Hiroshima Math. J.
40 (2010), 315–331

Caloric morphisms for rotation invariant metrics

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(Received March 17, 2009)
(Revised May 6, 2009)

Abstract. We determine all the caloric morphisms for rotation invariant (spherically symmetric) metrics, in the case where the space dimension is greater than two. We also treat the caloric morphisms on two dimensional spheres and hyperbolae.

1. Introduction

Let (M, g) be an n-dimensional Riemannian manifold. We denote by Δ_g the Laplace-Beltrami operator of (M, g), which is given in a local coordinate $(x_i)_{i=1}^n$ by

$$\Delta_g u = \sum_{i,j=1}^{n} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_j} \right),$$

where $|g| = \det(g_{ij})$ and (g^{ij}) denotes the inverse matrix of (g_{ij}).

We consider the heat equation on Riemannian manifolds.

Definition 1. A C^∞-function $u(t, x)$ defined on an open set $D \subset \mathbf{R} \times M$ is said to be caloric if u satisfies the heat equation

$$\frac{\partial u}{\partial t} - \Delta_g u = 0$$

on D. We call $H_g := \dfrac{\partial}{\partial t} - \Delta_g$ the heat operator on $\mathbf{R} \times M$.

In the following, we consider the caloric morphisms, the transformations which preserve the caloric functions. The precise definition is the following:

Definition 2. Let (M, g) and (N, h) be Riemannian manifolds, f a C^∞-mapping from a domain $D \subset \mathbf{R} \times M$ to $\mathbf{R} \times N$ and φ a strictly positive C^∞-

2000 *Mathematics Subject Classification*: Primary 35K99, Secondary 31B99, 35A30.

Key words and phrases: Caloric morphism, heat equation, Appell transformation, rotation invariant metric.

Partially supported by Grant-in-aid for Scientific Research (C) No. 19540161 and No. 19540193, Japan Society for the Promotion of Science.

funcion on D. The pair (f, φ) is said to be a caloric morphism if f and φ satisfy the following conditions:

(1) $f(D)$ is a domain in $\mathbf{R} \times N$;

(2) For any caloric function u defined on an open set E in $\mathbf{R} \times N$, the function $\varphi(t,x)(u \circ f)(t,x)$ is caloric on $f^{-1}(E)$.

The Appell transformation

$$D = \{(t,x); t > 0, x \in \mathbf{R}^n\}, \qquad f(t,x) = \left(-\frac{1}{t}, \frac{x}{t}\right), \qquad \varphi(t,x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$$

is the most important example of the caloric morphism on $\mathbf{R}^n$. The Appell transformation plays important roles, especially in the study of positive solutions, because it preserves positive solutions of the heat equation.

In [7], we defined the notion of caloric morphism on manifolds as a generalization of the Appell transformation, and obtained a characterization theorem of caloric morphisms. Note that the well-known harmonic morphism is considered as the case where φ is constant 1 (see e.g. [2] and [3]).

In [5], Leutwiler determined all the caloric morphisms for Euclidean space. He proved that every caloric morphism from a domain of $\mathbf{R}^{n+1}$ into $\mathbf{R}^{n+1}$ is a composition of the Appell transformation and parabolic similarities:

$$u(t,x) \mapsto u(\lambda^2 t + d, \lambda R x + v),$$

where $\lambda > 0$, $d \in \mathbf{R}$, R is an orthogonal matrix of degree n, and $v \in \mathbf{R}^n$.

This paper treats the following problem: determine all the caloric morphisms for rotation invariant (spherically symmetric) metrics on $\mathbf{R}^n$ $(n \geq 2)$ regarded as a Riemannian manifold.

Some partial results on this problem were obtained in [8]. We restricted ourselves there to consider radial metrics $g = \rho(|x|)g_0$, where g_0 is the Euclidean metric of $\mathbf{R}^n$, and we considered the caloric morphism whose mapping f is of form

$$(1) \qquad f(t,x) = (f_0(t), v(t)R(t)x) \qquad \text{or} \qquad f(t,x) = \left(f_0(t), v(t)R(t)\frac{x}{|x|^2}\right)$$

with $R(t) \in O(n)$, where $O(n)$ denotes the totality of orthogonal matrices. We have determined ρ, f and φ in this case.

The purpose of this paper is to solve the problem completely for rotation invariant metrics in the case $n \geq 3$. Clearly, if the metric is rotation invariant, then the time translations and the space rotations are caloric morphisms, which we call trivial caloric morphisms. In the course of determining all the caloric morphisms, it turns out that if a rotation invariant metric admits a non-trivial caloric morphism, then not only the mapping but also the metric is much restricted.

Theorem 1. *Let g be a rotation invariant metric on $M = \mathbf{R}^n \backslash \{0\}$ with $n \geq 3$. If there exists a non-trivial caloric morphism (f, φ) defined on a domain $D \subset \mathbf{R} \times M$ into $\mathbf{R} \times M$, then g satisfies one of the following up to isometry:*

(a) $$g = \rho(|x|)g_0 \qquad \text{with } \rho(\nu(t)r) = \frac{\lambda(t)}{\nu(t)^2}\rho(r),$$

where $\lambda(t)$ and $\nu(t)$ are positive C^∞ functions. In this case, the space component f^t has the form $f^t(x) = \nu(t)R(t)x$ for each t, where $R(t) \in O(n)$ is a C^∞ function of t.

(b) $$g = \rho(|x|)g_0 \qquad \text{with } \rho\left(\frac{\nu(t)}{r}\right) = \frac{\lambda(t)r^4}{\nu(t)^2}\rho(r),$$

where $\lambda(t)$ and $\nu(t)$ are positive C^∞ functions. In this case, the space component f^t has the form $f^t(x) = \nu(t)R(t)\dfrac{x}{|x|^2}$ for each t, where $R(t) \in O(n)$ is a C^∞ function.

(c) $$g = \frac{1}{(|x|^2 + q)^2} g_0,$$

where $q \in \mathbf{R}$. (For the precise definition of the space component and the trivial caloric morphism, see §2 and §3 below, respectively.)

In the above three cases (a), (b) and (c), the cases (a) and (b) have been already studied in [8]. We study the case (c) in §5 in this paper, where we treat constant curvature manifolds including 2-dimensional case. It is remarkable that the results of the semi-riemannian radial metric case are applied to the study of the case (c).

After that, we determine all the caloric morphisms for rotation invariant metrics in Theorems 2 and 3, stated in §6.

2. Preliminaries

In this section, we list necessary results on caloric morphisms for later use.

For each mapping $f : \mathbf{R} \times M \to \mathbf{R} \times N$, we sometimes need to separate the mapping f into the time component f_0 and the space component f^t as

$$f(t,x) = (f_0(t,x), f^t(x)), \qquad f_0(t,x) \in \mathbf{R},\ f^t(x) \in N.$$

The following theorem characterizes the caloric morphism.

Theorem A ([7, Theorem 2.1]). *Let (M, g) and (N, h) be Riemannian manifolds. For a smooth mapping f from a domain $D \subset \mathbf{R} \times M$ into $\mathbf{R} \times N$*

such that $f(D)$ is a domain and for a positive smooth function φ defined on D, the following three conditions are equivalent:

(1) *(f, φ) is a caloric morphism;*

(2) *There exists a positive smooth function λ on D such that*

$$H_g(\varphi \cdot u \circ f)(t,x) = \lambda(t,x)\varphi(t,x)((H_h u) \circ f)(t,x)$$

holds for every C^2-function u on $f(D)$;

(3) *(f, φ) satisfies the following equations:*

(E-1) $H_g \varphi = 0$,

(E-2) $H_g f_\alpha^t = 2g(\mathrm{grad}_g \log \varphi, \mathrm{grad}_g f_\alpha^t) + \sum_{\beta,\gamma=1}^{n} g(\mathrm{grad}_g f_\beta^t, \mathrm{grad}_g f_\gamma^t)(\Gamma_{\beta\gamma}^\alpha \circ f^t)$ $(\alpha = 1, \dots, n)$,

(E-3) $\mathrm{grad}_g f_0 = 0$, *which means f_0 is a function of t,*

(E-4) $g(\mathrm{grad}_g f_\alpha^t, \mathrm{grad}_g f_\beta^t) = \dfrac{df_0}{dt}(t)(h^{\alpha\beta} \circ f^t)$ $(\alpha, \beta = 1, \dots, n)$,

where we denote by grad_g *the gradient operator on M, we write $f^t = (f_1^t, \dots, f_n^t)$ in a local coordinate $(x_\alpha)_{\alpha=1}^n$ of (N, h), and $\Gamma_{\beta\gamma}^\alpha = \sum_{l=1}^{n} \frac{1}{2} h^{\alpha l}\left(\frac{\partial h_{\gamma l}}{\partial x_\beta} + \frac{\partial h_{\beta l}}{\partial x_\gamma} - \frac{\partial h_{\beta\gamma}}{\partial x_l}\right)$ is the Christoffel symbol of (N, h).*

Remark 1. Please note that in paper [7] the equation (E-2) has been miswritten. As a matter of fact, the sign of the second term of the right hand side in the equation (E-2) of Theorem 2.1 of [7] was incorrect. We have corrected it in the above.

Remark 2. By (E-3) in the above theorem, f_0 depends only on t for each caloric morphism. Hence $\lambda = f_0'$ and λ also depends only on t. So, we can write the mapping f in the form

$$f(t,x) = (f_0(t), f^t(x)), \qquad f_0 : \mathbf{R} \to \mathbf{R},\ f^t : M \to N.$$

Example 1. **(Isometry)** An isometry induces a caloric morphism. Let $\iota : M \to N$ be an isometry. Then $(t,x) \mapsto (t, \iota(x))$ is a caloric morphism: $\mathbf{R} \times M \to \mathbf{R} \times N$, where $\varphi = 1$.

(Time scaling) For a constant $p > 0$, the time scaling $(t,x) \mapsto (pt, x)$ gives a caloric morphism from a manifold with metric g to that with pg, where $\varphi = 1$.

(Time translation) For $d \in \mathbf{R}$, time translation $(t,x) \mapsto (t+d, x)$ is always a caloric morphism, where $\varphi = 1$. We call it the identity if $d = 0$.

We use the direct product of caloric morphisms, stated in [7, Proposition 2.1]. We summarize here some methods to construct new caloric morphisms.

Proposition 1. **(Direct product)** *Let I be an interval of* $\mathbf{R}$ *and Ω_j be a domain of a manifold M_j $(j = 1, 2)$. Consider two caloric morphisms (f, φ)*

from $I \times \Omega_1$ *to* $\mathbf{R} \times M_1$ *and* (h, ψ) *from* $I \times \Omega_2$ *to* $\mathbf{R} \times M_2$ *such that* $f_0 = h_0$. *We consider the map* (f_0, f^t, h^t) *from* $I \times \Omega_1 \times \Omega_2$ *to* $\mathbf{R} \times M_1 \times M_2$*:*

$$(t, x, y) \mapsto (f_0(t), f^t(x), h^t(y)),$$

and the function $\varphi\psi$ *on* $I \times \Omega_1 \times \Omega_2$*:*

$$(t, x, y) \mapsto \varphi(t, x)\psi(t, y).$$

Then the pair $((f_0, f^t, h^t), \varphi\psi)$ *is a caloric morphism. We remark that the assertion also holds for semi-Riemannian manifolds, the manifolds with metrics not-necessarily positive definite. (See* [7, Theorem 2.1]*.)*

(Composition) *The composition of caloric morphisms is also a caloric morphism. Let* M*,* N *and* L *be Riemannian manifolds and* D*,* E *be domains in* $\mathbf{R} \times M$*,* $\mathbf{R} \times N$*, respectively. If* $(f, \varphi) : D \to \mathbf{R} \times N$ *and* $(h, \psi) : E \to \mathbf{R} \times L$ *are caloric morphisms such that* $f(D) \subset E$*, then we can make a caloric morphism* $(F, \Phi) : D \to \mathbf{R} \times L$ *by putting* $(F, \Phi) = (h \circ f, \varphi \cdot (\psi \circ f))$*.*

(Conjugate by isometry) *Let* $\iota : M \to N$ *be an isometry. Then for each caloric morphism* (f, φ) *from* $E \subset \mathbf{R} \times N$ *to* $\mathbf{R} \times N$*, the pair* (f^*, φ^*) *with*

$$f^*(t, x) = (f_0(t), \iota^{-1} \circ f^t \circ \iota(x)), \qquad \varphi^*(t, x) = \varphi(t, \iota(x))$$

is a caloric morphism for $\mathbf{R} \times M$*.*

(Conjugate by time scaling) *A time scaling corresponds to a dilatation of the metric. For a fixed* $p > 0$*, we put*

$$A_p f(t, x) := (p f_0(t/p), f^{t/p}(x)), \qquad A_p \varphi(t, x) := \varphi(t/p, x).$$

Then, $(A_p f, A_p \varphi)$ *is a caloric morphism for the metric* pg *if* (f, φ) *is a caloric morphism for the metric* g*.*

3. Rotation invariant metrics on $\mathbf{R}^n \setminus \{0\}$

In the following, we consider caloric morphisms for rotation invariant metrics on the Euclidean spaces. Let g be a rotation invariant metric on $M = \mathbf{R}^n \setminus \{0\}$. Here, "rotation invariant" means that the metric is invariant under every orthogonal transformation $R \in O(n)$. Clearly, the Laplacian Δ_g is also rotation invariant if g is rotation invariant. Hence for all $C > 0$, $d \in \mathbf{R}$ and $R_0 \in O(n)$, the pair (f, φ) with

$$f(t, x) = (t + d, R_0 x), \qquad \varphi(t, x) = C$$

is a caloric morphism for any rotation invariant metric. We refer to these caloric morphisms as trivial caloric morphisms.

A metric g is rotation invariant if and only if there exist positive smooth functions $\mu_0(r)$ and $\mu_1(r)$ on $(0, \infty)$ such that

$$g = \mu_0(r)^2\sigma + \mu_1(r)^2(dr)^2$$

by using the polar coordinate, where σ denotes the surface metric of the unit sphere in $\mathbf{R}^n$. When $\mu_0(r) = r\mu_1(r)$, we say that g is a radial metric. In other words, $g = \mu_1(r)^2 g_0$ where $g_0 = r^2\sigma + (dr)^2$ denotes the Euclidean metric.

As a matter of fact, any rotation invariant metric is isometric to a radial metric by a change of variables.

Lemma 1. *Any rotation invariant metric is, at least locally, isometric to a radial metric.*

Proof. Let g be a rotation invariant metric. Using the polar coordinate, we write

$$g = \mu_0(r)^2\sigma + \mu_1(r)^2(dr)^2$$

with some positive smooth functions μ_0 and μ_1. By the change of variable

$$s(r) = \exp \int_1^r \mu_1(\tau)/\mu_0(\tau)d\tau,$$

we have

$$g = \mu_0(r)^2\sigma + \frac{\mu_0(r)^2}{s(r)^2}(ds)^2 = \rho(s)(s^2\sigma + (ds)^2),$$

where we put $\rho(s) = \mu_0(r(s))^2 s^{-2}$. □

Therefore we may assume that g is a radial metric of form $g = \rho(r)g_0$.

4. Proof of Theorem 1

To prove Theorem 1, we prepare a technical lemma.

Lemma 2. (i) *Let U be a non-empty open set in $\mathbf{R}^n$. Assume that $n \geq 2$ and C^1 functions $\mu(s)$ and $v(s)$ satisfy the equation*

$$\mu(|x - a|^2) + v(|x|^2) = 0, \qquad x \in U$$

with some $a \in \mathbf{R}^n$. If v is not constant on $\{|x|^2; x \in U\}$, then $a = 0$.

(ii) *Let U be a non-empty open set in $\mathbf{R}^n$. Assume that $n \geq 3$ and C^1 functions $\mu(s_1, s_2)$ and $v(s)$ satisfy the equation*

$$\mu(|x - a|^2, |x - b|^2) + v(|x|^2) = 0, \qquad x \in U$$

with some $a, b \in \mathbf{R}^n$, $a \neq b$. *If* $v'(|x_0|^2) \neq 0$ *at some point* $x_0 \in U$, *then there exist constants* α *and* β *with* $\alpha^2 + \beta^2 \neq 0$ *such that*

$$\alpha a + \beta b = 0 \tag{2}$$

$$\left(\alpha \frac{\partial \mu}{\partial s_2} - \beta \frac{\partial \mu}{\partial s_1}\right)(|x-a|^2, |x-b|^2) = 0, \qquad x \in U. \tag{3}$$

Proof. (i) Differentiating the both sides of the given equation by x_i and x_j $(i \neq j)$, we have

$$\begin{cases} (x_i - a_i)\mu'(|x-a|^2) + x_i v'(|x|^2) = 0, \\ (x_j - a_j)\mu'(|x-a|^2) + x_j v'(|x|^2) = 0. \end{cases}$$

Taking $x \in U$ so that $v'(|x|^2) \neq 0$, we see

$$\begin{vmatrix} a_i & x_i \\ a_j & x_j \end{vmatrix} = -\begin{vmatrix} x_i - a_i & x_i \\ x_j - a_j & x_j \end{vmatrix} = 0$$

for all $i \neq j$. Hence a and x are linearly dependent for all x in the non-empty open set $\{x \in U; v'(|x|^2) \neq 0\}$. This implies $a = 0$.

(ii) Differentiating the both sides of the given equation by $x_1, \dots, x_n$, we have

$$\frac{\partial \mu}{\partial s_1}(|x-a|^2, |x-b|^2)(x-a) + \frac{\partial \mu}{\partial s_2}(|x-a|^2, |x-b|^2)(x-b) + v'(|x|^2)x = 0,$$

i.e.,

$$\frac{\partial \mu}{\partial s_1} a + \frac{\partial \mu}{\partial s_2} b = \left(\frac{\partial \mu}{\partial s_1} + \frac{\partial \mu}{\partial s_2} + v'\right)x. \tag{4}$$

Suppose that

$$V := \left\{x \in U; \left(\frac{\partial \mu}{\partial s_1} + \frac{\partial \mu}{\partial s_2}\right)(|x-a|^2, |x-b|^2) + v'(|x|^2) \neq 0\right\}$$

is not empty. Then the above equality (4) implies that every x in an n-dimensional non-empty open set V is a linear combination of two vectors a and b. This is a contradiction because $n \geq 3$. Therefore $\left(\frac{\partial \mu}{\partial s_1} + \frac{\partial \mu}{\partial s_2}\right)(|x-a|^2, |x-b|^2) + v'(|x|^2) = 0$ on U, and hence

$$\frac{\partial \mu}{\partial s_1}(|x-a|^2, |x-b|^2)a + \frac{\partial \mu}{\partial s_2}(|x-a|^2, |x-b|^2)b = 0, \qquad x \in U \tag{5}$$

by (4). Putting $\alpha = \frac{\partial \mu}{\partial s_1}(|x_0-a|^2, |x_0-b|^2)$ and $\beta = \frac{\partial \mu}{\partial s_2}(|x_0-a|^2, |x_0-b|^2)$, we obtain (2). Also $(\alpha, \beta) \neq (0, 0)$ holds, because the assumption $v'(|x_0|^2) \neq 0$

implies $\alpha+\beta = \left(\frac{\partial\mu}{\partial s_1}+\frac{\partial\mu}{\partial s_2}\right)(|x_0-a|^2, |x_0-b|^2) = -v'(|x_0|^2) \neq 0$. Since $a \neq b$ and $(\alpha,\beta) \neq (0,0)$, (3) follows from (2) and (5). $\square$

Now we shall prove Theorem 1.

Proof of Theorem 1. By Lemma 1, we may assume that g is a radial metric $g = \rho(|x|)g_0$ with some positive smooth function ρ. First, we note that if ρ is constant, then g is the Euclidean metric, which corresponds to the case (c) with $q = 0$ by inversion $x \mapsto x/|x|^2$. Next we assume that ρ is not constant. Then the equation (E-4) implies

$$\frac{1}{\rho(|x|)}\sum_{i=1}^{n}\frac{\partial f_\alpha^t(x)}{\partial x_i}\frac{\partial f_\beta^t(x)}{\partial x_i} = \lambda(t)\delta_{\alpha\beta}\frac{1}{\rho(|f^t(x)|)}, \qquad \alpha,\beta = 1,\dots,n, \tag{6}$$

where $\lambda(t) = f_0'(t)$. The equality (6) shows that the space component f^t is conformal with respect to the Euclidean metric for each t. We fix t in the rest of the proof. By Liouville's theorem on conformal mappings (see [1, pp. 222–227]), every conformal mapping on $\mathbf{R}^n$ $(n \geq 3)$ is a similarity

$$x \mapsto vRx + v, \qquad (v > 0, R \in O(n), v \in \mathbf{R}^n)$$

or a composition of a similarity and an inversion with respect to a sphere

$$x \mapsto \frac{x-a}{|x-a|^2} + a \qquad (a \in \mathbf{R}^n).$$

Then $f^t(x)$ has the form

$$f^t(x) = vR(x-a)$$

or

$$f^t(x) = vR\left(\frac{x-a}{|x-a|^2} + b\right),$$

where $v = v(t) > 0$, $R = R(t) \in O(n)$ and $a = a(t) \in \mathbf{R}^n$, $b = b(t) \in \mathbf{R}^n$.

If $f^t(x) = vR(x-a)$, then (6) implies

$$v^2\rho(v|x-a|) = \lambda\rho(|x|). \tag{7}$$

Then by (i) of Lemma 2, we have $a = 0$ because ρ is assumed to be non-constant. Hence

$$f^t(x) = vRx \qquad \text{and} \qquad \rho(v|x|) = \frac{\lambda}{v^2}\rho(|x|).$$

Thus we have (a).

We proceed to the latter case $f^t(x) = \nu R\left(\frac{x-a}{|x-a|^2}+b\right)$. Then by differentiation we have

$$\sum_{i=1}^{n}\left(\frac{\partial f_\alpha^t(x)}{\partial x_i}\right)^2 = \nu^2|x-a|^{-4}, \qquad \alpha = 1,\dots,n.$$

First, we assume that $b=0$. Then (6) implies

$$\frac{\nu^2}{|x-a|^4}\rho\left(\frac{\nu}{|x-a|}\right) = \lambda\rho(|x|)$$

and hence, (i) of Lemma 2 shows $a=0$. Therefore $f^t(x) = \nu R\frac{x}{|x|^2}$ and $\rho\left(\frac{\nu}{|x|}\right) = \frac{\lambda|x|^4}{\nu^2}\rho(|x|)$ hold. Thus we have (b).

Next, we assume that $b \neq 0$. Then (6) implies

$$\frac{\nu^2}{|x-a|^4}\rho\left(\nu|b|\frac{\left|x-a+\frac{b}{|b|^2}\right|}{|x-a|}\right) = \lambda\rho(|x|) \tag{8}$$

because $\left|\frac{x-a}{|x-a|}+|x-a|b\right| = \left|\frac{b}{|b|}+|b|(x-a)\right|$. If we put $\mu(s_1,s_2) = \frac{\nu^2}{s_1^2}\rho\left(\nu|b|\frac{\sqrt{s_2}}{\sqrt{s_1}}\right)$ and $\tilde\rho(r) = \lambda\rho(\sqrt{r})$, then

$$\mu\left(|x-a|^2, \left|x-a+\frac{b}{|b|^2}\right|^2\right) - \tilde\rho(|x|^2) = 0.$$

Since ρ is not constant, (ii) of Lemma 2 shows that $\alpha a + \beta\left(a - \frac{b}{|b|^2}\right) = 0$ and $\alpha\frac{\partial\mu}{\partial s_2} - \beta\frac{\partial\mu}{\partial s_1} = 0$ with some $(\alpha,\beta)\neq(0,0)$. By the definition of μ, we obtain

$$\alpha\frac{\partial\mu}{\partial s_2} - \beta\frac{\partial\mu}{\partial s_1} = \frac{\nu^2}{2s_1^3}\left(\nu|b|\left(\alpha\frac{\sqrt{s_1}}{\sqrt{s_2}}+\beta\frac{\sqrt{s_2}}{\sqrt{s_1}}\right)\rho'\left(\nu|b|\frac{\sqrt{s_2}}{\sqrt{s_1}}\right) - 4\beta\rho\left(\nu|b|\frac{\sqrt{s_2}}{\sqrt{s_1}}\right)\right) = 0.$$

Putting $r = \nu|b|\frac{\sqrt{s_2}}{\sqrt{s_1}}$, we have

$$\left(\frac{\nu^2|b|^2\alpha}{r}+\beta r\right)\rho'(r) + 4\beta\rho(r) = 0.$$

If $\beta = 0$, then $\rho'(r) = 0$ holds for all r, which is a contradiction because ρ is not constant. Putting $q = \frac{\nu^2|b|^2\alpha}{\beta}$, we have the equation

$$\rho'(r) + \frac{4r}{r^2+q}\rho(r) = 0.$$

Therefore $\rho(r) = \dfrac{p}{(r^2+q)^2}$ with some $p > 0$, and hence $g = \dfrac{p}{(|x|^2+q)^2} g_0$, which is isometric to

$$\frac{1}{(|x|^2+q/p)^2} g_0.$$

Thus we have (c) and this completes the proof. □

5. Caloric morphisms on spheres and hyperbolae

In this section, we treat constant curvature cases, which correspond to (c) in Theorem 1. Only in this section, we let $n \geq 2$.

We begin with the following lemma.

Lemma 3. *Let (M, g) be a Riemannian manifold and (f, φ) be a caloric morphism defined on a domain $D \subset \mathbf{R} \times M$. If (M, g) has a non-zero constant curvature, then $f_0' \equiv 1$ and the space component $f^t(x)$ is a local isometry for each t.*

Proof. By (E-4) of Theorem A,

$$g(\mathrm{grad}_g\, f_\alpha^t, \mathrm{grad}_g\, f_\beta^t) = \frac{df_0}{dt}(t)(g^{\alpha\beta} \circ f^t).$$

Since (M, g) has a constant curvature $\kappa \neq 0$, the above equality implies

$$\kappa = \frac{df_0}{dt}(t)\kappa.$$

Therefore $f_0' \equiv 1$ and then

$$g(\mathrm{grad}_g\, f_\alpha^t, \mathrm{grad}_g\, f_\beta^t) = g^{\alpha\beta} \circ f^t,$$

which shows that f^t is an isometry for each t. □

First we consider the case where M is the n-dimensional sphere

$$\mathbf{S} = \{x \in \mathbf{R}^{n+1}; x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

Proposition 2. *Let (f, φ) be a caloric morphism defined on a domain $D \subset \mathbf{R} \times \mathbf{S}$. Then*

$$f(t, x) = (t + d, R_0 x) \qquad \text{and} \qquad \varphi(t, x) = C,$$

where d and C are constants with $C > 0$ and $R_0 \in O(n+1)$.

Proof. By Lemma 3, $f_0(t) = t + d$ and $f^t(x) = R(t)x$ where $d \in \mathbf{R}$ and $R(t) \in O(n+1)$. To complete the proof, it suffices to show that $R(t)$ is constant.

Consider the direct product (f^*, φ^*) of (f, φ) and a time translation induced by $f_0(t)$ on $\mathbf{R} \times \mathbf{R}$. More precisely, let $\mathbf{S}^* = \mathbf{S} \times \mathbf{R}$ be the direct product manifold and define the pair of the mapping $f^* : D \times \mathbf{R} \subset \mathbf{R} \times \mathbf{S}^* \to \mathbf{R} \times \mathbf{S}^*$ and the function φ^* on $D \times \mathbf{R}$ by

$$f^*(t, x, r) = (t + d, f^t(x), r) = (t + d, R(t)x, r) \qquad \text{and} \qquad \varphi^*(t, x, r) = \varphi(t, x).$$

Note that the space component f^{*t} of f^* is $f^{*t}(x, r) = (R(t)x, r)$. Then (f^*, φ^*) is a caloric morphism defined on the domain $D \times \mathbf{R} \subset \mathbf{R} \times \mathbf{S}^*$.

Now consider the mapping $\imath : \mathbf{S}^* \to \mathbf{R}^{n+1} \backslash \{0\}$

$$y = \imath(x, r) = e^r x, \qquad \imath^{-1}(y) = \left(\frac{y}{|y|}, \log|y|\right).$$

By $\imath$, the product metric $g^* := \tilde{\sigma} + dr^2$ of $\mathbf{S}^*$, where $\tilde{\sigma}$ denotes the induced metric of $\mathbf{S}$ in $\mathbf{R}^{n+1}$, corresponds to the radial metric

$$s^{-2}(s^2\tilde{\sigma} + ds^2) = s^{-2}g_0^*$$

of $\mathbf{R}^{n+1} \backslash \{0\}$ in the polar coordinate of y-space $s = |y|$, where g_0^* is the Euclidean metric of $\mathbf{R}^{n+1}$. The mapping $\imath$ gives the isometry from the cylinder $\{(x, r) \in \mathbf{R}^{n+2}; x_1^2 + \cdots + x_{n+1}^2 = 1\}$ to the punctured space $\{y \in \mathbf{R}^{n+1}; y \neq 0\}$ with metric $|y|^{-2}g_0^*$. Then the conjugate (f^{**}, φ^{**}) of (f^*, φ^*) by $\imath^{-1}$ is given by

$$\begin{aligned} f^{**}(t, y) &= (t + d, (\imath \circ f^{*t} \circ \imath^{-1})(y)) = \left(t + d, \imath\left(f^{*t}\left(\frac{y}{|y|}, \log|y|\right)\right)\right) \\ &= \left(t + d, \imath\left(R(t)\frac{y}{|y|}, \log|y|\right)\right) = \left(t + d, e^{\log|y|}R(t)\frac{y}{|y|}\right) \\ &= (t + d, R(t)y). \end{aligned}$$

On the other hand, (f^{**}, φ^{**}) is a caloric morphism for the radial metric $s^{-2}g_0^*$. Since $n + 1 \geq 3$, this corresponds to Case 2-a of Theorem 1 of [8]:

$$f^{**}(t, y) = (t + d, ce^{at}R_0 y), \qquad \varphi^{**}(t, y) = C|y|^{(1/2)pa} \exp\left(\frac{1}{4}pa^2 t\right)$$

with $p = 1$, $c = 1$ and $a = 0$, or Case 4:

$$f^{**}(t, y) = (\lambda t + d, \nu R_0 y), \qquad \varphi^{**}(t, y) = C$$

with $\lambda = 1$ and $\nu = 1$, or Case 6:

$$f^{**}(t, y) = (t + d, R_0 y), \qquad \varphi^{**}(t, y) = C,$$

where $R_0 \in O(n+1)$. Hence $R(t)$ is a constant matrix R_0. This completes the proof. $\square$

Next we consider the case where M is the hyperbola

$$\mathbf{H} := \{x \in \mathbf{R}^{n+1}; \langle x \rangle^2 := -x_1^2 - \cdots - x_n^2 + x_{n+1}^2 = 1\}$$

in the semi-Euclidean space $\mathbf{R}^{1,n}$, which is the real hyperbolic space. Here $\mathbf{R}^{1,n}$ denotes $\mathbf{R}^{n+1}$ with semi-Euclidean metric $g_{-1}^* := -dx_1^2 - \cdots - dx_n^2 + dx_{n+1}^2$.

Proposition 3. *Let (f, φ) be a caloric morphism defined on a domain $D \subset \mathbf{R} \times \mathbf{H}$. Then*

$$f(t,x) = (t+d, R_0 x) \qquad \textit{and} \qquad \varphi(t,x) = C,$$

where $R_0 \in O(1,n)$ and d and C are constants with $C > 0$. Here

$$O(1,n) = \{R \in M(n+1, \mathbf{R}); {}^t R J_{1,n} R = J_{1,n}\} \qquad \textit{with } J_{1,n} = \begin{pmatrix} -I_n & \mathbf{0} \\ {}^t\mathbf{0} & 1 \end{pmatrix}$$

is the Lorentz group.

Proof. The proof is similar to that of Proposition 2. For the completeness, we enter into details. By Lemma 3, $f_0(t) = t + d$ and $f^t(x) = R(t)x$, where $d \in \mathbf{R}$ and $R(t) \in O(1,n)$. We need to show that $R(t)$ is constant. Consider the direct product of (f, φ) and a time translation of $\mathbf{R} \times \mathbf{R}$, i.e., let $\mathbf{H}^* = \mathbf{H} \times \mathbf{R}$ be the direct product and define the pair of the mapping $f^* : D \times \mathbf{R} \subset \mathbf{R} \times \mathbf{H}^* \to \mathbf{R} \times \mathbf{H}^*$ and the function φ^* on $D \times \mathbf{R}$ by

$$f^*(t,x,r) = (t+d, f^t(x), r) = (t+d, R(t)x, r) \qquad \text{and} \qquad \varphi^*(t,x,r) = \varphi(t,x).$$

Then (f^*, φ^*) is a caloric morphism defined on the domain $D \times \mathbf{R} \subset \mathbf{R} \times \mathbf{H}^*$. Now consider the mapping $\iota : \mathbf{H}^* \to \{y \in \mathbf{R}^{1,n}; \langle y \rangle > 0\} : y = \iota(x,r) = e^r x$, by which the product metric $g^* := \tilde{\sigma}_{-1} + dr^2$ of $\mathbf{H}^*$ corresponds to the radial metric

$$s^{-2}(s^2 \tilde{\sigma}_{-1} + ds^2) = s^{-2} g_{-1}^*$$

on $\mathbf{R}^{1,n}$ in the polar coordinate of y-space $s = \langle y \rangle$. Then the conjugate (f^{**}, φ^{**}) of (f^*, φ^*) by ι^{-1} is $f^{**}(t,y) = (t+d, R(t)y)$. On the other hand, (f^{**}, φ^{**}) is a caloric morphism for the radial metric $s^{-2} g_{-1}^*$. The assumption $n + 1 \geq 3$ implies that this corresponds to Case 2-a of Theorem 1 in [9]:

$$f^{**}(t,y) = (t+d, ce^{at} R_0 y), \qquad \varphi^{**}(t,y) = C \langle y \rangle^{(1/2)pa} \exp\left(\frac{1}{4} pa^2 t\right)$$

with $p = 1$, $c = 1$ and $a = 0$, or Case 4-a:

$$f^{**}(t, y) = (\lambda t + d, vR_0 y), \qquad \varphi^{**}(t, y) = C$$

with $\lambda = 1$ and $v = 1$, or Case 5:

$$f^{**}(t, y) = (t + d, R_0 y), \qquad \varphi^{**}(t, y) = C,$$

where $R_0 \in O(1, n)$. Therefore, $R(t)$ is a constant matrix R_0. This completes the proof. $\square$

6. Determination of caloric morphisms

Now, we state our main theorem. Here g_0 denotes the Euclidean metric on $\mathbf{R}^n$.

THEOREM 2. *Let g be a rotation invariant metric on $M = \mathbf{R}^n \backslash \{0\}$ with $n \geq 3$. If (f, φ) is a caloric morphism defined on a domain of $\mathbf{R} \times M$, then one of the following cases occurs, up to isometry of M, time translation, constant multiple of φ and the conjugate by isometry and time scaling.*

CASE (a-1). *$g = |x|^q g_0$ with some constant $q \in \mathbf{R} \backslash \{-2\}$, and*

$$f(t, x) = \left(-\frac{1}{a^2 t}, \frac{x}{|at|^{2/(q+2)}} \right), \qquad \varphi(t, x) = \frac{1}{|t|^{n/2}} \exp\left[-\frac{|x|^{q+2}}{(q+2)^2 t} \right],$$

where $a > 0$ is a constant.

CASE (a-2). *$g = |x|^{-2} g_0$, $f(t, x) = (t, ce^{at} x)$ and $\varphi(t, x) = |x|^{a/2} e^{a^2 t/4}$, where $a, c \in \mathbf{R}$ with $c > 0$.*

CASE (a-3). *$g = \rho(|x|) g_0$, where $\rho(r)$ satisfies $\rho(vr) = v^q \rho(r)$ with some constants $v > 0$ and q, and $f(t, x) = (v^{q+2} t, vx)$, $\varphi(t, x) = 1$.*

CASE (b-1). *$g = |x|^{-2} g_0$ and*

$$f(t, x) = \left(t, ce^{at} \frac{x}{|x|^2} \right), \qquad \varphi(t, x) = \frac{1}{|x|^{a/2}} e^{(1/4) a^2 t},$$

where $a, c \in \mathbf{R}$ with $c > 0$.

CASE (b-2). *$g = \rho(|x|) g_0$, where $\rho(r)$ satisfies $\rho(v/r) = (\lambda r^4 / v^2) \rho(r)$ with some positive constants v and λ, and $f(t, x) = (\lambda t, vx/|x|^2)$, $\varphi(t, x) = 1$.*

CASE (c-1). $g = \dfrac{1}{(|x|^2 + 1)^2} g_0$ *and $f(t, x) = (t, x)$, $\varphi(t, x) = 1$ (identity).*

CASE (c-2). $g = \dfrac{1}{(|x|^2 - 1)^2} g_0$ *and* $f(t,x) = (t,x)$, $\varphi(t,x) = 1$ *(identity).*

CASE (c-3). $g = g_0$ *and*

$$f(t,x) = \left(-\frac{a^2}{t}, \frac{ax}{t}\right), \qquad \varphi(t,x) = \frac{1}{(4\pi|t|)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right),$$

or

$$f(t,x) = (a^2 t, a(x + tv)), \qquad \varphi(t,x) = \exp\left(\frac{|v|^2}{4} t + \frac{1}{2} v \cdot x\right),$$

where $a \neq 0$ *is a constant and* $v \in \mathbf{R}^n$, *and where* $v \cdot x$ *denotes the usual inner product of* v *and* x *in* $\mathbf{R}^n$.

PROOF. Our starting point is Theorem 1. We have three cases (a), (b) and (c) in Theorem 1. The cases (a) and (b) in Theorem 1 have already been treated in the paper [8] of one of the authors. Thus the remaining case is (c). In the case $q = 0$ in (c), the metric $g = |x|^{-4} g_0$ is isometric to the Euclidean metric g_0 by inversion $x \mapsto |x|^{-2} x$. The Euclidean case was solved by Leutwiler [5].

After the following preparation, the case $q \neq 0$ is reduced to Proposition 2 or 3 in §5 according as $q > 0$ or $q < 0$. In the case $q > 0$ in (c), the metric $g = (|x|^2 + q)^{-2} g_0$ is the metric induced from the sphere $\mathbf{S}_q := \{y_1^2 + \cdots + y_{n+1}^2 = 1/4q\}$ in the Euclidean space $\mathbf{R}^{n+1}$ by the mapping

$$\iota_s : x \mapsto \left(\frac{\sqrt{q}}{q + |x|^2} x, \frac{(-q + |x|^2)}{2\sqrt{q}(q + |x|^2)}\right) : M \to \mathbf{S}_q.$$

In the case $q < 0$ in (c), the manifold (M, g) is isometric to the hyperbola $\mathbf{H}_q := \{-y_1^2 - \cdots - y_n^2 + y_{n+1}^2 = 1/4q\}$ in the semi-Euclidean space $\mathbf{R}^{1,n}$ by the mapping

$$\iota_h : x \mapsto \left(\frac{\sqrt{|q|}}{|q| - |x|^2} x, \frac{(|q| + |x|^2)}{2\sqrt{|q|}(|q| - |x|^2)}\right) : M \to \mathbf{H}_q.$$

In both cases in the above, by the conjugate by time scaling in Proposition 1, we can choose $|q| = 1/4$ (or $|q| = 1$). □

Note that semi-Riemannian metrics actually appear in the case $q < 0$.

As is remarked in §3, every rotation invariant metric is represented as a radial metric. For reference, we restate our main theorem in radial metric form without any normalization.

THEOREM 3. *Let* $g = \rho(|x|)g_0$ *be a radial metric on* $M = \mathbf{R}^n \backslash \{0\}$ *with* $n \geq 3$. *If* (f, φ) *is a non-trivial caloric morphism defined on a domain of* $\mathbf{R} \times M$, *then one of the following cases occurs:*

CASE (a-1). $\rho(r) = pr^q$ *with some constant* $q \in \mathbf{R}\backslash\{-2\}$, $p > 0$, *and*

$$f(t,x) = \left(\frac{ct+d}{at+b}, \frac{R_0 x}{|at+b|^{2/(q+2)}} \right),$$

$$\varphi(t,x) = \frac{C}{|at+b|^{n/2}} \exp\left[-\frac{pa|x|^{q+2}}{(q+2)^2(at+b)} \right],$$

where constants $a, b, c, d, C \in \mathbf{R}$ *satisfy* $bc - ad = 1$, $C > 0$ *and* $R_0 \in O(n)$.

CASE (a-2). $\rho(r) = pr^{-2}$ *with some constant* $p > 0$ *and*

$$f(t,x) = (t+b, ce^{at}R_0 x), \qquad \varphi(t,x) = C|x|^{pa/2} e^{pa^2 t/4}.$$

CASE (a-3). $\rho(r)$ *satisfies* $\rho(vr) = v^q \rho(r)$ *with some constants* $v > 0$, $q \in \mathbf{R}$, *and*

$$f(t,x) = (v^{q+2}t + d, vR_0 x), \qquad \varphi(t,x) = C,$$

where $C > 0$, $d \in \mathbf{R}$ *and* $R_0 \in O(n)$.

CASE (b-1). $\rho(r) = pr^{-2}$ *with some constant* $p > 0$ *and*

$$f(t,x) = \left(t+d, ce^{at} \frac{R_0 x}{|x|^2} \right), \qquad \varphi(t,x) = C \frac{1}{|x|^{pa/2}} e^{pa^2 t/4}$$

where $a, c, d, C \in \mathbf{R}$ *with* $c, C > 0$ *and* $R_0 \in O(n)$.

CASE (b-2). $\rho(r)$ *satisfies* $\rho(v/r) = (\lambda r^4/v^2)\rho(r)$ *with some positive constants* v, λ, *and*

$$f(t,x) = \left(\lambda t + d, \frac{vR_0 x}{|x|^2} \right), \qquad \varphi(t,x) = C$$

where $C > 0$, $d \in \mathbf{R}$ *and* $R_0 \in O(n)$.

CASE (c-1). $\rho(r) = p(r^2+q)^{-2}$ *with* $q > 0$, *and* (f, φ) *is a caloric morphism induced by a time translation and an isometry* F_0 *which is a composition of inversions:*

$$f(t,x) = (t+d, F_0(x)), \qquad \varphi(t,x) = C$$

where $C > 0$ *and* $d \in \mathbf{R}$.

Case (c-2). *$\rho(r) = p(r^2 + q)^{-2}$ with $q < 0$, and (f, φ) is a caloric morphism induced by a time translation and an isometry F_0 which is a composition of inversions:*

$$f(t,x) = (t + d, F_0(x)), \qquad \varphi(t,x) = C$$

where $C > 0$ and $d \in \mathbf{R}$.

Case (c-3). *$g = pg_0$ and*

$$f(t,x) = \left(\frac{ct+d}{at+b}, \frac{R_0(x+tv+w)}{at+b}\right),$$

$$\varphi(t,x) = \begin{cases} \dfrac{C}{|at+b|^{n/2}} \exp\left[-\dfrac{p|aR_0x + aw - bv|^2}{4a(at+b)}\right], & a \neq 0, \\ C \exp\left[p\left(\dfrac{|v|^2}{4}t + \dfrac{1}{2}v \cdot R_0x\right)\right], & a = 0, \end{cases}$$

where constants $a, b, c, d, C \in \mathbf{R}$ satisfy $bc - ad = 1$, $C > 0$, and $v, w \in \mathbf{R}^n$, $R_0 \in O(n)$.

Finally, we remark that when $n = 2$, we have not determined the caloric morphism for all rotation invariant metrics.

Acknowledgments

The authors would like to express our gratitude to the referee and the editors for their valuable comments.

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