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The Gordian complex with pass moves is not homogeneous with respect to Conway polynomials

Dedicated to Professor Akio Kawauchi on the occasion of his sixtieth birthday

Yasutaka Nakanishi and Yoshiyuki Ohyama

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Abstract. After the works of Kauffman-Banchoff and Yamasaki, it is known that a local move called the pass move is strongly related to the Arf invariant, which is equivalent to the parity of the coefficient of the degree two term in the Conway polynomial. Our main result is the following: There exists a pair of knots such that their Conway polynomials coincide, and that the sets of Conway polynomials of knots obtained from them by a single pass move do not coincide.

1. Introduction

Erle [1] introduced a notion of operation for knots and links by an exchange of band crossing. Kauffman [3] studied the structure of knots and links by the move. We usually call the operation a *pass move*, which is defined to be a local move between two knot diagrams K_1 and K_2 which are identical except near one crossing point as in Fig. 1. Furthermore, we consider its spatial realization as follows: For two knots k_1 and k_2 represented by K_1 and K_2, k_1 and k_2 are said to be transformed into each other by a pass move.

Kauffman [3] shows that a Γ move as in Fig. 2 is realized by a single pass move. It is known by Kauffman-Banchoff [4] and Yamasaki [13] that a pass move keeps the Arf invariant, which is equivalent to the parity of the coefficient of z^2 term in the Conway polynomial, a_2 mod 2. Furthermore, two knots have the same Arf invariant if and only if the two knots can be transformed into each other by a finite sequence of pass moves. We can consider the *Gordian*

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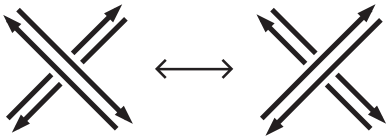

Fig. 1

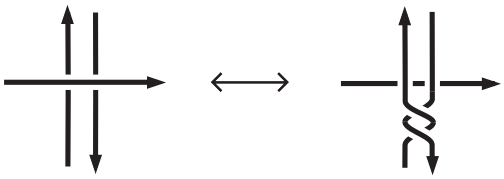

Fig. 2

complex with pass moves in a parallel manner to the Gordian complex in Hirasawa-Uchida [2]. We consider a knot as a 0-simplex (or vertex). For a positive integer m, we consider a set of $m+1$ knots, each pair of which can be transformed into each other by a single pass move, as an m-simplex.

It can be easily seen that every 0-simplex of the Gordian complex with pass moves has the degree ∞. It is shown in Section 5 that there are infinitely many knots with mutually distinct Conway polynomials, which can be transformed into a trivial knot by a single pass move. By the connected sum with the given knot, the proof is given. The following has an information on the Gordian complex with pass moves.

THEOREM 1. *Every* 0*-simplex of the Gordian complex with pass moves is a face of arbitrarily large dimensional simplex.*

The proof is given in Section 3.
The following is our main result.

THEOREM 2. *There exists a pair of knots* K_1, K_2 *such that* $\nabla_{K_1}(z) = \nabla_{K_2}(z)$, *and* $\nabla K_1^{\mathrm{P}} \neq \nabla K_2^{\mathrm{P}}$.

Here K^{P} means the set of knots obtained from a knot K by a single pass move. $\nabla\mathscr{K}$ means the set of the Conway polynomials $\{\nabla_K(z)\}_{K\in\mathscr{K}}$ for a set of knots $\mathscr{K}$. The proof is given in Section 4.

2. Surgical description

It is well-known that any knot can be transformed to a trivial knot by crossing-changes at suitable crossing points. Every crossing-change is obtained by a ± 1 surgery along a small trivial knot around the crossing point with linking number 0. Levine [6] and Rolfsen [10, 11] introduced a *surgery description* of a knot and a *surgical view* of Alexander matrix and Alexander polynomial as follows:

Proposition 3. *Let K be a knot, K_0 a trivial knot. Then, there exist n disjoint solid tori $T_1, \dots, T_n$ in $S^3 - K_0$ and a homeomorphism ϕ from $S^3 - {}^\circ T_1 \cup \cdots \cup {}^\circ T_n$ to itself such that*

(1) $\phi(K_0) = K$,

(2) $T_1 \cup \cdots \cup T_n$ *is a trivial link,*

(3) $\mathrm{lk}(T_i, K_0) = \mathrm{lk}(T_i, K) = 0$ *for each i, and*

(4) $\phi(\partial T_i) = \partial T_i$ *and* $\mathrm{lk}(\mu_i', T_i) = 1$ *where $\mu_i \subset \partial T_i$ is a meridian of T_i and $\mu_i' = \phi^{-1}(\mu_i)$.*

From a surgery description, we have a surgical view of Alexander matrix of the knot as follows:

Proposition 4. *Let K be a knot. Then, K has an Alexander matrix $M_K = (m_{ij}(t))$ of the following form:*

(1) $m_{ij}(t) = m_{ji}(t^{-1})$, *and* (2) $|m_{ij}(1)| = \delta_{ij}$,

where $\delta_{ij} = 1$ (if $i = j$), 0 (if $i \neq j$) is Kronecker's delta.

Here, the size of M_K is given by the number n in Proposition 3. The Alexander polynomial of a knot K is given by the determinant of an Alexander matrix of K up to units.

3. Proof of Theorem 1

In a parallel manner to the argument in [9], we consider the case that the given knot is a trivial knot. Otherwise, by considering the connected sum with the given knot, we can obtain the required result.

The knot in Fig. 3 is a trivial knot, which can be considered as the tangle T with ears as in the right side. Let K_i be the knot in Fig. 4, which is

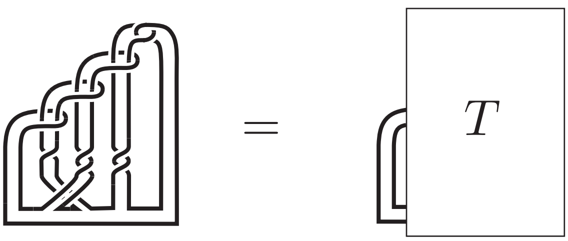

Fig. 3

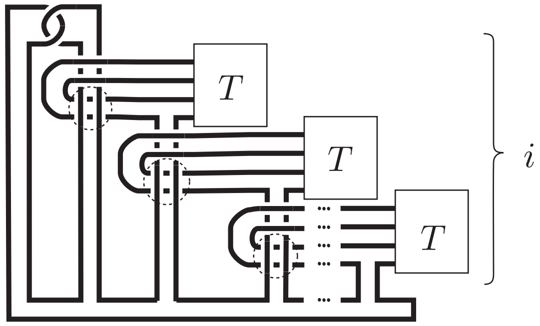

Fig. 4

constructed by using i copies of T's in Fig. 3. The knot K_0 is a trivial knot and the family $\{K_0, K_1, \dots, K_n\}$ satisfies the condition that each pair in the family can be transformed into each other by a single pass move at one of the dotted circles. The calculation of the HOMFLY polynomials in [9] shows that the knots in the family are mutually distinct. The proof is complete. $\square$

4. Proof of Theorem 2

It is known that there is a close relationship between the Alexander polynomial $\Delta_K(t)$ and the Conway polynomial $\nabla_K(z)$ for a knot K: $\Delta_K(t) = \nabla_K(t^{-1/2} - t^{1/2})$. The proof of Theorem 2 is given by a modification of the proof of the following Theorem 5.

THEOREM 5. *For j polynomials with variables z, $\nabla_i(z) = 1 + a_2 z^2 + a_4^{(i)} z^4 + \cdots + a_{2\ell_j}^{(i)} z^{2\ell_j}$ $(1 \le i \le j)$, there exists a pair of knots K_1 and K_2 such that $\nabla_{K_1}(z) = \nabla_{K_2}(z)$, $\nabla K_1^{\mathrm{P}} \not\ni \nabla_1(z), \dots, \nabla_j(z)$, and $\nabla K_2^{\mathrm{P}} \ni \nabla_1(z), \dots, \nabla_j(z)$.*

The reason why the coefficients of the z^2 term in ∇_i's are identical follows from a technical argument.

PROOF. Let $\Delta_i(t) = \nabla_i(t^{-1/2} - t^{1/2})$ $(1 \le i \le j)$. It is also known that any Alexander polynomial can be realized by a knot with unknotting number 1 by Kondo [5] and Sakai [12]. For the polynomial $\nabla_{j+1}(z) = 1 - ja_2 z^2$, let $\Delta_{j+1}(t) = \nabla_{j+1}(t^{-1/2} - t^{1/2})$. Let K^* be a knot with unknotting number 1 and $\Delta_{K^*}(t) = \prod_{i=1}^{j+1} \Delta_i(t)^2$. For the polynomial $\nabla_{j+2}(z) = 1 - (a_2 \pm 1)z^2$, let $\Delta_{j+2}(t) = \nabla_{j+2}(t^{-1/2} - t^{1/2})$. Let K^{**} be a knot with unknotting number 1 and $\Delta_{K^{**}}(t) = \Delta_{j+2}(t)$. Let $K_1 = K^* \# K^* \# K^* \# K^* \# K^* \# K^{**}$. Then, K_1 has a surgical view of Alexander matrix of the following form:

$$\begin{pmatrix} \prod \Delta_i^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \prod \Delta_i^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \prod \Delta_i^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \prod \Delta_i^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \prod \Delta_i^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_{j+2}(t) \end{pmatrix}.$$

Here, $\prod \Delta_i^2$ means $\prod_{i=1}^{j+1} \Delta_i(t)^2$.

A pass move is realized by a ± 1 surgery along the three component trivial link having linking number 0 with the given knot as in Fig. 5.

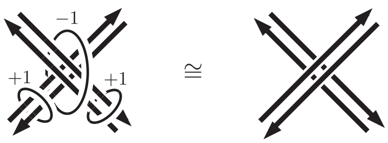

Fig. 5

If K_1' is obtained from K_1 by a single pass move, then K_1' is obtained from K_1 by three surgeries. Therefore, K_1' has a surgical view of Alexander matrix of the following form:

$$\begin{pmatrix} \prod \Delta_i^2 & 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & \prod \Delta_i^2 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & \prod \Delta_i^2 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & \prod \Delta_i^2 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & \prod \Delta_i^2 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \Delta_{j+2}(t) & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \end{pmatrix}.$$

If $\Delta_{K_1'}(t) = \Delta_i(t)$, then we have the determinant of the above matrix is $\pm\Delta_i(t)$. In the case $\Delta_i(t) \neq 1$, we consider the equation modulo $\Delta_i(t)^2$, which becomes a contradiction. In the case $\Delta_i(t) = 1$, we take another nontrivial $\Delta_{i'}(t)$ and consider the equation modulo $\Delta_{i'}(t)^2$, which becomes a contradiction. Therefore, we have $\nabla K_1^{\mathrm{P}} \not\ni \nabla_1(z), \nabla_2(z), \dots, \nabla_j(z)$.

Let K_2 be a knot with unknotting number 1 and $\Delta_{K_2}(t) = \Delta_{K_1}(t)$. By the following Lemma, it can be seen that $\nabla K_2^{\mathrm{P}} \ni \nabla_1(z), \nabla_2(z), \dots, \nabla_j(z)$. Hence the proof is complete. $\square$

LEMMA 6. *Let K be a knot with unknotting number* 1. *For a set of integers $a'_2 = a_2(K) \pm 2$, and arbitrary integers a'_{2i} $(i = 2, 3, \dots, \ell)$, there exists a knot $K' \in K^{\mathrm{P}}$ with $\nabla_{K'}(z) = 1 + a'_2 z^2 + a'_4 z^4 + \cdots + a'_{2\ell} z^{2\ell}$.*

PROOF. Since K is a knot with unknotting number 1, there exists a crossing point at which the crossing-change yields a trivial knot. We consider such a crossing point as in the left of Fig. 6. We transform this part of K to the right of Fig. 6 by a single Γ move. Here, $m_2, \dots, m_\ell$ are numbers of left-handed full-twists, respectively. In a negative case, it means $|m_i|$ right-handed full-twists. By the parallel argument to that in Murakami [7], the difference of the Conway polynomials is $2z^2 - (m_2 + 1)z^4 \cdots + (-1)^{\ell-2}(m_{\ell-1} + 1)z^{2\ell-2} + (-1)^{\ell-1} m_\ell z^{2\ell}$. The proof is complete. $\square$

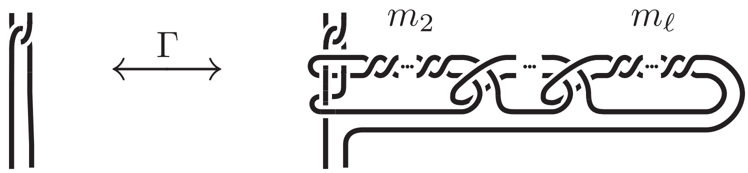

Fig. 6

5. Questions

5.1. The Gordian complex with pass moves has two components, one of which contains all knots with the Arf invariant 0, and the other all knots with the Arf invariant 1. We denote the former subcomplex by $\mathscr{K}_0$, and the latter by $\mathscr{K}_1$. The following question is natural and open.

QUESTION 7. *Are $\mathscr{K}_0$ and $\mathscr{K}_1$ isomorphic as complexes?*

If isomorphic, it is an interesting problem that which kind of knots in $\mathscr{K}_1$ are corresponding to a trivial knot.

5.2. There are infinitely many knots with mutually distinct Conway polynomials, which can be transformed into a trivial knot by a single pass move. We raise the following question:

QUESTION 8. *For a set of an even integer a_2 and arbitrary integers a_i $(i = 4, \dots, 2\ell)$, does there exist a knot K with $\nabla_K(t) = 1 + a_2 z^2 + a_4 z^4 + \cdots + a_{2\ell} z^{2\ell}$, which can be transformed into a trivial knot by a single pass move?*

We will construct knots in the cases (1) $a_2 = 0$, (2) $a_2 = \pm 2$, (3) $a_2 = 2p$ and all the other integers a_i $(i = 4, \dots, 2\ell)$ are divisible by 2, and (4) $a_2 = 2p$

and all the other integers a_i $(i = 4, \dots, 2\ell)$ are divisible by p. We remark that the construction in the case (4) is strongly inspired by the idea of Nakamura [8].

Let K be a knot as illustrated in Fig. 7. Here, $b, b_2, -b_4, \dots, (-1)^{n-1}b_{2\ell}$ are numbers of right-handed full-twists, respectively.

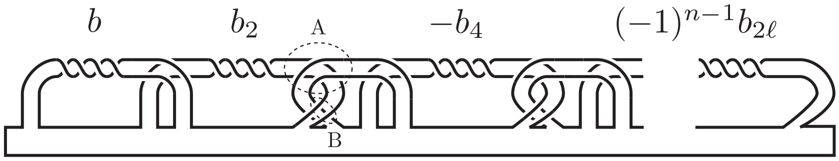

Fig. 7

We calculate a Seifert matrix V as follows:

$$V = \begin{pmatrix} b & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -b_4 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (-1)^{\ell-1}b_{2\ell} \end{pmatrix}$$

Therefore, the Alexander polynomial $\Delta(t) = t^{\ell} + bb_2t^{\ell-1}(t-1)^2 + bb_4t^{\ell-2}(t-1)^4 + \cdots + bb_{2\ell}(t-1)^{2\ell}$, and then the Conway polynomial $\nabla(z) = 1 + bb_2z^2 + bb_4z^4 + \cdots + bb_{2\ell}z^{2\ell}$.

In the case (1) $a_2 = 0$, we consider that $b = 1, b_2 = 0, b_4 = a_4, \dots, b_{2\ell} = a_{2\ell}$. Operate a single pass move at the part surrounded by dotted circle A, and the knot is transformed into a trivial knot.

In the case (2) $a_2 = \pm 2$, we consider that $b = \pm 1, b_2 = 2, b_4 = \pm a_4, \dots, b_{2\ell} = \pm a_{2\ell}$. Operate a single Γ move at the part surrounded by dotted circle B, and the knot is transformed into a trivial knot. Otherwise, the argument parallel to that in the proof of Lemma in Section 4 gives the proof.

In the case (3) $a_2 = 2p$ and all the other integers a_i $(i = 4, \dots, 2\ell)$ are divisible by 2, we consider that $b = 2, b_2 = p, b_4 = a_4/2, \dots, b_{2\ell} = a_{2\ell}/2$. Operate a single pass move at the left-most band and remove 2 full-twists, and the knot is transformed into a trivial knot.

In the case (4) $a_2 = 2p$ and all the other integers a_i $(i = 4, \dots, 2\ell)$ are divisible by p, we consider that $b = p, b_2 = 2, b_4 = a_4/p, \dots, b_{2\ell} = a_{2\ell}/p$. Operate a single Γ move at the part surrounded by dotted circle B, and the knot is transformed into a trivial knot.

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Yasutaka Nakanishi
Department of Mathematics
Graduate School of Science
Kobe University
Rokko, Nada-ku, Kobe 657-8501*, Japan*
E-mail: nakanisi@math.kobe-u.ac.jp

Yoshiyuki Ohyama
Department of Mathematics
College of Arts and Sciences
Tokyo Woman's Christian University
Zempukuji, Suginami-ku, Tokyo 167-8585*, Japan*
E-mail: ohyama@lab.twcu.ac.jp