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Negative slope algorithm and multiplicative Rauzy induction of 3-interval exchange transformations

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Abstract. The induced transformation on a properly chosen interval of a 2-interval exchange transformation is a 2-interval exchange transformation again. This induction is explicitly integrated by using the simple continued fraction algorithm. Therefore, we can say that the simple continued fraction algorithm acts well as a "multiplicative Rauzy induction" on the family of 2-interval exchange transformations. Now, we have the following question; On what kind of family of 3-interval exchange transformations does the negative slope algorithm act well as a multiplicative Rauzy induction? The purpose of this paper is to give the answer to this question.

1. Introduction

There exists an essential relationship, called the multiplicative Rauzy induction, between the simple continued fraction algorithm and a certain family of 2-interval exchange transformations. In order to describe this relationship more precisely, we start with the following definition.

Definition 1. For each $\alpha \notin \mathbf{Q}$ with $0 < \alpha < 1$, let I_α denote the interval $[-\alpha, 1)$ or $(-\alpha, 1]$. The transformation $R_\alpha : I_\alpha \to I_\alpha$ is defined to be the 2-interval exchange transformation given by

$$R_\alpha(x) = \begin{cases} x + \alpha & \text{if } -\alpha \le x < 1 - \alpha \\ x - 1 & \text{if } 1 - \alpha \le x < 1 \end{cases}$$

or

$$R_\alpha(x) = \begin{cases} x + \alpha & \text{if } -\alpha < x \le 1 - \alpha \\ x - 1 & \text{if } 1 - \alpha < x \le 1 \end{cases}$$

according as $I_\alpha = [-\alpha, 1)$ or $I_\alpha = (-\alpha, 1]$.

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We consider the 2-interval exchange transformation R_α on $I_\alpha = [-\alpha, 1)$ and the subinterval $J^{(1)} = [-\alpha, 1 - a_1\alpha)$ of I_α, where a_1 is given by $a_1 = \lfloor \frac{1}{\alpha} \rfloor$. We note that α and a_1 are related by means of the simple continued fraction algorithm T as $T\alpha = \frac{1}{\alpha} - a_1$. Furthermore, let $(R_\alpha)_{J^{(1)}}$ be the induced transformation of R_α into the interval $J^{(1)}$, that is,

$$(R_\alpha)_{J^{(1)}}(x) := R_\alpha^{k(x)}(x)$$

where $k(x) := \min\{k \mid R_\alpha^k(x) \in J^{(1)}, k \geq 1\}$. Then, we have the following theorem.

Theorem 1. *Let α_1 be the image of α by the simple continued fraction algorithm T, that is, $\alpha_1 = T\alpha = \frac{1}{\alpha} - a_1$. Then, $(R_\alpha)_{J^{(1)}}$ is isomorphic to R_{α_1} on $(-\alpha_1, 1]$ and an isomorphism $\varphi : [-\alpha, 1 - a_1\alpha) \to (-\alpha_1, 1]$ is given by $\varphi(x) = -\frac{1}{\alpha}x$, that is, we have*

$$R_{\alpha_1}(\varphi(x)) = \varphi((R_\alpha)_{J^{(1)}}(x)). \tag{1}$$

The relation (1) in Theorem 1 enables us to say that the simple continued fraction algorithm acts well as a multiplicative Rauzy induction on the family of 2-interval exchange transformations given in Definition 1. Here the terminology "multiplicative Rauzy induction" means that we use the Rauzy inductions ([6]) multiplicatively. By using Theorem 1, we can recognize the behavior of the recurrent rule of the orbit of the origin $\{R_\alpha^n(0) \mid n \geq 0\}$ (see Corollary 1 in the section 2). Now we have the following question; Consider the negative slope algorithm instead of the simple continued fraction algorithm. Then, on what kind of family of 3-interval exchange transformations does the negative slope algorithm act well as a multiplicative Rauzy induction? We arrive at the answer to this question in Theorem 3, that is, the negative slope algorithm acts well as a multiplicative Rauzy induction on the family of 2-interval exchange transformations with 3-partition (see Figure 3 and Theorem 3).

2. The simple continued fraction algorithm and the family of 2-interval exchange transformations

In this paper, the interval that we consider is only of the type $I = [a, b)$ or $I = (a, b]$. The difference between $[a, b)$ and $(a, b]$ is characterized by the sign function *sgn* as follows:

$$sgn[a, b) = +1, \qquad sgn(a, b] = -1.$$

Let $\{J_1, J_2, \ldots, J_N\}$ be a partition of the interval I, that is,

$$\bigcup_{k=1}^{N} J_k = I \qquad \text{and} \qquad J_i \cap J_j = \emptyset \quad \text{for } i \neq j$$

and $\{J_1, J_2, \ldots, J_N\}$ has the property $sgn\, I = sgn\, J_k$ for all $k \in \{1, 2, \ldots, N\}$.

From now on, we give a sketch of the proof of Theorem 1. To prove Theorem 1, it is enough to check the following structure. Let I_α, I_1, and I_2 be

$$I_\alpha := I_1 \cup I_2, \qquad I_1 := [-\alpha, 1-\alpha), \qquad I_2 := [1-\alpha, 1)$$

and let $J^{(1)}$, $J_1^{(1)}$, and $J_2^{(1)}$ be

$$J^{(1)} = J_1^{(1)} \cup J_2^{(1)}, \quad J_1^{(1)} := [1-(a_1+1)\alpha, 1-a_1\alpha), \quad J_2^{(1)} := [-\alpha, 1-(a_1+1)\alpha)$$

where $a_1 = \left\lfloor \frac{1}{\alpha} \right\rfloor$ (see Figure 1). Then we see that

$$\{J_2^{(1)}, J_1^{(1)}, R_\alpha(J_1^{(1)}), R_\alpha^2(J_1^{(1)}), \ldots, R_\alpha^{a_1}(J_1^{(1)})\} \tag{2}$$

is a partition of $[-\alpha, 1)$. Moreover, we denote by $\{J_1^{(1)'}, J_2^{(1)'}\}$ the partition of $[-\alpha, 1-a_1\alpha)$ where $R_\alpha^{a_1+1}(J_1^{(1)}) = J_1^{(1)'}$ and $R_\alpha(J_2^{(1)}) = J_2^{(1)'}$. We know that the

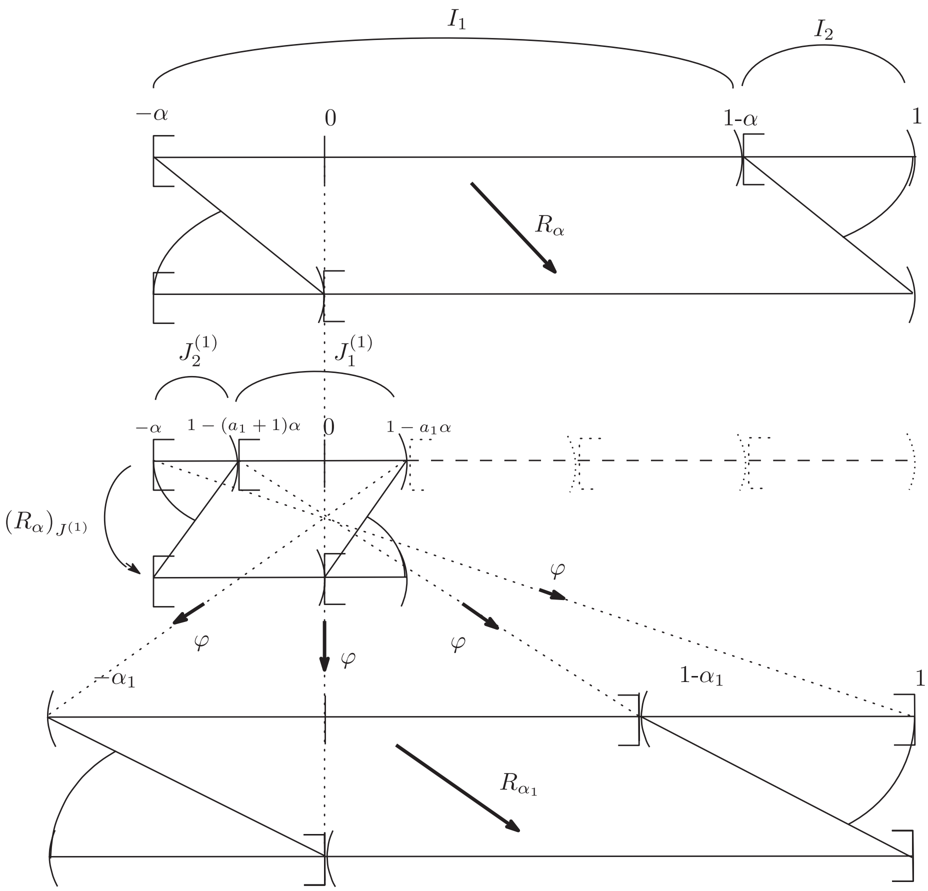

Fig. 1. The induced transformation $(R_\alpha)_{J^{(1)}}$ and the 2-interval exchange transformation R_{α_1}.

signs of these intervals are $+1$. Let $(R_\alpha)_{J^{(1)}}$ be the induced transformation of R_α into $J^{(1)}$, that is,

$$(R_\alpha)_{J^{(1)}}(x) := R_\alpha^{k(x)}(x)$$

where $k(x) := \min\{k \mid R_\alpha^k(x) \in J^{(1)}, k \geq 1\}$. Then, we see that

$$(R_\alpha)_{J^{(1)}}(J_1^{(1)}) = R_\alpha^{a_1+1}(J_1^{(1)}), \qquad (R_\alpha)_{J^{(1)}}(J_2^{(1)}) = R_\alpha(J_2^{(1)}),$$

so we write

$$J_1^{(1)'} = (R_\alpha)_{J^{(1)}}(J_1^{(1)}), \qquad J_2^{(1)'} = (R_\alpha)_{J^{(1)}}(J_2^{(1)}).$$

Furthermore, let $\varphi(x)$ be the map from $J^{(1)}$ to $I_{\alpha_1} = (-\alpha_1, 1]$ given by $\varphi(x) = -\frac{1}{\alpha}x$ where $\alpha_1 = \frac{1}{\alpha} - a_1$, $sgn\, J^{(1)} = +1$ and $sgn\, I_{\alpha_1} = -1$. Then, we see that

$$\varphi(J_2^{(1)}) = (1-\alpha_1, 1], \qquad \varphi(J_1^{(1)}) = (-\alpha_1, 1-\alpha_1],$$

$$\varphi(J_1^{(1)'}) = (0, 1], \qquad \varphi(J_2^{(1)'}) = (-\alpha_1, 0].$$

Therefore we have

$$R_{\alpha_1}(\varphi(x)) = \varphi((R_\alpha)_{J^{(1)}}(x)) \qquad \text{for } x \in J^{(1)}$$

where the 2-interval exchange transformation R_{α_1} is defined on I_{α_1} with $sgn\, I_{\alpha_1} = -1$ (see Figure 1).

Here, we recall the simple continued fraction algorithm T. For $0 < \alpha < 1$, $\alpha \notin \mathbf{Q}$, we define a map $T : (0,1) \to (0,1)$ as follows:

$$T(\alpha) := \frac{1}{\alpha} - a(\alpha)$$

where $a(\alpha) = \lfloor \frac{1}{\alpha} \rfloor$. For the integer valued function $a(\alpha)$, we put

$$a_1 := a(\alpha) = \left\lfloor \frac{1}{\alpha} \right\rfloor, \qquad a_n = a_n(\alpha) := a(T^{n-1}(\alpha)) = \left\lfloor \frac{1}{T^{n-1}(\alpha)} \right\rfloor.$$

Then we know the following properties:

(1) for $0 < \alpha < 1$, $\alpha \notin \mathbf{Q}$, there exists an infinite sequence $(a_1 a_2 \ldots)$ and we see that

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + T^n(\alpha)}}}} = \frac{p_n + p_{n-1}T^n(\alpha)}{q_n + q_{n-1}T^n(\alpha)}$$

where (q_n, p_n) is given by

$$\begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} := \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix};$$

(2) let $\alpha_n := T^n\alpha$, then we see that

$$q_n\alpha - p_n = (-1)^n\alpha\alpha_1 \dots \alpha_n.$$

Now let us define the intervals $J^{(2n-1)}$, $J_1^{(2n-1)}$, $J_2^{(2n-1)}$, $J^{(2n)}$, $J_1^{(2n)}$, $J_2^{(2n)}$ as follows:

$$J^{(2n-1)} := [-\alpha\alpha_1 \dots \alpha_{2(n-1)}, \alpha\alpha_1 \dots \alpha_{2n-1})$$
$$J_1^{(2n-1)} := [-\alpha\alpha_1 \dots \alpha_{2(n-1)}(1-\alpha_{2n-1}), \alpha\alpha_1 \dots \alpha_{2n-1})$$
$$J_2^{(2n-1)} := [-\alpha\alpha_1 \dots \alpha_{2(n-1)}, -\alpha\alpha_1 \dots \alpha_{2(n-1)}(1-\alpha_{2n-1}))$$

and

$$J^{(2n)} := [-\alpha\alpha_1 \dots \alpha_{2n}, \alpha\alpha_1 \dots \alpha_{2n-1})$$
$$J_1^{(2n)} := [-\alpha\alpha_1 \dots \alpha_{2n}, \alpha\alpha_1 \dots \alpha_{2n-1}(1-\alpha_{2n}))$$
$$J_2^{(2n)} := [\alpha\alpha_1 \dots \alpha_{2n-1}(1-\alpha_{2n}), \alpha\alpha_1 \dots \alpha_{2n})$$

(see Figure 2).

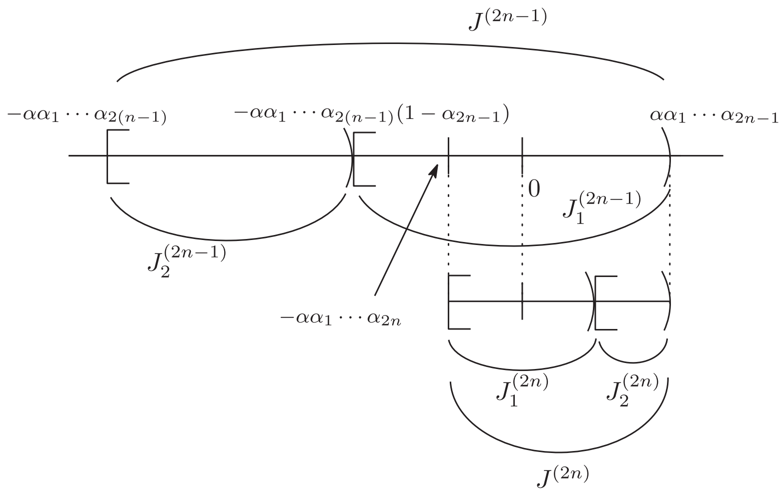

Fig. 2. The intervals $J^{(2n-1)}$, $J_1^{(2n-1)}$, $J_2^{(2n-1)}$, $J^{(2n)}$, $J_1^{(2n)}$, and $J_2^{(2n)}$.

Let us define the substitution σ_a as follows:

$$\sigma_a : 1 \to \overbrace{1 \ldots 1}^{a} 2$$
$$2 \to 1$$

$$\sigma_{a_1} \circ \sigma_{a_2} \circ \cdots \circ \sigma_{a_n}(1) = s_1 s_2 \ldots s_{q_n+p_n}$$

$$\sigma_{a_1} \circ \sigma_{a_2} \circ \cdots \circ \sigma_{a_n}(2) = \sigma_{a_1} \circ \cdots \circ \sigma_{a_{n-1}}(1) = s_1 \ldots s_{q_{n-1}+p_{n-1}}.$$

Then we have the following theorem.

THEOREM 2. (1) *The induced transformation* $(R_\alpha)_{J^{(1)}}$ *of* R_α *into* $J^{(n)}$ *is isomorphic to* R_{α_n} *on the interval* I_{α_n} *with sgn* $I_{\alpha_n} = (-1)^n$ *and an isomorphism* φ_n *is given by*

$$\begin{array}{cccc} \varphi_n : & J^{(n)} & \to & I_{\alpha_n} \\ & \Psi & & \Psi \\ & x & \mapsto & \dfrac{(-1)^n}{\alpha\alpha_1 \ldots \alpha_{n-1}} x. \end{array}$$

(2) *For the sequence* $s_1 s_2 \ldots s_{q_n+p_n}$, *we see that*

$$R_\alpha^{k-1} J_1^{(n)} \subset I_{s_k}, \qquad k = 1, 2, \ldots, q_n + p_n$$

$$R_\alpha^{k-1} J_2^{(n)} \subset I_{s_k}, \qquad k = 1, 2, \ldots, q_{n-1} + p_{n-1}.$$

(3) *Put*

$$J_1^{(n)'} := R_\alpha^{q_n+p_n} J_1^{(n)} \qquad \text{and} \qquad J_2^{(n)'} := R_\alpha^{q_{n-1}+p_{n-1}} J_2^{(n)},$$

then we see that

$$J_1^{(n)'} \cup J_2^{(n)'} = J^{(n)} \qquad \text{and} \qquad J_1^{(n)'} \cap J_2^{(n)'} = \varnothing.$$

Then we have an interesting result as a corollary of Theorem 2 in the following.

COROLLARY 1. *Let* $0 < \alpha < 1$ *be an irrational number, then we have*

$$R_\alpha^{k-1}(0) \in I_{s_k}, \qquad k = 1, 2, \ldots,$$

where $s_1 s_2 \ldots s_k \ldots$ *is given by*

$$\lim_{n\to\infty} \sigma_{a_1} \circ \sigma_{a_2} \circ \cdots \circ \sigma_{a_n}(1) = s_1 s_2 \ldots s_k \ldots.$$

In the next section, we will consider the family of 3-interval exchange transformations on which the negative slope algorithm acts well as a multiplicative Rauzy induction.

3. Definitions of the negative slope algorithm and a family of 3-interval exchange transformations

We introduce a map S on $\mathbf{X} := [0,1)^2 \setminus \{(\alpha,\beta) \mid \alpha+\beta = 1\}$, which is called the negative slope algorithm ([1], [2], [3], [4], [5]), as follows.

Definition 2. By using the integer valued functions

$$(n(\alpha,\beta), m(\alpha,\beta)) = \begin{cases} \left(\left\lfloor \frac{\beta}{(\alpha+\beta)-1} \right\rfloor, \left\lfloor \frac{\alpha}{(\alpha+\beta)-1} \right\rfloor\right) & \text{if } \alpha+\beta > 1 \\ \left(\left\lfloor \frac{1-\beta}{1-(\alpha+\beta)} \right\rfloor, \left\lfloor \frac{1-\alpha}{1-(\alpha+\beta)} \right\rfloor\right) & \text{if } \alpha+\beta < 1 \end{cases}$$

and

$$\varepsilon(\alpha,\beta) = \begin{cases} -1 & \text{if } \alpha+\beta > 1 \\ +1 & \text{if } \alpha+\beta < 1 \end{cases},$$

let us define the algorithm S, called the negative slope algorithm, by

$$S(\alpha,\beta) := \begin{cases} \left(\frac{\beta}{(\alpha+\beta)-1} - n(\alpha,\beta), \frac{\alpha}{(\alpha+\beta)-1} - m(\alpha,\beta)\right) & \text{if } \alpha+\beta > 1 \\ \left(\frac{1-\beta}{1-(\alpha+\beta)} - n(\alpha,\beta), \frac{1-\alpha}{1-(\alpha+\beta)} - m(\alpha,\beta)\right) & \text{if } \alpha+\beta < 1 \end{cases}$$

and denote $(\alpha_1,\beta_1) := S(\alpha,\beta)$.

For each $(\alpha,\beta) \in \mathbf{X}$, we have the sequence of vectors $((\varepsilon_1(\alpha,\beta), n_1(\alpha,\beta), m_1(\alpha,\beta)), \dots)$ by setting

$$\begin{pmatrix} \varepsilon_k(\alpha,\beta) \\ n_k(\alpha,\beta) \\ m_k(\alpha,\beta) \end{pmatrix} := \begin{pmatrix} \varepsilon(S^{k-1}(\alpha,\beta)) \\ n(S^{k-1}(\alpha,\beta)) \\ m(S^{k-1}(\alpha,\beta)) \end{pmatrix}. \tag{3}$$

Remark 1. *Let $(\alpha_k,\beta_k) = S^k(\alpha,\beta)$ denote the image of (α,β) by the k-fold iteration S^k of S. Then, we say that the iteration of the negative slope algorithm S at $(\alpha,\beta) \in \mathbf{X}$ stops if there exists $k_0 \geq 0$ such that $x_{k_0} = 0$, $y_{k_0} = 0$, or $x_{k_0} + y_{k_0} = 1$. In this paper, we treat only the point (α,β) at which the iteration of the negative slope algorithm does not stop.*

Remark 2 ([1]). *We note that $n_k, m_k \geq 1$ for $k \geq 1$ and for any such sequence $((\varepsilon_i, n_i, m_i))_{i\geq 1}$, there exists $(\alpha,\beta) \in \mathbf{X}$ such that $(\varepsilon_i(\alpha,\beta), n_i(\alpha,\beta), m_i(\alpha,\beta)) = (\varepsilon_i, n_i, m_i)$ unless there exists $k \geq 1$ such that $(\varepsilon_i, m_i) = (+1,1)$ for all $i \geq k$ or $(\varepsilon_i, n_i) = (+1,1)$ for all $i \geq k$.*

We introduce a projective representation of S as follows.

We put

$$A_{(+1,n,m)} := \begin{pmatrix} n & n-1 & 1-n \\ m-1 & m & 1-m \\ -1 & -1 & 1 \end{pmatrix}, \quad A_{(-1,n,m)} := \begin{pmatrix} -n & -n+1 & n \\ -m+1 & -m & m \\ 1 & 1 & -1 \end{pmatrix}$$

for $m, n \geq 1$, then we have

$$A_{(+1,n,m)}^{-1} = \begin{pmatrix} 1 & 0 & n-1 \\ 0 & 1 & m-1 \\ 1 & 1 & n+m-1 \end{pmatrix}, \qquad A_{(-1,n,m)}^{-1} = \begin{pmatrix} 0 & 1 & m \\ 1 & 0 & n \\ 1 & 1 & n+m-1 \end{pmatrix}.$$

We identify $(\alpha, \beta) \in \mathbf{X}$ with $c^t(\alpha, \beta, 1)$ for $c \neq 0$, then we identify (α_1, β_1) $(= S(\alpha, \beta))$ with

$$cA_{(\varepsilon_1(\alpha,\beta), n_1(\alpha,\beta), m_1(\alpha,\beta))}{}^t(\alpha, \beta, 1) \qquad \text{for } c \neq 0$$

and its local inverse is given by

$$A_{(\varepsilon_1(\alpha,\beta), n_1(\alpha,\beta), m_1(\alpha,\beta))}^{-1}.$$

On the negative slope algorithm S, the following fundamental fact related to the periodicity of the sequence $(S^k(\alpha, \beta))_{k \geq 0}$ is known and it will be used in Corollary 5.

COROLLARY 2 ([4]). *Suppose that the iteration of the negative slope algorithm S at $(\alpha, \beta) \in \mathbf{X}$ does not stop. Then the sequence $(S^k(\alpha, \beta))_{k \geq 0}$ is purely periodic if and only if α and β are in the same quadratic extension of $\mathbf{Q}$ and (α^*, β^*) is in $(-\infty, 0)^2$ where α^* denotes the algebraic conjugate of α.*

Let us introduce the substitutions $\sigma_{(+1,n,m)}$ and $\sigma_{(-1,n,m)}$ from $\{1, 2, 3\}$ to $\bigcup_{n \geq 0}^{\infty} \{1, 2, 3\}^n$ by

$$\sigma_{(+1,n,m)} : \begin{array}{l} 1 \to 31 \\ 2 \to 32 \\ 3 \to (32)^{m-1} 3 (31)^{n-1} \end{array}, \qquad \sigma_{(-1,n,m)} : \begin{array}{l} 1 \to 32 \\ 2 \to 31 \\ 3 \to (31)^m 2 (32)^{n-1}. \end{array} \tag{4}$$

Then the incidence matrices $L_{\sigma_{(\varepsilon,n,m)}}$ of the substitutions $\sigma_{(\varepsilon,n,m)}$ are given by

$$L_{\sigma_{(\varepsilon,n,m)}} = A_{(\varepsilon,n,m)}^{-1}.$$

Suppose that the iteration of the negative slope algorithm S at $(\alpha, \beta) \in \mathbf{X}$ does not stop. Let us define a family of 3-interval exchange transformations $R_{\alpha,\beta}$ as follows.

DEFINITION 3. Put the intervals $I_{\alpha,\beta}$, I_3, I_2 and I_1 with $sgn = +1$ by

$$I_{\alpha,\beta} := [-\beta, 1+\alpha), \quad I_3 := [-\beta, 1-\beta), \quad I_2 := [1-\beta, 1), \quad I_1 := [1, 1+\alpha)$$

and define the interval exchange transformation $R_{\alpha,\beta}$ on $I_{\alpha,\beta}$ by

$$R_{\alpha,\beta}(x) := \begin{cases} x + (\alpha+\beta) & \text{if } x \in I_3 \\ x - 1 & \text{if } x \in I_2 \\ x - 1 & \text{if } x \in I_1. \end{cases}$$

(see Figure 3).

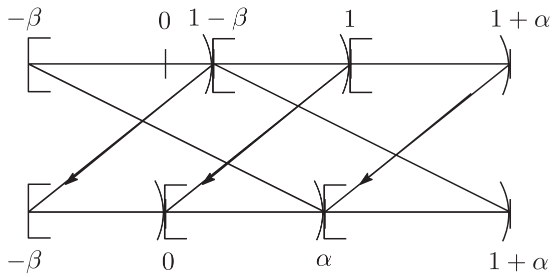

Fig. 3. The interval exchange transformation $R_{\alpha,\beta}$.

Theorem 3. *Let us introduce the subinterval $J^{(\alpha,\beta)} \subset I_{\alpha,\beta}$ with sgn $= +1$ by*

$$J^{(\alpha,\beta)} = \begin{cases} [-\beta + (n-1)(\alpha+\beta-1), (1-\beta) - (m-1)(\alpha+\beta-1)) & \text{if } \varepsilon = -1 \\ [-\beta + (m-1)(1-(\alpha+\beta)), (1-\beta) - (n-1)(1-(\alpha+\beta))) & \text{if } \varepsilon = +1 \end{cases}$$

where ε, n, m are given by Definition 2. The case of sgn $= -1$ is defined analogously (see Figure 8 and Figure 13 in the section 4). Then we have the following properties.

(1) *The induced transformation $(R_{\alpha,\beta})_{J^{(\alpha,\beta)}}$ of the interval exchange transformation $R_{\alpha,\beta}$ into the interval $J^{(\alpha,\beta)}$ is isomorphic to R_{α_1,β_1} and the isomorphism $\varphi_{(\alpha,\beta)} : J^{(\alpha,\beta)} \to I_{\alpha_1,\beta_1}$ with sgn $I_{\alpha_1,\beta_1} = \varepsilon_1$ is given by $\varphi_{(\alpha,\beta)}(x) = \frac{x}{1-\alpha-\beta}$, that is, the following relation holds:*

$$(R_{\alpha_1,\beta_1} \circ \varphi)(x) = (\varphi \circ (R_{\alpha,\beta})_{J^{(\alpha,\beta)}})(x).$$

(2) *More precisely, let J_i and J'_i, $i = 1,2,3$ be the decomposition of $J^{(\alpha,\beta)}$ given by*

$$\begin{aligned} J_1 &:= [-\beta + (n-1)(\alpha+\beta-1), -(\alpha+\beta-1)) \\ J_2 &:= [-(\alpha+\beta-1), 1-\beta-m(\alpha+\beta-1)) \\ J_3 &:= [1-\beta-m(\alpha+\beta-1), 1-\beta-(m-1)(\alpha+\beta-1)) \quad \text{if } \varepsilon = -1 \\ J'_1 &:= [-\beta + n(\alpha+\beta-1), 0) \\ J'_2 &:= [0, \alpha - m(\alpha+\beta-1)) \\ J'_3 &:= [-\beta + (n-1)(\alpha+\beta-1), -\beta + n(\alpha+\beta-1)) \end{aligned} \tag{5}$$

and

$$J_1 = [1-(\alpha+\beta), (1-\beta) - (n-1)(1-(\alpha+\beta)))$$

$$J_2 = [-(1-\alpha) + (m+1)(1-(\alpha+\beta)), 1-(\alpha+\beta))$$

$$J_3 = [-(1-\alpha) + m(1-(\alpha+\beta)), -(1-\alpha) + (m+1)(1-(\alpha+\beta)))$$

if $\varepsilon = +1$

$$J_1' := [0, (1-\beta) - n(1-(\alpha+\beta))) \tag{6}$$

$$J_2' := [-(1-\alpha) + m(1-(\alpha+\beta)), 0)$$

$$J_3' := [(1-\beta) - n(1-(\alpha+\beta)), (1-\beta) - (n-1)(1-(\alpha+\beta))).$$

Let us denote the substitution $\sigma_{(\varepsilon,n,m)}$ *as* (4) *by*

$$\sigma_{(\varepsilon,n,m)} = \begin{cases} 1 \to s_1^{(1)} s_{l_1}^{(1)} = s_1^{(1)} s_2^{(1)} \\ 2 \to s_1^{(2)} s_{l_2}^{(2)} = s_1^{(2)} s_2^{(2)} \\ 3 \to s_1^{(3)} s_2^{(3)} \dots s_{l_3}^{(3)} \end{cases}$$

where

$$l_1 = l_2 = 2 \qquad \text{and} \qquad l_3 = \begin{cases} m+n+(m+n-1) & \text{if } \varepsilon = -1 \\ (n-1)+(m-1)+(n+m-1) & \text{if } \varepsilon = +1, \end{cases}$$

then there exist J_1, J_2 *and* J_3, *which are given in* (5) *and* (6), *such that*

(I) $\{J_1, R_{\alpha,\beta}J_1, J_2, R_{\alpha,\beta}J_2, J_3, R_{\alpha,\beta}J_3, \dots, R_{\alpha,\beta}^k J_3, \dots, R_{\alpha,\beta}^{l_3-1} J_3\}$ *is a partition of* $I_{\alpha,\beta}$;

(II) $R_{\alpha,\beta}^{k-1} J_i \subset I_{s_k^{(i)}}$, $1 \le k \le l_i$;

(III) $R_{\alpha,\beta}^{l_i} J_i = J_i'$.

This theorem says that the negative slope algorithm acts well on the family of 3-interval exchange transformations given by Definition 3 as a multiplicative Rauzy induction. Let us define

$$J^{(n)} := \varphi_{(\alpha,\beta)}^{-1} \circ \varphi_{(\alpha_1,\beta_1)}^{-1} \circ \cdots \circ \varphi_{(\alpha_{n-1},\beta_{n-1})}^{-1}(I_{\alpha_n,\beta_n})$$

and $\varphi_n : J^{(n)} \to I_{\alpha_n,\beta_n}$ by

$$\varphi_n(x) := \varphi_{(\alpha_{n-1},\beta_{n-1})} \circ \cdots \circ \varphi_{(\alpha,\beta)}(x).$$

Then we know that $sgn\, I_{\alpha_n,\beta_n}$ of R_{α_n,β_n} is given by $sgn\, I_{\alpha_n,\beta_n} = \varepsilon_1\varepsilon_2\dots\varepsilon_n$. Therefore we have the following corollaries.

COROLLARY 3. *The induced transformation* $(R_{\alpha,\beta})_{J^{(n)}}$ *of* $R_{\alpha,\beta}$ *into* $J^{(n)}$ *is isomorphic to* R_{α_n,β_n} *under the isomorphism* φ_n, *that is, the following relation holds:*

$$R_{\alpha_n,\beta_n}(\varphi_n(x)) = \varphi_n((R_{\alpha,\beta})_{J^{(n)}}(x)) \qquad \text{for } x \in J^{(n)}.$$

COROLLARY 4. *Put*

$$\lim_{k\to\infty} \sigma_{(\varepsilon_1,n_1,m_1)} \dots \sigma_{(\varepsilon_k,n_k,m_k)}(3) = s_1 s_2 \dots. \tag{7}$$

Then we see that

$$R_{\alpha,\beta}^{n-1}(0) \in I_{s_n}$$

for all $n \in \mathbf{N}$.

By Theorem 3, we have the following corollary.

COROLLARY 5. *Suppose the sequence* $(S^k(\alpha,\beta))_{k\geq 0}$ *is purely periodic with the period* $l \geq 1$, *that is,* $S^l(\alpha,\beta) = (\alpha,\beta)$. *Then we see that the sequence* $s_1 s_2 \dots$ *given by* (7) *is a fixed point of the substitution* σ^*, *that is,*

$$\sigma^*(s_1 s_2 \dots) = s_1 s_2 \dots$$

where σ^* *is given by*

$$\sigma^* = \sigma_{(\varepsilon_1, n_1, m_1)} \dots \sigma_{(\varepsilon_l, n_l, m_l)}.$$

4. The proof of the main theorem

In order to prove Theorem 3, we need the following lemma.

LEMMA 1. *In the case when* $\alpha + \beta > 1$, *i.e.* $\varepsilon = -1$, *and* $\alpha, \beta < 1$, *we have the following.*

(1) *The interval* $I = [-\beta, 1+\alpha)$ *is decomposed into* I_i, $i = 1,2,3$. *We see that* $0 \in I_3$, $\alpha \in I_2$, *and* $|I_1| = \alpha$, $|I_2| = \beta$, $|I_3| = 1$ *(see Figure* 4*).*

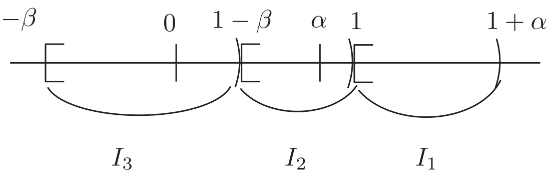

Fig. 4. The decomposition of I into I_i, $i = 1,2,3$.

(2-1) *By using* $m = \left\lfloor \frac{\alpha}{\alpha+\beta-1} \right\rfloor$, *we can decompose the interval* I_1 *into* m *intervals of length* $\alpha + \beta - 1$ *and the interval of length* $\alpha - m(\alpha+\beta-1)$ *as in Figure* 5, *where the length of a* ○*-marked interval is* $\alpha+\beta-1$ *and the length of a* □*-marked interval is* $\alpha - m(\alpha+\beta-1)$.

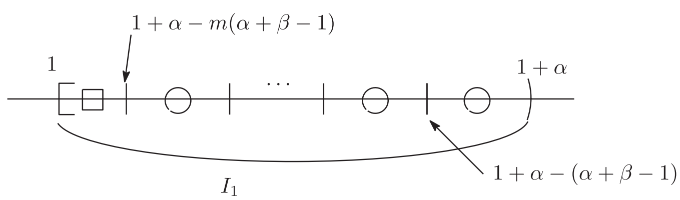

Fig. 5. The decomposition of I_1 into ○, □-marked intervals.

(2-2) *By using* $n = \left\lfloor \frac{\beta}{\alpha+\beta-1} \right\rfloor$, *we can decompose the interval* I_2 *into* n *intervals of length* $\alpha+\beta-1$ *and the interval of length* $\beta - n(\alpha+\beta-1)$ *as in Figure* 6, *where the length of a* $\triangle$*-marked interval is* $\beta - n(\alpha+\beta-1)$.

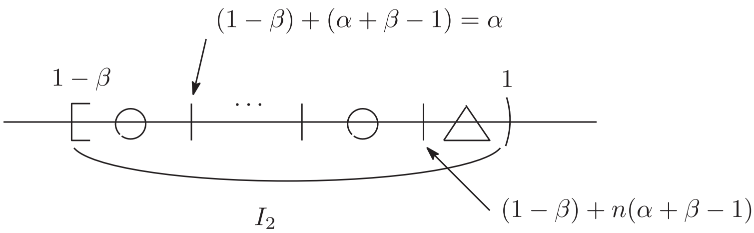

Fig. 6. The decomposition of I_2 into $\bigcirc, \triangle$-marked intervals.

(2-3) *We can decompose the interval* $I_3 = [-\beta, 1-\beta)$ *into* $(m+n-1)$ *intervals of length* $\alpha+\beta-1$, *the interval of length* $\alpha - m(\alpha+\beta-1)$ *and the interval of length* $\beta - n(\alpha+\beta-1)$ *as marked intervals in Figure* 7.

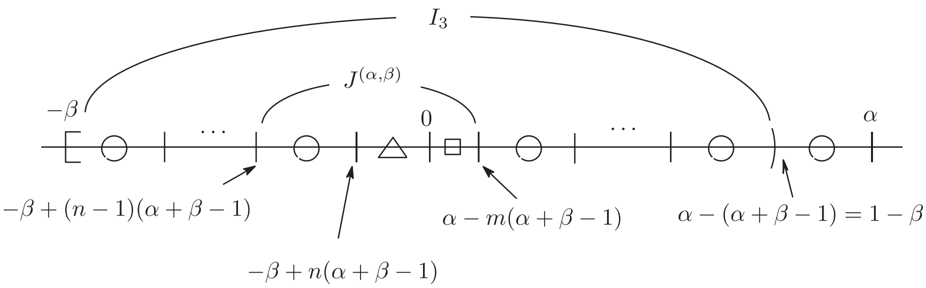

Fig. 7. The decomposition of I_3 into $\bigcirc, \triangle, \square$-marked intervals.

(2-4) *The interval* $J^{(\alpha,\beta)} = [-\beta+(n-1)(\alpha+\beta-1), \alpha - m(\alpha+\beta-1))$ *is decomposed by* (2-3) *as* (5). *We show the figure of the decompositions of* $[-\beta, 1+\alpha)$ *into the marked intervals* $\bigcirc$, $\triangle$ *and* $\square$ *(see Figure* 8*).*

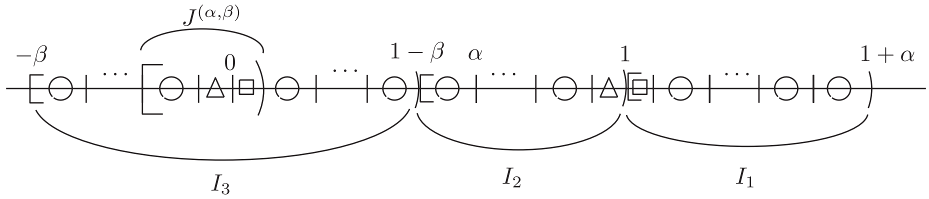

Fig. 8. The decomposition of I into $\bigcirc, \triangle, \square$-marked intervals.

LEMMA 2. *Let $R_{\alpha,\beta}$ be the interval exchange transformation given by Definition 3 and assume that $\alpha+\beta>1$, i.e. $\varepsilon=-1$. Let J_i, J_i', $i=1,2,3$ be the intervals given by (5) where we know*

$$|J_i| = |J_i'|, \qquad i=1,2,3.$$

Moreover, for the sets $R^k_{\alpha,\beta}(J_i)$, we know the following fact:

(1) $J_1 \subset I_3$
$R_{\alpha,\beta}(J_1) = [\alpha+(n-1)(\alpha+\beta-1), 1) \subset I_2$
$R^2_{\alpha,\beta}(J_1) = [-\beta+n(\alpha+\beta-1), 0) = J_1' \subset J^{(\alpha,\beta)}$

(2) $J_2 \subset I_3$
$R_{\alpha,\beta}(J_2) = [1, 1+\alpha-m(\alpha+\beta-1)) \subset I_1$
$R^2_{\alpha,\beta}(J_2) = [0, \alpha-m(\alpha+\beta-1)) = J_2' \subset J^{(\alpha,\beta)}$

(3) $J_3 \subset I_3$
$R_{\alpha,\beta}(J_3) = [1+\alpha-m(\alpha+\beta-1), 1+\alpha-(m-1)(\alpha+\beta-1)) \subset I_1$
$R^2_{\alpha,\beta}(J_3) = [\alpha-m(\alpha+\beta-1), \alpha-(m-1)(\alpha+\beta-1)) \subset I_3$
$\vdots$
$R^{2k-1}_{\alpha,\beta}(J_3) = [1+\alpha-(m-k+1)(\alpha+\beta-1), 1+\alpha-(m-k)(\alpha+\beta-1)) \subset I_1$
$R^{2k}_{\alpha,\beta}(J_3) = [\alpha-(m-k+1)(\alpha+\beta-1), \alpha-(m-k)(\alpha+\beta-1)) \subset I_3$
$\vdots$
$R^{2m-1}_{\alpha,\beta}(J_3) = [1+\alpha-(\alpha+\beta-1), 1+\alpha) \subset I_1$
$R^{2m}_{\alpha,\beta}(J_3) = [1-\beta, \alpha) \subset I_2$
$R^{2m+1}_{\alpha,\beta}(J_3) = [-\beta, -1+\alpha) \subset I_3$
$R^{2m+2}_{\alpha,\beta}(J_3) = [\alpha, \alpha+(\alpha+\beta-1)) \subset I_2$
$\vdots$
$R^{2m+2l-1}_{\alpha,\beta}(J_3) = [-\beta+(l-1)(\alpha+\beta-1), -\beta+l(\alpha+\beta-1)) \subset I_3$
$R^{2m+2l}_{\alpha,\beta}(J_3) = [\alpha+(l-1)(\alpha+\beta-1), \alpha+l(\alpha+\beta-1)) \subset I_2$
$\vdots$
$R^{2m+2n-2}_{\alpha,\beta}(J_3) = [\alpha+(n-2)(\alpha+\beta-1), \alpha+(n-1)(\alpha+\beta-1)) \subset I_2$
$R^{2m+2n-1}_{\alpha,\beta}(J_3) = [-\beta+(n-1)(\alpha+\beta-1), -\beta+n(\alpha+\beta-1)) = J_3' \subset J^{(\alpha,\beta)}$.

PROOF. From the definitoin of J_1 and I_3, it is easy to see that

$$J_1 = [-\beta + (n-1)(\alpha+\beta-1), -(\alpha+\beta-1)) \subset I_3.$$

Therefore, from the fact that $J_1 \subset I_3$, we know that $R_{\alpha,\beta}(J_1) = J_1 + (\alpha+\beta) = [\alpha + (n-1)(\alpha+\beta-1), 1) \subset I_2$. So, we obtain the assertions of (1). The other assertions of the lemma are obtained by analogous discussions.

We give a lemma which is an anologous of Lemma 1.

LEMMA 3. *In the case when $\alpha+\beta<1$, i.e. $\varepsilon=+1$, and $\alpha,\beta<1$, we have the following.*

(1)′ *The interval $I = [-\beta, 1+\alpha)$ is decomposed into I_i, $i=1,2,3$ (see Figure 9) and we see that $0 \in I_3$, $\alpha \in I_3$, and $|I_1| = \alpha$, $|I_2| = \beta$, $|I_3| = 1$ where the length of a ○-marked interval is $1-(\alpha+\beta)$ in Figure 9.*

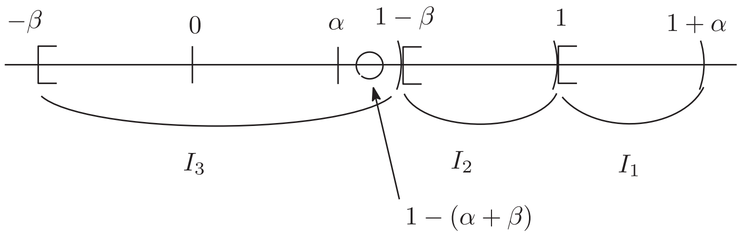

Fig. 9. The decomposition of I into I_i, $i = 1,2,3$.

(2-1)′ *By using $n = \left\lfloor \frac{1-\beta}{1-(\alpha+\beta)} \right\rfloor$, we can decompose the interval I_1 into $(n-1)$ intervals of length $1-(\alpha+\beta)$ and the interval of length $1-\beta-n(1-(\alpha+\beta))$ as in Figure 10, where the length of a ○-marked interval is $1-(\alpha+\beta)$ and the length of a △-marked interval is $1-\beta-n(1-(\alpha+\beta))$.*

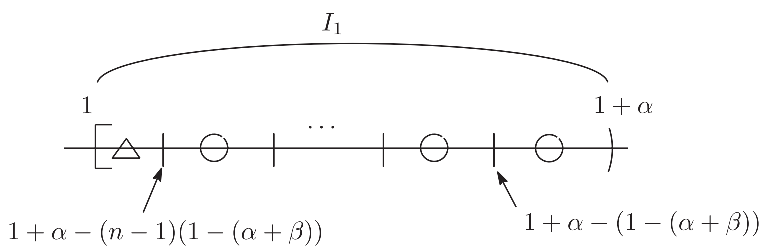

Fig. 10. The decomposition of I_1 into ○, △-marked intervals.

(2-2)′ *By using* $m = \left\lfloor \frac{1-\alpha}{1-(\alpha+\beta)} \right\rfloor$, *we can decompose the interval* I_2 *into* $(m-1)$ *intervals of length* $1-(\alpha+\beta)$ *and the interval of length* $1-\alpha-m(1-(\alpha+\beta))$ *as in Figure* 11, *where the length of a* ○-*marked interval is* $1-(\alpha+\beta)$ *and the length of a* □-*marked interval is* $1-\alpha-m(1-(\alpha+\beta))$.

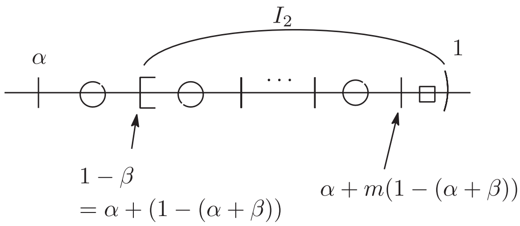

Fig. 11. The decomposition of I_2 into ○, □-marked intervals.

(2-3)′ *We can decompose the interval* $I_3 = [-\beta, 1-\beta)$ *into* $(m+n-1)$ *intervals of length* $1-(\alpha+\beta)$, *the interval of length* $1-\beta-n(1-(\alpha+\beta))$ *and the interval of length* $1-\alpha-m(1-(\alpha+\beta))$ *as marked intervals in Figure* 12.

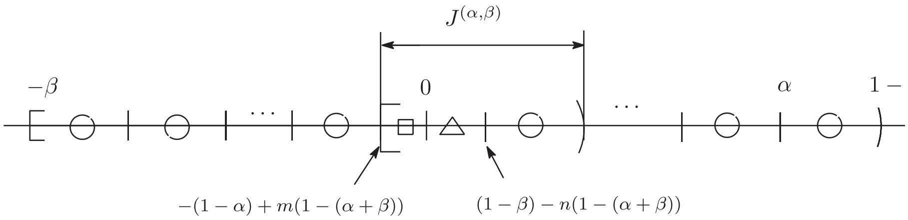

Fig. 12. The decomposition of I_3 into ○, △, □-marked intervals.

(2-4)′ *The interval*

$$J^{(\alpha,\beta)} = [-(1-\alpha)+m(1-(\alpha+\beta)), (1-\beta)-(n-1)(1-(\alpha+\beta)))$$

is decomposed by (2-3)′ *as* (6). *Finally, the figure of the decomposition of* $[-\beta, 1+\alpha)$ *into the marked intervals* ○, △ *and* □ *is Figure* 13.

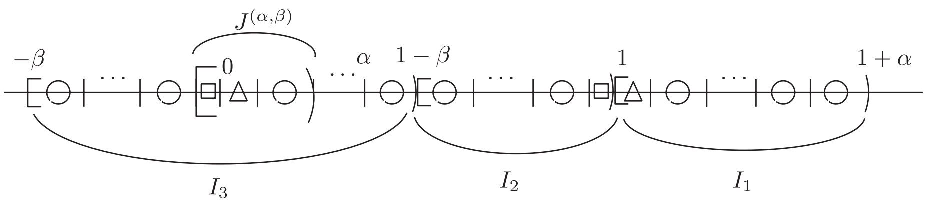

Fig. 13. The decomposition of I into $\bigcirc, \triangle, \square$-marked intervals.

LEMMA 4. *Let $R_{\alpha,\beta}$ be the interval exchange transformation given by Definition 3 and assume that $\alpha+\beta<1$, i.e. $\varepsilon=+1$. Let J_i, J_i', $i=1,2,3$ be the intervals given by (6) where we know*

$$|J_i| = |J_i'|, \qquad i=1,2,3.$$

Then for the sets $R^k_{(\alpha,\beta)}(J_i)$, we know the following fact:

(1) $J_1 \subset I_3$
$R_{\alpha,\beta}(J_1) = [1, 1+\alpha-(n-1)(1-\alpha-\beta)) \subset I_1$
$R^2_{\alpha,\beta}(J_1) = [0, (1-\beta)-n(1-\alpha-\beta)) = J_1' \subset J^{(\alpha,\beta)}$

(2) $J_2 \subset I_3$
$R_{\alpha,\beta}(J_2) = [\alpha+m(1-\alpha-\beta), 1) \subset I_2$
$R^2_{\alpha,\beta}(J_2) = [-(1-\alpha)+m(1-\alpha-\beta), 0) = J_2' \subset J^{(\alpha,\beta)}$

(3) $J_3 \subset I_3$
$R_{\alpha,\beta}(J_3) = [\alpha+(m-1)(1-\alpha-\beta), \alpha+m(1-\alpha-\beta)) \subset I_2$
$R^2_{\alpha,\beta}(J_3) = [-\beta+(m-2)(1-\alpha-\beta), -\beta+(m-1)(1-\alpha-\beta)) \subset I_3$
$\vdots$
$R^{2k}_{\alpha,\beta}(J_3) = [-\beta+(m-k-1)(1-\alpha-\beta), -\beta+(m-k)(1-\alpha-\beta)) \subset I_3$
$R^{2k+1}_{\alpha,\beta}(J_3) = [\alpha+(m-k-1)(1-\alpha-\beta), \alpha+(m-k)(1-\alpha-\beta)) \subset I_2$
$\vdots$
$R^{2m-3}_{\alpha,\beta}(J_3) = [1-\beta, 1-\beta+(1-\alpha-\beta)) \subset I_2$
$R^{2m-2}_{\alpha,\beta}(J_3) = [-\beta, -\beta+(1-\alpha-\beta)) \subset I_3$
$R^{2m-1}_{\alpha,\beta}(J_3) = [\alpha, \alpha+(1-\alpha-\beta)) \subset I_3$
$R^{2m}_{\alpha,\beta}(J_3) = [1+\alpha-(1-\alpha-\beta), \alpha+1) \subset I_1$
$R^{2m+1}_{\alpha,\beta}(J_3) = [\alpha-(1-\alpha-\beta), \alpha) \subset I_3$
$R^{2m+2}_{\alpha,\beta}(J_3) = [1+\alpha-2(1-\alpha-\beta), 1+\alpha-(1-\alpha-\beta)) \subset I_1$
$\vdots$
$R^{2m+2l-1}_{\alpha,\beta}(J_3) = [\alpha-l(1-\alpha-\beta), \alpha-(l-1)(1-\alpha-\beta)) \subset I_3$
$R^{2m+2l}_{\alpha,\beta}(J_3) = [1+\alpha-(l+1)(1-\alpha-\beta), 1+\alpha-l(1-\alpha-\beta)) \subset I_1$
$\vdots$
$R^{2m+2n-4}_{\alpha,\beta}(J_3) = [1+\alpha-(n-1)(1-\alpha-\beta), 1+\alpha-(n-2)(1-\alpha-\beta)) \subset I_1$

$R_{\alpha,\beta}^{2m+2n-3}(J_3) = [(1-\beta) - n(1-\alpha-\beta), (1-\beta) - (n-1)(1-\alpha-\beta))$
$= J_3' \subset J^{(\alpha,\beta)}$

Now let us give the proof of Theorem 3. From Lemma 1 and Lemma 2, we obtain that under the assumption $\alpha+\beta > 1$, i.e. $\varepsilon = -1$,

$$(R_{\alpha,\beta})_{J^{(\alpha,\beta)}} = D^{(-1)}$$

where the interval exchange transformation $D^{(-1)} : J^{(\alpha,\beta)} \to J^{(\alpha,\beta)}$ is given by

$$D^{(-1)}(J_1) = J_1', \qquad D^{(-1)}(J_2) = J_2', \qquad D^{(-1)}(J_3) = J_3'.$$

Let us define $\varphi_{\alpha,\beta} : J^{(\alpha,\beta)} \to \mathbf{R}$ by

$$\varphi_{\alpha,\beta}(x) := \frac{1}{1-(\alpha+\beta)}x,$$

then the endpoints of the interval J_i, $i = 1,2,3$ of $J^{(\alpha,\beta)}$ are given by

$$\{-\beta + (n-1)(\alpha+\beta-1), -(\alpha+\beta-1), 1-\beta-m(\alpha+\beta-1),$$
$$1-\beta-(m-1)(\alpha+\beta-1)\}$$

and they are mapped by $\varphi_{\alpha,\beta}$ to

$$\{-\beta_1, 1-\beta_1, 1, 1+\alpha_1\}$$

bijectively. Therefore, we know that

$$\varphi_{\alpha,\beta}(J^{(\alpha,\beta)}) = (-\beta_1, 1+\alpha_1] = I_{\alpha_1,\beta_1}$$

with $sgn\, I_{\alpha_1,\beta_1} = -1$ and the induced transformation $(R_{\alpha,\beta})_{J^{(\alpha,\beta)}}$ of $R_{\alpha,\beta}$ into $J^{(\alpha,\beta)}$ is isomorphic to R_{α_1,β_1} under the isomorphism $\varphi_{\alpha,\beta}$.

Analogously, from Lemma 3 and Lemma 4, we obtain that under the assumption $\alpha+\beta < 1$, i.e. $\varepsilon = +1$,

$$(R_{\alpha,\beta})_{J^{(\alpha,\beta)}} = D^{(+1)}$$

where the interval exchange transformation $D^{(+1)} : J^{(\alpha,\beta)} \to J^{(\alpha,\beta)}$ is given by

$$D^{(+1)}(J_1) = J_1', \qquad D^{(+1)}(J_2) = J_2', \qquad D^{(+1)}(J_3) = J_3'$$

where the endpoints of the intervals J_i, $i = 1,2,3$ of $J^{(\alpha,\beta)}$ are given by

$$\left\{ \begin{matrix} -(1-\alpha) + (m+1)(1-(\alpha+\beta)), -(1-\alpha) + m(1-(\alpha+\beta)), \\ 1-(\alpha+\beta), (1-\beta) - (n-1)(1-(\alpha+\beta)) \end{matrix} \right\}.$$

Let us define $\varphi_{\alpha,\beta} : J^{(\alpha,\beta)} \to \mathbf{R}$ by

$$\varphi_{\alpha,\beta}(x) := \frac{1}{1-(\alpha+\beta)}x,$$

then we see that the endpoints of the interval J_i are mapped by $\varphi_{\alpha,\beta}$ to

$$\{-\beta_1, 1-\beta_1, 1, 1+\alpha_1\}$$

bijectively. Therefore, we know that

$$\varphi_{\alpha,\beta}(J^{(\alpha,\beta)}) = [-\beta_1, 1+\alpha_1) = I_{\alpha_1,\beta_1}$$

with $sgn\, I_{\alpha_1,\beta_1} = +1$ and the induced transformation $(R_{\alpha,\beta})_{J^{(\alpha,\beta)}}$ of $R_{\alpha,\beta}$ into $J^{(\alpha,\beta)}$ is isomorphic to R_{α_1,β_1} under the isomorphism $\varphi_{\alpha,\beta}$.

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