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An asymptotic result for second order linear nonautonomous neutral delay differential equations

Ch. G. PHILOS and I. K. PURNARAS

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ABSTRACT. In this paper, we establish a fundamental asymptotic result for the solutions to second order linear nonautonomous neutral delay differential equations. By the use of a solution of the corresponding generalized characteristic equation, we transform the second order neutral delay differential equation into a first order neutral delay differential equation and then we utilize its generalized characteristic equation.

1. Introduction and preliminaries

Our aim in this paper is to establish an asymptotic result for the solutions to second order linear neutral delay differential equations with variable coefficients and constant delays. An analogous asymptotic criterion for the solutions to second order linear nonautonomous (non-neutral) delay differential equations has recently been obtained by the authors [13]. Our work in the present paper is essentially motivated by the results in the papers by Dix and the authors [3, 4], the authors [9–13], and Yeniçerioğlu [14]. (Some more details on the results in these papers may be found in [13].) For some results closely related to the ones contained in the above mentioned articles, the reader may look at the references cited in [11, 12].

The basic idea in the present work is essentially originated in an old but very interesting asymptotic result due to Driver [5] concerning the solutions of linear differential systems with small delays (see, also, Arino and Pituk [1]). Another idea employed in this paper is that of transforming the second order neutral delay differential equation into a first order neutral delay differential equation, by the use of a solution of the corresponding generalized characteristic equation. In the case of second order linear autonomous delay differential equations, this idea is originated in [14] (see, also, [12]).

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For the basic theory of delay and neutral delay differential equations, we refer to the books by Diekmann *et al.* [2], Driver [6], Hale [7], and Hale and Verduyn Lunel [8].

Consider the neutral delay differential equation

$$x''(t) + c(t)x''(t-\sigma) = a(t)x(t) + b(t)x(t-\tau), \tag{1.1}$$

where *c, a, and b are continuous real-valued functions on the interval* $[0,\infty)$*, and* σ *and* τ *are positive real constants.* Throughout the paper, by r we will denote the positive real number defined by $r = \max\{\sigma, \tau\}$.

By a *solution* of the neutral delay differential equation (1.1), we mean a continuously differentiable real-valued function x defined on the interval $[-r,\infty)$, which is twice continuously differentiable on $[-\sigma,\infty)$ and satisfies (1.1) for all $t \geq 0$.

Together with the neutral delay differential equation (1.1), we specify an *initial condition* of the form

$$x(t) = \phi(t) \qquad \text{for } -r \leq t \leq 0, \tag{1.2}$$

where *the initial function* ϕ *is a given continuously differentiable real-valued function on the initial interval* $[-r, 0]$.

The neutral delay differential equation (1.1) together with the initial condition (1.2) constitute an *initial value problem* (*IVP*, for short). It is well-known (see, for example, Diekmann *et al.* [2], Hale [7], or Hale and Verduyn Lunel [8]; see, also, Driver [6] for the non-neutral case) that there exists a unique solution x of (1.1) which satisfies (1.2); this unique solution x will be called the *solution* of the initial value problem (1.1) and (1.2) or, more briefly, the *solution* of the IVP (1.1) and (1.2).

Along with the neutral delay differential equation (1.1), we associate the following equation

$$\begin{aligned}\lambda'(t) + \lambda^2(t) + c(t)[\lambda'(t-\sigma) + \lambda^2(t-\sigma)] \exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds\right] \\ = a(t) + b(t) \exp\left[-\int_{t-\tau}^{t} \lambda(s)ds\right],\end{aligned} \tag{1.3}$$

which will be called the *generalized characteristic equation* of (1.1). Equation (1.3) is obtained from (1.1) by looking for solutions of the form $x(t) = \exp[\int_0^t \lambda(s)ds]$ for $t \geq -r$, where λ is a continuous real-valued function on the interval $[-r,\infty)$, which is continuously differentiable on $[-\sigma,\infty)$.

A *solution* of the generalized characteristic equation (1.3) is a continuous real-valued function λ defined on the interval $[-r,\infty)$, which is continuously differentiable on $[-\sigma,\infty)$ and satisfies (1.3) for all $t \geq 0$.

For a given solution λ of the generalized characteristic equation (1.3), we consider the (first order) neutral delay differential equation

$$(1.4)\qquad z'(t) + c(t)z'(t-\sigma)\exp\left[-\int_{t-\sigma}^{t}\lambda(s)ds\right] + 2\lambda(t)z(t) + 2c(t)\lambda(t-\sigma)z(t-\sigma)\exp\left[-\int_{t-\sigma}^{t}\lambda(s)ds\right]$$
$$= c(t)[\lambda'(t-\sigma)+\lambda^2(t-\sigma)]\left[\int_{t-\sigma}^{t}z(s)ds\right]\exp\left[-\int_{t-\sigma}^{t}\lambda(s)ds\right] - b(t)\left[\int_{t-\tau}^{t}z(s)ds\right]\exp\left[-\int_{t-\tau}^{t}\lambda(s)ds\right].$$

By a *solution* of the neutral delay differential equation (1.4), we mean a continuous real-valued function z defined on the interval $[-r,\infty)$, which is continuously differentiable on $[-\sigma,\infty)$ and satisfies (1.4) for all $t \geq 0$.

With the neutral delay differential equation (1.4), we associate the equation

$$(1.5)\qquad \mu(t) + 2\lambda(t) + c(t)[\mu(t-\sigma)+2\lambda(t-\sigma)]\exp\left\{-\int_{t-\sigma}^{t}[\lambda(s)+\mu(s)]ds\right\}$$
$$= c(t)[\lambda'(t-\sigma)+\lambda^2(t-\sigma)]\left\{\int_{t-\sigma}^{t}\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\} \times \exp\left[-\int_{t-\sigma}^{t}\lambda(s)ds\right]$$
$$- b(t)\left\{\int_{t-\tau}^{t}\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\}\exp\left[-\int_{t-\tau}^{t}\lambda(s)ds\right],$$

which is said to be the *generalized characteristic equation* of (1.4). The last equation is obtained from (1.4) by seeking solutions of the form $z(t) = \exp[\int_0^t \mu(s)ds]$ for $t \geq -r$, where μ is a continuous real-valued function on the interval $[-r,\infty)$.

A *solution* of the generalized characteristic equation (1.5) is a continuous real-valued function μ defined on the interval $[-r,\infty)$, which satisfies (1.5) for all $t \geq 0$.

The main result of the paper (Theorem 3.1) is established in Section 3. An auxiliary result (Proposition 2.1) used in proving Theorem 3.1, is given in Section 2. In Section 4, an example demonstrating the applicability of our main result is presented. The possibility of extending our results to more general second order linear nonautonomous neutral delay differential equations is discussed in Section 5.

2. An auxiliary result

For our convenience, we introduce some notation. For a given solution λ of the generalized characteristic equation (1.3), we define

$$\Phi(\lambda;\phi)(t) = \left\{\phi(t)\exp\left[-\int_0^t \lambda(s)ds\right]\right\}' \qquad \text{for } -r \le t \le 0. \tag{2.1}$$

Clearly, $\Phi(\lambda;\phi)$ is a continuous real-valued function on the initial interval $[-r, 0]$.

Proposition 2.1 below will be used in proving the main result of the paper, i.e., Theorem 3.1 given in the next section. This proposition essentially establishes a transformation (via a solution of the generalized characteristic equation (1.3)) of the second order neutral delay differential equation (1.1) into the first order neutral delay differential equation (1.4).

PROPOSITION 2.1. *Let* λ *be a solution of the generalized characteristic equation* (1.3), *and define* $\Phi(\lambda;\phi)$ *by* (2.1).

Then a continuously differentiable real-valued function x *defined on the interval* $[-r,\infty)$ *is the solution of the IVP* (1.1) *and* (1.2) *if and only if the function* z *defined by*

$$z(t) = \left\{x(t)\exp\left[-\int_0^t \lambda(s)ds\right]\right\}' \qquad \textit{for } t \ge -r \tag{2.2}$$

is the solution of the neutral delay differential equation (1.4) *which satisfies the initial condition*

$$z(t) = \Phi(\lambda;\phi)(t) \qquad \textit{for } -r \le t \le 0. \tag{2.3}$$

PROOF. Consider the solution x of the IVP (1.1) and (1.2), and define

$$y(t) = x(t)\exp\left[-\int_0^t \lambda(s)ds\right] \qquad \text{for } t \ge -r. \tag{2.4}$$

Then, by taking into account the fact that λ is a solution of the generalized characteristic equation (1.3), we obtain, for every $t \ge 0$,

$$\begin{aligned}
&[x''(t) + c(t)x''(t-\sigma) - a(t)x(t) - b(t)x(t-\tau)]\exp\left[-\int_0^t \lambda(s)ds\right] \\
&\quad = \{y''(t) + 2\lambda(t)y'(t) + [\lambda'(t) + \lambda^2(t)]y(t)\} \\
&\qquad + c(t)\{y''(t-\sigma) + 2\lambda(t-\sigma)y'(t-\sigma) + [\lambda'(t-\sigma) + \lambda^2(t-\sigma)]y(t-\sigma)\} \\
&\qquad \times \exp\left[-\int_{t-\sigma}^t \lambda(s)ds\right] - a(t)y(t) - b(t)y(t-\tau)\exp\left[-\int_{t-\tau}^t \lambda(s)ds\right]
\end{aligned}$$

$$
\begin{aligned}
&= y''(t) + 2\lambda(t)y'(t) + [\lambda'(t) + \lambda^2(t) - a(t)]y(t) \\
&\quad + c(t)\{y''(t-\sigma) + 2\lambda(t-\sigma)y'(t-\sigma) + [\lambda'(t-\sigma) + \lambda^2(t-\sigma)]y(t-\sigma)\} \\
&\quad \times \exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds\right] - b(t)y(t-\tau)\exp\left[-\int_{t-\tau}^{t} \lambda(s)ds\right] \\
&= y''(t) + c(t)y''(t-\sigma)\exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds\right] \\
&\quad + 2\lambda(t)y'(t) + 2c(t)\lambda(t-\sigma)y'(t-\sigma)\exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds\right] \\
&\quad + \left\{-c(t)[\lambda'(t-\sigma) + \lambda^2(t-\sigma)]\exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds\right]\right. \\
&\qquad \left. + b(t)\exp\left[-\int_{t-\tau}^{t} \lambda(s)ds\right]\right\} y(t) \\
&\quad + c(t)[\lambda'(t-\sigma) + \lambda^2(t-\sigma)]y(t-\sigma)\exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds\right] \\
&\quad - b(t)y(t-\tau)\exp\left[-\int_{t-\tau}^{t} \lambda(s)ds\right] \\
&= y''(t) + c(t)y''(t-\sigma)\exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds\right] \\
&\quad + 2\lambda(t)y'(t) + 2c(t)\lambda(t-\sigma)y'(t-\sigma)\exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds\right] \\
&\quad - c(t)[\lambda'(t-\sigma) + \lambda^2(t-\sigma)][y(t) - y(t-\sigma)]\exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds\right] \\
&\quad + b(t)[y(t) - y(t-\tau)]\exp\left[-\int_{t-\tau}^{t} \lambda(s)ds\right].
\end{aligned}
$$

Thus, the fact that x satisfies (1.1) for all $t \geq 0$ is equivalent to the fact that y satisfies

$$
\begin{aligned}
(2.5) \qquad & y''(t) + c(t)y''(t-\sigma)\exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds\right] \\
&\quad + 2\lambda(t)y'(t) + 2c(t)\lambda(t-\sigma)y'(t-\sigma)\exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds\right] \\
&= c(t)[\lambda'(t-\sigma) + \lambda^2(t-\sigma)][y(t) - y(t-\sigma)]\exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds\right] \\
&\quad - b(t)[y(t) - y(t-\tau)]\exp\left[-\int_{t-\tau}^{t} \lambda(s)ds\right]
\end{aligned}
$$

for all $t \geq 0$. Moreover, we observe that, in terms of the function y, the initial condition (1.2) is equivalently written as

$$y(t) = \phi(t) \exp\left[-\int_0^t \lambda(s)ds\right] \qquad \text{for } -r \leq t \leq 0. \tag{2.6}$$

Furthermore, we see that y satisfies (2.5) for all $t \geq 0$ if and only if it satisfies the equation

$$\begin{aligned} y''(t) &+ c(t)y''(t-\sigma) \exp\left[-\int_{t-\sigma}^t \lambda(s)ds\right] \\ &+ 2\lambda(t)y'(t) + 2c(t)\lambda(t-\sigma)y'(t-\sigma) \exp\left[-\int_{t-\sigma}^t \lambda(s)ds\right] \\ &= c(t)[\lambda'(t-\sigma) + \lambda^2(t-\sigma)]\left[\int_{t-\sigma}^t y'(s)ds\right] \exp\left[-\int_{t-\sigma}^t \lambda(s)ds\right] \\ &- b(t)\left[\int_{t-\tau}^t y'(s)ds\right] \exp\left[-\int_{t-\tau}^t \lambda(s)ds\right] \end{aligned} \tag{2.7}$$

for all $t \geq 0$. Next, we set

$$z(t) = y'(t) \qquad \text{for } t \geq -r. \tag{2.8}$$

Then we observe that y satisfies (2.7) for all $t \geq 0$ if and only if z satisfies (1.4) for all $t \geq 0$, i.e., if and only if z is a solution of the neutral delay differential equation (1.4). Moreover, by the use of the function z, the initial condition (2.6) takes the following equivalent form

$$z(t) = \left\{\phi(t) \exp\left[-\int_0^t \lambda(s)ds\right]\right\}' \qquad \text{for } -r \leq t \leq 0. \tag{2.9}$$

Hence, it has been established that x is the solution of the IVP (1.1) and (1.2) if and only if z is the solution of the neutral delay differential equation (1.4) which satisfies the initial condition (2.9). Because of (2.4), it is clear that (2.8) coincides with (2.2). Also, by taking into account the definition of $\Phi(\lambda;\phi)$ by (2.1), we see that (2.9) coincides with the initial condition (2.3). The proof of our proposition is complete.

3. The main result

Our main result in the present paper is the following theorem.

Theorem 3.1. *Let λ be a solution of the generalized characteristic equation* (1.3). *Furthermore, let μ be a solution of the generalized characteristic equation* (1.5). *Assume that*

$$
\begin{aligned}
(3.1)\quad \limsup_{t\to\infty}\Bigg(&|c(t)|[1+|\mu(t-\sigma)+2\lambda(t-\sigma)|\sigma]\exp\left\{-\int_{t-\sigma}^{t}[\lambda(s)+\mu(s)]ds\right\}\\
&+|c(t)|\,|\lambda'(t-\sigma)+\lambda^2(t-\sigma)|\left\{\int_{t-\sigma}^{t}(t-s)\exp\left[-\int_s^t\mu(u)du\right]ds\right\}\\
&\times\exp\left[-\int_{t-\sigma}^{t}\lambda(s)ds\right]\\
&+|b(t)|\left\{\int_{t-\tau}^{t}(t-s)\exp\left[-\int_s^t\mu(u)du\right]ds\right\}\exp\left[-\int_{t-\tau}^{t}\lambda(s)ds\right]\Bigg)<1.
\end{aligned}
$$

Then the solution x of the IVP (1.1) *and* (1.2) *satisfies*

$$
(3.2)\qquad \lim_{t\to\infty}\left(\left\{x(t)\exp\left[-\int_0^t\lambda(s)ds\right]\right\}'\exp\left[-\int_0^t\mu(s)ds\right]\right)=L(\lambda,\mu;\phi),
$$

where $L(\lambda,\mu;\phi)$ is some real number depending on λ, μ and determined by ϕ, and

$$
(3.3)\qquad \lim_{t\to\infty}\left(\left\{x(t)\exp\left[-\int_0^t\lambda(s)ds\right]\right\}'\exp\left[-\int_0^t\mu(s)ds\right]\right)'=0.
$$

Before we proceed to the proof of Theorem 3.1, we will present a particular result, as a consequence of this theorem. Let λ and μ be as in Theorem 3.1, and assume that (3.1) holds. Then from Theorem 3.1 it follows that the solution x of the IVP (1.1) and (1.2) is such that

$$
\lim_{t\to\infty}\left([x'(t)-\lambda(t)x(t)]\exp\left\{-\int_0^t[\lambda(s)+\mu(s)]ds\right\}\right)
$$

exists (as a real number). Hence, the solution x satisfies

$$
|x'(t)-\lambda(t)x(t)|\le K(\lambda,\mu;\phi)\exp\left\{\int_0^t[\lambda(s)+\mu(s)]ds\right\}\qquad \text{for all } t\ge -r,
$$

where $K(\lambda,\mu;\phi)$ is some positive real constant. Thus, we have arrived at the next result:

Let λ and μ be as in Theorem 3.1, *and assume that* (3.1) *holds. Then, for the solution x of the IVP* (1.1) *and* (1.2), *we have:*

(i) *$x'-\lambda x$ is bounded if $\limsup_{t\to\infty}\int_0^t[\lambda(s)+\mu(s)]ds<\infty$;*

(ii) *$x'-\lambda x$ tends to zero at ∞ if $\lim_{t\to\infty}\int_0^t[\lambda(s)+\mu(s)]ds=-\infty$.*

Proof of Theorem 3.1. Let x be the solution of the IVP (1.1) and (1.2), and define the function z by (2.2). By Proposition 2.1, the fact that x is the solution of the IVP (1.1) and (1.2) is equivalent to the fact that z is the solution of the neutral delay differential equation (1.4) which satisfies the initial condition (2.3), where $\Phi(\lambda;\phi)$ is defined by (2.1).

Set

$$w(t) = z(t) \exp\left[-\int_0^t \mu(s)ds\right] \qquad \text{for } t \geq -r. \tag{3.4}$$

Then, as μ is a solution of the generalized characteristic equation (1.5), we derive, for every $t \geq 0$,

$$\begin{aligned}
&\Bigg\{z'(t) + c(t)z'(t-\sigma) \exp\left[-\int_{t-\sigma}^t \lambda(s)ds\right] \\
&\quad + 2\lambda(t)z(t) + 2c(t)\lambda(t-\sigma)z(t-\sigma) \exp\left[-\int_{t-\sigma}^t \lambda(s)ds\right] \\
&\quad - c(t)[\lambda'(t-\sigma) + \lambda^2(t-\sigma)]\left[\int_{t-\sigma}^t z(s)ds\right] \exp\left[-\int_{t-\sigma}^t \lambda(s)ds\right] \\
&\quad + b(t)\left[\int_{t-\tau}^t z(s)ds\right] \exp\left[-\int_{t-\tau}^t \lambda(s)ds\right]\Bigg\} \exp\left[-\int_0^t \mu(s)ds\right] \\
&= [w'(t) + \mu(t)w(t)] + c(t)[w'(t-\sigma) + \mu(t-\sigma)w(t-\sigma)] \\
&\quad \times \exp\left\{-\int_{t-\sigma}^t [\lambda(s) + \mu(s)]ds\right\} + 2\lambda(t)w(t) + 2c(t)\lambda(t-\sigma)w(t-\sigma) \\
&\quad \times \exp\left\{-\int_{t-\sigma}^t [\lambda(s) + \mu(s)]ds\right\} - c(t)[\lambda'(t-\sigma) + \lambda^2(t-\sigma)] \\
&\quad \times \left\{\int_{t-\sigma}^t w(s) \exp\left[-\int_s^t \mu(u)du\right] ds\right\} \exp\left[-\int_{t-\sigma}^t \lambda(s)ds\right] \\
&\quad + b(t)\left\{\int_{t-\tau}^t w(s) \exp\left[-\int_s^t \mu(u)du\right] ds\right\} \exp\left[-\int_{t-\tau}^t \lambda(s)ds\right] \\
&= w'(t) + c(t)w'(t-\sigma) \exp\left\{-\int_{t-\sigma}^t [\lambda(s) + \mu(s)]ds\right\} + [\mu(t) + 2\lambda(t)]w(t) \\
&\quad + c(t)[\mu(t-\sigma) + 2\lambda(t-\sigma)]w(t-\sigma) \exp\left\{-\int_{t-\sigma}^t [\lambda(s) + \mu(s)]ds\right\} \\
&\quad - c(t)[\lambda'(t-\sigma) + \lambda^2(t-\sigma)]\left\{\int_{t-\sigma}^t w(s) \exp\left[-\int_s^t \mu(u)du\right] ds\right\} \\
&\quad \times \exp\left[-\int_{t-\sigma}^t \lambda(s)ds\right] \\
&\quad + b(t)\left\{\int_{t-\tau}^t w(s) \exp\left[-\int_s^t \mu(u)du\right] ds\right\} \exp\left[-\int_{t-\tau}^t \lambda(s)ds\right]
\end{aligned}$$

$$
\begin{aligned}
&= w'(t) + c(t)w'(t-\sigma)\exp\left\{-\int_{t-\sigma}^{t}[\lambda(s)+\mu(s)]ds\right\} \\
&\quad + \Bigg(-c(t)[\mu(t-\sigma)+2\lambda(t-\sigma)]\exp\left\{-\int_{t-\sigma}^{t}[\lambda(s)+\mu(s)]ds\right\} \\
&\quad + c(t)[\lambda'(t-\sigma)+\lambda^2(t-\sigma)]\left\{\int_{t-\sigma}^{t}\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\} \\
&\quad \times \exp\left[-\int_{t-\sigma}^{t}\lambda(s)ds\right] \\
&\quad - b(t)\left\{\int_{t-\tau}^{t}\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\}\exp\left[-\int_{t-\tau}^{t}\lambda(s)ds\right]\Bigg)w(t) \\
&\quad + c(t)[\mu(t-\sigma)+2\lambda(t-\sigma)]w(t-\sigma)\exp\left\{-\int_{t-\sigma}^{t}[\lambda(s)+\mu(s)]ds\right\} \\
&\quad - c(t)[\lambda'(t-\sigma)+\lambda^2(t-\sigma)]\left\{\int_{t-\sigma}^{t}w(s)\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\} \\
&\quad \times \exp\left[-\int_{t-\sigma}^{t}\lambda(s)ds\right] \\
&\quad + b(t)\left\{\int_{t-\tau}^{t}w(s)\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\}\exp\left[-\int_{t-\tau}^{t}\lambda(s)ds\right] \\
&= w'(t) + c(t)w'(t-\sigma)\exp\left\{-\int_{t-\sigma}^{t}[\lambda(s)+\mu(s)]ds\right\} \\
&\quad - c(t)[\mu(t-\sigma)+2\lambda(t-\sigma)][w(t)-w(t-\sigma)]\exp\left\{-\int_{t-\sigma}^{t}[\lambda(s)+\mu(s)]ds\right\} \\
&\quad + c(t)[\lambda'(t-\sigma)+\lambda^2(t-\sigma)]\left\{\int_{t-\sigma}^{t}[w(t)-w(s)]\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\} \\
&\quad \times \exp\left[-\int_{t-\sigma}^{t}\lambda(s)ds\right] \\
&\quad - b(t)\left\{\int_{t-\tau}^{t}[w(t)-w(s)]\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\}\exp\left[-\int_{t-\tau}^{t}\lambda(s)ds\right].
\end{aligned}
$$

So, z is a solution of the neutral delay differential equation (1.4) if and only if w satisfies

$$w'(t) + c(t)w'(t-\sigma)\exp\left\{-\int_{t-\sigma}^{t}[\lambda(s)+\mu(s)]ds\right\}$$
$$= c(t)[\mu(t-\sigma) + 2\lambda(t-\sigma)][w(t) - w(t-\sigma)]$$
$$\times \exp\left\{-\int_{t-\sigma}^{t}[\lambda(s)+\mu(s)]ds\right\} - c(t)[\lambda'(t-\sigma)+\lambda^2(t-\sigma)]$$
$$\times \left\{\int_{t-\sigma}^{t}[w(t)-w(s)]\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\}\exp\left[-\int_{t-\sigma}^{t}\lambda(s)ds\right]$$
$$+ b(t)\left\{\int_{t-\tau}^{t}[w(t)-w(s)]\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\}\exp\left[-\int_{t-\tau}^{t}\lambda(s)ds\right] \tag{3.5}$$

for all $t \geq 0$. Moreover, we see that the initial condition (2.3) can be written in the following equivalent form

$$w(t) = \Phi(\lambda;\phi)(t)\exp\left[-\int_{0}^{t}\mu(s)ds\right] \qquad \text{for } -r \leq t \leq 0. \tag{3.6}$$

We have thus proved that x is the solution of the IVP (1.1) and (1.2) if and only if w satisfies (3.5) for all $t \geq 0$ and also w satisfies the initial condition (3.6).

By the definitions of the functions z and w by (2.2) and (3.4), respectively, we have

$$w(t) = \left\{x(t)\exp\left[-\int_{0}^{t}\lambda(s)ds\right]\right\}'\exp\left[-\int_{0}^{t}\mu(s)ds\right] \qquad \text{for } t \geq -r.$$

Thus, by taking into account (3.2) and (3.3), we conclude that all we have to prove is that

$$\lim_{t\to\infty} w(t) \qquad \text{exists (as a real number)} \tag{3.7}$$

and

$$\lim_{t\to\infty} w'(t) = 0. \tag{3.8}$$

We notice that, when (3.7) is true, $\lim_{t\to\infty} w(t)$ depends on λ, μ and is determined by the solution x, i.e., it is determined by the initial function ϕ. The proof of the theorem will be accomplished by establishing (3.7) and (3.8).

From (3.5) it follows immediately that

$$
\begin{aligned}
(3.9)\qquad & w'(t) + c(t)w'(t-\sigma)\exp\left\{-\int_{t-\sigma}^{t}[\lambda(s)+\mu(s)]ds\right\} \\
&= c(t)[\mu(t-\sigma)+2\lambda(t-\sigma)]\left[\int_{t-\sigma}^{t} w'(s)ds\right] \\
&\times \exp\left\{-\int_{t-\sigma}^{t}[\lambda(s)+\mu(s)]ds\right\} - c(t)[\lambda'(t-\sigma)+\lambda^2(t-\sigma)] \\
&\times \left\{\int_{t-\sigma}^{t}\left[\int_{s}^{t} w'(u)du\right]\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\}\exp\left[-\int_{t-\sigma}^{t}\lambda(s)ds\right] \\
&+ b(t)\left\{\int_{t-\tau}^{t}\left[\int_{s}^{t} w'(u)du\right]\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\}\exp\left[-\int_{t-\tau}^{t}\lambda(s)ds\right]
\end{aligned}
$$

for all $t \geq r - \sigma$. Assumption (3.1) implies the existence of an integer $m \geq 1$ such that

$$
\begin{aligned}
\theta(\lambda,\mu) \equiv \sup_{t\geq mr-\sigma} & \Bigg(|c(t)|[1+|\mu(t-\sigma)+2\lambda(t-\sigma)|\sigma]\exp\left\{-\int_{t-\sigma}^{t}[\lambda(s)+\mu(s)]ds\right\} \\
&+ |c(t)|\,|\lambda'(t-\sigma)+\lambda^2(t-\sigma)|\left\{\int_{t-\sigma}^{t}(t-s)\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\} \\
&\times \exp\left[-\int_{t-\sigma}^{t}\lambda(s)ds\right] \\
&+ |b(t)|\left\{\int_{t-\tau}^{t}(t-s)\exp\left[-\int_{s}^{t}\mu(u)du\right]ds\right\}\exp\left[-\int_{t-\tau}^{t}\lambda(s)ds\right]\Bigg) < 1.
\end{aligned}
$$

Clearly, $\theta(\lambda,\mu) \geq 0$. Assume that $\theta(\lambda,\mu) = 0$. Then from the definition of $\theta(\lambda,\mu)$ it follows immediately that $c(t) = b(t) = 0$ for $t \geq mr - \sigma$, and consequently (3.9) guarantees that $w' = 0$ on $[mr - \sigma, \infty)$. This means that w is equal to a real constant on $[mr - \sigma, \infty)$. In this case, (3.7) and (3.8) are always valid. So, in what follows, we may (and do) suppose that $\theta(\lambda,\mu) > 0$. Hence, we have

$$
(3.10)\qquad 0 < \theta(\lambda,\mu) < 1.
$$

Furthermore, we see that the maximum of $|w'|$ on the interval $[(m-1)r - \sigma, mr - \sigma]$ depends on λ, μ, and x; so, it depends on λ, μ, and ϕ. Define

$$
M(\lambda,\mu;\phi) = \max\{|w'(t)| : (m-1)r - \sigma \leq t \leq mr - \sigma\}.
$$

Then

$$(3.11) \qquad |w'(t)| \le M(\lambda, \mu; \phi) \qquad \text{for } (m-1)r - \sigma \le t \le mr - \sigma.$$

We shall prove that $M(\lambda, \mu; \phi)$ is a bound of w' on the whole interval $[(m-1)r - \sigma, \infty)$, i.e., that

$$(3.12) \qquad |w'(t)| \le M(\lambda, \mu; \phi) \qquad \text{for all } t \ge (m-1)r - \sigma.$$

For this purpose, we consider an arbitrary positive real number ε. Then (3.11) gives

$$(3.13) \qquad |w'(t)| < M(\lambda, \mu; \phi) + \varepsilon \qquad \text{for } (m-1)r - \sigma \le t \le mr - \sigma.$$

We claim that

$$(3.14) \qquad |w'(t)| < M(\lambda, \mu; \phi) + \varepsilon \qquad \text{for every } t \ge (m-1)r - \sigma.$$

Otherwise, because of (3.13), there exists a point $\xi > mr - \sigma$ so that

$$|w'(t)| \le M(\lambda, \mu; \phi) + \varepsilon \quad \text{for } (m-1)r - \sigma \le t < \xi, \qquad \text{and}$$
$$|w'(\xi)| = M(\lambda, \mu; \phi) + \varepsilon.$$

Then, by taking into account the definition of $\theta(\lambda, \mu)$ and by using (3.10), from (3.9) we get

$$\begin{aligned}
&M(\lambda, \mu; \phi) + \varepsilon \\
&\quad = |w'(\xi)| \\
&\quad \le |c(\xi)|\,|w'(\xi - \sigma)| \exp\left\{-\int_{\xi-\sigma}^{\xi} [\lambda(s) + \mu(s)]ds\right\} \\
&\qquad + |c(\xi)|\,|\mu(\xi - \sigma) + 2\lambda(\xi - \sigma)| \left[\int_{\xi-\sigma}^{\xi} |w'(s)|ds\right] \exp\left\{-\int_{\xi-\sigma}^{\xi} [\lambda(s) + \mu(s)]ds\right\} \\
&\qquad + |c(\xi)|\,|\lambda'(\xi - \sigma) + \lambda^2(\xi - \sigma)| \left\{\int_{\xi-\sigma}^{\xi} \left[\int_{s}^{\xi} |w'(u)|du\right] \exp\left[-\int_{s}^{\xi} \mu(u)du\right] ds\right\} \\
&\qquad \times \exp\left[-\int_{\xi-\sigma}^{\xi} \lambda(s)ds\right] \\
&\qquad + |b(\xi)| \left\{\int_{\xi-\tau}^{\xi} \left[\int_{s}^{\xi} |w'(u)|du\right] \exp\left[-\int_{s}^{\xi} \mu(u)du\right] ds\right\} \exp\left[-\int_{\xi-\tau}^{\xi} \lambda(s)ds\right]
\end{aligned}$$

$$\leq \left(|c(\xi)|[1 + |\mu(\xi - \sigma) + 2\lambda(\xi - \sigma)|\sigma] \exp\left\{ -\int_{\xi-\sigma}^{\xi} [\lambda(s) + \mu(s)]ds \right\} \right.$$

$$+ |c(\xi)|\,|\lambda'(\xi - \sigma) + \lambda^2(\xi - \sigma)| \left\{ \int_{\xi-\sigma}^{\xi} (\xi - s) \exp\left[-\int_s^{\xi} \mu(u)du \right] ds \right\}$$

$$\times \exp\left[-\int_{\xi-\sigma}^{\xi} \lambda(s)ds \right] + |b(\xi)| \left\{ \int_{\xi-\tau}^{\xi} (\xi - s) \exp\left[-\int_s^{\xi} \mu(u)du \right] ds \right\}$$

$$\left. \times \exp\left[-\int_{\xi-\tau}^{\xi} \lambda(s)ds \right] \right) [M(\lambda, \mu; \phi) + \varepsilon]$$

$$\leq \theta(\lambda, \mu)[M(\lambda, \mu; \phi) + \varepsilon]$$

$$< M(\lambda, \mu; \phi) + \varepsilon.$$

This is a contradiction, which establishes our claim, i.e., that (3.14) holds true. Since (3.14) is valid for all real numbers $\varepsilon > 0$, we conclude that (3.12) is always satisfied. Furthermore, by using (3.12), from (3.9) we obtain, for every $t \geq mr - \sigma$,

$$|w'(t)| \leq |c(t)|\,|w'(t - \sigma)| \exp\left\{ -\int_{t-\sigma}^{t} [\lambda(s) + \mu(s)]ds \right\}$$

$$+ |c(t)|\,|\mu(t - \sigma) + 2\lambda(t - \sigma)| \left[\int_{t-\sigma}^{t} |w'(s)|ds \right] \exp\left\{ -\int_{t-\sigma}^{t} [\lambda(s) + \mu(s)]ds \right\}$$

$$+ |c(t)|\,|\lambda'(t - \sigma) + \lambda^2(t - \sigma)| \left\{ \int_{t-\sigma}^{t} \left[\int_s^t |w'(u)|du \right] \exp\left[-\int_s^t \mu(u)du \right] ds \right\}$$

$$\times \exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds \right]$$

$$+ |b(t)| \left\{ \int_{t-\tau}^{t} \left[\int_s^t |w'(u)|du \right] \exp\left[-\int_s^t \mu(u)du \right] ds \right\} \exp\left[-\int_{t-\tau}^{t} \lambda(s)ds \right]$$

$$\leq \left(|c(t)|[1 + |\mu(t - \sigma) + 2\lambda(t - \sigma)|\sigma] \exp\left\{ -\int_{t-\sigma}^{t} [\lambda(s) + \mu(s)]ds \right\} \right.$$

$$+ |c(t)|\,|\lambda'(t - \sigma) + \lambda^2(t - \sigma)| \left\{ \int_{t-\sigma}^{t} (t - s) \exp\left[-\int_s^t \mu(u)du \right] ds \right\}$$

$$\times \exp\left[-\int_{t-\sigma}^{t} \lambda(s)ds \right]$$

$$\left. + |b(t)| \left\{ \int_{t-\tau}^{t} (t - s) \exp\left[-\int_s^t \mu(u)du \right] ds \right\} \exp\left[-\int_{t-\tau}^{t} \lambda(s)ds \right] \right) M(\lambda, \mu; \phi).$$

So, in view of the definition of $\theta(\lambda, \mu)$, it holds that

$$|w'(t)| \le \theta(\lambda, \mu) M(\lambda, \mu; \phi) \quad \text{for all } t \ge mr - \sigma. \tag{3.15}$$

By using (3.9) and taking into account the definition of $\theta(\lambda, \mu)$ as well as having in mind (3.12) and (3.15), one can prove, by an easy induction, that

$$|w'(t)| \le [\theta(\lambda, \mu)]^n M(\lambda, \mu; \phi) \quad \text{for all } t \ge (m - 1 + n)r - \sigma \quad (n = 0, 1, 2, \dots). \tag{3.16}$$

In particular, from (3.16) it follows that

$$|w'(t)| \le [\theta(\lambda, \mu)]^n M(\lambda, \mu; \phi) \quad \text{for all } t \ge (m - 1 + n)r \quad (n = 0, 1, 2, \dots). \tag{3.17}$$

Working exactly as in the last part of the proof of the main result in the previous authors' paper [13] (see, also, the proofs of the main results in [3, 4]), we can take into account (3.10) and use (3.17) to conclude that (3.7) and (3.8) hold true.

The proof of the theorem is completed.

4. An example

In this section, we will present an example, which demonstrates the applicability of our main result, i.e., of Theorem 3.1.

EXAMPLE 4.1. Consider the neutral delay differential equation (1.1) with

$$c(t) = -\frac{1}{2}, \qquad a(t) = \frac{1}{2(t+3)}, \qquad b(t) = -\frac{1}{2(t+1)}, \qquad \text{for } t \ge 0$$

and

$$\sigma = 1, \qquad \tau = 2,$$

i.e., the neutral equation

$$x''(t) - \frac{1}{2} x''(t-1) = \frac{1}{2(t+3)} x(t) - \frac{1}{2(t+1)} x(t-2). \tag{4.1}$$

In this case, the generalized characteristic equation (1.3) becomes

$$\lambda'(t) + \lambda^2(t) - \frac{1}{2}[\lambda'(t-1) + \lambda^2(t-1)] \exp\left[-\int_{t-1}^{t} \lambda(s) ds\right] = \frac{1}{2(t+3)} - \frac{1}{2(t+1)} \exp\left[-\int_{t-2}^{t} \lambda(s) ds\right]. \tag{4.2}$$

We immediately see that (4.2) has the solution

$$\lambda(t) = \frac{1}{t+3} \qquad \text{for } t \geq -2.$$

For this solution λ of (4.2), the generalized characteristic equation (1.5) is written as follows

$$\mu(t) + \frac{2}{t+3} - \frac{1}{2}\left[\mu(t-1) + \frac{2}{t+2}\right]\frac{t+2}{t+3}\exp\left[-\int_{t-1}^{t}\mu(s)ds\right] \tag{4.3}$$
$$= \frac{1}{2(t+3)}\int_{t-2}^{t}\exp\left[-\int_{s}^{t}\mu(u)du\right]ds.$$

It is easy to verify that (4.3) admits the solution

$$\mu(t) = 0 \qquad \text{for } t \geq 0.$$

Furthermore, we see that assumption (3.1) takes the form

$$\limsup_{t\to\infty}\left[\frac{1}{2}\left(1 + \frac{2}{t+2}\right)\frac{t+2}{t+3} + \frac{1}{t+3}\right] < 1.$$

This inequality holds true, i.e., condition (3.1) is always satisfied. Now, with the neutral delay differential equation (4.1), we associate the initial condition

$$x(t) = \phi(t) \qquad \text{for } -2 \leq t \leq 0, \tag{4.4}$$

where ϕ is a given continuously differentiable real-valued function on $[-2, 0]$. By applying Theorem 3.1, we conclude that the solution x of the IVP (4.1) and (4.4) satisfies

$$\lim_{t\to\infty}\left[\frac{x(t)}{t+3}\right]' = \ell(\phi),$$

where $\ell(\phi)$ is some real number determined by ϕ, and

$$\lim_{t\to\infty}\left[\frac{x(t)}{t+3}\right]'' = 0.$$

5. Discussion

Let us consider the case of second order linear nonautonomous neutral delay differential equations involving the first order derivative of the unknown function. More precisely, consider the neutral delay differential equation

$$x''(t) + c(t)x''(t-\sigma) + p(t)x'(t) + q(t)x'(t-\rho) = a(t)x(t) + b(t)x(t-\tau), \tag{5.1}$$

where c, p, q, a, and b are continuous real-valued functions on the interval $[0, \infty)$, and σ, ρ, and τ are positive real constants. As it concerns equation (5.1), r stands for the positive real number defined by $r = \max\{\sigma, \rho, \tau\}$. The techniques applied in proving our results (i.e., Proposition 2.1 and Theorem 3.1) can equally well be employed to prove analogous results for the more general case of the initial value problem (5.1) and (1.2).

Next, we consider the more general case of the second order linear nonautonomous neutral delay differential equation

$$(5.2) \qquad x''(t) + \sum_{i \in I} c_i(t)x''(t - \sigma_i) + p(t)x'(t) + \sum_{j \in J} q_j(t)x'(t - \rho_j) = a(t)x(t) + \sum_{k \in K} b_k(t)x(t - \tau_k),$$

where I, J, and K are initial segments of natural numbers, c_i for $i \in I$, p, q_j for $j \in J$, a, and b_k for $k \in K$ are continuous real-valued functions on the interval $[0, \infty)$, and σ_i for $i \in I$, ρ_j for $j \in J$, and τ_k for $k \in K$ are positive real constants. As customary, it is assumed that $\sigma_{i_1} \neq \sigma_{i_2}$ for $i_1, i_2 \in I$ with $i_1 \neq i_2$, $\rho_{j_1} \neq \rho_{j_2}$ for $j_1, j_2 \in J$ with $j_1 \neq j_2$, and $\tau_{k_1} \neq \tau_{k_2}$ for $k_1, k_2 \in K$ with $k_1 \neq k_2$. As it concerns the neutral delay differential equation (5.2), we use the notation $r = \max\left\{\max_{i \in I} \sigma_i, \max_{j \in J} \rho_j, \max_{k \in K} \tau_k\right\}$ (r is a positive real number). By using the methods applied in obtaining the results in this paper, we can derive analogous results for the solution of the more general initial value problem (5.2) and (1.2).

We would be especially interested in the possibility of generalizing our results in the case of second order linear neutral delay differential equations with variable coefficients and variable delays. In case the delays are variable and bounded, this seems easy to be achieved. However, the general case of variable delays seems to be somewhat more difficult. Furthermore, it would be interesting to generalize our results for second order linear nonautonomous neutral delay differential equations with infinitely many distributed type delays.

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Ch. G. Philos
Department of Mathematics
University of Ioannina
P. O. Box 1186
451 10 *Ioannina*
Greece
E-mail: cphilos@cc.uoi.gr

I. K. Purnaras
Department of Mathematics
University of Ioannina
P. O. Box 1186
451 10 *Ioannina*
Greece
E-mail: ipurnara@cc.uoi.gr