Hiroshima Math. J.
40 (2010), 75–99

On the fundamental group of the complement of linear torus curves of maximal contact

Masayuki Kawashima

(Received February 5, 2009)
(Revised October 7, 2009)

Abstract. In this paper, we compute the fundamental group of the complement of linear torus curves of maximal contact and we show that it is isomorphic to that of generic linear torus curves. As an application, we give new two Zariski triples.

1. Introduction

Let C be a curve of degree d in $\mathbf{P}^2$. We are interested in two important invariants of C: the Alexander polynomial $\Delta_C(t)$ and the fundamental group of the complement $\pi_1(\mathbf{P}^2 \backslash C)$. A plane curve $C \subset \mathbf{P}^2$ is called a *curve of* (p, q) *torus type* with $p > q \geq 2$, if p, q are positive integers that divide d and there is a defining polynomial F of C of the form $F(X, Y, Z) = F_{d/q}(X, Y, Z)^q - F_{d/p}(X, Y, Z)^p$ where $F_{d/q}$, $F_{d/p}$ are homogeneous polynomials of X, Y, Z of degree d/q and d/p respectively. This is an important class of plane curves of degree d. For a given curve of torus type, we consider the intersection locus $\{F_{d/q} = F_{d/p} = 0\} = \{P_1, \dots, P_k\}$ and the local intersection numbers $n_j := I(F_{d/q}, F_{d/p}; P_j)$ for $j = 1, \dots, k$. By the Bézout theorem, we have the equality $\sum_{i=1}^k n_i = \frac{d^2}{pq}$. We call $\mathscr{I} = \{n_1, \dots, n_k\}$ *the intersection partition* of C. Consider the pencil $C(\tau)$:

$$C(\tau) = \{F(X, Y, Z, \tau) = \tau F_{d/q}(X, Y, Z)^q - (1 - \tau) F_{d/p}(X, Y, Z)^p = 0\}, \quad \tau \in \mathbf{C}.$$

We assume that the curve $\{F_{d/q} = 0\}$ is non-singular at each P_j for $j = 1, \dots, k$. A singular point $P \in C$ is called *inner* if $P \in \{F_{d/q} = F_{d/p} = 0\}$. Otherwise, P is called an *outer singularity*. We say that C is a *tame* torus curve if C has no outer singularities. By the Bertini theorem (p. 137 in [3]), $C(\tau)$ is a tame curve for a generic τ (namely except for a finite number of exceptional values of $\tau's$) and the topology of $(\mathbf{P}^2, C(\tau))$ does not depend on the particular choice of a generic τ.

2000 *Mathematics Subject Classification*. Primary 14HA30; Secondary 14H20, 14H45.
Key words and phrases. Torus curve, maximal contact.

We say that a curve C of (p,q) torus type is *a torus curve of a maximal contact* if $\{F_{d/q}=F_{d/p}=0\}=\{\xi_0\}$ and $\{F_{d/q}=0\}$ is smooth at ξ_0. In this case, the singularity (C,ξ_0) is topologically equivalent to the Brieskorn-Pham singularity $B_{d^2/q,q}$ where we use the notation $B_{m,n}:=\{(x,y)\in\mathbf{C}^2 \mid x^m+y^n=0\}$.

Now we are interested in the Alexander polynomial $\Delta_C(t)$ and the fundamental group $\pi_1(\mathbf{P}^2\backslash C)$. These two invariants are difficult to compute in general but there is a convenient criterion for their computation. Suppose there is a family of reduced curves $C(s)$, $s\in U\subset\mathbf{C}$ of degree d (U is an open neighborhood of the origin) such that for $s\neq 0$, the topology of $C(s)$ is independent of s but $C(0)$ has a bigger singularities. (See [7] for the definition.) Then we have a degeneration principle: $\Delta_{C(s)}(t)\mid\Delta_{C(0)}(t)$ *and* $\pi_1(\mathbf{P}^2\backslash C(0))$ *is mapped surjectively onto* $\pi_1(\mathbf{P}^2\backslash C(s))$.

Let $\mathcal{M}(p,q;d,\mathcal{I})$ be the space of pairs of polynomials $(F_{d/q},F_{d/p})$ such that the intersection partition of $\{F_{d/q}=F_{d/p}=0\}$ is equal to $\mathcal{I}=\{n_1,\dots,n_k\}$ and the curve $\{F_{d/q}=0\}$ is smooth at each intersection points. To such a pair, we associate a generic torus curve

$$C(\tau)=\{\tau F_{d/q}(X,Y,Z)^q-(1-\tau)F_{d/p}(X,Y,Z)^p=0\},\qquad \tau\in\mathbf{C}.$$

This moduli space $\mathcal{M}(p,q;d,\mathcal{I})$ has a canonical topology and a structure of an algebraic variety. (In fact, let $P(n)$ be the affine space of the homogeneous polynomials of degree n in three variables X, Y, Z. Then we can identify $C(\tau)$ as a point $(F_{d/q},F_{d/p},\tau)\in P(d/q)\times P(d/p)\times\mathbf{C}$ so that the moduli space can be considered as an algebraic subset of $P(d/q)\times P(d/p)\times\mathbf{C}$.) Putting the degeneration principle into the consideration, we have the following basic problems.

(1) Is $\mathcal{M}(p,q;d,\mathcal{I})$ connected? (Or equivalently is the corresponding moduli space irreducible?)

(2) For a given two partition $\mathcal{I}$, $\mathcal{I}'$ such that $\mathcal{I}'$ is a finer partition than $\mathcal{I}$, and two generic curves $C\in\mathcal{M}(p,q;d,\mathcal{I})$ and $C'\in\mathcal{M}(p,q;d,\mathcal{I}')$, is there a degeneration family $C(s)$, $s\in U$ such that $C(s)=C'$ and $C(0)=C$?

(3) (Sandwich principle) Let C be a generic curve in $\mathcal{M}(p,q;d,\mathcal{I})$. We consider two particular partitions: $\mathcal{I}_g=\{1,\dots,1\}$ and $\mathcal{I}_m=\{d^2/pq\}$. Are there families of degenerations $C(s)$, $s\in U$ and $D(s)$, $s\in U$ with the following properties?

(a) $C(s)\in\mathcal{M}(p,q;d,\mathcal{I}_g)$ for $s\neq 0$ and $C(0)=C$.

(b) $D(s)\in\mathcal{M}(p,q;d,\mathcal{I})$ for $s\neq 0$, $D(1)=C$ and $D(0)\in\mathcal{M}(p,q;d,\mathcal{I}_m)$.

(c) Let $C_g=C(1)\in\mathcal{M}(p,q;d,\mathcal{I}_g)$ and $C_m=D(0)\in\mathcal{M}(p,q;d,\mathcal{I}_m)$.

$$\Delta_{C_g}(t)=\Delta_{C_m}(t),\qquad \pi_1(\mathbf{P}^2\backslash C_g)\cong\pi_1(\mathbf{P}^2\backslash C_m).$$

If these properties are satisfied, we have

$$\Delta_C(t) = \Delta_{C_g}(t), \qquad \pi_1(\mathbf{P}^2 \backslash C) \cong \pi_1(\mathbf{P}^2 \backslash C_g).$$

We call the above two partitions $\mathscr{I}_g$ and $\mathscr{I}_m$ are called *the generic partition* and *the maximal partition*. The generic partition $\mathscr{I}_g$ means geometrically that the associated curves intersect transversely at each intersection point. On the other hand, the maximal partition $\mathscr{I}_m$ means that the associated curves intersect only one point.

So far, there exist very few known results. In fact, we only know the following.

- For the generic partition $\mathscr{I}_g$, the moduli space $\mathscr{M}(p,q;d,\mathscr{I}_g)$ is irreducible and for a generic $C \in \mathscr{M}(p,q;d,\mathscr{I}_g)$,

$$\Delta_C(t) = \frac{(t^{pq/r}-1)^r(t-1)}{(t^p-1)(t^q-1)}$$

where $r = \gcd(p,q)$ and the fundamental group is given by:

$$\pi_1(\mathbf{P}^2 \backslash C) \cong G(p,q,d/p).$$

The group $G(p,q,d/p)$ was introduced in [5]. It is known that $G(p,q,d/p)$ has a cyclic group $\mathscr{Z} := \mathbf{Z}/a\mathbf{Z}$, $a = \frac{dr}{pq}$ as the center and the quotient group $G(p,q,d/p)/\mathscr{Z}$ is isomorphic to $(\mathbf{Z}/(p/r)\mathbf{Z}) * (\mathbf{Z}/(q/r)\mathbf{Z}) * F(r-1)$ where $F(n)$ is the free group of rank n (Oka [5]. See also [1, 2]).

- For the case of curves of $(3,2)$ torus type of degree 6, the moduli spaces $\mathscr{M}(3,2;6,\mathscr{I})$ are irreducible for any intersection partition $\mathscr{I}$ and the above properties (1), (2) and (3) hold true and we have isomorphisms:

$$\Delta_C(t) = t^2 - t + 1, \qquad \pi_1(\mathbf{P}^2 \backslash C) \cong \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$$

for any generic $C \in \mathscr{M}(3,2;6,\mathscr{I})$ ([9]).

- Let $\mathscr{N}_{d/p}^{d/p}(p,q;d)$ be the subspace of $\mathscr{M}(p,q;d,\mathscr{I}_m)$ defined by the following: $(F_{d/q}, F_{d/p}) \in \mathscr{N}_{d/p}^{d/p}(p,q;d)$ if and only if $(F_{d/q}, F_{d/p}) \in \mathscr{M}(p,q;d,\mathscr{I}_m)$ and $I(T_{\xi_0}, F_{d/p}; \xi_0) = d/p$ where ξ_0 is the intersection point $\{F_{d/q} = F_{d/p} = 0\}$ and $\{F_{d/p} = 0\}$ is smooth at ξ_0 and T_{ξ_0} is the tangent line of $\{F_{d/p} = 0\}$ at ξ_0 ([1]). The moduli space $\mathscr{N}_{d/p}^{d/p}(p,q;d)$ is irreducible and the normal forms are explicitly obtained (Lemma 2 of [1]). Take a generic curve $C \in \mathscr{N}_{d/p}^{d/p}(p,q;d)$. Then

$$\Delta_C(t) = \frac{(t^{pq/r}-1)^r(t-1)}{(t^p-1)(t^q-1)}.$$

A typical such curve can be

$$C : (y^{d/q} + y - x^{d/p})^q + \tau(y - x^{d/p})^p = 0, \qquad \tau \in \mathbf{C}^*.$$

It is also shown that the moduli space $\mathcal{M}(d/2, q; d, \mathcal{I}_m) = \mathcal{N}_2^2(d/2, q; d)$ is irreducible where q divides d (Lemma 2 of [1]).

We can ask the next question: *Is $\mathcal{M}(p, q; d, \mathcal{I}_m)$ is irreducible?* If this is true, the above result of [1] determines the Alexander polynomial $\Delta_C(t)$ for any generic $C \in \mathcal{M}(p, q; d, \mathcal{I}_m)$.

In this paper, we consider the following special class of torus curves of torus type (pq, q) and of degree pq. The defining polynomial of C in the affine coordinates takes the following form:

$$C : f(x, y) = f_p(x, y)^q - \ell(x, y)^{pq} = 0$$

where $\ell(x, y)$ is a linear form. We say such a curve C *a linear torus curve of type* (pq, q). We associate to C the following two curves $C_p := \{f_p = 0\}$ and $L := \{\ell = 0\}$. If C is a linear torus curve of type (pq, q), then C generically consists of q smooth irreducible curves of degree p as

$$f = \prod_{j=1}^{q}(f_p - \zeta^j \ell^p), \qquad \text{where } \zeta := \exp\left(\frac{2\pi\sqrt{-1}}{q}\right)$$

and the inner singularities of C are situated at the intersection $C_p \cap L$. For the generic partition $\mathcal{I}_g$ and $C \in \mathcal{M}(pq, q; pq, \mathcal{I}_g)$, we call C *a generic linear torus curve*. If C is a generic linear torus curve of type (pq, q), the fundamental group is given by

$$\pi_1(\mathbf{P}^2 \backslash C) \cong F(q-1) * \mathbf{Z}/p\mathbf{Z}$$

and the Alexander polynomial $\Delta_C(t)$ is given by ([5, 1])

$$\Delta_C(t) = \frac{(t^{pq} - 1)^{q-1}(t-1)}{t^q - 1}.$$

Let C be a tame (pq, q) linear torus curve of a maximal contact with degree pq. Then C has q components of degree p which intersect at one point with intersection multiplicity p^2 each other. In this paper, we compute that fundamental groups of $\mathbf{P}^2 \backslash C$ and $\mathbf{C}^2 \backslash C$ and also the Alexander polynomial $\Delta_C(t)$.

Our main result is the following:

THEOREM 1. *Let C be a tame (pq, q) linear torus curve of a maximal contact. Then the fundamental group $\pi_1(\mathbf{P}^2 \backslash C)$ is isomorphic to that of generic linear torus curves. Namely*

$$\pi_1(\mathbf{C}^2 \backslash C) \cong \langle g_1, \dots, g_q, \omega \mid \omega = g_1 \dots g_q, [g_j, \omega^p] = e, j = 1, \dots, q \rangle$$

$$\pi_1(\mathbf{P}^2 \backslash C) \cong \langle g_1, \dots, g_q, \omega \mid \omega^p = e, \omega = g_1 \dots g_q \rangle \cong F(q-1) * \mathbf{Z}/p\mathbf{Z}$$

where $[g_j, \omega^p]$ is the commutator of g_j and ω^p. The Alexander polynomial $\Delta_C(t)$ is equal to that of the generic curve. Namely it is given by the following:

$$\Delta_C(t) = \frac{(t^{pq}-1)^{q-1}(t-1)}{t^q - 1}.$$

We also show the irreducibility of the moduli space $\mathcal{M}(pq, q; pq, \mathcal{I})$ for an arbitrary intersection partition $\mathcal{I}$ (Proposition 2). Thus one of the important application of Theorem 1 is the following.

Corollary 1. *Let C be a generic curve in $\mathcal{M}(pq, q; pq, \mathcal{I})$ for an arbitrary partition $\mathcal{I}$. Then the fundamental group $\pi_1(\mathbf{P}^2 \backslash C)$ is isomorphic to $F(q-1) * \mathbf{Z}/p\mathbf{Z}$ and the Alexander polynomial $\Delta_C(t)$ is given by the following:*

$$\Delta_C(t) = \frac{(t^{pq}-1)^{q-1}(t-1)}{t^q - 1}.$$

As a second application, we will give new two Zariski triples. See §4.

2. Preliminaries

2.1. Van Kampen-Zariski Pencil method. Let C be a reduced plane curve of degree d in $\mathbf{P}^2$. To compute the fundamental groups $\pi_1(\mathbf{P}^2 \backslash C)$ and $\pi_1(\mathbf{C}^2 \backslash C)$, we use the so-called *van Kampen-Zariski pencil method*. We recall it briefly in the following ([7]). We fix a point $B_0 \in \mathbf{P}^2 \backslash C$ and we consider the set of lines $\mathcal{L} = \{L_s \mid s \in \mathbf{P}^1\}$ through B_0 and $\mathcal{L}$ is called a *pencil*. Taking a linear change of coordinates if necessary, we may assume that $B_0 = [1:0:0]$ and L_s is defined by $L_s = \{Y - sZ = 0\}$ in $\mathbf{P}^2$ where (X, Y, Z) is the fixed homogeneous coordinates. Take $L_\infty = \{Z = 0\}$ as the line at infinity and assume that L_∞ intersects transversely C. We consider the affine coordinates $(x, y) = (X/Z, Y/Z)$ on $\mathbf{C}^2 = \mathbf{P}^2 - L_\infty$. Let $F(X, Y, Z)$ be the defining homogeneous polynomial of C and let $f(x, y) = F(x, y, 1)$ be the affine equation of C. We use the following notations:

$$C^a = C \cap \mathbf{C}^2, \qquad L_s^a = L_s \cap \mathbf{C}^2.$$

We identify L_s and L_s^a with $\mathbf{P}^1$ and $\mathbf{C}$ respectively and the pencil line L_s^a is defined by $\{y = s\}$ in the affine coordinates (x, y). We use x as the coordinates of L_s^a.

A pencil line L_s is called *singular with respect to* C if L_s passes through a singular point of C or L_s is tangent to C. Otherwise, we call L_s is *generic*. Hereafter we assume that L_∞ is generic and B_0 is not contained in C.

Let $\mathbf{C}_y$ be the space of the parameters of the pencil with coordinates y and let $\Sigma = \{s \in \mathbf{C}_y \mid L_s$ is a singular pencil line$\}$ and suppose that $\Sigma = \{s_1, \dots, s_k\} \subset \mathbf{C}_y$. We fix a generic pencil line L_{s_0} (so $s_0 \in \mathbf{C}_y \backslash \Sigma$) and put $L_{s_0}^a \cap C^a = \{Q_1, \dots, Q_d\}$ where d is the degree of C. We take a base point $*_0 \in L_{s_0}^a \backslash L_{s_0}^a \cap C^a$ on the real axis that is sufficiently near to B_0 and $*_0 \neq B_0$. We take a large disk $\Delta_R \subset L_{s_0}^a$ such that $L_{s_0}^a \cap C^a \subset \Delta_R$ and $*_0 \notin \Delta_R$. We may assume that $\Delta_R = \{(x, s_0) \in L_{s_0}^a \mid |x| \leq R\}$ with a sufficient large R. We orient the boundary of Δ_R counter-clockwise and we put $\Xi = \partial \Delta_R$. Join the circle Ξ to the base point by a line segment L connecting $*_0$ and Ξ along the real axis. Let Ω be the class of this loop $L \circ \Xi \circ L^{-1}$ in $\pi_1(L_{s_0}^a \backslash L_{s_0}^a \cap C; *_0)$. We take free generators $g_1, \dots, g_d$ of $\pi_1(L_{s_0}^a \backslash L_{s_0}^a \cap C; *_0)$ so that g_i goes around Q_i counter-clockwise along a small circle and we assume that $\omega = g_d \dots g_1$, taking a suitable ordering of $g_1, \dots, g_d$ if necessary.

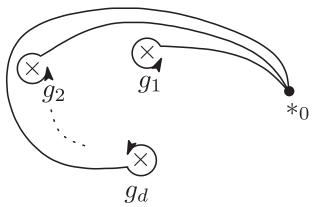


Fig. 1. $L_{s_0} \cap C$

Hereafter we denote a small lasso oriented in the counter clockwise direction by a bullet with a path in the following figures. Thus $\longrightarrow\!\!\bullet$ indicates $\longrightarrow\!\!\circlearrowleft$.

The fundamental group $\pi_1(\mathbf{C}_y \backslash \Sigma; s_0)$ acts on $\pi_1(L_{s_0}^a \backslash L_{s_0}^a \cap C; *_0)$. We call this action *the monodromy action* of $\pi_1(\mathbf{C}_y \backslash \Sigma; s_0)$. For details, we refer to [7, 6]. Note that $\pi_1(L_{s_0}^a \backslash L_{s_0}^a \cap C; *_0)$ is a free group of rank d with generators $g_1, \dots, g_d$. The result of the action of $\sigma \in \pi_1(\mathbf{C}_y \backslash \Sigma; s_0)$ on $g \in \pi_1(L_{s_0}^a \backslash L_{s_0}^a \cap C; *_0)$ is denoted by g^σ.

Let $\mathscr{M}$ be the normal subgroup of $\pi_1(L_{s_0}^a \backslash L_{s_0}^a \cap C; *_0)$ that is normally generated by

$$\mathscr{R} = \{g^{-1} g^\sigma \mid g \in \pi_1(L_{s_0}^a \backslash L_{s_0}^a \cap C; *_0), \qquad \sigma \in \pi_1(\mathbf{C}_y \backslash \Sigma; s_0)\}$$

and we call $\mathscr{M}$ the *group of the monodromy relations*. Put

$$\mathscr{M}(\sigma_i) = \{g_j^{-1} g_j^{\sigma_i} \mid j = 1, \dots, d\}.$$

Then it is easy to see that the group $\mathscr{M}$ is normally generated by $\bigcup_{j=1}^{k} \mathscr{M}(\sigma_i)$. By the definition, we have the relation

$$g_j = g_j^{\sigma_i} \qquad R(\sigma_i)$$

in the quotient group $\pi_1(L_{s_0}^a \backslash L_{s_0}^a \cap C; *_0)/\mathscr{M}$. We call $R(\sigma_i)$ *the monodromy relation for* σ_i. Let $j : L_{s_0}^a \backslash L_{s_0}^a \cap C \to \mathbf{C}^2 \backslash C^a$ and $\imath : \mathbf{C}^2 \backslash C^a \to \mathbf{P}^2 \backslash C$ be the respective inclusions.

Proposition 1 ([12, 11, 10]). *Under the above situations, the following hold.*

(1) *The canonical homomorphism* $j_\# : \pi_1(L_{s_0}^a \backslash L_{s_0}^a \cap C; *_0) \to \pi_1(\mathbf{C}^2 \backslash C^a; *_0)$ *is surjective and the kernel* Ker $j_\#$ *is equal to* $\mathscr{M}$. *Thus we have the isomorphism:*

$$\pi_1(\mathbf{C}^2 \backslash C^a; *_0) \cong \pi_1(L_{s_0}^a \backslash L_{s_0}^a \cap C; *_0)/\mathscr{M}.$$

(2) ([4]) *The canonical homomorphism* $\imath_\# : \pi_1(\mathbf{C}^2 \backslash C^a; *_0) \to \pi_1(\mathbf{P}^2 \backslash C; *_0)$ *is surjective and the kernel* Ker $\imath_\#$ *is generated by a single element* $\omega = g_d \dots g_1$ *which is in the center of* $\pi_1(\mathbf{C}^2 \backslash C^a)$ *and* Ker $\imath_\# = \langle \omega \rangle \cong \mathbf{Z}$. *Thus we have an isomorphism*

$$\pi_1(\mathbf{P}^2 \backslash C; *_0) \cong \pi_1(\mathbf{C}^2 \backslash C^a; *_0)/\langle \omega \rangle$$

3. Proof of Theorem 1

Let (x, y) be affine coordinates such that $x = X/Z, y = Y/Z$ on $\mathbf{C}^2 := \mathbf{P}^2 \backslash \{Z = 0\}$.

3.1. Construction of curves. In this section, we construct a linear torus curve C of a maximal contact and investigate its local properties. First we introduce a plane curve $D_\alpha = \{g_\alpha(x, y) = 0\}$ of degree p where the defining polynomial $g_\alpha(x, y)$ is defined by

$$g_\alpha(u, y) = u - \psi(y, \alpha), \qquad \psi(y, \alpha) = y - \alpha y^p, \qquad \alpha \in \mathbf{C}^*.$$

Now we consider the p-fold cyclic covering ([6]) defined by

$$\varphi_p : \mathbf{C}^2 \to \mathbf{C}^2, \qquad \varphi_p(x, y) = (u, y), \qquad u = x^p.$$

To distinguish two affine planes, we denote the source space of φ_p by $\mathbf{C}_s^2$ with coordinates (x, y) and the target space of φ_p by $\mathbf{C}_t^2$ with coordinates (u, y). Hereafter we simply denote $\mathbf{C}^2$ instead of $\mathbf{C}_s^2$.

Let $C_\alpha := \varphi^{-1}(D_\alpha)$ be the pull-back of D_α by φ_p and let $f_\alpha(x, y) = g_\alpha(x^p, y)$ be the defining polynomial of C_α. Note that

$$f_\alpha(x, y) = x^p - \psi(y, \alpha).$$

By the defining equation of C_α, we see that the set of parameters that correspond to the singular pencil lines for C_α is given by

$$\Sigma_\alpha := \{y \in \mathbf{C}_y \mid \psi(y, \alpha) = 0\}$$

(cf. [5]). Fix a complex number γ such that $\gamma^{p-1} = 1/\alpha$. Then we factorize $\psi(y, \alpha)$ as follows:

$$\psi(y, \alpha) = \alpha y \prod_{k=0}^{p-2} (\gamma \xi^k - y), \qquad \xi := \exp\left(\frac{2\pi\sqrt{-1}}{p-1}\right).$$

Then we can see that $O = (0, 0)$ and $Q_k = (0, \gamma\xi^k)$ for $k = 0, \dots, p-2$ are flex points of C_α of flex order $p-2$ and their tangent lines are nothing but the singular pencil lines through these points and they are given by $y = 0$ and $y = \gamma\xi^k$ respectively.

Now we are ready to define a reduced curve C. Take q non-zero mutually distinct complex numbers $\alpha_1, \dots, \alpha_q$ and put $D_j = \{g_{\alpha_j}(x, y) = 0\}$ for $j = 1, \dots, q$ and put $D = \bigcup_{i=1}^{q} D_j$. Then put $C_j = \varphi_p^{-1}(D_j)$ for $= 1, \dots, q$ and finally we define

$$C = \varphi_p^{-1}(D) = C_1 \cup \cdots \cup C_q.$$

The defining polynomials $f_j(x, y)$ and $f(x, y)$ of C_j and C respectively are given as follows.

$$f_j(x, y) = x^p - \psi(y, \alpha_j), \qquad f(x, y) = \prod_{j=1}^{q} f_j(x, y).$$

Put $U = \{(\alpha_1, \dots, \alpha_q) \in \mathbf{C}^{*q} \mid \alpha_i \neq \alpha_j, \text{ for any } i \neq j\}$. It is known that the embedded topology of $C \subset \mathbf{C}^2$ does not depend on the choice of $(\alpha_1, \dots, \alpha_q) \in U$ (see [2]).

LEMMA 1. *The reduced curve C can be a (pq, q) linear torus curve of a maximal contact for a certain choice of $(\alpha_1, \dots, \alpha_q)$.*

PROOF. We take $(\alpha_1, \dots, \alpha_q) = (1, \zeta, \dots, \zeta^{p-1}) \in U$, then we claim that $C = \{f(x, y) = 0\}$ is a (pq, q)-linear torus curve of a maximal contact. Indeed, $f(x, y)$ takes the form:

$$\begin{aligned} f(x, y) &= \prod_{j=1}^{q} (\zeta^{j-1} y^p - y + x^p) \\ &= (y^p)^q - (y - x^p)^q \\ &= y^{pq} - (y - x^p)^q. \end{aligned}$$

This expression shows that C is a (pq, q) linear torus curve of a maximal contact. □

For practical computations, we suppose hereafter that $\alpha_1, \dots, \alpha_q$ are real numbers such that $\alpha_1 > \cdots > \alpha_q > 0$. Let γ_j be a real positive number such that $\gamma_j^{p-1} = 1/\alpha_j$ for $j = 1, \dots, q$. By the assumption $\alpha_1 > \cdots > \alpha_q > 0$, we have

$$0 < \gamma_1 < \cdots < \gamma_q.$$

As $C_j \cap C_i = \{O\}$ for any $j \neq i$, the possible singular pencil $L_s = \{y = s\}$ is either $\{y = 0\}$ or L_s is tangent to one of C_j outside of O.

Lemma 2. *Under the above situation, the local data of C for the calculation of the fundamental group of $\mathbf{P}^2 \backslash C$ is the following.*

(1) *Singular pencil lines are $y = 0$ and $y = \gamma_j \xi^k$ for $j = 1, \dots, q$ and $k = 0, \dots, p-2$. The pencil lines $y = \gamma_j \xi^k$ is tangent to C_j at $Q_{j,k} := (0, \gamma_j \xi^k)$.*

(2) *Two curves C_j and C_i $(j \neq i)$ intersect only at $O \in \mathbf{C}^2$ and $I(C_j, C_i; O) = p^2$.*

(3) *The singularity type C at O is given by $(C, O) \sim B_{p^2 q, q}$.*

3.2. Calculation of the fundamental group $\pi_1(\mathbf{P}^2 \backslash C)$ and $\pi_1(\mathbf{C}^2 \backslash C)$. For the calculations of the fundamental groups $\pi_1(\mathbf{P}^2 \backslash C)$ and $\pi_1(\mathbf{C}^2 \backslash C)$, we use the van Kampen-Zariski pencil method. We take the base point $B_0 = [1 : 0 : 0]$ in $\mathbf{P}^2$ and consider the pencil $\mathscr{L} = \{L_s \mid s \in \mathbf{C}\}$ through B_0 with $L_s = \{Y = sZ\}$. The line at infinity L_∞ is given by $\{Z = 0\}$. Then L_∞ is generic with respect to C. Affine pencil is $\mathscr{L}^a = \{L_s^a\}_{s \in \mathbf{C}}$ with $L_s^a = \{y = s\}$. (By abuse of notation, we consider this pencil $\mathscr{L} = \{L_s \mid s \in \mathbf{C}\}$ in $\mathbf{C}_t^2$ and $\mathbf{C}_s^2$.) By Lemma 2, the set $\Sigma \subset \mathbf{C}_y$ of parameters that correspond to singular pencil lines for C is given as follows:

$$\Sigma := \{0, \gamma_j \xi^k \in \mathbf{C}_y \mid k = 0, \dots, p-2, j = 1, \dots, q\}.$$

Take the base point γ_0 of $\mathbf{C}_y \backslash \Sigma$ on the real axis so that $0 < \gamma_0 < \gamma_1$. As

$$\psi(\gamma_0, \alpha_j) - \psi(\gamma_0, \alpha_i) = (\alpha_i - \alpha_j)\gamma_0^p > 0 \qquad \text{if } i < j,$$

we have

$$0 < \psi(\gamma_0, \alpha_1) < \psi(\gamma_0, \alpha_2) < \cdots < \psi(\gamma_0, \alpha_q).$$

We take the base point $*_0 = (\tau_0, \gamma_0)$ where τ_0 is a sufficiently large positive number. As C is the pull-back of D by the p-fold cyclic covering $\varphi_p : (x, y) \mapsto (x^p, y)$, the monodromy relations for $\pi_1(L_{\gamma_0}^a \backslash L_{\gamma_0}^a \cap C; *_0)$ are essentially obtained by taking lifting the monodromy relations for $\pi_1(L_{\gamma_0}^a \backslash (L_{\gamma_0}^a \cap D) \cup \{0\}; *_t)$ by φ_p

where the base point $*_t$ is a real point defined by $*_t = (\tau_0^p, \gamma_0)$ ([6]). This is the basic idea for the computation of the fundamental groups.

We first take loops $b_1, \dots, b_q$ of $\pi_1(L_{\gamma_0}^a \backslash (L_{\gamma_0}^a \cap D) \cup \{0\}, *_t)$ and put $\tau := b_1 \dots b_q$ as in Figure 2.

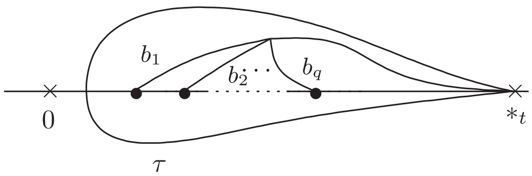


Fig. 2. The loops $b_1, \dots, b_q$ in $\{y = \gamma_0\} \cap \mathbf{C}_t^2$

Let $a'_{i,j}$ and ω'_i be the pull-back of b_j and τ by φ_p respectively starting from $*_i := (\eta^i \tau_0, \gamma_0)$ with $i = 0, \dots, p-1$ where $\eta := \exp(2\pi\sqrt{-1}/p)$ and let $a_{i,j}$ and ω_i be the loop $\ell_i \circ a'_{i,j} \circ \ell_i^{-1}$ and $\ell_i \circ \omega'_i \circ \ell_i^{-1}$ where ℓ_i is the arc of the circle $|x| = \tau_0$ from $*_0$ to $*_i$ as in Figure 3. Hereafter we identify $a'_{i,j}$ and $a_{i,j}$ in this way.

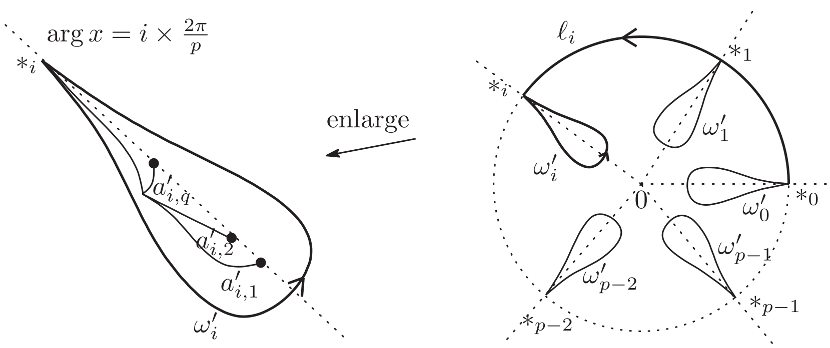


Fig. 3. The loops $a_{i,j}$ and ω_i in $\{y = \gamma_0\} \cap \mathbf{C}_s^2$

First we see the monodromy relations on the real axis in $\mathbf{C}_y$ that correspond to singular pencil lines $y = 0$ and $y = \gamma_j$ for $j = 1, \dots, q$. To see these monodromy relations, we consider following loops σ_0 and σ_j in $\mathbf{C}_y$ for $j = 1, \dots, q$. First we define the loop σ_0. Let K_0 be the line segment from γ_0 to $0 - \varepsilon$ on the real axis and let S_0 be the circle $|y| = \varepsilon$ where the circle is always oriented counter-clockwise. Then σ_0 is defined as the loop (see Figure 4)

$$\sigma_0 := K_0 \circ S_0 \circ K_0^{-1}.$$

Next we define loops σ_j for $j = 1, \dots, q$. Let S_j be the loop that is represented by the circle $|y - \gamma_j| = \varepsilon$ oriented counter clockwise. Let K_j be the modified line segment from γ_0 to $\gamma_j - \varepsilon$. The segment $[\gamma_i - \varepsilon, \gamma_i + \varepsilon]$ is replaced by the lower half circle of S_i. Then σ_j is defined as the loop (see Figure 4)

$$\sigma_j := K_j \circ S_j \circ K_j^{-1}.$$

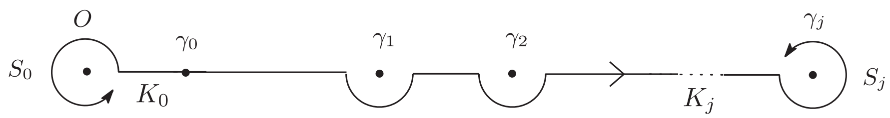


Fig. 4. Loops in $\mathbf{C}_y$

Case 1: First we see the monodromy relations at $y = 0$. By the definitions of C_j's and Lemma 2, the origin O is a flex point of C_j such that $\{y = 0\}$ is the tangent line for $j = 1, \dots, q$ and C_i and C_j intersect with intersection multiplicity p^2 at O for each $i \neq j$ and the topological type of C at O is $B_{p^2q,q}$. To see that monodromy relations, we look at the Puiseux parametrization of each component C_j at O. Consider that curves D_j and D whose defining polynomials are $g_j(x, y) = x - \psi(y, \alpha_j)$ and $g(x, y) = \prod_{j=1}^{q} g_j(x, y)$ respectively. By the definitions, $\psi(y, \alpha_j) = y(y - \alpha_j y^{p-1})$, $f_j(x, y) = g_j(x^p, y)$, we have $x^p = y(1 - \alpha_j y^{p-1})$. By the generalized binomial theorem, we can solve $x^p = y(1 - \alpha_j y^{p-1})$ as follows.

$$(1) \quad C_j : \begin{cases} x = \varphi_j(t), \quad \varphi_j(t) = t\left(1 - \dfrac{\alpha_j}{p} t^{p(p-1)} + \cdots\right), \; j = 1, \dots, q. \\ y = t^p, \end{cases}$$

$$(2) \quad \frac{\varphi_j(t)}{t} - \frac{\varphi_i(t)}{t} = \frac{1}{p}(\alpha_i - \alpha_j) t^{p(p-1)} + \cdots, \; j \neq i.$$

Note that the leading term of $\varphi_j(t)$ is t which is independent of index $j = 1, \dots, q$. The topological behavior of the centers of the generators, pq points $C \cap \{y = \varepsilon \exp(\sqrt{-1}\theta)\}$, looks like the movements of satellites around planets with $0 \leq \theta \leq 2\pi$. For a fixed y, there are p choices of t so that $y = t^p$. We take t so that $0 \leq \arg t \leq 2\pi/p$. Thus planets are the points $P_i = (t\eta^i, t^p)$ for $i = 0, \dots, p - 1$ and the satellites around P_i are $\{(\varphi_j(t\eta^i), t^p) \mid j = 1, \dots, q\}$ where $\eta = \exp(2\pi\sqrt{-1}/p)$.

Above conditions (1) and (2) say that p planets moves an arc of the angle $2\pi/p$ centered at the origin when $t = \varepsilon^{1/p} \exp(\sqrt{-1}\theta/p)$ moves from $\theta = 0$ to 2π. Then the satellites, which are the center of loops $\{a_{i,j} \mid j = 1, \dots, q\}$, are rotated $(p-1)$-times around P_i simultaneously for $i = 0, \dots, p-1$. Hence we have the monodromy relations:

$$(1\text{-}1)\quad a_{i,j} = a_{i,j}^{\sigma_0} = \begin{cases} \omega_{i+1}^{p-1} a_{i+1,j} \omega_{i+1}^{-(p-1)} & 0 \le i \le p-2, \\ \Omega \omega_1^{p-1} a_{0,j} (\Omega \omega_1^{p-1})^{-1} & i = p-1, \end{cases} \qquad j = 1, \dots, q$$

where $a_{i,j}^{\sigma_0}$ is the monodromy action by σ_0 on $a_{i,j}$. See Figure 5 for the case $p = 3$ and $q = 2$.

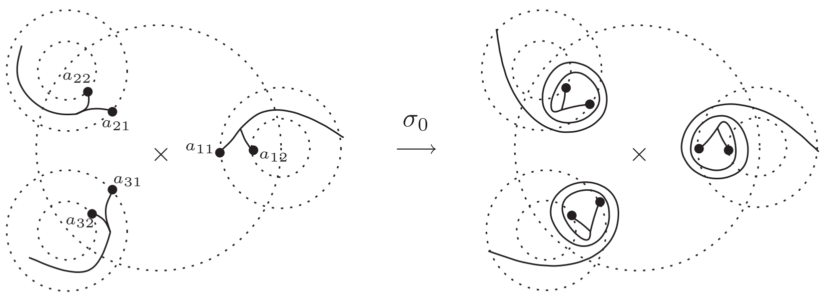


Fig. 5. The case $p = 3$ and $q = 2$

On the other hand, we get the relation $\omega_1 = \omega_2 = \cdots = \omega_p$ when $y = \varepsilon \exp(2\pi\sqrt{-1}\theta)$ moves around the origin once. Hence we have

$$\Omega = \omega^p, \qquad \omega := \omega_1 = \omega_2 = \cdots = \omega_p.$$

We can rewrite the relations (1-1) as follows:

$$(1\text{-}2)\quad a_{i,j} = \begin{cases} \omega^{p-1} a_{i+1,j} \omega^{-(p-1)} & 0 \le i \le p-2, \\ \omega^{2p-1} a_{0,j} \omega^{-(2p-1)} & i = p-1, \end{cases} \qquad j = 1, \dots, q.$$

Case 2: Next we consider the monodromy relations at $y = \gamma_j$ for $j \ge 1$. In this case, the pencil line L_{γ_j} is tangent to C_j and $C_j \cap L_{\gamma_j} = \{Q_{j,0}\} = \{(0, \gamma_j)\}$ is a flex point of C_j of flex order $p-2$. On the other hand, the pencil line L_{γ_j} is generic with respect to other C_i for $i \ne j$.

First we consider the case $j = 1$. Recall that the defining polynomial of C_i is

$$f_i(x, y) = x^p - \psi(y, \alpha_i) = x^p - y \prod_{k=0}^{p-2} (\gamma_i \xi^k - y), \qquad i = 1, \dots, q.$$

We take the local coordinates $(x, y_1) := (x, y - \gamma_1)$ centered at $Q_{1,0}$. By an easy calculation,

$$f_i(x, y_1 + \gamma_1) = 0 \Leftrightarrow x^p = \begin{cases} (1-p)y_1 + H_1(y_1), & i = 1, \\ \frac{\gamma_1}{\alpha_1}(\alpha_1 - \alpha_i) + H_i(y_1) & i \neq 1 \end{cases} \tag{1}$$

and $\alpha_1 - \alpha_i > 0$ where $\mathrm{ord}_{y_1} H_1 \geq 2$ and $\mathrm{ord}_{y_1} H_i \geq 1$ for $i \geq 2$. The first coefficients $1 - p$ and $\frac{\gamma_1}{\alpha_1}(\alpha_1 - \alpha_i)$ are obtained from the equalities:

$$\begin{cases} \psi(y_1 + \gamma_1, \alpha_1) = -(y_1 + \gamma_1)y_1 \prod_{k \geq 1}(\gamma_1 \xi^k - y_1 - \gamma_1), \\ \psi(y_1 + \gamma_1, \alpha_i) = (y_1 + \gamma_1) - \alpha_i(y_1 + \gamma_1)^p \end{cases}$$

and

$$\begin{cases} \left.\dfrac{d\psi(y_1 + \gamma_1, \alpha_1)}{dy_1}\right|_{y_1 = 0} = \left.\dfrac{d\psi(y, \alpha_1)}{dy}\right|_{y = \gamma_1} = 1 - p, \\ \psi(\gamma_1, \alpha_i) = \frac{\gamma_1}{\alpha_1}(\alpha_1 - \alpha_i). \end{cases}$$

Now we consider the monodromy relations at $y = \gamma_1$. First, the action of σ_1 on $b_1, \ldots, b_q$ is sketched as in Figure 6. Thus we see that the generators that are topologically deformed are $\{a_{i,1} \mid i = 0, \ldots, p-1\}$ under the rotation $y_1 = -\varepsilon \exp(\sqrt{-1}\theta)$ with $0 \leq \theta \leq 2\pi$. The other generators are unchanged. Namely $a_{i,j}^{\sigma_1} = a_{i,j}$ for $i = 0, \ldots, p-1$ and $j \geq 2$. To simplify the monodromy relations, we introduce an element $g_1 := b_2 \ldots b_q$. Then $\tau = b_1 g_1$. See Figure 6.

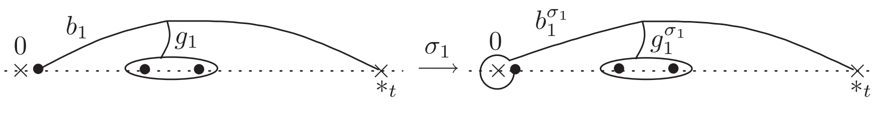


Fig. 6. $\{y = \gamma_1 - \varepsilon\} \cap \mathbf{C}_t^2$

Let $g_{i,1}$ be the pull-back of g_1 starting from $*_i$ for $i = 0, \ldots, p-1$. More precisely, $g_{i,1} = a_{i,2} \ldots a_{i,q}$ and $\omega_i = a_{i,1} g_{i,1}$. When $y_1 = y - \gamma_1 = -\varepsilon \exp(\sqrt{-1}\theta)$ moves from $\theta = 0$ to 2π, the generators $a_{0,1}, \ldots, a_{p-1,1}$ moves an arc of the angle $2\pi/p$ centered at the origin (the lifts of $b_1^{\sigma_1}$) and the other generators do not move. Thus we have following monodromy relations:

$$\text{(2-1)} \quad a_{i,1} = a_{i,1}^{\sigma_1} = \begin{cases} g_{i+1,1}^{-1} a_{i+1,1} g_{i+1,1} & 0 \leq i \leq p-2, \\ \Omega g_{0,1}^{-1} a_{0,1} (\Omega g_{0,1}^{-1})^{-1} & i = p-1 \end{cases}$$

and $g_{i,1}^{\sigma_1} = g_{i,1}$ for $i = 0, \ldots, p-1$. See Figure 7.

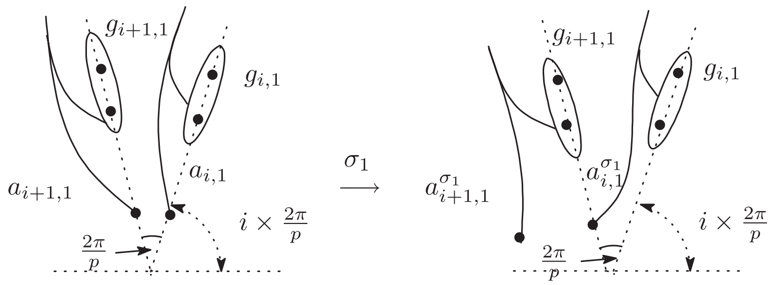


Fig. 7. The action of σ_1

By the previous argument, we have $\omega_1 = \omega$ and $\Omega = \omega^p$. Hence we can rewrite the relations (2-1) as follows:

$$\text{(2-2)} \quad a_{i,1} = a_{i,1}^{\sigma_1} = \begin{cases} \omega^{-1} a_{i+1,1} \omega & 0 \le i \le p-2, \\ \omega^{p-1} a_{0,1} \omega^{-(p-1)} & i = p-1. \end{cases}$$

Now we consider the case $j \ge 2$. First we deform the pencil from γ_0 to $\gamma_j - \varepsilon$ along K_j. Note that

$$\psi(y, \alpha_k) \begin{cases} <0 & k < j, \\ >0 & k > j \end{cases}$$

where $y \in [\gamma_{j-1} + \varepsilon, \gamma_j - \varepsilon]$. Thus the generators $b_1, \dots, b_q$ are deformed as in Figure 8 where $\psi_j := \psi(y, \alpha_j)$.

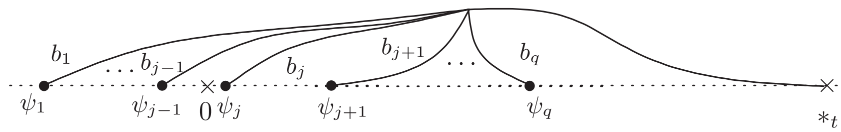


Fig. 8. $\{y = \gamma_j - \varepsilon\} \cap \mathbf{C}_t^2$

When y moves along $S_j : |y - \gamma_j| = \varepsilon$, the single root of $g_{\alpha_j}(x, y) = 0$ that is near the origin goes around the origin once and the other roots $g_{\alpha_k}(x, y) = 0$ $(k \ne j)$ do not move as in Figure 9 where $\psi_j := \psi(y, \alpha_j)$.

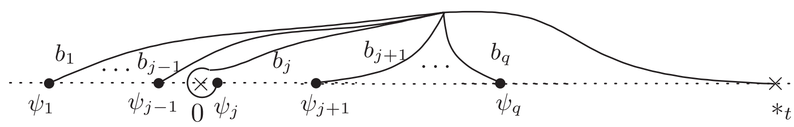


Fig. 9. $\{y = \gamma_j - \varepsilon\} \cap \mathbf{C}_t^2$

This implies, by taking p-fold covering, the corresponding generators $a_{0,j},\dots,a_{p-1,j}$ of b_j moves an arc of the angle $2\pi/p$ centered at the origin. To see it more precisely, we put new loops:

$$h_j = \begin{cases} e & j=1, \\ b_1\dots b_{j-1} & 2\le j\le q \end{cases}, \qquad g_j = \begin{cases} b_{j+1}\dots b_q & 1\le j\le q-1, \\ e & j=q. \end{cases}$$

By the definitions, we have $\tau = h_j b_j g_j$. See Figure 10.

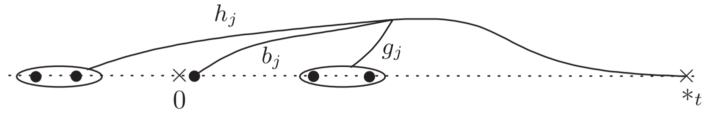


Fig. 10. New loops

We take the local coordinates $(x, y_j) := (x, y-\gamma_j)$ centered at $Q_{j,0}$. Then

$$f_i(x, y_j+\gamma_j) = 0 \Leftrightarrow x^p = \begin{cases} (1-p)y_j + H_j(y_j) & i=j, \\ \frac{\gamma_j}{\alpha_j}(\alpha_j-\alpha_i) + H_i(y_j) & i\ne j \end{cases}$$

where $\operatorname{ord}_{y_j} H_j \ge 2$ and $\operatorname{ord}_{y_j} H_i \ge 1$ for $i\ne j$. By the assumption, we have $\alpha_j-\alpha_i>0$ or $\alpha_j-\alpha_i<0$ corresponding to either $i>j$ or $i<j$ respectively. Thus we can see that the generators that are deformed under this monodromy are $\{a_{i,j}\mid i=0,\dots,p-1\}$ when y_j moves around the circle $|y_j|=\varepsilon$. Thus $a_{i,k}^{\sigma_j}=a_{i,k}$ for $k\ne j$. Let $h_{i,j}$ and $g_{i,j}$ be the pull-back of h_j and g_j respectively. By the definition, we have

$$h_{i,j} = a_{i,1}\dots a_{i,j-1}$$

$$g_{i,j} = a_{i,j+1}\dots a_{i,q}$$

where $h_{i,1}=e$ and $g_{i,q}=e$ and we put $\omega_i = h_{i,j}a_{i,j}g_{i,j}$.

When y moves around the circle $|y-\gamma_j|=\varepsilon$ once, the generators $a_{0,j},\dots,a_{p-1,j}$ moves an arc of the angle $2\pi/p$ centered at the origin. Thus we have following monodromy relations:

$$\text{(2-3)}\quad a_{i,j} = a_{i,j}^{\sigma_j} = \begin{cases} (g_{i+1,j}h_{i,j})^{-1}a_{i+1,j}g_{i+1,j}h_{i,j} & 0\le i\le p-2, \\ h_{p-1,j}^{-1}\Omega g_{0,j}^{-1}a_{0,j}(h_{p-1,j}^{-1}\Omega g_{0,j}^{-1})^{-1} & i=p-1 \end{cases}$$

and $g_{i,j}^{\sigma_j}=g_{i,j}$ and $h_{i,j}^{\sigma_j}=h_{i,j}$ for $i=0,\dots,p-1$. See Figure 11.

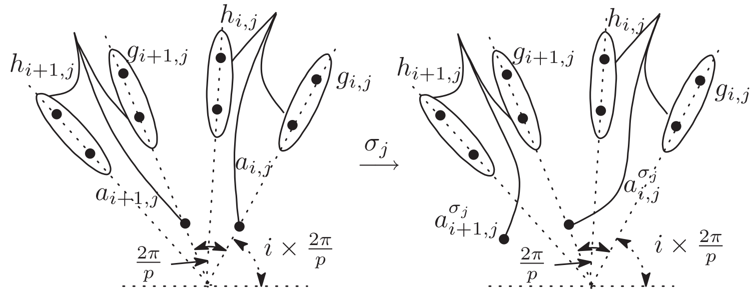


Fig. 11. The action of σ_j

Other cases: Finally we read the monodromy relations at $y = \gamma$ where $\gamma \in \Sigma$ with $\gamma \neq 0, \gamma_1, \dots, \gamma_q$. Recall that the set $\Sigma \subset \mathbf{C}_y$ of parameters that correspond to singular pencils are given by

$$\Sigma = \{0, \gamma_j \xi^k \in \mathbf{C}_y \mid k = 0, \dots, p-2, j = 1, \dots, q\}, \qquad \xi = \exp\left(\frac{2\pi\sqrt{-1}}{p-1}\right).$$

Then the pencil line $L_{\gamma_j \xi^k} = \{y = \gamma_j \xi^k\}$ is singular with respect to C_j and $C_j \cap L_{\gamma_j \xi^k} = \{Q_{j,k}\} = \{(0, \gamma_j \xi^k)\}$ is a flex point of C_j of flex order $p-2$ for $k = 1, \dots, p-2$. Note that the pencil line $L_{\gamma_j \xi^k}$ is generic with respect to other C_i for $i \neq j$.

First we consider the case $k = 1$. That is, we consider the monodromy relations at $y = \gamma_j \xi$. We take a path L_1 which connects γ_0 and $\gamma_0 \xi$ as in Figure 12.

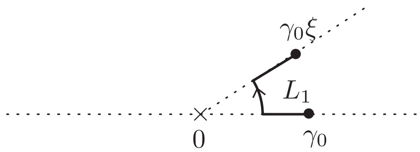


Fig. 12. The loop L_1

Then the loops $b_1, \dots, b_q$ are deformed as in the left side of Figure 13. We take new loops $c_1, \dots, c_q$ as in the right side of Figure 13. Here $\xi *_t = (\xi \omega_0^p, \xi \gamma_0)$.

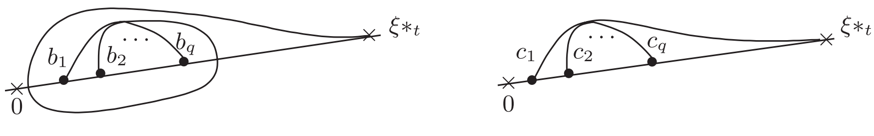


Fig. 13. New loops $c_1, \dots, c_q$

They are related by the following.

$$c_j = \tau^{-1} b_j \tau, \qquad j = 1, \dots, q. \tag{2}$$

Let $d_{0,j}, \dots, d_{p-1,j}$ be the pull-back of c_j by φ_p for $j = 1, \dots, q$. Then the relation (2) implies

$$d_{i,j} = \omega^{-1} a_{i,j} \omega. \tag{3}$$

Now we consider the loops $\sigma_1^{(1)}, \dots, \sigma_q^{(1)}$ in $\mathbf{C}_y$ with base point $\gamma_0 \xi$ as in Figure 14.

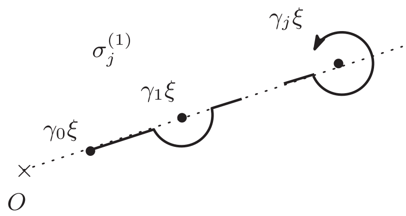


Fig. 14. The loop $\sigma_j^{(1)}$

We will see that the monodromy relations are exactly as (2-3). To see this assertion, we take the modified coordinates $(\tilde{x}, \tilde{y})$ defined by

$$\tilde{x} := \exp\left(\frac{-2\pi\sqrt{-1}}{p(p-1)}\right) x, \qquad \tilde{y} := \xi y.$$

In these coordinates, the loops $\sigma_1^{(1)}, \dots, \sigma_q^{(1)}$ coincide with $\sigma_1, \dots, \sigma_q$ and C_j is defined by the same equality:

$$C_j : \tilde{x}^p = \tilde{y}(1 - \alpha_j \tilde{y}^{p-1}).$$

The situation of loops $c_1, \dots, c_q$ are the same with that of $b_1, \dots, b_q$ and the situation of loops $d_{i,j}$, $i = 0, \dots, p-1$, $j = 1, \dots, q$ are the same with that of $a_{i,j}$, $i = 0, \dots, p-1$, $j = 1, \dots, q$. Therefore we obtain the relations

$$\text{(2-3)}' \quad d_{i,j} = \begin{cases} (\tilde{g}_{i+1,j} \tilde{h}_{i,j})^{-1} d_{i+1,j} \tilde{g}_{i+1,j} \tilde{h}_{i,j} & 0 \le i \le p-2 \\ \tilde{h}_{p-1,j}^{-1} \tilde{\Omega} \tilde{g}_{0,j}^{-1} d_{0,j} (\tilde{h}_{p-1,j}^{-1} \tilde{\Omega} \tilde{g}_{0,j}^{-1})^{-1} & i = p-1 \end{cases}, \qquad j = 1, \dots, q$$

where $\tilde{h}_{i,j} := d_{i,1} \dots d_{i,j-1}$, $\tilde{g}_{i,j} := d_{i,j+1} \dots d_{i,q}$ and $\tilde{\Omega} := \omega^{-1} \Omega \omega$. Now we claim the following.

Lemma 3. *The relation* (2-3)$'$ *is the same with the relation* (2-3).

Proof. First we consider the relation $d_{i,j} = (\tilde{g}_{i+1,j} \tilde{h}_{i,j})^{-1} d_{i+1,j} \tilde{g}_{i+1,j} \tilde{h}_{i,j}$ in (2-3)$'$. By the relation (3), we have

$$\tilde{h}_{i,j} = \omega^{-1} h_{i,j} \omega, \qquad \tilde{g}_{i,j} = \omega^{-1} g_{i,j} \omega.$$

Thus $d_{i,j} = (\tilde{g}_{i+1,j}\tilde{h}_{i,j})^{-1} d_{i+1,j}\tilde{g}_{i+1,j}\tilde{h}_{i,j}$ can be translated as follows

$$\begin{aligned} d_{i,j} = \omega^{-1} a_{i,j}\omega &= (\tilde{g}_{i+1,j}\tilde{h}_{i,j})^{-1} d_{i+1,j}\tilde{g}_{i+1,j}\tilde{h}_{i,j} \\ &= ((\omega^{-1} g_{i+1,j}\omega)(\omega^{-1} h_{i,j}\omega))^{-1}(\omega^{-1} a_{i+1,j}\omega)(\omega^{-1} g_{i+1,j}\omega)(\omega^{-1} h_{i,j}\omega) \\ &= \omega^{-1}(g_{i+1,j}h_{i,j})^{-1} a_{i+1,j} g_{i+1,j} h_{i,j}\omega \end{aligned}$$

which implies (2-3). For the relation $d_{i,j} = \tilde{h}_{p-1,j}^{-1}\tilde{\Omega}\tilde{g}_{0,j}^{-1} d_{0,j}(\tilde{h}_{p-1,j}^{-1}\tilde{\Omega}\tilde{g}_{0,j}^{-1})^{-1}$, the argument is the same. This completes the proof. □

Next we consider general cases $k \geq 2$. That is, we consider the monodromy relations at $y = \gamma_j \xi^k$. Then we take a path L_k which connects γ_0 and $\gamma_0 \xi^k$:

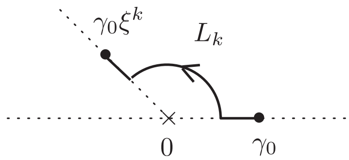


Fig. 15. The loop L_k

By the exact same arguments as in the case $k = 1$, we see that no new monodromy relations are necessary.

3.3. The group structures of $\pi_1(\mathbf{C}^2\backslash C)$ and $\pi_1(\mathbf{P}^2\backslash C)$. In this section, we consider the group structures of $\pi_1(\mathbf{P}^2\backslash C)$ and $\pi_1(\mathbf{C}^2\backslash C)$. First by previous considerations, we have proved that

$$\pi_1(\mathbf{C}^2\backslash C) = \langle \omega, a_{i,j}, i = 0, \dots, p-1, j = 1, \dots, q, \mid (1\text{-}2), (2\text{-}3), (S)\rangle \tag{4}$$

where

$$(1\text{-}2) \quad a_{i,j} = \begin{cases} \omega^{p-1} a_{i+1,j}\omega^{-(p-1)} & 0 \leq i \leq p-2, \\ \omega^{2p-1} a_{0,j}(\omega^{2p-1})^{-1} & i = p-1, \end{cases}, \qquad j = 1, \dots, q$$

$$(2\text{-}3) \quad a_{i,j} = \begin{cases} (g_{i+1,j}h_{i,j})^{-1} a_{i+1,j} g_{i+1,j} h_{i,j} & 0 \leq i \leq p-2 \\ h_{p-1,j}^{-1}\Omega g_{0,j}^{-1} a_{0,j}(h_{p-1,j}^{-1}\Omega g_{0,j}^{-1})^{-1} & i = p-1 \end{cases}, \qquad j = 1, \dots, q$$

$$\omega = a_{0,1} \dots a_{0,q}. \tag{S}$$

Note that last relations in (1-2) and (2-3) are unnecessary as they follow from previous relations. By the definitions of $g_{i,j}$ and $h_{i,j}$ and (1-2), we have the following inductive relations.

$$\begin{cases} h_{i+1,j} = \omega^{-(p-1)} h_{i,j} \omega^{p-1},\ g_{i+1,j} = \omega^{-(p-1)} g_{i,j} \omega^{p-1} \\ g_{i+1,j} h_{i,j} = \omega^{-(p-1)} g_{i,j} \omega^{p-1} h_{i,j} \end{cases} \tag{5}$$

First we examine the relation (2-3) for a fixed $i \le p-2$ using (1-2). The case $j = 1$ gives the equality:

$$\begin{aligned} a_{i,1} &= (g_{i+1,1})^{-1} a_{i+1,1} g_{i+1,1} \qquad (\text{as } h_{i,1} = e) \\ &= (\omega^{-(p-1)} g_{i,1} \omega^{p-1})^{-1} (\omega^{-(p-1)} a_{i,1} \omega^{p-1})(\omega^{-(p-1)} g_{i,1} \omega^{p-1}) \\ &= \omega^{-(p-1)} g_{i,1}^{-1} a_{i,1} g_{i,1} \omega^{p-1} \\ &= \omega^{-p} a_{i,1} \omega^{p}. \end{aligned}$$

This implies ω^p and $a_{i,1}$ commute. Now by the induction on j, we show that

$$[a_{i,j}, \omega^p] = e, \qquad j = 1, \dots, q \tag{R_i}$$

where $[a,b] = aba^{-1}b^{-1}$. In fact, assuming $a_{i,1}, \dots, a_{i,j-1}$ commute with ω^p, we get

$$\begin{aligned} a_{i,j} &= (g_{i+1,j} h_{i,j})^{-1} a_{i+1,j} g_{i+1,j} h_{i,j} \\ &= (\omega^{-(p-1)} g_{i,j} \omega^{p-1} h_{i,j})^{-1} (\omega^{-(p-1)} a_{i,j} \omega^{p-1})(\omega^{-(p-1)} g_{i,j} \omega^{p-1} h_{i,j}) \\ &= h_{i,j}^{-1} \omega^{-(p-1)} g_{i,j}^{-1} a_{i,j} g_{i,j} \omega^{p-1} h_{i,j} \\ &= h_{i,j}^{-1} \omega^{-(p-1)} g_{i,j}^{-1} (h_{i,j}^{-1} h_{i,j}) a_{i,j} g_{i,j} \omega^{p-1} h_{i,j} \\ &= h_{i,j}^{-1} \omega^{-p} h_{i,j} a_{i,j} h_{i,j}^{-1} \omega^{p} h_{i,j} \qquad (\text{as } [h_{i,j}, \omega^p] = e) \\ &= \omega^{-p} a_{i,j} \omega^{p}. \end{aligned}$$

Thus we get $[a_{i,j}, \omega^p] = e$ for all $j = 1, \dots, q$.

The relation (R_i) for $i = 0, \dots, p-1$ implies ω^p is in the center of $\pi_1(\mathbf{C}^2 \backslash C)$. Using relations (1-2) and (R_i), we have

$$a_{i+1,j} = \omega a_{i,j} \omega^{-1}, \qquad i = 0, \dots, p-2,\ j = 1, \dots, q.$$

Thus we get

$$a_{i,j} = \omega^i a_{0,j} \omega^{-i}, \qquad i = 0, \dots, p-1,\ j = 1, \dots, q. \tag{6}$$

Hence we can take $a_{0,1}, \dots, a_{0,q}$ as generators. They satisfy the relations

$$[a_{0,j}, \omega^p] = e, \qquad j = 1, \dots, q. \tag{R_0}$$

It is easy to see (and we have seen implicitly in the above discussions) that the relations (1-2) and (2-3) follow from (R_0), (S) and (5). Thus we have shown

$$\begin{aligned}
\pi_1(\mathbf{C}^2 \backslash C) &= \langle a_{i,j}(i = 0, \dots, p-1, j = 1, \dots, q), \omega \,|\, (1\text{-}1), (2\text{-}3), (S), (5) \rangle \\
&\cong \langle a_{0,1}, \dots, a_{0,q}, \omega \,|\, (R_0), (S) \rangle \\
\pi_1(\mathbf{P}^2 \backslash C) &\cong \langle a_{0,1}, \dots, a_{0,q}, \omega \,|\, \omega^p = e, (R_0), (S) \rangle \\
&\cong \langle a_{0,1}, \dots, a_{0,q}, \omega \,|\, \omega^p = e, (S) \rangle \\
&\cong \langle a_{0,1}, \dots, a_{0,q-1}, \omega \,|\, \omega^p = e \rangle \\
&\cong F(q-1) * \mathbf{Z}/p\mathbf{Z}.
\end{aligned}$$

This completes the proof of Theorem 1.

4. Applications

This section is a joint work with Mutsuo Oka. We give some applications of the main result.

4.1. Degeneration of linear torus curve. Consider a linear torus curve

$$C : f_p(x, y)^q + \ell(x, y)^{pq} = 0 \tag{7}$$

where $(f_p, \ell) \in \mathcal{M}(pq, q; pq, \mathcal{I})$. We assume that C is a generic member of the linear system $C(\tau)$ defined by

$$\tau f_p(x, y)^q + (1 - \tau)\ell(x, y)^{pq} = 0.$$

Let $C_p \cap L = \{P_1, \dots, P_k\}$ and put $m_i = I(C_p, L; P_i)$ for $i = 1, \dots, k$ so that $\mathcal{I} = \{m_1 \dots, m_k\}$. We always assume that C_p is smooth at each P_i for $i = 1, \dots, k$.

PROPOSITION 2. *The moduli space $\mathcal{M}(pq, q; pq, \mathcal{I})$ is irreducible.*

PROOF. We may assume that $L = \{y = 0\}$. Then by the assumption on the intersection partition, we can write

$$f_p(x, 0) = (x - \alpha_1)^{m_1} \dots (x - \alpha_k)^{m_k}$$

with mutually distinct complex numbers $\alpha_1, \dots, \alpha_k$ up to a multiplication of a non-zero constant. Thus f_p takes the form

$$f_p(x, y) = yf_{p-1}(x, y) + (x - \alpha_1)^{m_1} \dots (x - \alpha_k)^{m_k},$$

where $f_{p-1}(x, y)$ is a polynomial of degree $p - 1$. By the assumption,

$$\frac{\partial f_p}{\partial y}(\alpha_i, 0) = f_{p-1}(\alpha_i, 0) \neq 0, \qquad \text{if } m_i \geq 2.$$

By a small perturbation of $f_{p-1}(x, y)$, we may also assume that $f_{p-1}(x, 0)$ is a polynomial of degree $p - 1$. This description implies the irreducibility of $\mathscr{M}(pq, q; pq, \mathscr{I})$. In fact, we only show the connectivity of the moduli space $\mathscr{M}(pq, q; pq, \mathscr{I})$. Take another linear torus curve

$$\begin{aligned} C' : g_p(x, y)^q - y^{pq} = 0 \qquad &\text{where} \\ g_p(x, y) = yg_{p-1}(x, y) + (x - \beta_1)^{m_1} &\dots (x - \beta_k)^{m_k}, \end{aligned}$$

with $(g_p, y) \in \mathscr{M}(pq, q; pq, \mathscr{I})$. We consider the linear family

$$\begin{aligned} f_p(x, y, s) = y(sf_{p-1}(x, y) &+ (1 - s)g_{p-1}(x, y)) \\ &+ \prod_{i=1}^{k} (x - (s\alpha_i + (1 - s)\beta_i))^{m_i}, \qquad s \in \mathbf{C}. \end{aligned}$$

Consider the polynomial

$$h_i(s) = sf_{p-1}(s\alpha_i + (1 - s)\beta_i, 0) + (1 - s)g_{p-1}(s\alpha_i + (1 - s)\beta_i, 0).$$

As $h_i(0) = g_{p-1}(\beta_i, 0)$ and $h_i(1) = f_{p-1}(\alpha_i, 0)$, $h_i(s)$ is a non-zero polynomial in s. Consider the set $A_i \subset \mathbf{C}$ define by $A_i = \{s \in \mathbf{C} \mid h_i(s) = 0\}$. As $h_i(s)$ is a non-zero polynomial in s, A_i is a finite set. Put $A = \bigcup_{i=1}^{k} A_i$. Thus we can take a path in the parameter space $\mathbf{C}$ from $s = 1$ to $s = 0$ avoiding the exceptional set A. This shows the connectedness of the moduli space $\mathscr{M}(pq, q; pq, \mathscr{I})$. $\square$

Lemma 4. *For any $(f_p, \ell) \in \mathscr{M}(pq, q; pq, \mathscr{I})$, there is a degeneration family C_t, $t \in U$ with $1 \in U$ so that C_1 is the linear torus curve that corresponds to (f_p, ℓ) and C_0 is a linear torus curve of the maximal contact.*

Proof. We assume that $L = \{y = 0\}$. The assumption implies that

$$f_p(x, y) = yf_{p-1}(x, y) + (x - \alpha_1)^{m_1} \dots (x - \alpha_k)^{m_k}.$$

We may assume for simplicity that $f_{p-1}(0,0) \neq 0$ and we consider the family of curves $C_{p,t}$ defined by $\{f_p(x, y, t) = 0\}$ where

$$f_p(x, y, t) := yf_{p-1}(x, y) + (x - t\alpha_1)^{m_1} \dots (x - t\alpha_k)^{m_k}, \qquad t \in \mathbf{C}.$$

Then $C_{p,0}$ is defined by

$$f_p(x, y, 0) = yf_{p-1}(x, y) + x^p = 0$$

and we see that $(f_p(x, y, 0), y) \in M(pq, q; pq, \mathscr{I}_m)$. Consider the corresponding linear torus curve

$$C_t : f_p(x, y, t)^q - c_1 y^{pq} = 0, \qquad c_1 \in \mathbf{C}^*.$$

First choosing a generic c_1 and fixing c_1, we may assume that C_1 and C_0 have only smooth components. There exists at most a finite number of $t = t_1, \dots, t_s$ such that C_t has some singular points by the Bertini theorem ([3]). Then we may simply consider the restriction of the family over $U := \mathbf{C} \backslash \{t_1, \dots, t_s\}$. This gives a desired degeneration. □

It is easy to show that we can also degenerate a linear torus curve with the generic partition $(1, \dots, 1)$ to our curve C. (Essentially we use the degeneration $\gamma(x, s) = \gamma_1(x, s) \dots \gamma_k(x, s)$ where $\gamma_j(x) = (x - \alpha_i)^{m_i} - \varepsilon s$, $i = 1, \dots, k$ for a sufficiently small $\varepsilon > 0$.) Thus by the degeneration principle ([7]) and Theorem 1, we obtain Corollary 1.

4.2. Zariski triples. Consider a pair of smooth curves C_1, C_2 of degree p and let $\mathscr{I}$ be the intersection partition $\{I(C_1, C_2; P) \mid P \in C_1 \cap C_2\}$ of p^2. The topology of $C_1 \cup C_2$ is not determined by $\mathscr{I}$. For example, consider the case $\mathscr{I} = \{p^2\}$. In [1], they showed that there are at least β configurations with different topologies where β is the number of positive integers n such that $1 \leq n < p$ and n divides p. The defining polynomial of $C^{(n)}$ can be written as

$$C^{(n)} : f_n^{2p/n}(x, y) + f_p^2(x, y) = 0.$$

These curves $\{C^{(n)} \mid 1 \leq n < p, n \mid p\}$ come from torus curves of different types. More precisely, the curve $C^{(n)}$ belongs to the moduli space $\mathscr{M}(2p/n, 2; 2p, \{pn\})$ where n is an positive integer such that $1 \leq n < p$ and n divides p. (In [8], Oka has proved that there exists another configuration whose complement has an abelian fundamental group for $p = 3, 4, 5$.)

The same discussion works for non-maximal partitions. For simplicity, we consider the case $p = 6$.

Suppose that the intersection partition $\mathscr{I}$ is $\{18, 18\}$ of 36. First we consider a linear torus curve $C^{(1)} \in \mathscr{M}(12, 2; 12, \mathscr{J})$ with $\mathscr{J} = \{3, 3\}$ that is associated with $(f_6^{(1)}, L) \in \mathscr{M}(12, 2; 12, \mathscr{J})$ with

$$C^{(1)} : f_6^{(1)}(x, y)^2 - L(x, y)^{12} = 0 \qquad \text{where}$$
$$f_6^{(1)}(x, y) = (x^2 - 1)^3 + y + y^6, \qquad L(x, y) = y.$$

Then the Alexander polynomial of $C^{(1)}$ is given by:

$$\Delta_{C^{(1)}}(t) = \frac{(t^{12} - 1)(t - 1)}{t^2 - 1}.$$

Next we consider a $(2, 6)$ torus curve $C^{(2)}(s) \in \mathscr{M}(6, 2; 12, 2\mathscr{J})$ of degree 12 defined by

$$C^{(2)}(s) : f_6^{(2)}(x, y, s)^2 - f_2(x, y)^6 = 0 \qquad \text{where}$$
$$f_6^{(2)}(x, y, s) = (y - s)^6 + y - x^2, \qquad f_2(x, y) = y - x^2, \qquad s \in \mathbf{C}.$$

This family degenerates into a maximal contact curve $C^{(2)}(0)$. Thus by the sandwich principle, the Alexander polynomial of $C^{(2)}(s)$ is given by

$$\Delta_{C^{(2)}(s)}(t) = \frac{(t^6 - 1)(t - 1)}{t^2 - 1}.$$

The third one is a $(2, 4)$ torus curve $C^{(3)}(s) \in \mathscr{M}(4, 2; 12, 3\mathscr{J})$ defined by

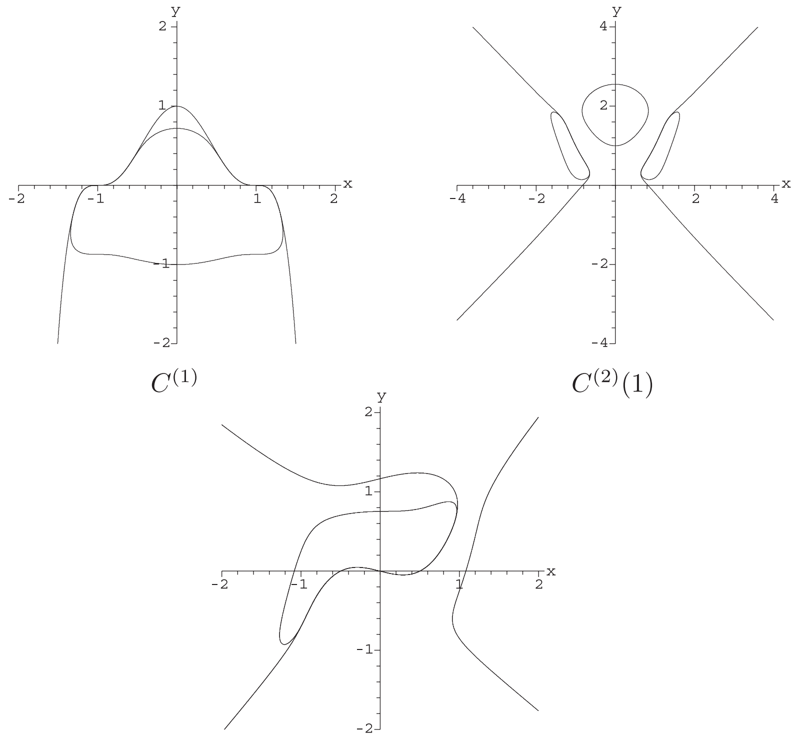
$$C^{(3)}(s) : f_6^{(3)}(x, y, s)^2 - f_3(x, y, s)^4 = 0 \qquad \text{with}$$
$$\begin{aligned} f_6^{(3)}(x, y, s) &= y^6 - 3s^2xy^5 + 6s^4x^2y^4 - 5xs^6(2x^2 - 3s^2)y^3 \\ &\quad + s^{10}(7s^2x - 6y)x^2y - s^{14}(8x^2 - 9s^2)x^2 - s^{18} - f_3(x, y, s), \end{aligned}$$
$$f_3(x, y, s) = y - x(x - s)(x + s).$$

This family degenerates into a maximal contact curve $C^{(3)}(0)$. Thus by the sandwich principle, the Alexander polynomial of $C^{(3)}(s)$ is given by

$$\Delta_{C^{(3)}(s)}(t) = \frac{(t^4 - 1)(t - 1)}{t^2 - 1}.$$

Therefore the triple $\{C^{(1)}, C^{(2)}(1), C^{(3)}(1)\}$ is a Zariski triple which are distinguished by the Alexander polynomials. Their graphs are as in Figure 16.

In Figure 16, $C^{(1)}$, $C^{(2)}(1)$ and $C^{(3)}(1/2)$ have two irreducible components which are tangent at $(\pm 1, 0)$, $(\pm 1, 1)$ and $(\pm 1/2, 0)$ with the respective intersection number 18 respectively.



$C^{(1)}$ $C^{(2)}(1)$

$C^{(3)}(1/2)$

Fig. 16.

Case 2. Next we consider the case $\mathcal{I} = \{12, 12, 12\}$. We consider the following three torus curves: $(D^{(1)}(s), D^{(2)}(s), D^{(3)}(s))$ where $D^{(1)}(s) \in \mathcal{M}(12, 2; 12, \mathcal{J})$, $D^{(2)}(s) \in \mathcal{M}(6, 2; 12, 2\mathcal{J})$ and $D^{(3)}(s) \in \mathcal{M}(4, 2; 12, 3\mathcal{J})$ where $\mathcal{J} = \{2, 2, 2\}$. They are defined by

$$D^{(1)}(s) : g_6^{(1)}(x, y, s)^2 - L(x, y)^{12} = 0 \qquad \text{with}$$

$$g_6^{(1)}(x, y, s) = y^6 - y + x^2(x - s)^2(x + s)^2, \qquad L(x, y) = y,$$

$$D^{(2)}(s) : g_6^{(2)}(x, y, s)^2 - g_2(x, y, s)^6 = 0 \qquad \text{with}$$

$$g_6^{(2)}(x, y, s) = y^6 + s^2(2y^2 - ys^2 + s^4)x^2y^2 - s^8(x^2 - s^2)y - g_2(x, y)$$

$$g_2(x, y, s) = y - x^2 + s^2,$$

$$D^{(3)}(s) : g_6^{(3)}(x, y, s)^2 - g_3(x, y, s)^4 = 0 \qquad \text{with}$$

$$g_6^{(3)}(x, y, s) = y^6 + g_3(x, y), \qquad g_3(x, y, s) = y - x^3 + s^2x.$$

As three families degenerate into maximal contact curves in $\mathcal{N}_1^1(12,2;12)$, $\mathcal{N}_2^2(6,2;12)$ and $\mathcal{N}_3^3(4,2;12)$ respectively, their topology are distinguished by the Alexander polynomials. Thus the triple $\{D^{(1)}(1), D^{(2)}(1), D^{(3)}(1)\}$ is a Zariski triple.

Acknowledgement

I would like to express my deepest gratitude to Professor Mutsuo Oka for suggesting me the present problem and helping me to prepare this paper.

References

[1] B. Audoubert, C. Nguyen, and M. Oka. On Alexander polynomials of torus curves. J. Math. Soc. Japan, 57(4):935–957, 2005.

[2] B. Audoubert, T. C. Nguyen, and M. Oka. Erratum to: "On Alexander polynomials of torus curves" [J. Math. Soc. Japan **57** (2005), no. 4, 935–957; mr2183580]. J. Math. Soc. Japan, 59(3):911–912, 2007.

[3] P. Griffiths and J. Harris. Principles of algebraic geometry. Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[4] M. Oka. On the fundamental group of the complement of a reducible curve in P^2. J. London Math. Soc. (2), 12(2):239–252, 1975/76.

[5] M. Oka. On the fundamental group of the complement of certain plane curves. J. Math. Soc. Japan, 30(4):579–597, 1978.

[6] M. Oka. Two transforms of plane curves and their fundamental groups. J. Math. Sci. Univ. Tokyo, 3:399–443, 1996.

[7] M. Oka. A survey on Alexander polynomials of plane curves. Singularités Franco-Japonaise, Séminaire et congrès, 10:209–232, 2005.

[8] M. Oka. Geometry of pencil of plane curves via Taylor expansions. In Singularities II, volume 475 of Contemp. Math., pages 151–167. Amer. Math. Soc., Providence, RI, 2008.

[9] M. Oka and D. Pho. Fundamental group of sextic of torus type. In A. Libgober and M. Tibar, editors, Trends in Singularities, pages 151–180. Birkhäuser, Basel, 2002.

[10] E. van Kampen. On the fundamental group of an algebraic curve. Amer. J. Math., 55:255–260, 1933.

[11] O. Zariski. On the Poincaré group of a rational plane curves. Amer. J. Math., 58:607–619, 1929.

[12] O. Zariski. On the problem of existence of algebraic functions of two variables possessing a given branch curve. Amer. J. Math., 51:305–328, 1929.

Masayuki Kawashima
Department of Mathematics
Tokyo University of Science
Wakamiya-cho 26, *Shinjuku-ku*
Tokyo 162-0827, *Japan*
E-mail: kawashima@ma.kagu.tus.ac.jp