{2-t}}{\sqrt{y^2-4}\sqrt{z^2-t-2}}$$

$$\operatorname{Im} q_{AB} = -\frac{\sqrt{2-t}}{\sqrt{z^2-t-2}}, \qquad \operatorname{Im} q_{BA} = \frac{\sqrt{2-t}}{\sqrt{z^2-t-2}},$$

the ratio of the angle subtended by $I_{\{A,B\}}$ and the one subtended by $J_{\{A,B\}}$ satisfies

$$\frac{|I_{\{A,B\}}|}{|J_{\{A,B\}}|} \geq \min\left\{ \frac{\arcsin\left(\frac{\sqrt{2-t}}{\sqrt{z^2-t-2}}\right)}{\arcsin\left(\frac{x\sqrt{2-t}}{\sqrt{x^2-4}\sqrt{z^2-t-2}}\right)}, \frac{\arcsin\left(\frac{\sqrt{2-t}}{\sqrt{z^2-t-2}}\right)}{\arcsin\left(\frac{y\sqrt{2-t}}{\sqrt{y^2-4}\sqrt{z^2-t-2}}\right)} \right\}.$$

As in Section 3 this yields $|I_{\{a,b\}}| > c|J_{\{a,b\}}|$ for all positively oriented pairs of neighbors $\{a,b\}$ in G, where c is a constant defined as in (3.5), and we can show that the linear measure of the infinitesimal Birman-Series set is 0 and deduce a variation of (1.1) in [8, Corollary 1.10]

$$\sum_{\gamma} \arcsin\left(\frac{\cosh(|\delta|/4)}{\sqrt{\sinh^2(|\gamma|/2) + \cosh^2(|\delta|/4)}}\right) = \frac{\pi}{2},$$

where δ is the geodesic homotopic to the boundary curve and γ runs over all simple closed geodesics passing through the Weierstrass points other than the intersection of γ_a and γ_b.

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