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The best constant of Sobolev inequality corresponding to Dirichlet-Neumann boundary value problem for $(-1)^M(d/dx)^{2M}$

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ABSTRACT. We clarified the variational meaning of the special values $\zeta(2M)$ $(M = 1, 2, 3, \dots)$ of Riemann zeta function $\zeta(s)$. These are essentially the best constant of Sobolev inequality. In the background we consider Dirichlet-Neumann boundary value problem for a differential operator $(-1)^M(d/dx)^{2M}$. Its Green function is found and expressed in terms of the well-known Bernoulli polynomial. The supremum of the diagonal value of Green function is equal to the best constant for corresponding Sobolev inequality. Discrete version of the corresponding Sobolev inequality is also presented.

1. Introduction and results

Sobolev inequalities

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{W^{m,p}(\Omega)} \qquad (\Omega \subset \mathbf{R}^n)$$

play crucial roles in the development of theory of differential equations. However, one can rarely find explicit forms of the best constants among such C. In Watanabe et al. [10] and Kametaka et al. [2], a systematic way to find the best constant of Sobolev inequality was discovered in the case $p = 2$, $q = \infty$, where Green functions for suitable boundary value problem are obtained and their aspect as reproducing kernels is investigated. It should be noted that Talenti [9] found the best constant in another special case $q = np/(n-p)$, $\Omega = \mathbf{R}^n$.

Let us first survey our results [5]. For $M = 1, 2, 3, \dots,$ given Sobolev spaces

$$H(\mathrm{X}, M) = \{u(x) \,|\, u(x), u^{(M)}(x) \in L^2(0,1), u(x) \in A(\mathrm{X})\},$$

$$A(\mathrm{P}) : u^{(i)}(1) - u^{(i)}(0) = 0 \quad (0 \le i \le M-1), \qquad \int_0^1 u(x)dx = 0,$$

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$$A(\mathrm{D}) : u^{(2i)}(0) = u^{(2i)}(1) = 0 \quad (0 \le i \le [(M-1)/2]),$$

$$A(\mathrm{N}) : u^{(2i+1)}(0) = u^{(2i+1)}(1) = 0 \quad (0 \le i \le [(M-2)/2]), \qquad \int_0^1 u(x)dx = 0,$$

where the boundary conditions for $u(x)$ in $A(\mathrm{N})$ are not required when $M = 1$, we have found the best constants of the corresponding Sobolev inequalities, which are expressed by using Riemann zeta function as follows:

$$C(\mathrm{P}, M) = 2^{-(2M-1)}\pi^{-2M}\zeta(2M),$$

$$C(\mathrm{D}, M) = 2^{-(2M-1)}(2^{2M} - 1)\pi^{-2M}\zeta(2M),$$

$$C(\mathrm{N}, M) = 2\pi^{-2M}\zeta(2M).$$

The key to finding the best constants is Green functions of suitable boundary value problems for $(-1)^M(d/dx)^{2M}$.

We here introduce Sobolev space

$$H = H(M) = \{u(x) \,|\, u(x), u^{(M)}(x) \in L^2(0,1), u^{(2i)}(0) = 0 \ (0 \le i \le [(M-1)/2]), \\ u^{(2i+1)}(1) = 0 \ (0 \le i \le [(M-2)/2])\} \tag{1.1}$$

where the boundary conditions for $u(x)$ at $x = 1$ are not required when $M = 1$, Sobolev inner product

$$(u, v)_M = \int_0^1 u^{(M)}(x)\bar{v}^{(M)}(x)dx, \tag{1.2}$$

Sobolev energy

$$\|u\|_M^2 = \int_0^1 |u^{(M)}(x)|^2 dx \tag{1.3}$$

and Sobolev functional

$$S(u) = S(M; u) = \left(\sup_{0 \le y \le 1} |u(y)|\right)^2 \Big/ \|u\|_M^2. \tag{1.4}$$

Sesquilinear form $(\cdot\,,\cdot)_M$ is proved to be an inner product of H afterwards. H is Hilbert space with an inner product $(\cdot\,,\cdot)_M$.

The purpose of this paper is to find the supremum of Sobolev functional $S(u)$. Our conclusion is as follows:

Theorem 1.1. *Let $G(x, y) = G(M; x, y)$ be Green function defined later in Theorem* 3.1.

(1) *The supremum* $C_0 = \sup_{u \in H, u \notin 0} S(u)$ *is given by*

$$C_0 = C(M) = \max_{0 \le y \le 1} G(y, y) = G(1,1) = 2(2^{2M} - 1)\pi^{-2M}\zeta(2M) \quad (1.5)$$

and is attained by putting $u(x) = cG(x,1)$, *where c is an arbitrarily fixed complex number. We here list explicit forms of* $C(M)$.

$$C(1) = 1, \quad C(2) = 1/3, \quad C(3) = 2/15, \quad C(4) = 17/315,$$
$$C(5) = 62/2835, \quad C(6) = 1382/155925, \quad C(7) = 21844/6081075,$$
$$C(8) = 929569/638512875, \quad \ldots.$$

(2) *The infimum* $\inf_{u \in H, u \notin 0} S(u)$ *is equal to zero.*

Concerning the infimum, we can prove easily as follows:

$$S(\sin(\pi(n+1/2)x)) = 2(\pi(n+1/2))^{-2M} \to 0 \qquad (n \to \infty).$$

The above Theorem 1.1(1) is rewritten equivalently as follows:

Theorem 1.2. *For any function* $u(x) \in H$, *there exists a positive constant C which is independent of* $u(x)$ *such that Sobolev inequality*

$$\left(\sup_{0 \le y \le 1} |u(y)|\right)^2 \le C \int_0^1 |u^{(M)}(x)|^2 dx \quad (1.6)$$

holds. Among such C the best constant C_0 *is the same as Theorem* 1.1(1). *If we replace C by* C_0 *in* (1.6), *then the equality holds for* $u(x) = cG(x,1)$ $(0 < x < 1)$ *for any complex number c.*

The engineering meaning of Sobolev inequality is that the square of the maximal bending of a string ($M = 1$) [3] or a beam ($M = 2$) is estimated from above by the constant multiple of the potential energy.

This paper consists of seven sections. In section 2, we present Bernoulli polynomial [1, 7, 11], which plays an important role in this paper. In section 3, we present a boundary value problem for $(-1)^M (d/dx)^{2M}$ with Dirichlet-Neumann boundary condition. In section 4, we show that Green function $G(x, y)$ is expressed in terms of Bernoulli polynomial. In section 5, it is clarified that Green function $G(x, y)$ is a reproducing kernel for H and $(\cdot,\cdot)_M$. Section 6 is devoted to the proof of Theorem 1.2. Finally, in section 7, we present a discrete version of Theorem 1.2 ($M = 1$).

2. Bernoulli polynomial

As a preparation, we explain briefly about Bernoulli polynomials and their properties which are required in this paper.

Bernoulli polynomials $b_j(x)$ defined by the following relation:

$$\begin{cases} b_0(x) = 1 \\ b_j'(x) = b_{j-1}(x), \ \int_0^1 b_j(x)dx = 0 \quad (j = 1, 2, 3, \dots). \end{cases}$$

Here we list explicit forms of $b_j(x)$ $(j = 0, 1, \dots, 8)$.

$$b_0(x) = 1, \qquad b_1(x) = x - \frac{1}{2}, \qquad b_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12},$$

$$b_3(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x, \qquad b_4(x) = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}x^2 - \frac{1}{720},$$

$$b_5(x) = \frac{1}{120}x^5 - \frac{1}{48}x^4 + \frac{1}{72}x^3 - \frac{1}{720}x,$$

$$b_6(x) = \frac{1}{720}x^6 - \frac{1}{240}x^5 + \frac{1}{288}x^4 - \frac{1}{1440}x^2 + \frac{1}{30240},$$

$$b_7(x) = \frac{1}{5040}x^7 - \frac{1}{1440}x^6 + \frac{1}{1440}x^5 - \frac{1}{4320}x^3 + \frac{1}{30240}x,$$

$$b_8(x) = \frac{1}{40320}x^8 - \frac{1}{10080}x^7 + \frac{1}{8640}x^6 - \frac{1}{17280}x^4 + \frac{1}{60480}x^2 - \frac{1}{1209600}.$$

They are also defined by the following generating function:

$$\frac{e^{xt}}{t^{-1}(e^t - 1)} = \sum_{j=0}^{\infty} b_j(x)t^j \qquad (|t| < 2\pi).$$

Bernoulli polynomial $b_j(x)$ is j-th polynomial with respect to x. We list up the properties of Bernoulli polynomial $b_j(x)$ [4, 5].

$$b_j(1 - x) = (-1)^j b_j(x) \qquad (j = 0, 1, 2, \dots). \tag{2.1}$$

$$b_j(1) - b_j(0) = \begin{cases} 1 & (j = 1) \\ 0 & (j \neq 1). \end{cases} \tag{2.2}$$

$$b_{2j+1}(0) = \begin{cases} -1/2 & (j = 0) \\ 0 & (j = 1, 2, 3, \dots). \end{cases} \tag{2.3}$$

$$b_{2j+1}(1/2) = 0 \qquad (j = 0, 1, 2, \dots). \tag{2.4}$$

$$(-1)^{j+1} b_{2j}(0) = B_j/(2j)! \qquad (j = 0, 1, 2, \dots). \tag{2.5}$$

In (2.5), B_j is Bernoulli number defined by the following recurrence relation:

$$\begin{cases} \displaystyle\sum_{j=0}^{n-1} (-1)^j \binom{2n}{2j} B_j = -n \quad (n = 1, 2, 3, \dots) \\ B_0 = -1. \end{cases}$$

Next we derive Fourier expansion formula of $b_j(\{x\})$, where

$$\{x\} = x - [x], \qquad [x] = \sup\{n \in \mathbf{Z} \mid n \leq x\},$$

denotes a decimal part of a real number x. $\{x\}$ is a periodic function of x with period 1. For $j = 1, 2, 3, \dots,$ we have

$$b_j(\{x\}) = -\sum_{k \neq 0} (\sqrt{-1}2\pi k)^{-j} \exp(\sqrt{-1}2\pi k x)$$

that is to say

$$b_{2j}(\{x\}) = (-1)^{j+1} 2 \sum_{k=1}^{\infty} (2\pi k)^{-2j} \cos(2\pi k x), \tag{2.6}$$

$$b_{2j+1}(\{x\}) = (-1)^{j+1} 2 \sum_{k=1}^{\infty} (2\pi k)^{-(2j+1)} \sin(2\pi k x). \tag{2.7}$$

For $j = 0, 1, 2, \dots,$ the relation

$$(-1)^{j+1} b_{2j}(0) = 2 \sum_{k=1}^{\infty} (2\pi k)^{-2j} = \frac{2}{(2\pi)^{2j}} \zeta(2j), \tag{2.8}$$

$$b_{2j}(1/2) = -(1 - 2^{-(2j-1)}) b_{2j}(0) \tag{2.9}$$

follows from the above Fourier expansion of $b_j(\{x\})$. In (2.8), $\zeta(z)$ is Riemann ζ-function. The following lemma concerning Bernoulli polynomials plays important roles hereafter.

Lemma 2.1 ([4, 5]). $u(x) = (-1)^{j+1} b_{2j}(x)$ $(j = 1, 2, 3, \dots)$ *satisfy the following properties:*

$$\max_{0 \leq x \leq 1} u(x) = u(0) = u(1) > 0, \tag{2.10}$$

$$\min_{0 \leq x \leq 1} u(x) = u(1/2) < 0, \tag{2.11}$$

$$\max_{0 \leq x \leq 1} |u(x)| = u(0) = u(1), \tag{2.12}$$

$$u'(x) < 0 \qquad (0 < x < 1/2), \tag{2.13}$$

$$u'(x) > 0 \qquad (1/2 < x < 1). \tag{2.14}$$

3. Dirichlet-Neumann boundary value problem

For any bounded continuous function $f(x)$ on an interval $0 < x < 1$, we consider the following Dirichlet-Neumann boundary value problem:

BVP(M)

$$\begin{cases} (-1)^M u^{(2M)} = f(x) & (0 < x < 1) \quad (3.1) \\ u^{(2i)}(0) = u^{(2i+1)}(1) = 0 & (0 \le i \le M-1). \quad (3.2) \end{cases}$$

For later convenience sake, we introduce monomials $\{E_j(x)\}$ defined by

$$E_j(x) = \begin{cases} x^j/j! & (j = 0, 1, 2, \dots) \\ 0 & (j = -1, -2, -3, \dots). \end{cases}$$

Note that $E_j'(x) = E_{j-1}(x)$ $(j = 0, \pm 1, \pm 2, \dots)$. We also use the abbreviation $E_j = E_j(1)$ $(j = 0, \pm 1, \pm 2, \dots)$. We prepare the next lemma.

LEMMA 3.1. *For any $N \times N$ regular matrix $\boldsymbol{A}$ and $N \times 1$ matrices $\boldsymbol{a}$ and $\boldsymbol{b}$, we have the following equality:*

$$ {}^t\boldsymbol{a}\boldsymbol{A}^{-1}\boldsymbol{b} = -\left| \begin{array}{c|c} \boldsymbol{A} & \boldsymbol{b} \\ \hline {}^t\boldsymbol{a} & 0 \end{array} \right| \Big/ |\boldsymbol{A}|. \tag{3.3}$$

Concerning the uniqueness and existence of the solution to BVP(M), we have the following theorem:

THEOREM 3.1. *For any bounded continuous function $f(x)$ on an interval $0 < x < 1$, BVP(M) has a unique classical solution $u(x)$ expressed as*

$$u(x) = \int_0^1 G(x, y) f(y) dy \qquad (0 < x < 1). \tag{3.4}$$

Green function $G(x, y) = G(M; x, y)$ $(0 < x, y < 1)$ is given by the following three equivalent expressions:

$$(1) \quad G(x, y) = \frac{(-1)^M}{2}\left[E_{2M-1}(|x-y|) + \left| \begin{array}{c|c} E_{2(j-i)} & E_{2(M-1-i)}(1-y) \\ \hline E_{2j+1}(x) & 0 \end{array} \right| + \left| \begin{array}{c|c} E_{2(j-i)} & E_{2(M-1-i)+1}(y) \\ \hline E_{2j}(1-x) & 0 \end{array} \right| \right] \tag{3.5}$$

where $0 \le i, j \le M-1$. *Moreover, we have* $|E_{2(j-i)}| = 1$.

$$\text{(2)} \quad G(x,y) = (-1)^{M+1} 4^{2M-1} \left[b_{2M}\left(\frac{|x-y|}{4}\right) - b_{2M}\left(\frac{x+y}{4}\right) + b_{2M}\left(\frac{1}{2} - \frac{x+y}{4}\right) - b_{2M}\left(\frac{1}{2} - \frac{|x-y|}{4}\right) \right]. \tag{3.6}$$

$$\text{(3)} \quad G(x,y) = 2\sum_{j=0}^{\infty} (\pi(j+1/2))^{-2M} \sin(\pi(j+1/2)x) \sin(\pi(j+1/2)y). \tag{3.7}$$

Proof of Theorem 3.1(1). We suppose that BVP(M) has a classical solution $u(x)$. Introducing new functions $u_i = u^{(i)}$ $(0 \le i \le 2M-1)$ and matrices

$$\boldsymbol{u} = {}^t(u_0, \ldots, u_{2M-1}), \qquad \boldsymbol{e} = {}^t(0, \ldots, 0, 1),$$

$$\boldsymbol{N} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \qquad (2M \times 2M \ \text{nilpotent matrix}),$$

one can rewrite BVP(M) as follows:

$$\begin{cases} \boldsymbol{u}' = \boldsymbol{N}\boldsymbol{u} + \boldsymbol{e}(-1)^M f(x) \quad (0 < x < 1) & (3.8) \\ u_{2i}(0) = u_{2i+1}(1) = 0 \quad (0 \le i \le M-1). & (3.9) \end{cases}$$

Let $\boldsymbol{E}(x)$ be an upper-triangular matrix given by

$$\boldsymbol{E}(x) = \exp(x\boldsymbol{N}) = (E_{j-i})(x),$$

which is a fundamental solution to the initial-value problem $\boldsymbol{E}' = \boldsymbol{N}\boldsymbol{E}$, $\boldsymbol{E}(0) = \boldsymbol{I}$. Solving (3.8), we have

$$\boldsymbol{u}(x) = \boldsymbol{E}(x)\boldsymbol{u}(0) + \int_0^x \boldsymbol{E}(x-y)\boldsymbol{e}(-1)^M f(y)dy,$$

$$\boldsymbol{u}(x) = \boldsymbol{E}(x-1)\boldsymbol{u}(1) - \int_x^1 \boldsymbol{E}(x-y)\boldsymbol{e}(-1)^M f(y)dy,$$

or equivalently, for $0 \le i \le 2M-1$,

$$u_i(x) = \sum_{j=0}^{2M-1} E_{j-i}(x)u_j(0) + \int_0^x (-1)^M E_{2M-1-i}(x-y)f(y)dy,$$

$$u_i(x) = \sum_{j=0}^{2M-1} E_{j-i}(x-1)u_j(1) - \int_x^1 (-1)^M E_{2M-1-i}(x-y)f(y)dy.$$

Employing the boundary conditions $u_{2i}(0) = u_{2i+1}(1) = 0$ $(0 \le i \le M-1)$, we have

$$u_{2i}(x) = \sum_{j=0}^{M-1} E_{2(j-i)+1}(x)u_{2j+1}(0) + \int_0^x (-1)^M E_{2(M-1-i)+1}(x-y)f(y)dy,$$

$$u_{2i+1}(x) = \sum_{j=0}^{M-1} E_{2(j-i)}(x)u_{2j+1}(0) + \int_0^x (-1)^M E_{2(M-1-i)}(x-y)f(y)dy,$$

$$u_{2i}(x) = \sum_{j=0}^{M-1} E_{2(j-i)}(x-1)u_{2j}(1) - \int_x^1 (-1)^M E_{2(M-1-i)+1}(x-y)f(y)dy,$$

$$u_{2i+1}(x) = \sum_{j=0}^{M-1} E_{2(j-i-1)+1}(x-1)u_{2j}(1) - \int_x^1 (-1)^M E_{2(M-1-i)}(x-y)f(y)dy$$

for $0 \le i \le M-1$. In particular, we have

$$u_0(x) = \sum_{j=0}^{M-1} E_{2j+1}(x)u_{2j+1}(0) + \int_0^x (-1)^M E_{2M-1}(x-y)f(y)dy, \quad (3.10)$$

$$u_0(x) = \sum_{j=0}^{M-1} E_{2j}(x-1)u_{2j}(1) - \int_x^1 (-1)^M E_{2M-1}(x-y)f(y)dy. \quad (3.11)$$

Using the boundary conditions $u_{2i}(0) = u_{2i+1}(1) = 0$ $(0 \le i \le M-1)$ again, we have

$$0 = u_{2i}(0) = \sum_{j=0}^{M-1} E_{2(j-i)}(-1)u_{2j}(1) - \int_0^1 (-1)^M E_{2(M-1-i)+1}(-y)f(y)dy,$$

$$0 = u_{2i+1}(1) = \sum_{j=0}^{M-1} E_{2(j-i)}(1)u_{2j+1}(0) + \int_0^1 (-1)^M E_{2(M-1-i)}(1-y)f(y)dy.$$

Solving the above linear system of equations with respect to $u_{2i}(1)$, $u_{2i+1}(0)$ $(0 \le i \le M-1)$, we have

$$(u_{2i})(1) = \int_0^1 (-1)^M (E_{2(j-i)})^{-1}(-1)(E_{2(M-1-i)+1})(-y)f(y)dy, \quad (3.12)$$

$$(u_{2i+1})(0) = -\int_0^1 (-1)^M (E_{2(j-i)})^{-1}(1)(E_{2(M-1-i)})(1-y)f(y)dy. \quad (3.13)$$

Substituting (3.12) and (3.13) into (3.10) and (3.11), we have

$$u_0(x) = -\int_0^1 (-1)^M (E_{2j+1})(x)(E_{2(j-i)})^{-1}(1)(E_{2(M-1-i)})(1-y)f(y)dy$$
$$+ \int_0^x (-1)^M E_{2M-1}(|x-y|)f(y)dy,$$
$$u_0(x) = \int_0^1 (-1)^M (E_{2j})(x-1)(E_{2(j-i)})^{-1}(-1)(E_{2(M-1-i)+1})(-y)f(y)dy$$
$$+ \int_x^1 (-1)^M E_{2M-1}(|x-y|)f(y)dy.$$

Note that $E_{2M-1}(-x) = -E_{2M-1}(x)$. Taking an average of the above two expressions, we have obtained the following expression for a solution $u(x) = u_0(x)$ to BVP(M):

$$u(x) = \int_0^1 G(x,y)f(y)dy \qquad (0 < x < 1), \tag{3.14}$$

where $G(x,y)$ represents Green function given by

$$G(x,y) = \frac{(-1)^M}{2}[E_{2M-1}(|x-y|) - (E_{2j+1})(x)(E_{2(j-i)})^{-1}(1)(E_{2(M-1-i)})(1-y)$$
$$+ (E_{2j})(x-1)(E_{2(j-i)})^{-1}(-1)(E_{2(M-1-i)+1})(-y)] \qquad (0 < x, y < 1).$$

Owing to the relation $E_i(-x) = (-1)^i E_i(x)$, $G(x,y)$ is rewritten as

$$G(x,y) = \frac{(-1)^M}{2}[E_{2M-1}(|x-y|) - (E_{2j+1})(x)(E_{2(j-i)})^{-1}(E_{2(M-1-i)})(1-y)$$
$$- (E_{2j})(1-x)(E_{2(j-i)})^{-1}(E_{2(M-1-i)+1})(y)] \qquad (0 < x, y < 1), \tag{3.15}$$

where $E_j = E_j(1)$. Applying Lemma 3.1 to (3.15), we have (1) of Theorem 3.1.

Since the right-hand side of (3.14) includes only a data function $f(x)$, the solution to BVP(M) is unique. From the next theorem, we can show that $u(x)$ defined by (3.14) satisfies BVP(M), which guarantees the existence of the solution.

Theorem 3.2. *Green function $G(x,y) = G(M;x,y)$ satisfies the following properties:*

(1) $\partial_x^{2M} G(x,y) = 0 \qquad (0 < x, y < 1, x \neq y).$ (3.16)

(2) $\partial_x^{2i} G(x,y)|_{x=0} = \partial_x^{2i+1} G(x,y)|_{x=1} = 0 \quad (0 \leq i \leq M-1, 0 < y < 1).$ (3.17)

(3) $\partial_x^i G(x,y)|_{y=x-0} - \partial_x^i G(x,y)|_{y=x+0}$

$$= \begin{cases} 0 & (0 \le i \le 2M-2) \\ (-1)^M & (i = 2M-1) \ \ (0 < x < 1). \end{cases} \tag{3.18}$$

(4) $\partial_x^i G(x,y)|_{x=y+0} - \partial_x^i G(x,y)|_{x=y-0}$

$$= \begin{cases} 0 & (0 \le i \le 2M-2) \\ (-1)^M & (i = 2M-1) \ \ (0 < y < 1). \end{cases} \tag{3.19}$$

(5) $G(x,y) > 0 \qquad (0 < x, y < 1).$ (3.20)

PROOF OF THEOREM 3.2(1)~(4). Operating ∂_x^k $(1 \le k \le 2M)$ on both sides of (3.5), we have

$$\partial_x^k G(x,y) = \frac{(-1)^M}{2}\Bigg[(\mathrm{sgn}(x-y))^k E_{2M-1-k}(|x-y|) + \left|\begin{array}{c|c} E_{2(j-i)} & E_{2(M-1-i)}(1-y) \\ \hline E_{2j+1-k}(x) & 0 \end{array}\right| + (-1)^k \left|\begin{array}{c|c} E_{2(j-i)} & E_{2(M-1-i)+1}(y) \\ \hline E_{2j-k}(1-x) & 0 \end{array}\right|\Bigg]. \tag{3.21}$$

Putting $k = 2M$ in (3.21) and using $E_j(x) = 0$ $(j < 0)$, we have (3.16). For $0 \le k \le M-1$, we have

$$(-1)^M 2\partial_x^{2k} G(x,y)|_{x=0} = E_{2(M-1-k)+1}(y) + \left|\begin{array}{c|c} E_{2(j-i)} & E_{2(M-1-i)}(1-y) \\ \hline E_{2(j-k)+1}(0) & 0 \end{array}\right| + \left|\begin{array}{c|c} E_{2(j-i)} & E_{2(M-1-i)+1}(y) \\ \hline E_{2(j-k)} & 0 \end{array}\right| = E_{2(M-1-k)+1}(y) + \left|\begin{array}{c|c} E_{2(j-i)} & E_{2(M-1-i)+1}(y) \\ \hline 0 \ \cdots \ 0 & -E_{2(M-1-k)+1}(y) \end{array}\right| = 0$$

and

$$(-1)^M 2\partial_x^{2k+1} G(x,y)|_{x=1} = E_{2(M-1-k)}(1-y) + \left|\begin{array}{c|c} E_{2(j-i)} & E_{2(M-1-i)}(1-y) \\ \hline E_{2(j-k)} & 0 \end{array}\right| - \left|\begin{array}{c|c} E_{2(j-i)} & E_{2(M-1-i)+1}(y) \\ \hline E_{2(j-k-1)+1}(0) & 0 \end{array}\right|$$

$$= E_{2(M-1-k)}(1-y) + \left| \begin{array}{ccc|c} & E_{2(j-i)} & & E_{2(M-1-i)}(1-y) \\ \hline 0 & \cdots & 0 & -E_{2(M-1-k)}(1-y) \end{array} \right| = 0$$

where we have used $E_k(0) = 0$ $(k \neq 0)$, 1 $(k = 0)$ and $|E_{2(j-i)}| = 1$. Hence we have (3.17). For $0 \leq k \leq 2M-1$, we have

$$\partial_x^k G(x,y)|_{y=x-0} - \partial_x^k G(x,y)|_{y=x+0} = \frac{(-1)^M}{2}(1-(-1)^k)E_{2M-1-k}(0)$$
$$= \begin{cases} 0 & (0 \leq k \leq 2M-2) \\ (-1)^M & (k = 2M-1) \end{cases} \quad (0 < x < 1),$$

which proves (3.18). (3.19) follows from (3.18). The positivity (3.20) is shown later.

Concerning the uniqueness of Green function, we have the following theorem:

THEOREM 3.3. *The smooth function $G(x,y)$ on an open set $0 < x, y < 1$, $x \neq y$ satisfying properties* (3.16), (3.17) *and* (3.18) *is unique.*

PROOF OF THEOREM 3.3. Suppose that we have another function $\tilde{G}(x,y)$ satisfying the same properties (3.16), (3.17) and (3.18). For any function $f(x)$,

$$u(x) = \int_0^1 \tilde{G}(x,y)f(y)dy \qquad (0 < x < 1)$$

satisfies BVP(M). From Theorem 3.1, we have

$$\int_0^1 \tilde{G}(x,y)f(y)dy = \int_0^1 G(x,y)f(y)dy \qquad (0 < x < 1).$$

This shows $\tilde{G}(x,y) = G(x,y)$ $(0 < x, y < 1)$.

PROOF OF THEOREM 3.1(2), (3). (3) follows from (2) by Fourier series expansion of Bernoulli polynomial (2.6). In order to prove (2), it is enough to show that $G(x,y)$ defined by (3.6) satisfies the properties (3.16), (3.17) and (3.18). Differentiating $G(x,y)$ with respect to x, we have

$$\partial_x^i G(x,y) = (-1)^{M+1}4^{2M-1-i}\left[(\mathrm{sgn}(x-y))^i b_{2M-i}\left(\frac{|x-y|}{4}\right) - b_{2M-i}\left(\frac{x+y}{4}\right) + (-1)^i b_{2M-i}\left(\frac{1}{2}-\frac{x+y}{4}\right) - (-1)^i(\mathrm{sgn}(x-y))^i b_{2M-i}\left(\frac{1}{2}-\frac{|x-y|}{4}\right)\right]$$
$$(0 < x, y < 1, x \neq y, 0 \leq i \leq 2M). \tag{3.22}$$

(3.16) is shown by putting $i = 2M$ in the above equality. For $0 \le i \le M - 1$ we have

$$\partial_x^{2i} G(x, y) = (-1)^{M+1} 4^{2(M-1-i)+1} \left[b_{2(M-i)}\left(\frac{|x-y|}{4}\right) - b_{2(M-i)}\left(\frac{x+y}{4}\right) + b_{2(M-i)}\left(\frac{1}{2} - \frac{x+y}{4}\right) - b_{2(M-i)}\left(\frac{1}{2} - \frac{|x-y|}{4}\right) \right]. \tag{3.23}$$

Putting $x = 0$ in (3.23), we have

$$\partial_x^{2i} G(x, y)|_{x=0} = 0 \qquad (0 < y < 1,\ 0 \le i \le M - 1).$$

For $0 \le i \le M - 1$ we have

$$\partial_x^{2i+1} G(x, y) = (-1)^{M+1} 4^{2(M-1-i)} \left[\operatorname{sgn}(x-y) b_{2(M-1-i)+1}\left(\frac{|x-y|}{4}\right) - b_{2(M-1-i)+1}\left(\frac{x+y}{4}\right) - b_{2(M-1-i)+1}\left(\frac{1}{2} - \frac{x+y}{4}\right) + \operatorname{sgn}(x-y) b_{2(M-1-i)+1}\left(\frac{1}{2} - \frac{|x-y|}{4}\right) \right]. \tag{3.24}$$

Putting $x = 1$ in (3.24), we have

$$\partial_x^{2i+1} G(x, y)|_{x=1} = 0 \qquad (0 < y < 1,\ 0 \le i \le M - 1).$$

Hence we have (3.17). Putting $y = x \mp 0$ in (3.22) and taking their difference, we have

$$\begin{aligned} &\partial_x^i G(x, y)|_{y=x-0} - \partial_x^i G(x, y)|_{y=x+0} \\ &\quad = (-1)^{M+1}(1 - (-1)^i) 4^{2M-1-i} [b_{2M-i}(0) - (-1)^i b_{2M-i}(1/2)] \\ &\quad = \begin{cases} 0 & (0 \le i \le 2M - 2) \\ (-1)^M & (i = 2M - 1) \end{cases} \quad (0 < x < 1), \end{aligned} \tag{3.25}$$

where we have used (2.3) and (2.4). This completes the proof of Theorem 3.1(2), (3).

Proof of Theorem 3.2(5). We start with the expression (3.6). Noting that

$$0 \le \frac{|x-y|}{4} < \frac{x+y}{4} < \frac{1}{2}, \quad 0 < \frac{1}{2} - \frac{x+y}{4} < \frac{1}{2} - \frac{|x-y|}{4} \le \frac{1}{2} \qquad (0 < x, y < 1)$$

and using (2.13), we have Theorem 3.2(5).

From Theorem 3.1(2), it is easy to see the following remark:

Remark 3.1.

$$G(x,0)=0 \qquad (0 \le x \le 1). \tag{3.26}$$

$$\partial_x^{2M} G(x,1)=0 \qquad (0 < x < 1). \tag{3.27}$$

$$\partial_x^{2k} G(x,1)|_{x=0}=0 \qquad (0 \le k \le M-1). \tag{3.28}$$

$$\partial_x^{2k+1} G(x,1)|_{x=1}=\begin{cases} 0 & (0 \le k \le M-2) \\ (-1)^{M-1} & (k=M-1). \end{cases} \tag{3.29}$$

We here list concrete forms of the functions $G(x,y)=G(M;x,y)$ $(M=1,2,3,4)$ and related functions.

$$G(1;x,y)=\frac{1}{2}[x+y-|x-y|], \qquad G(1;y,y)=y, \qquad G(1;1,1)=1.$$

$$G(2;x,y)=\frac{1}{12}[-[(x+y)^3-|x-y|^3]+3[(x+y)^2-|x-y|^2]],$$

$$G(2;y,y)=\frac{1}{3}[-2y^3+3y^2], \qquad G(2;1,1)=\frac{1}{3}.$$

$$G(3;x,y)=\frac{1}{240}[(x+y)^5-|x-y|^5-5[(x+y)^4-|x-y|^4] \\ +20[(x+y)^2-|x-y|^2]],$$

$$G(3;y,y)=\frac{1}{15}[2y^5-5y^4+5y^2], \qquad G(3;1,1)=\frac{2}{15}.$$

$$G(4;x,y)=\frac{1}{10080}[-[(x+y)^7-|x-y|^7]+7[(x+y)^6-|x-y|^6] \\ -70[(x+y)^4-|x-y|^4]+336[(x+y)^2-|x-y|^2]],$$

$$G(4;y,y)=\frac{1}{315}[-4y^7+14y^6-35y^4+42y^2], \qquad G(4;1,1)=\frac{17}{315}.$$

4. The method of reflection

In this section, we derive the solution to BVP(M) starting from Dirichlet boundary problem BVP(D, M). We call this procedure "the method of reflection". The latter half, we show that the relationship between BVP(D, M) and BVP(M).

In [5, 6], we have proved the following theorem:

Theorem 4.1. *For any bounded continuous function* $f(x)$ *on an interval* $0 < x < 1$, *Dirichlet boundary value problem*

$$\mathrm{BVP(D}, M)$$

$$\begin{cases} (-1)^M u^{(2M)} = f(x) & (0 < x < 1) \\ u^{(2i)}(0) = u^{(2i)}(1) = 0 & (0 \le i \le M-1) \end{cases}$$

has a unique classical solution $u(x)$ *expressed as*

$$u(x) = \int_0^1 G(\mathrm{D}; x, y) f(y) dy \qquad (0 < x < 1).$$

Green function $G(\mathrm{D}; x, y)$ *is given by*

$$G(\mathrm{D}; x, y) = (-1)^{M+1} 2^{2M-1} \left[b_{2M}\left(\frac{|x-y|}{2}\right) - b_{2M}\left(\frac{x+y}{2}\right) \right] \qquad (0 < x, y < 1). \tag{4.1}$$

If we extend the domain of definition $0 < x < 1$ to $0 < x < 2$, then we have the following theorem.

Theorem 4.2. *For any bounded continuous function* $f(x)$ *on an interval* $0 < x < 2$, *Dirichlet boundary value problem*

$$\begin{cases} (-1)^M u^{(2M)} = f(x) & (0 < x < 2) & (4.2) \\ u^{(2i)}(0) = u^{(2i)}(2) = 0 & (0 \le i \le M-1) & (4.3) \end{cases}$$

has a unique classical solution $u(x)$ *given by*

$$u(x) = \int_0^2 2^{2M-1} G\left(\mathrm{D}; \frac{x}{2}, \frac{y}{2}\right) f(y) dy \qquad (0 < x < 2). \tag{4.4}$$

We impose the following additional condition on inhomogeneous term $f(x)$:

$$f(x) = f(2-x) \qquad (1 < x < 2). \tag{4.5}$$

For this $f(x)$ $(0 < x < 2)$, the solution $u(x)$ to (4.2) and (4.3) is given by (4.4). If $0 < x < 1$, we have

$$u(x) = \int_0^2 2^{2M-1} G\left(\mathrm{D}; \frac{x}{2}, \frac{y}{2}\right) f(y) dy = I_1 + I_2,$$

$$I_1 = \int_0^1 2^{2M-1} G\left(\mathrm{D}; \frac{x}{2}, \frac{y}{2}\right) f(y) dy,$$

$$I_2 = \int_1^2 2^{2M-1} G\left(\mathrm{D}; \frac{x}{2}, \frac{y}{2}\right) f(y) dy.$$

Applying (4.5) to I_2, we have

$$I_2 = \int_1^2 2^{2M-1} G\left(\mathrm{D}; \frac{x}{2}, \frac{y}{2}\right) f(2-y)dy = \int_0^1 2^{2M-1} G\left(\mathrm{D}; \frac{x}{2}, 1 - \frac{y}{2}\right) f(y)dy.$$

Thus we have

$$u(x) = \int_0^1 G(x, y) f(y)dy \qquad (0 < x < 1),$$

where $G(x, y)$ is given by

$$G(x, y) = 2^{2M-1}\left[G\left(\mathrm{D}; \frac{x}{2}, \frac{y}{2}\right) + G\left(\mathrm{D}; \frac{x}{2}, 1 - \frac{y}{2}\right)\right] \qquad (0 < x, y < 1). \quad (4.6)$$

For the second term of the right-hand side of (4.6), we have

$$\begin{aligned}
&(-1)^{M+1} 2^{-(2M-1)} G\left(\mathrm{D}; \frac{x}{2}, 1 - \frac{y}{2}\right) \\
&= b_{2M}\left(\frac{1}{2} - \frac{x+y}{4}\right) - b_{2M}\left(\frac{1}{2} + \frac{x-y}{4}\right) \\
&= \left\{\begin{array}{ll} b_{2M}\left(\frac{1}{2} - \frac{x+y}{4}\right) - b_{2M}\left(\frac{1}{2} - \frac{|x-y|}{4}\right) & (0 < x < y < 1) \\ b_{2M}\left(\frac{1}{2} - \frac{x+y}{4}\right) - b_{2M}\left(\frac{1}{2} + \frac{|x-y|}{4}\right) & (0 < y < x < 1) \end{array}\right\} \\
&= b_{2M}\left(\frac{1}{2} - \frac{x+y}{4}\right) - b_{2M}\left(\frac{1}{2} - \frac{|x-y|}{4}\right) \qquad (0 < x, y < 1)
\end{aligned}$$

where we have used $b_{2j}(1/2 - x) = b_{2j}(1/2 + x)$ from (2.1). So we have

$$\begin{aligned}
G\left(\mathrm{D}; \frac{x}{2}, \frac{y}{2}\right) &= (-1)^{M+1} 2^{2M-1}\left[b_{2M}\left(\frac{|x-y|}{4}\right) - b_{2M}\left(\frac{x+y}{4}\right)\right], \\
G\left(\mathrm{D}; \frac{x}{2}, 1 - \frac{y}{2}\right) &= (-1)^{M+1} 2^{2M-1}\left[b_{2M}\left(\frac{1}{2} - \frac{x+y}{4}\right)\right. \\
&\qquad \left. - b_{2M}\left(\frac{1}{2} - \frac{|x-y|}{4}\right)\right] \qquad (0 < x, y < 1).
\end{aligned}$$

This shows that (4.6) is equivalent to (3.6). Therefore, (4.6) satisfies (3.16), (3.17) and (3.18) of Theorem 3.2. So we have the following relation:

$$G(x,y) = 2^{2M-1}\left[G\left(\mathrm{D}; \frac{x}{2}, \frac{y}{2}\right) + G\left(\mathrm{D}; \frac{x}{2}, 1-\frac{y}{2}\right)\right]$$
$$= (-1)^{M+1}4^{2M-1}\left[b_{2M}\left(\frac{|x-y|}{4}\right) - b_{2M}\left(\frac{x+y}{4}\right)\right.$$
$$\left. + b_{2M}\left(\frac{1}{2}-\frac{x+y}{4}\right) - b_{2M}\left(\frac{1}{2}-\frac{|x-y|}{4}\right)\right] \qquad (0 < x, y < 1). \quad (4.7)$$

Next, we investigate the diagonal values of Green functions $G(\mathrm{D}; y, y)$ and $G(y, y)$, which are given by

$$G(\mathrm{D}; y, y) = (-1)^{M+1}2^{2M-1}[b_{2M}(0) - b_{2M}(y)],$$
$$G(y,y) = 2^{2M-1}\left[G\left(\mathrm{D}; \frac{y}{2}, \frac{y}{2}\right) + G\left(\mathrm{D}; \frac{y}{2}, 1-\frac{y}{2}\right)\right]$$
$$= (-1)^{M+1}4^{2M-1}\left[b_{2M}(0) - b_{2M}\left(\frac{y}{2}\right) + b_{2M}\left(\frac{1}{2}-\frac{y}{2}\right)\right.$$
$$\left. - b_{2M}\left(\frac{1}{2}\right)\right] \qquad (0 < y < 1).$$

From Lemma 2.1, it is shown that $G(\mathrm{D}; y, y)$ attains its maximum at $y = 1/2$ and $G(y, y)$ attains its maximum at $y = 1$. As a conclusion, we have obtained the following theorem:

THEOREM 4.3.

$$C(\mathrm{D}, M) = \max_{0 \le y \le 1} G(\mathrm{D}; y, y) = G(\mathrm{D}; 1/2, 1/2)$$
$$= (-1)^{M+1}2^{2M-1}[b_{2M}(0) - b_{2M}(1/2)]$$
$$= (-1)^{M+1}(2^{2M}-1)b_{2M}(0) = 2^{-(2M-1)}(2^{2M}-1)\pi^{-2M}\zeta(2M),$$
$$C(M) = \max_{0 \le y \le 1} G(y, y) = G(1, 1) = 2^{2M}G(\mathrm{D}; 1/2, 1/2)$$
$$= (-1)^{M+1}4^{2M-1}2[b_{2M}(0) - b_{2M}(1/2)]$$
$$= (-1)^{M+1}2^{2M}(2^{2M}-1)b_{2M}(0) = 2(2^{2M}-1)\pi^{-2M}\zeta(2M).$$

From this theorem, we obtained the relation $C(M) = 2^{2M}C(\mathrm{D}, M)$.

5. Reproducing kernel

In this section, it is shown that Green function $G(x, y)$ is a reproducing kernel for a set of Hilbert space H and its inner product $(\cdot, \cdot)_M$ introduced in section 1.

Theorem 5.1. (1) *For any* $u(x) \in H$, *we have the following reproducing relation:*

$$u(y) = (u, G(\cdot, y))_M = \int_0^1 u^{(M)}(x)\partial_x^M G(x, y)dx \qquad (0 \le y \le 1). \qquad (5.1)$$

This means that Green function $G(x, y)$ *is a reproducing kernel for* H *and* $(\cdot\,,\cdot)_M$.

$$(2) \quad G(y, y) = \int_0^1 |\partial_x^M G(x, y)|^2 dx \qquad (0 \le y \le 1). \qquad (5.2)$$

Proof of Theorem 5.1. For functions $u = u(x)$ and $v = v(x) = G(x, y)$ with y arbitrarily fixed in $0 \le y \le 1$, we have

$$u^{(M)}v^{(M)} - u(-1)^M v^{(2M)} = \left(\sum_{j=0}^{M-1} (-1)^{M-1-j} u^{(j)} v^{(2M-1-j)} \right)'.$$

Integrating this with respect to x on intervals $0 < x < y$ and $y < x < 1$, we have

$$\int_0^1 u^{(M)}(x)v^{(M)}(x)dx - \int_0^1 u(x)(-1)^M v^{(2M)}(x)dx$$
$$= \left[\sum_{j=0}^{M-1} (-1)^{M-1-j} u^{(j)}(x) v^{(2M-1-j)}(x) \right] \left\{ \Big|_{x=0}^{x=y-0} + \Big|_{x=y+0}^{x=1} \right\}$$
$$= \sum_{j=0}^{M-1} (-1)^{M-1-j} [u^{(j)}(1)v^{(2M-1-j)}(1) - u^{(j)}(0)v^{(2M-1-j)}(0)]$$
$$+ \sum_{j=0}^{M-1} (-1)^{M-1-j} u^{(j)}(y)[v^{(2M-1-j)}(y-0) - v^{(2M-1-j)}(y+0)].$$

The first term on the right-hand side is rewritten as

$$\sum_{j=0}^{M-1} (-1)^{M-1-j} [u^{(j)}(1)v^{(2M-1-j)}(1) - u^{(j)}(0)v^{(2M-1-j)}(0)]$$
$$= \sum_{j=0}^{[(M-1)/2]} (-1)^{M-1} [u^{(2j)}(1)v^{(2(M-1-j)+1)}(1) - u^{(2j)}(0)v^{(2(M-1-j)+1)}(0)]$$
$$+ \sum_{j=0}^{[(M-2)/2]} (-1)^{M} [u^{(2j+1)}(1)v^{(2(M-1-j))}(1) - u^{(2j+1)}(0)v^{(2(M-1-j))}(0)].$$

Using (3.16), (3.17) and (3.19) in Theorem 3.2, we have (1) $(0 < y < 1)$. Using (3.26) in Remark 3.1, we have (1) $(y = 0)$. Using (3.27), (3.28) and (3.29) in Remark 3.1, we have (1) $(y = 1)$. (2) follows from (1) by putting $u(x) = G(x, y)$ in (5.1). We have proved Theorem 5.1.

6. Sobolev inequality

In this section, we give a proof of Theorem 1.2, from which Theorem 1.1 is derived simultaneously.

Applying Schwarz inequality to (5.1) and using (5.2), we have

$$\begin{aligned} |u(y)|^2 &\le \int_0^1 \left|\partial_x^M G(x, y)\right|^2 dx \int_0^1 \left|u^{(M)}(x)\right|^2 dx \\ &= G(y, y) \int_0^1 \left|u^{(M)}(x)\right|^2 dx. \end{aligned}$$

Noting that $C_0 = \max_{0 \le y \le 1} G(y, y) = G(1, 1)$, we have following Sobolev inequality:

$$\left(\sup_{0 \le y \le 1} |u(y)|\right)^2 \le C_0 \int_0^1 |u^{(M)}(x)|^2 dx. \tag{6.1}$$

This inequality shows that $(\cdot\,,\cdot)_M$ is positive definite. It should be noted that it requires Schwarz inequality but does not require "positive definiteness" of the inner product for the purpose of proving (6.1).

In the second place, we apply this inequality to $u(x) = G(x, 1) \in H$ and have

$$\left(\sup_{0 \le y \le 1} |G(y, 1)|\right)^2 \le C_0 \int_0^1 |\partial_x^M G(x, 1)|^2 dx = C_0^2.$$

Together with a trivial inequality

$$C_0^2 = G^2(1, 1) \le \left(\sup_{0 \le y \le 1} |G(y, 1)|\right)^2,$$

we finally obtain

$$\left(\sup_{0 \le y \le 1} |G(y, 1)|\right)^2 = C_0 \int_0^1 |\partial_x^M G(x, 1)|^2 dx, \tag{6.2}$$

which completes the proof of Theorem 1.2.

7. Discrete Sobolev inequality $(M = 1)$

In this section, we consider a discrete version of the result obtained in previous sections.

We assume that $N = 2, 3, 4, \dots$. We consider the following set of simultaneous equations:

$$\begin{cases} -u(i+1) + 2u(i) - u(i-1) = f(i) & (1 \le i \le N) \\ u(0) = 0,\ u(N+1) - u(N) = 0, \end{cases}$$

which are regarded as a discrete version of BVP(1). This is rewritten equivalently as

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$$

where

$$\boldsymbol{A} = (a_{ij}) = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{pmatrix},$$

$$\boldsymbol{u} = {}^t(u(1), \dots, u(N)), \qquad \boldsymbol{f} = {}^t(f(1), \dots, f(N)) \in \boldsymbol{C}^N.$$

Let $\boldsymbol{\delta}_j$ be a vector defined by

$$\boldsymbol{\delta}_j = {}^t(0, \dots, 0, \overset{\overset{j}{\vee}}{1}, 0, \dots, 0) \qquad (1 \le j \le N).$$

We also introduce an ordinary unitary inner product

$$(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{v}^* \boldsymbol{u} = {}^t\bar{\boldsymbol{v}}\boldsymbol{u} = \sum_{j=1}^{N} \bar{v}(j) u(j),$$

Sobolev inner product

$$(\boldsymbol{u}, \boldsymbol{v})_A = (\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{v}^* \boldsymbol{A}\boldsymbol{u} = \sum_{i,j=1}^{N} \bar{v}(i) a_{ij} u(j)$$

and Sobolev energy

$$\|\boldsymbol{u}\|_A^2 = (\boldsymbol{u}, \boldsymbol{u})_A = \sum_{i,j=1}^{N} \bar{u}(i) a_{ij} u(j).$$

$(\cdot, \cdot)_A$ is proved to be an inner product of $\mathbf{C}^N$ afterwards.

The conclusion of this section is as follows:

Theorem 7.1. *For any $\boldsymbol{u} \in \boldsymbol{C}^N$, there exists a positive constant C which is independent of $\boldsymbol{u}$ such that the discrete Sobolev inequality* [8]

$$\left(\max_{1 \le j \le N} |u(j)| \right)^2 \le C \|\boldsymbol{u}\|_A^2 \tag{7.1}$$

holds. Among such C *the best constant is* $C_0 = N$. *If we replace* C *by* C_0, *then the equality holds for* $\boldsymbol{u} = \boldsymbol{G}\boldsymbol{\delta}_N$ *where* $\boldsymbol{G} = \boldsymbol{A}^{-1}$ *is given by the following expression:*

$$\boldsymbol{G} = (g_{ij}) = (\min\{i, j\}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & N \end{pmatrix}. \tag{7.2}$$

(7.2) is easily proved by using Gauss's sweeping-out method.

THEOREM 7.2. (1) *For any* $\boldsymbol{u} \in \boldsymbol{C}^N$, *we have the following reproducing relation:*

$$u(j) = (\boldsymbol{u}, \boldsymbol{G}\boldsymbol{\delta}_j)_A \qquad (1 \le j \le N). \tag{7.3}$$

$$(2) \quad g_{jj} = (\boldsymbol{G}\boldsymbol{\delta}_j, \boldsymbol{G}\boldsymbol{\delta}_j)_A \qquad (1 \le j \le N). \tag{7.4}$$

PROOF OF THEOREM 7.2. Noting that $\boldsymbol{G}^* = \boldsymbol{G}$, we have

$$(\boldsymbol{u}, \boldsymbol{G}\boldsymbol{\delta}_j)_A = (\boldsymbol{A}\boldsymbol{u}, \boldsymbol{G}\boldsymbol{\delta}_j) = \boldsymbol{\delta}_j^* \boldsymbol{G}^* \boldsymbol{A}\boldsymbol{u} = \boldsymbol{\delta}_j^* \boldsymbol{u} = u(j).$$

Applying $\boldsymbol{u} = \boldsymbol{G}\boldsymbol{\delta}_j \in \mathbf{C}^N$ to (1), we have

$$(\boldsymbol{G}\boldsymbol{\delta}_j, \boldsymbol{G}\boldsymbol{\delta}_j)_A = (\boldsymbol{A}\boldsymbol{G}\boldsymbol{\delta}_j, \boldsymbol{G}\boldsymbol{\delta}_j) = (\boldsymbol{\delta}_j, \boldsymbol{G}\boldsymbol{\delta}_j) = \boldsymbol{\delta}_j^* \boldsymbol{G}^* \boldsymbol{\delta}_j = \boldsymbol{\delta}_j^* \boldsymbol{G}\boldsymbol{\delta}_j = g_{jj}.$$

This shows Theorem 7.2.

PROOF OF THEOREM 7.1. Applying Schwarz inequality to (7.3) and using (7.4), we have

$$|u(j)|^2 \le \|\boldsymbol{u}\|_A^2 \|\boldsymbol{G}\boldsymbol{\delta}_j\|_A^2 = g_{jj} \|\boldsymbol{u}\|_A^2.$$

Taking the maximum with respect to $1 \le j \le N$, we have the following discrete Sobolev inequality:

$$\left(\max_{1 \le j \le N} |u(j)| \right)^2 \le C_0 \|\boldsymbol{u}\|_A^2, \qquad C_0 = \max_{1 \le j \le N} g_{jj} = g_{NN} = N. \tag{7.5}$$

This inequality shows that $(\cdot\,,\cdot)_A$ is positive definite.

In the second place, we apply this inequality to $\boldsymbol{u} = \boldsymbol{G}\boldsymbol{\delta}_N \in \mathbf{C}^N$. Then we have

$$\left(\max_{1 \le j \le N} |g_{jN}| \right)^2 \le C_0 \|\boldsymbol{G}\boldsymbol{\delta}_N\|_A^2 = C_0^2.$$

Combining this and trivial inequality

$$C_0^2 = g_{NN}^2 \leq \left(\max_{1 \leq j \leq N} |g_{jN}| \right)^2,$$

we obtain

$$\left(\max_{1 \leq j \leq N} |g_{jN}| \right)^2 = C_0 \| \boldsymbol{G}\boldsymbol{\delta}_N \|_A^2, \tag{7.6}$$

which completes the proof of Theorem 7.1.

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