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A note on singular integral operators with non doubling measures

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Abstract. Let μ be a Radon measure on $\mathbf{R}^d$ which satisfies $\mu(B(x,r)) \le Cr^n$ for any $x \in \mathbf{R}^d$ and $r > 0$ and some fixed positive constants C and n with $0 < n \le d$. In this paper, the relationship between the $L^p(\mu)$ boundedness and certain weak type endpoint estimates are considered for the singular integral operators.

1. Introduction

In the last several years, there has been significant progress on the study of the function spaces with non doubling measures and the boundedness of singular integral operators on these spaces (see [1–3], [5–7]). Let μ be a Radon measure on $\mathbf{R}^d$ which satisfies the following growth condition

$$\mu(B(x,r)) \le Cr^n, \qquad \text{for all } x \in \mathbf{R}^d \text{ and } r > 0, \tag{1}$$

where $C > 0$, $0 < n \le d$ are constants, and $B(x,r)$ is the ball centered at x with radius r. Let $K(x,y)$ be a locally μ-integrable function on $\mathbf{R}^d \times \mathbf{R}^d \backslash \{x = y\}$ which satisfies

$$|K(x,y)| \le C|x-y|^{-n}, \qquad x, y \in \mathbf{R}^d,\ x \ne y \tag{2}$$

and

$$|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)| \le C\frac{|y-y'|^\gamma}{|x-y|^{n+\gamma}}, \qquad \text{if } |x-y| \ge 2|y-y'|, \tag{3}$$

where $0 < \gamma \le 1$ and C are positive constants. For $\epsilon > 0$, define the truncated operator T_ϵ by

$$T_\epsilon f(x) = \int_{|x-y|\ge\epsilon} K(x,y) f(y)\, d\mu(y). \tag{4}$$

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For a fixed p with $1 < p < \infty$, we note that if the operators $\{T_\epsilon\}_{\epsilon>0}$ are bounded on $L^p(\mu)$ uniformly, then there is an operator T bounded on $L^2(\mu)$ which is the weak limit as $\epsilon \to 0$ of some subsequence of the uniformly bounded operators $\{T_\epsilon\}_{\epsilon>0}$. We say that the operator T is bounded on $L^2(\mu)$ if T_ϵ is bounded on $L^2(\mu)$ with a bound independent of ϵ. In the remarkable work [6], Tolsa established the Calderón-Zygmund decomposition and showed that if T is bounded on $L^2(\mu)$, then T is also bounded from $L^1(\mu)$ to weak $L^1(\mu)$. The purpose of this paper is to show that the $L^p(\mu)$ boundedness of T can be deduced from some weak type estimates. Our main result can be stated as follows.

Theorem. *Let $0 < r < 1 < p_0 < \infty$, and let Φ be a Young function such that*

$$\Phi(t_1 t_2) \leq C\Phi(t_1)\Phi(t_2), \qquad \textit{for any } t_1, t_2 \geq 0$$

and

$$\int_0^1 \Phi(1/t) t^{p_0-1}\, dt + \int_1^\infty \Phi(1/t) t^{r-1}\, dt < \infty.$$

Let $K(x,y)$ be a locally μ-integrable function on $\mathbf{R}^d \times \mathbf{R}^d \backslash \{x = y\}$ that satisfies (2) *and* (3)*, and let T_ϵ be the operator defined by* (4)*. Suppose that there is a positive constant C independent of ϵ, such that, for any $\lambda > 0$ and any bounded function f with compact support,*

$$\mu(\{x \in \mathbf{R}^d : |T_\epsilon f(x)| > \lambda\}) \leq C \int_{\mathbf{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x).$$

Then T_ϵ is bounded on $L^p(\mu)$ with a bound independent of ϵ for any p with $1 < p < \infty$.

As an application of Theorem, we have

Corollary. *Let $K(x,y)$ be a locally μ-integrable function on $\mathbf{R}^d \times \mathbf{R}^d \backslash \{x = y\}$ that satisfies* (2) *and* (3)*, and let T_ϵ be the operator defined by* (4)*. Suppose that, for some $1 \leq p_0 < \infty$ and $0 < \gamma < \infty$, there is a positive constant C independent of ϵ, such that, for any $\lambda > 0$ and any bounded function f with compact support,*

$$\mu(\{x \in \mathbf{R}^d : |T_\epsilon f(x)| > \lambda\}) \leq C \int_{\mathbf{R}^d} \left(\frac{|f(x)|}{\lambda}\right)^{p_0} \log^\gamma\left(2 + \frac{|f(x)|}{\lambda}\right) d\mu(x).$$

Then T_ϵ is bounded on $L^p(\mu)$ uniformly for any p with $1 < p < \infty$.

Remark. Even if μ is the Lebesgue measure, both Theorem and Corollary are new.

Throughout this paper, C denotes a constant that is independent of the main parameters involved but whose value may differ from line to line. By a cube $Q \subset \mathbf{R}^n$ we mean a closed cube whose sides are parallel to the axes and we denote its side length by l_Q. Let β_d be a fixed positive constant such that $\beta_d > 2^n$. For a cube Q, we say that Q is doubling if $\mu(2Q) \le \beta_d \mu(Q)$, and $\tilde{Q}$ is the smallest doubling cube of the family $\{2^k Q\}_{k \ge 0}$, where αQ denotes the cube concentric with Q and having side length αl_Q. For two cubes $Q_1 \subset Q_2$, set

$$S_{Q_1,Q_2} = 1 + \sum_{k=1}^{N_{Q_1,Q_2}} \frac{\mu(2^k Q_1)}{l_{2^k Q_1}^n},$$

where N_{Q_1,Q_2} is the least integer k such that $l_{2^k Q_1} \ge l_{Q_2}$. For a locally integrable function f, $M^\# f$ denotes the sharp maximal function of f defined by Tolsa [5], that is,

$$M^\# f(x) = \sup_{Q \ni x} \frac{1}{\mu(\frac{3}{2}Q)} \int_Q |f(y) - m_{\tilde{Q}}(f)| d\mu(y) + \sup_{\substack{x \in Q \subset R \\ Q, R \text{ are doubling}}} \frac{|m_Q(f) - m_R(f)|}{S_{Q,R}},$$

where $m_Q(f)$ is the mean value of f on the cube Q. For $0 < r < \infty$, let $M_r^\# f = (M^\#(|f|^r))^{1/r}$. For $0 < s < \infty$ and $\eta > 1$, $M_{s,\eta}$ is the non central maximal operator defined by

$$M_{s,\eta} f(x) = \sup_{Q \ni x} \left(\frac{1}{\mu(\eta Q)} \int_Q |f(y)|^s d\mu(y) \right)^{1/s}.$$

It is well known that $M_{s,\eta}$ is bounded on $L^p(\mu)$ provided that $s < p < \infty$.

2. Proof of Theorem

We begin with some preliminary lemmas.

Lemma 1 (see [4]). *Let $1 < p_0 < \infty$, and let T be a sublinear operator on the set of bounded functions with compact support to the set of μ-measurable functions on $\mathbf{R}^d$. Suppose that T satisfies*

(i) *there is a positive constant C such that*

$$\|Tf\|_{L^\infty(\mu)} \le C\|f\|_{L^\infty(\mu)},$$

(ii) *for a Young function $\Phi : [0, \infty) \to [0, \infty)$ which satisfies*

$$\int_0^1 t^{p_0 - 1} \Phi(1/t) dt < \infty,$$

there is a positive constant C such that for $\lambda > 0$,

$$\mu(\{x \in \mathbf{R}^d : |Tf(x)| > \lambda\}) \le C \int_{\mathbf{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x).$$

Then T is bounded on $L^{p_0}(\mu)$.

The following lemma will be useful in the proof of our theorem, and is of independent interest.

LEMMA 2. *Let $0 < r < 1$. Under the assumptions of Theorem, there is a positive constant C which is independent of ϵ such that for any cube Q and bounded function f with compact support contained in Q,*

$$\frac{1}{\mu(Q)} \int_Q |T_\epsilon f(x)|^r d\mu(x) \le C\|f\|_{\Phi, Q, \mu}^r,$$

where $\|\cdot\|_{\Phi, Q, \mu}$ is the Luxemburg norm defined by

$$\|f\|_{\Phi, Q, \mu} = \inf\left\{t > 0 : \frac{1}{\mu(Q)} \int_Q \Phi\left(\frac{|f(x)|}{t}\right) d\mu(x) \le 1\right\}.$$

PROOF. By homogeneity, we may assume that $\|f\|_{\Phi, Q, \mu} = 1$ which in turn implies that

$$\frac{1}{\mu(Q)} \int_Q \Phi(|f(x)|) d\mu(x) \le 1.$$

A straightforward computation shows that

$$\begin{aligned}
\int_Q |T_\varepsilon f(x)|^r d\mu(x) &= r\int_0^1 \lambda^{r-1} \mu(\{x \in Q : |T_\varepsilon f(x)| > \lambda\}) d\lambda \\
&\quad + r\int_1^\infty \lambda^{r-1} \mu(\{x \in Q : |T_\varepsilon f(x)| > \lambda\}) d\lambda \\
&\le C\mu(Q) + \int_Q \int_1^\infty \lambda^{r-1} \Phi(1/\lambda) d\lambda \Phi(|f(x)|) d\mu(x) \\
&\le C\mu(Q).
\end{aligned}$$

The desired estimate then follows directly.

LEMMA 3. *Let $K(x, y)$ be a locally μ-integrable function on $\mathbf{R}^d \times \mathbf{R}^d \backslash \{x = y\}$ which satisfies* (2) *and* (3), *and let T_ϵ be the operator defined by* (4). *Then there is a positive constant C which is independent of ϵ such that for any bounded function f with compact support,*

$$|m_Q(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus(4/3)Q})|^r) - m_R(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus(4/3)R})|^r)| \le CS_{Q,R}^r\|f\|_{L^\infty(\mu)}^r,$$

where $Q \subset R$, Q is an arbitrary cube and R is a doubling cube.

Proof. Denote $N_{Q,R} + 1$ simply by N. Recalling that R is doubling, we obtain $\tilde{R} = R$. We write

$$\begin{aligned}
&|m_Q(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus(4/3)Q})|^r) - m_R(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus(4/3)R})|^r)| \\
&\quad \le |m_Q(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus(4/3)Q})|^r) - m_Q(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})|^r)| \\
&\qquad + |m_Q(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})|^r) - m_R(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})|^r)| \\
&\qquad + |m_R(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus(4/3)R})|^r) - m_R(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})|^r)| \\
&\quad \le m_Q(|T_\epsilon(f\chi_{2^N Q\setminus(4/3)Q})|^r) \\
&\qquad + |m_Q(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})|^r) - m_R(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})|^r)| \\
&\qquad + m_R(|T_\epsilon(f\chi_{2^N Q\setminus(4/3)R})|^r) \\
&\quad = \mathrm{I} + \mathrm{J} + \mathrm{H}.
\end{aligned}$$

The estimate for H is easy. Indeed, by the growth condition (1), we see that for each fixed $y \in R$,

$$|T_\epsilon(f\chi_{2^N Q\setminus(4/3)R})(y)| \le Cl_R^{-n}\mu(2^N Q)\|f\|_{L^\infty(\mu)} \le C\|f\|_{L^\infty(\mu)},$$

which implies that

$$\mathrm{H} \le C\|f\|_{L^\infty(\mu)}.$$

Recall that $N_{Q,R}$ is the least integer k such that $2^k Q \supset R$. A simple computation involving the regularity condition (3) shows that for any $x \in Q$ and $y \in R$,

$$\begin{aligned}
&||T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})(x)|^r - |T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})(y)|^r| \\
&\quad \le |T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})(x) - T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})(y)|^r \\
&\quad \le \left(\|f\|_{L^\infty(\mu)}\int_{\mathbf{R}^d\setminus 2^N Q}|K(x,z) - K(y,z)|d\mu(z)\right)^r \\
&\quad \le C\|f\|_{L^\infty(\mu)}^r.
\end{aligned}$$

This gives the desired estimate for J in view of the estimate

$$|m_Q(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})|^r) - m_R(|T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})|^r)|$$
$$\leq \frac{1}{\mu(Q)\mu(R)}\int_Q\int_R ||T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})(x)|^r - |T_\epsilon(f\chi_{\mathbf{R}^d\setminus 2^N Q})(y)|^r| d\mu(x)d\mu(y).$$

On the other hand, we can verify that for any $y \in Q$,

$$\begin{aligned}|T_\epsilon(f\chi_{2^N Q\setminus(4/3)Q})(y)| &\leq \|f\|_{L^\infty(\mu)}\left(\int_{2^N Q\setminus 2^{N-1}Q}\frac{1}{|y-z|^n}\,d\mu(z)\right.\\ &\quad\left.+\int_{2^{N-1}Q\setminus 2Q}\frac{1}{|y-z|^n}\,d\mu(z)+\int_{2Q\setminus(4/3)Q}\frac{1}{|y-z|^n}\,d\mu(z)\right)\\ &\leq C\|f\|_{L^\infty(\mu)}\left(\frac{\mu(2^N Q)}{l^n_{2^{N-1}Q}}+\sum_{k=1}^{N-1}\frac{\mu(2^k Q)}{l^n_{2^{k-1}Q}}+\frac{\mu(2Q)}{l^n_{(4/3)Q}}\right)\\ &\leq C\|f\|_{L^\infty(\mu)}S_{Q,R},\end{aligned}$$

and so

$$\mathrm{I} \leq C\|f\|^r_{L^\infty(\mu)}S^r_{Q,R}.$$

This completes the proof of Lemma 3.

Lemma 4. *Let $1 < p < \infty$. There is a constant C such that if f satisfies*

$$\int_{\mathbf{R}^d} f(x)d\mu(x) = 0$$

and

$$\int_0^R \lambda^{p-1}\mu(\{x \in \mathbf{R}^d : |f(x)| > \lambda\})d\lambda < \infty, \qquad \textit{for any } R > 0 \tag{5}$$

then

$$\|Nf\|_{L^p(\mu)} \leq C\|M^\# f\|_{L^p(\mu)}, \tag{6}$$

where N is the non centered doubling maximal operator defined by

$$Nh(x) = \sup_{x\in Q,\, Q \text{ is doubling}} \frac{1}{\mu(Q)}\int_Q |h(x)|d\mu(x).$$

Proof. By Lemma 6.2 in [5], we see that if $f \in L^p(\mu)$, then

$$\|Nf\|_{L^p(\mu)} \leq C\|M^\# f\|_{L^p(\mu)}.$$

Suppose that $f \in L^1(\mu)$ satisfies (5), and let

$$f_k(x) = f(x), \quad \text{if } |f(x)| \leq k; \qquad f_k(x) = \frac{f(x)}{|f(x)|}k, \quad \text{if } |f(x)| > k.$$

It is easy to verify that

$$\begin{aligned}\int_{\mathbf{R}^d} |f_k(x)|^p d\mu(x) &= p\int_0^k \lambda^{p-1}\mu(\{x\in\mathbf{R}^d : |f_k(x)|>\lambda\})d\lambda \\ &\le p\int_0^k \mu(\{x\in\mathbf{R}^d : |f(x)|>\lambda\})\lambda^{p-1}\,d\lambda<\infty,\end{aligned}$$

and so $f_k \in L^p(\mu)$. Therefore, the same argument used in [6, p. 130] yields that

$$\|Nf_k\|_{L^p(\mu)} \le C\|M^{\#}f_k\|_{L^p(\mu)}.$$

Letting $k\to\infty$, we then obtain our estimate (6).

Proof of Theorem. At first, we claim that there is a positive constant C such that for any bounded function f with compact support,

$$\|M_r^{\#}(Tf)\|_{L^\infty(\mu)} \le C\|f\|_{L^\infty(\mu)}. \tag{7}$$

In order to show this, we may assume that $\|f\|_{L^\infty(\mu)}=1$. Note that our assumption together with Lemma 2 implies that $|Tf|^r$ is locally integrable if f is bounded with compact support. By Lemma 3, we see that the proof of (7) can be reduced to proving that there is a positive constant C such that

$$\frac{1}{\mu(\frac{3}{2}Q)}\int_Q \big|\,|T_\epsilon f(x)|^r - m_{\tilde{Q}}(|T_\epsilon f|^r)\big|d\mu(x)\le C. \tag{8}$$

For each fixed cube Q, set

$$h_Q = m_Q(|T_\epsilon f\chi_{\mathbf{R}^d\setminus(4/3)Q}|^r).$$

If we can show that for any cube Q,

$$\frac{1}{\mu(\frac{3}{2}Q)}\int_Q \big|\,|T_\epsilon f(x)|^r - h_Q\big|d\mu(x)\le C, \tag{9}$$

then

$$\begin{aligned}&\frac{1}{\mu(\frac{3}{2}Q)}\int_Q \big|\,|T_\epsilon f(x)|^r - m_{\tilde{Q}}(|T_\epsilon f|^r)\big|d\mu(x)\\ &\quad\le \frac{1}{\mu(\frac{3}{2}Q)}\int_Q \big|\,|T_\epsilon f(x)|^r - h_Q\big|d\mu(x) + |h_Q-h_{\tilde{Q}}| + |h_{\tilde{Q}} - m_{\tilde{Q}}(|T_\epsilon f|^r)|\\ &\quad\le C + CS_{Q,\tilde{Q}}^r + \frac{1}{\mu(\frac{3}{2}\tilde{Q})}\left|\int_{\tilde{Q}}(|T_\epsilon f(x)|^r - h_{\tilde{Q}})d\mu(x)\right|\\ &\quad\le C,\end{aligned}$$

and the inequality (8) follows.

Now we prove (9). For a fixed cube Q and a bounded function f with compact support such that $\|f\|_{L^\infty(\mu)} = 1$, we decompose f as

$$f(x) = f(x)\chi_{(4/3)Q}(x) + f(x)\chi_{\mathbf{R}^d\setminus(4/3)Q}(x) = f_1(x) + f_2(x).$$

Write

$$\begin{aligned}\frac{1}{\mu(\frac{3}{2}Q)}\int_Q ||T_\epsilon f(x)|^r - h_Q|d\mu(x) &\le \frac{1}{\mu(\frac{3}{2}Q)}\int_Q ||T_\epsilon f(x)|^r - |T_\epsilon f_2(x)|^r|d\mu(x)\\ &\quad + \frac{1}{\mu(\frac{3}{2}Q)}\int_Q ||T_\epsilon f_2(x)|^r - h_Q|d\mu(x)\\ &\le \frac{1}{\mu(\frac{3}{2}Q)}\int_Q |T_\epsilon f_1(x)|^r d\mu(x)\\ &\quad + \frac{1}{\mu(\frac{3}{2}Q)}\int_Q ||T_\epsilon f_2(x)|^r - h_Q|d\mu(x).\end{aligned}$$

Our hypothesis along with Lemma 2 yields that

$$\frac{1}{\mu(\frac{3}{2}Q)}\int_Q |T_\epsilon f_1(x)|^r d\mu(x) \le C.$$

On the other hand, for any $x, y \in Q$, an easy computation yields

$$\begin{aligned}||T_\epsilon f_2(x)|^r - |T_\epsilon f_2(y)|^r| &\le |T_\epsilon f_2(x) - T_\epsilon f_2(y)|^r\\ &\le \left(\int_{\mathbf{R}^d\setminus(4/3)Q} |K(x,z) - K(y,z)|d\mu(z)\right)^r\\ &\le C.\end{aligned}$$

Thus, for any $x \in Q$, we have

$$||T_\epsilon f_2(x)|^r - h_Q| = \frac{1}{\mu(Q)}\left|\int_Q (|T_\epsilon f_2(y)|^r - |T_\epsilon f_2(x)|^r)d\mu(y)\right| \le C,$$

which implies that

$$\frac{1}{\mu(\frac{3}{2}Q)}\int_Q ||T_\epsilon f_2(x)|^r - h_Q|d\mu(x) \le C.$$

The inequality (9) then holds.

We can now complete the proof of Theorem. Obviously, it suffices to show that T is bounded on $L^{p_0}(\mu)$. Repeating the argument used in [1, p. 471], for any $h \in L^1_{loc}(\mu)$ and $\lambda > 0$, we see that

$$\lambda\mu(\{x \in \mathbf{R}^d : M_{r,3/2}h(x) > \lambda\}) \leq C \sup_{\tau \geq C\lambda} \tau\mu(\{x \in \mathbf{R}^d : |h(x)| > C\tau\}).$$

This together with the fact that

$$M_r^{\#}h(x) \leq M_{r,3/2}h(x)$$

yields that

$$\begin{aligned}\mu(\{x \in \mathbf{R}^d : M_r^{\#}(Tf)(x) > \lambda\}) &\leq C\lambda^{-1} \sup_{\tau \geq C\lambda} \tau \int_{\mathbf{R}^d} \Phi\left(\frac{|f(x)|}{\tau}\right) d\mu(x) \\ &\leq C\lambda^{-1} \sup_{\tau \geq C\lambda} \tau\Phi(\lambda\tau^{-1}) \int_{\mathbf{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \\ &\leq C \int_{\mathbf{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x).\end{aligned}$$

Thus, by Lemma 1, we have

$$\|M_r^{\#}(Tf)\|_{L^{p_0}(\mu)} \leq C\|f\|_{L^{p_0}(\mu)}. \tag{10}$$

Observe that for any bounded function f with compact support and $R > 0$,

$$\begin{aligned}&\int_0^R \lambda^{p_0/r-1}\mu(\{x \in \mathbf{R}^d : |Tf(x)|^r > \lambda\}) d\lambda \\ &= \int_0^{R^{1/r}} r\lambda^{p_0-1}\mu(\{x \in \mathbf{R}^d : |Tf(x)| > \lambda\}) d\lambda \\ &\leq C \int_{\mathbf{R}^d} \phi(|f(x)|) d\mu(x) \int_0^{R^{1/r}} \lambda^{p_0-1}\phi\left(\frac{1}{\lambda}\right) d\lambda \\ &< \infty.\end{aligned}$$

Therefore, it follows from the inequality (10), Lemma 4 and the last inequality that

$$\begin{aligned}\|Tf\|_{L^{p_0}(\mu)}^r &\leq \|N(|Tf|^r)\|_{L^{p_0/r}(\mu)} \leq \|M^{\#}(|Tf|^r)\|_{L^{p_0/r}(\mu)} \\ &= \|M_r^{\#}(Tf)\|_{L^{p_0}(\mu)}^r \leq C\|f\|_{L^{p_0}(\mu)}^r.\end{aligned}$$

This completes the proof of Theorem.

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