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Existence of affine α-resolvable PBIB designs with some constructions

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Abstract. A mathematical topic using the property of resolvability and affine resolvability was introduced in 1850 and the designs having such concept have been statistically discussed since 1939. Their combinatorial structure on existence has been discussed richly since 1942. This concept was generalized to α-resolvability and affine α-resolvability in 1963. These arguments are mostly done for a class of balanced incomplete block designs. The present paper will make the combinatorial investigation on affine α-resolvable partially balanced incomplete block designs with two associate classes. The characterization of parameters in a closed form will be given and then existence problems with construction methods will be discussed. Comprehensive and useful results on combinatorics are obtained. Several methods of construction are newly presented with some illustrations.

1. Introduction

Though Yates [45, 46] has pointed out some statistical advantages of resolvable designs and their original form had appeared earlier in the mathematical literature as the Kirkman school girl problem [33] formulated in 1850, the interest in resolvable balanced incomplete block (BIB) designs was greatly enhanced by a combinatorial paper by Bose [4]. Further statistical usefulness of affine resolvable block designs can be found in Bailey, Monod and Morgan [1], and Caliński and Kageyama [9, 11].

Such concept was generalized to α-resolvability and affine α-resolvability by Shrikhande and Raghavarao [41]. A block design BD(v, b, r, k) is said to be *α-resolvable* if the b blocks of size k each can be grouped into t sets (called α-resolution sets) of β blocks each ($b = \beta t$) such that in each α-resolution set every treatment (or point) is replicated α times ($r = \alpha t$). An α-resolvable BD is said to be *affine α-resolvable* if every two distinct blocks from the same α-resolution set intersect in the same number, say, q_1, of treatments, whereas every two blocks belonging to different α-resolution sets intersect in the same

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number, say, q_2, of treatments. It follows (see [22, 42]) that for an affine α-resolvable $\mathrm{BD}(v, b = \beta t, r = \alpha t, k)$ with block intersection numbers q_1 and q_2, the following relations $q_1 = k(\alpha - 1)/(\beta - 1)$ and $q_2 = k\alpha/\beta = k^2/v$ hold. Note that both of q_1 and q_2 must be nonnegative integers. An integral expression of q_1 without α and β in terms of design parameters only is meaningful.

When $\alpha = 1$, the definition of (affine) 1-resolvability coincides with that by Bose [4]. Hence a 1-resolvable or an affine 1-resolvable design is simply called a resolvable or an affine resolvable design, respectively.

The constructions of (affine) α-resolvable BIB designs or partially balanced incomplete block (PBIB) designs with their combinatorial properties have been discussed in literature (see, for example, [1, 2, 9, 10, 12, 14, 15, 22, 23, 29, 30, 37, 38, 41, 44]).

In this paper, some combinatorial investigation on affine α-resolvable PBIB designs are dealt with. Their topics are concerned with the characterization of parameters in a closed form and existence problems with construction methods. Comprehensive and useful results on combinatorics are obtained. Several methods of construction are newly presented with practical affine resolvable block designs.

2. Preliminaries

Several definitions on technical terms are described in this section.

Definition 2.1. A balanced incomplete block (BIB) design with parameters v, b, r, k, λ is defined as an arrangement of v treatments into b blocks of k $(< v)$ treatments each such that

(1) each treatment occurs at most once in a block,

(2) each treatment occurs in exactly r different blocks,

(3) every pair of treatments occurs together in exactly λ blocks.

This is denoted by $\mathrm{BIB}(v, b, r, k, \lambda)$ or $\mathrm{BIB}(v, k, \lambda)$. The parameter λ is called a coincidence number of the design.

It is known that $vr = bk$, $\lambda(v - 1) = r(k - 1)$ and $b \geq v$ hold. In particular, when $b = v$, the BIB design is said to be symmetric. It is also known that in an α-resolvable BIB design with $b = \beta t$ and $r = \alpha t$, $b \geq v + t - 1$ holds, and $b = v + t - 1$ is a necessary and sufficient condition for an α-resolvable BIB design to be affine α-resolvable with the block intersection number $q_1 = k(\alpha - 1)/(\beta - 1) = k + \lambda - r$ (cf. [22, 42]).

In defining a 2-associate PBIB design with two distinct coincidence numbers λ_1 and λ_2 different from a BIB design, the concept of an association scheme for a set of v treatments is needed.

Given v treatments $1, 2, \ldots, v$, a relation satisfying the following conditions is said to have an association scheme with two associate classes:

(1) Any two treatments are either 1st or 2nd associates, the relation of association being symmetric, that is, if the treatment x is ith associate of the treatment y, then y is ith associate of x, $i = 1, 2$.

(2) Each treatment x has n_i ith associates, the number n_i being independent of x, $i = 1, 2$.

(3) If any two treatments x and y are ith associates, then the number of treatments that are jth associates of x and ℓth associates of y is $p^i_{j\ell}$ and is independent of the pair of ith associates x and y, $i, j, \ell = 1, 2$.

Definition 2.2. Given an association scheme with two associate classes for a set of v treatments, a 2-associate PBIB design is defined as an arrangement of v treatments into b blocks of size k $(< v)$ each such that

(1) each treatment occurs at most once in a block,

(2) each treatment occurs in exactly r different blocks,

(3) if two treatments are ith associates, then they occur together in exactly λ_i blocks, the number λ_i being independent of the particular pair of ith associates, $i = 1, 2$.

Like a BIB design, when $b = v$, the PBIB design is said to be symmetric. It holds that in a 2-associate PBIB design, $vr = bk$, $n_1 + n_2 = v - 1$, $n_1\lambda_1 + n_2\lambda_2 = r(k - 1)$. Conventionally let every treatment be the 0th associate of itself and of no other treatment, and then it is seen that $n_0 = 1$ and $\lambda_0 = r$.

From Definitions 2.1 and 2.2, when $\lambda_1 = \lambda_2$, a PBIB design becomes a BIB design. Though a symmetric BIB design cannot possess a property of affine α-resolvability, it is remarkable that there exists an affine α-resolvable symmetric PBIB design.

The known "2-associate" PBIB designs have been mainly classified into the following types depending on association schemes, i.e., *group divisible*, *triangular*, *Latin-square* (L_2), *cyclic* (see [7]).

Definition 2.3. A 2-associate PBIB design is said to be group divisible (GD) if there are $v = mn$ treatments which can be divided into m groups of n treatments each, such that any two treatments of the same group are 1st associates and any two treatments from different groups are 2nd associates. Here $m, n \geq 2$, $n_1 = n - 1$ and $n_2 = n(m - 1)$.

The GD designs are further classified into three subclasses: *Singular* (S) if $r - \lambda_1 = 0$; *Semi-Regular* (SR) if $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$; *Regular* if $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$. By a relation $n_1\lambda_1 + n_2\lambda_2 = r(k - 1)$, it holds that $(rk - v\lambda_2) - (r - \lambda_1) = n(\lambda_1 - \lambda_2)$. The last relation shows that for an SGD design $\lambda_1 > \lambda_2$, while for an SRGD design $\lambda_2 > \lambda_1$.

Definition 2.4. A 2-associate PBIB design is said to be triangular if there are $v = n(n-1)/2$ treatments which are arranged into an $n \times n$ array such that

(1) the position in the principal diagonal are left blank,

(2) the $n(n-1)/2$ positions above the principal diagonal are filled by the numbers $1, 2, \dots, n(n-1)$ corresponding to the treatments,

(3) the $n(n-1)/2$ positions below the principal diagonal are filled so that the array is symmetric about the principal diagonal,

(4) for any treatment x 1st associates are exactly those that occur in the same row or in the same column as x, otherwise they are 2nd associates.

Here $n \geq 4$, $n_1 = 2(n-2)$ and $n_2 = (n-2)(n-3)/2$.

Definition 2.5. A 2-associate PBIB design is said to be L_2 (Latin-sqaure) if there are $v = s^2$ treatments which are arranged into an $s \times s$ array such that any two treatments in the same row or in the same column of the array are 1st associates, otherwise they are 2nd associates. Here $s \geq 2$, $n_1 = 2(s-1)$ and $n_2 = (s-1)^2$.

Definition 2.6. A 2-associate PBIB design with v treatments is said to be cyclic if the set of 1st associates of ith treatment is $(i + d_1, i + d_2, \dots, i + d_{n_1})$ mod v, where the elements d_j satisfy the following conditions:

(1) The elements d_j are all different and $0 < d_j < v$ for $j = 1, 2, \dots, n_1$.

(2) Among the $n_1(n_1 - 1)$ differences $d_j - d_{j'}$ each of the $d_1, d_2, \dots, d_{n_1}$ occurs p_{11}^1 times and each of the $e_1, e_2, \dots, e_{n_2}$ occurs p_{11}^2 times, where d_j, $e_{j'}$ are all nonzero distinct and $\{d_1, d_2, \dots, d_{n_1}, e_1, e_2, \dots, e_{n_2}\} \subseteq \{1, 2, \dots, v\}$.

(3) For each d_i in a set $D = (d_1, d_2, \dots, d_{n_1})$, there exists d_k in D such that $d_k = -d_i$.

Here $n_1 = n_2 = (v-1)/2$.

It is shown ([36]) that all cyclic association schemes have the parameters $v = 4t + 1$ being a prime and $n_1 = n_2 = 2t$ for a positive integer t. Thus the cyclic design may exist only for a prime v being the number of treatments.

Definition 2.7. In a $\mathrm{BD}(v, b, r, k)$, the $v \times b$ incidence matrix $N = (n_{ij})$ is defined such that n_{ij} is the number of times ith treatment occurs in jth block. Hence $r = \sum_{j=1}^{b} n_{ij}$ for all i and $k = \sum_{i=1}^{v} n_{ij}$ for all j. In this paper $n_{ij} = 0$ or 1 for all $i = 1, 2, \dots, v$ and $j = 1, 2, \dots, b$ (called a binary design) as seen, for example, from (1) of Definitions 2.1 and 2.2.

Two results will be needed for our further argument.

Lemma 2.1 (cf. [26, 42]). *In an affine α-resolvable $BD(v, b = \beta t, r = \alpha t, k)$ with the incidence matrix N, the matrix $N'N$ has eigenvalues rk, $k\{1 - (\alpha - 1)/(\beta - 1)\}$ and 0, with multiplicities 1, $b - t$ and $t - 1$, respectively.*

Lemma 2.2 (cf. [35]). *The matrices XY and YX have the same nonzero eigenvalues with the same multiplicities, where the matrices X and Y are of appropriate sizes.*

Finally, a known equivalence result on existence of an affine α-resolvable BD is described. This can be seen from the complementation of a design.

Lemma 2.3. *The existence of an affine α-resolvable $BD(v, b = \beta t, r = \alpha t, k)$ with block intersection numbers q_1 and q_2 is equivalent to the existence of an affine $(\beta - \alpha)$-resolvable $BD(v^* = v, b^* = b, r^* = (\beta - \alpha)t, k^* = v - k)$ with block intersection numbers $q_1^* = v - 2k + q_1$ and $q_2^* = v - 2k + q_2$.*

3. Affine α-resolvable PBIB designs

The present section is devoted to the comprehensive combinatorial investigation on a property of affine α-resolvability in a 2-associate PBIB design.

In literature there are much combinatorial discussions on α-resolvable PBIB designs (see, for example, [5, 24, 25, 26, 27, 30]). However, there are not many papers on "affine" α-resolvable PBIB designs. As was mentioned in Section 2, there are several types of 2-associate PBIB designs. Among them, two types are at first considered here.

Let us take a class of cyclic PBIB designs (see Definition 2.6). In this case the following can be seen.

Theorem 3.1. *There does not exist an affine α-resolvable cyclic 2-associate PBIB design for any $\alpha \geq 1$.*

Proof. In the cyclic design, it is known (see a few lines after Definition 2.6) that the number of treatments is $v = 4t + 1$ being a prime. On the other hand, the affine α-resolvability requires that $q_2 = k^2/v$ is an integer. Now since v is a prime and $v > k$, q_2 is not an integer. Hence the proof is complete. ■

Next take a class of triangular PBIB designs with $v = n(n-1)/2$ (see Definition 2.4). No example has been found for an affine α-resolvable triangular design for $\alpha \geq 1$ in literature. Recently the following has been shown.

Theorem 3.2 ([25, 27]). *There does not exist an affine α-resolvable triangular design for $1 \leq \alpha \leq 10$.*

Then Kageyama [25] has conjectured that there does not exist an affine α-resolvable triangular design for any $\alpha \geq 1$. Since the attractive result on existence could not be further obtained, the existence problem of affine α-resolvable triangular designs will not be discussed in this paper.

As of today, a cyclic design forms the only class of 2-associate PBIB designs which do not possess entirely a property of affine α-resolvability in design theory. A class of triangular designs may be the next such candidate.

For further argument, the following lemma is useful. This can be derived by use of Lemmas 2.1 and 2.2.

Lemma 3.1 (cf. [25]). *In a 2-associate PBIB design, having the incidence matrix N, with parameters v, b, r, k, λ_i, θ_i, ρ_i, $i = 0, 1, 2$, where $\lambda_0 = r$, $\theta_0 = rk$, $\rho_0 = 1$, θ_1 and θ_2 are the nonnegative eigenvalues (other than rk) of NN' with respective multiplicities ρ_1 and ρ_2, when $\theta_1 > 0$ and $\theta_2 > 0$, the design does not possess a property of affine α-resolvability.*

Remark 3.1. Similarly to λ_i as in Definition 2.2 (3), the eigenvalues θ_i are corresponding to ith associates of an association scheme, $i = 0, 1, 2$ (cf. [10, 39]). Since in a cyclic 2-associate PBIB design all the eigenvalues of NN' are positive (see, pp. 126 and 129 in [39]), Lemma 3.1 can yield the same result as in Theorem 3.1.

The following result plays a crucial role to characterize affine α-resolvable 2-associate PBIB designs in this paper.

Theorem 3.3. *Let N be the $v \times b$ incidence matrix of an affine α-resolvable 2-associate PBIB design with parameters v, $b = \beta t$, $r = \alpha t$, k, λ_1, λ_2, $q_1 = k(\alpha - 1)/(\beta - 1)$ and $q_2 = k^2/v$, and further let θ_i be eigenvalues of NN' with multiplicities ρ_i, $i = 0, 1, 2$, where $\theta_0 = rk$ and $\rho_0 = 1$. Then, when $\theta_i > 0$ and $\theta_{i'} = 0$, $i \neq i' \in \{1, 2\}$, $q_1 = k - \theta_i$ and $b = t + \rho_i$ hold.*

Proof. By Lemma 2.1, $N'N$ has the only nonzero eigenvalue (other than rk) $k\{1 - (\alpha - 1)/(\beta - 1)\}$, which is equal to $k - q_1$, with multiplicity $b - t$. Then (i) when $\theta_1 > 0$ and $\theta_2 = 0$, Lemma 2.2 implies that $k - q_1 = \theta_1$ and $b - t = \rho_1$, while (ii) when $\theta_1 = 0$ and $\theta_2 > 0$, Lemma 2.2 implies that $k - q_1 = \theta_2$ and $b - t = \rho_2$. On account of Lemma 3.1 note that a case of $\theta_1 > 0$ and $\theta_2 > 0$ does not occur in this design. ■

Remark 3.2. In Theorem 3.3, if $\theta_1 = \theta_2 = 0$, i.e., NN' has the only one nonzero eigenvalue rk, then the design is orthogonal and hence $N = \mathbf{1}_v \mathbf{1}_b'$, which is not incomplete (cf. [10, Chapters 6 and 7]), where $\mathbf{1}_s$ is an $s \times 1$ column vector all of whose elements are 1. Hence the orthogonal design is not a PBIB design, but a randomized block design.

REMARK 3.3. In 2-associate PBIB designs, the PBIB design with $\lambda_1 = \lambda_2$ becomes a BIB design and hence, as eigenvalues of NN', $\theta_1 = r - \lambda$ only other than rk. Therefore, by Theorem 3.3, in an affine α-resolvable BIB design $q_1 = k + \lambda - r$ holds (see the statements after Definition 2.1).

The largest, simplest and perhaps most important class of 2-associate PBIB designs is known as GD (group divisible). In a GD design the eigenvalues of NN' have $\theta_1 = rk - v\lambda_2$ and $\theta_2 = r - \lambda_1$ (other than rk) with respective multiplicities $\rho_1 = m - 1$ and $\rho_2 = m(n-1)$. Hence by Definition 2.3 and Lemma 3.1 the following has been provided.

THEOREM 3.4 (cf. [26]). *There does not exist an affine α-resolvable regular GD design for any $\alpha \geq 1$.*

By Remark 3.2, other two subclasses (i.e., SGD and SRGD) of GD designs will be discussed in subsequent Sections 3.1 to 3.4 below.

3.1. Affine α-resolvable SGD designs

By Definition 2.3, the present section is devoted to a GD design with $r = \lambda_1$, i.e., of singular type. Note that $\lambda_1 > \lambda_2$ in an SGD design.

It is known ([6]) that the existence of an SGD$(v = mn, b, r = \lambda_1, k, \lambda_1, \lambda_2)$ is equivalent to the existence of a BIB$(v^*, b^*, r^*, k^*, \lambda^*)$, where $v = nv^*$, $b = b^*$, $r = r^*$, $k = nk^*$, $\lambda_1 = r^*$, $\lambda_2 = \lambda^*$, $m = v^*$, $n = n$. This result can be obtained from replacing each treatment of the BIB design by a group of n treatments for $n \geq 2$. It is obvious that the present replacement procedure preserves a property of affine α-resolvability between a BIB design and an SGD design. Hence the following result has been established.

THEOREM 3.1.1. *The existence of an affine α-resolvable SGD$(v = nv^*$, $b = b^* = \beta t, r = r^* = \alpha t, k = nk^*, \lambda_1 = r^*, \lambda_2 = \lambda^*; m = v^*, n = n)$ with $q_1 = nk^*(\alpha - 1)/(\beta - 1)$ and $q_2 = n(k^*)^2/v^*$ is equivalent to the existence of an affine α-resolvable BIB$(v^*, b^* = \beta t, r^* = \alpha t, k^*, \lambda^*)$ with $q_1^* = k(\alpha - 1)/(\beta - 1)$ and $q_2^* = k^2/v$.*

Now an integral expression of q_1 is derived like $q_1 = k + \lambda - r$ in an affine α-resolvable BIB design as in Remark 3.3.

COROLLARY 3.1.1. *In an affine α-resolvable SGD design, $q_1 = k(\alpha - 1)/(\beta - 1) = k - \lambda_1 k + v\lambda_2$ holds.*

PROOF. Since $\theta_1 = rk - v\lambda_2$ and $\theta_2 = r - \lambda_1 = 0$, by Theorem 3.3 we have $q_1 = k - \theta_1 = k - rk + v\lambda_2$. ∎

Now the parameters of an affine α-resolvable SGD design with parameters $v = mn$, $b = \beta t$, $r = \alpha t$, k, λ_1, λ_2, $q_1 = k(\alpha - 1)/(\beta - 1)$ and $q_2 = k^2/v$ are characterized. The following can be shown.

THEOREM 3.1.2. *The parameters of an affine α-resolvable SGD design are given by*

$$v = mn, \qquad b = \frac{\beta(m-1)}{\beta - 1}, \qquad r = \frac{\alpha(m-1)}{\beta - 1}, \qquad k = \frac{\alpha mn}{\beta}, \qquad \lambda_1 = \frac{\alpha(m-1)}{\beta - 1},$$

$$\lambda_2 = \frac{\alpha(\alpha m - \beta)}{\beta(\beta - 1)}; \qquad t = \frac{m-1}{\beta - 1}, \qquad q_2 = \frac{\alpha^2 mn}{\beta^2},$$

where $\alpha m/\beta$ is an integer.

PROOF. Since eigenvalues of NN' are $rk - v\lambda_2$ and $r - \lambda_1 = 0$ with respective multiplicities $m - 1$ and $m(n-1)$, it follows from Theorem 3.3 that $b - t = m - 1$, i.e., $t = (m-1)/(\beta - 1)$ which also implies that $m > \beta$. Then we obtain the expression of parameters as $v = mn$, $b = \beta t = \beta(m-1)/(\beta - 1)$, $r = \alpha t = \alpha(m-1)/(\beta - 1)$, $k = vr/b = \alpha mn/\beta$, $\lambda_1 = r = \alpha(m-1)/\beta$. Furthermore, by a relation $r(k-1) = n_1\lambda_1 + n_2\lambda_2$, we get $\lambda_2 = \alpha(\alpha m - \beta)/[\beta(\beta - 1)]$. Also by Theorem 3.1.1, $k/n = \alpha m/\beta$ must be an integer. ■

Thus, all parameters of an affine α-resolvable SGD design can be expressed in terms of m, n, α and β.

3.2. Table of affine resolvable SGD designs with $v \le 100$ and $r, k \le 20$

There are a number of affine α-resolvable SGD designs with parameters $v = mn$, $b = \beta t$, $r = \alpha t$, k, λ_1, λ_2, q_1, q_2. We here restrict ourselves to the case of $\alpha = 1$. Even so, by Lemma 2.3, some of other affine α-resolvable SGD designs can be constructed for some $\alpha \ge 2$. Now, since $q_2 = k^2/v$, by Theorem 3.1.2 we have the expression of parameters as

$$v = mn, \qquad b = \frac{\beta(m-1)}{\beta - 1}, \qquad r = \frac{m-1}{\beta - 1}, \qquad k = \frac{mn}{\beta},$$

$$\lambda_1 = \frac{m-1}{\beta - 1}, \qquad \lambda_2 = \frac{m - \beta}{\beta(\beta - 1)}, \qquad q_2 = \frac{mn}{\beta^2},$$

where m/β is an integer. Since $m > \beta$, according to the value $m/\beta (\ge 2)$, we now systematically search the designs with admissible parameters (i.e., of satisfying necessary conditions for the existence) within the scope of $v \le 100$ and $r, k \le 20$. (Note that in Clatworthy [12] $r, k \le 10$.) In fact, there are

Table 3.2. Affine resolvable SGD designs

No.	m	n	v	b	r	k	λ_1	λ_2	q_2	Source 1	Source 2	Remark
1	4	2	8	6	3	4	3	1	2	$K1+\{2\}$		S6
2	4	3	12	6	3	6	3	1	3	$K1+\{3\}$		S27
3	4	4	16	6	3	8	3	1	4	$K1+\{4\}$		S61
4	4	5	20	6	3	10	3	1	5	$K1+\{5\}$		S106
5	4	6	24	6	3	12	3	1	6	$K1+\{6\}$		
6	4	7	28	6	3	14	3	1	7	$K1+\{7\}$		
7	4	8	32	6	3	16	3	1	8	$K1+\{8\}$		
8	4	9	36	6	3	18	3	1	9	$K1+\{9\}$		
9	4	10	40	6	3	20	3	1	10	$K1+\{10\}$		
10	6	2	12	10	5	6	5	2	3	Non-E	$BIB(6,3,2)+\{2\}$	※1
11	6	4	24	10	5	12	5	2	6	Non-E	$BIB(6,3,2)+\{4\}$	※1
12	6	6	36	10	5	18	5	2	9	Non-E	$BIB(6,3,2)+\{6\}$	※1
13	8	2	16	14	7	8	7	3	4	$K5+\{2\}$		S63
14	8	3	24	14	7	12	7	3	6	$K5+\{3\}$		
15	8	4	32	14	7	16	7	3	8	$K5+\{4\}$		
16	8	5	40	14	7	20	7	3	10	$K5+\{5\}$		
17	9	2	18	12	4	6	4	1	2	$K6+\{2\}$		S37
18	9	3	27	12	4	9	4	1	3	$K6+\{3\}$		S91
19	9	4	36	12	4	12	4	1	4	$K6+\{4\}$		
20	9	5	45	12	4	15	4	1	5	$K6+\{5\}$		
21	9	6	54	12	4	18	4	1	6	$K6+\{6\}$		
22	10	2	20	18	9	10	9	4	5	Non-E	$BIB(10,5,4)+\{2\}$	※1
23	10	4	40	18	9	20	9	4	10	Non-E	$BIB(10,5,4)+\{4\}$	※1
24	12	2	24	22	11	12	11	5	6	$K12+\{2\}$		
25	12	3	36	22	11	18	11	5	9	$K12+\{3\}$		
26	14	2	28	26	13	14	13	6	7	Non-E	$BIB(14,7,6)+\{2\}$	※1
27	15	3	45	21	7	15	7	2	5	Non-E	Non-E	※2
28	16	2	32	20	5	8	5	1	2	$K17+\{2\}$		S74
29	16	2	32	30	15	16	15	7	8	$K18+\{2\}$		
30	16	3	48	20	5	12	5	1	3	$K17+\{3\}$		
31	16	4	64	20	5	16	5	1	4	$K17+\{4\}$		
32	16	5	80	20	5	20	5	1	5	$K17+\{5\}$		
33	18	2	36	34	17	18	17	8	9	Non-E	$BIB(18,9,8)+\{2\}$	※1
34	20	2	40	38	19	20	19	9	10	$K25+\{2\}$		
35	25	2	50	30	6	10	6	1	2	$K28+\{2\}$		S121
36	25	3	75	30	6	15	6	1	3	$K28+\{3\}$		
37	25	4	100	30	6	20	6	1	4	$K28+\{4\}$		
38	27	2	54	39	13	18	13	4	6	$K30+\{2\}$		
39	36	2	72	42	7	12	7	1	2	Non-E	Non-E	※3
40	40	2	80	52	13	20	13	3	5	Non-E	$BIB(40,10,3)$?	※4
41	49	2	98	56	8	14	8	1	2	$K40+\{2\}$		

41 parameters' combinations, all of which have explicit information on the existence of affine α-resolvability. By Theorem 3.1.1 the existence problem completely depends on the existence status of the corresponding affine resolvable $\mathrm{BIB}(v^* = v/n, b^* = b, r^* = r = \lambda_1, k^* = k/n, \lambda^* = \lambda_2)$ whose combinatorics has been discussed widely in literature (cf. [14, 21, 40]). For example, the existence of a "self-complementary" (i.e., $v = 2k$) affine resolvable SGD design with parameters $v = mn$, $b = 2(m-1)$, $r = m-1$, $k = mn/2$, $\lambda_1 = m-1$, $\lambda_2 = (m-2)/2$, $q_1 = 0$, $q_2 = mn/4$ is equivalent to the existence of an affine resolvable $\mathrm{BIB}(v^* = m, b^* = 2(m-1), r^* = m-1, k^* = m/2, \lambda^* = (m-2)/2)$ for even m.

In Table 3.2, the admissible parameters of affine resolvable SGD designs are listed along with existence information. The designs are numbered in the ascending order of m and for the same m in the order of n. Since $q_1 = 0$, the parameter is not listed. "Non-E" means the nonexistence of the design, $Kx + \{y\}$ in Source 1 means that the design is constructed through an affine resolvable BIB design of No. x in Kageyama [21] in which each treatment is replaced by a group of y new treatments. In Source 2, when an affine resolvable SGD design does not exist, the status on existence of the corresponding BIB design, i.e., an SGD design, which is not affine resolvable, is described.

The column of Remark shows some information below:

For example, S6 denotes an SGD design number from Table IV of Clatworthy [12]. An actual affine resolvable solution is also given there.

※1: Though a $\mathrm{BIB}(v^* = v/n, b^* = b, r^* = r, k^* = k/n, \lambda^* = \lambda_2)$ exists, $(k^*)^2/v^*$ is not an integer. Hence the corresponding affine resolvable solution does not exist.

※2: A $\mathrm{BIB}(v = 15, b = 21, r = 7, k = 5, \lambda = 2)$ does not exist ([43]). Hence an affine resolvable solution does not exist.

※3: A $\mathrm{BIB}(v = 36, b = 42, r = 7, k = 6, \lambda = 1)$ does not exist ([43]). Hence an affine resolvable solution does not exist.

※4: In a $\mathrm{BIB}(v = 40, b = 52, r = 13, k = 10, \lambda = 3)$, k^2/v is not an integer and hence such an affine resolvable solution of a design of No. 40 does not exist, but the existence as a BIB design (or an SGD design) is in doubt.

3.3. Affine α-resolvable SRGD designs

In this section an affine α-resolvable SRGD design with parameters $v = mn$, $b = \beta t$, $r = \alpha t$, k, λ_1, λ_2, $q_1 = k(\alpha - 1)/(\beta - 1)$ and $q_2 = k^2/v$, in which $rk - v\lambda_2 = 0$, is considered. Note that $\lambda_2 > \lambda_1$ in an SRGD design.

Now an integral expression of q_1 is derived like $q_1 = k + \lambda - r$ in an affine α-resolvable BIB design and as in Corollary 3.1.1.

Corollary 3.3.1. *In an affine α-resolvable SRGD design,* $q_1 = k(\alpha - 1)/(\beta - 1) = k + \lambda_1 - r$ *holds.*

Proof. Since $\theta_1 = rk - v\lambda_2 = 0$ and $\theta_2 = r - \lambda_1$, Theorem 3.3 implies that $q_1 = k + \lambda_1 - r$. ■

Furthermore, a typical result is remarked.

Lemma 3.3.1 ([6]). *In an SRGD design, k is divisible by m.*

Next the following characterization of design parameters is obtained.

Theorem 3.3.1. *The parameters of an affine α-resolvable SRGD design are given by*

$$v = mn, \quad b = \frac{\beta m(n-1)}{\beta - 1}, \quad r = \frac{\alpha m(n-1)}{\beta - 1}, \quad k = \frac{\alpha mn}{\beta}, \quad \lambda_1 = \frac{\alpha m(\alpha n - \beta)}{\beta(\beta - 1)},$$

$$\lambda_2 = \frac{\alpha^2 m(n-1)}{\beta(\beta - 1)}; \qquad t = \frac{m(n-1)}{\beta - 1}, \qquad q_2 = \frac{\alpha^2 mn}{\beta^2},$$

where $\alpha n/\beta$ *is an integer.*

Proof. Since eigenvalues of NN' are $rk - v\lambda_2 = 0$ and $r - \lambda_1$ with respective multiplicities $m - 1$ and $m(n-1)$, by Theorem 3.3 it holds that $b - t = m(n-1)$, i.e., $b = v + t - m$ which also implies that $t = m(n-1)/(\beta - 1)$. Then it follows that $v = mn$, $b = \beta t = \beta m(n-1)/(\beta - 1)$, $r = \alpha t = \alpha m(n-1)/(\beta - 1)$, $k = vr/b = \alpha mn/\beta$, $\lambda_2 = rk/v = \alpha^2 m(n-1)/[\beta(\beta - 1)]$. Furthermore, from a relation $r(k-1) = n_1\lambda_1 + n_2\lambda_2$, we get $\lambda_1 = \alpha m(\alpha n - \beta)/[\beta(\beta - 1)]$. Also by Lemma 3.3.1, $k/m = \alpha n/\beta$ must be an integer. ■

Thus, all parameters of an affine α-resolvable SRGD design can be expressed in terms of m, n, α and β.

There are 14 affine resolvable SRGD designs listed by Clatworthy [12], among of which 12 designs are symmetric. That is, only two affine resolvable "nonsymmetric" SRGD designs are available within the scope of parameters (i.e., $r, k \le 10$) in Clatworthy [12].

When the SRGD design is symmetric, we have $t = m$ and $n = \beta$. Hence Theorem 3.3.1 yields the following.

Corollary 3.3.2. *The parameters of an affine α-resolvable symmetric SRGD design are given by*

$$v = b = mn, \qquad r = k = \alpha m, \qquad \lambda_1 = \frac{\alpha m(\alpha - 1)}{n - 1}, \qquad \lambda_2 = \frac{\alpha^2 m}{n};$$

$$t = m, \qquad \beta = n.$$

All the existing affine α-resolvable symmetric SRGD designs satisfy $m = n$. In this case Corollary 3.3.2 yields the following since $n = \beta$.

COROLLARY 3.3.3. *The parameters of an affine α-resolvable symmetric SRGD design with $m = n$ are given by*

$$v = b = m^2, \qquad r = k = \alpha m, \qquad \lambda_1 = \frac{\alpha m(\alpha - 1)}{m - 1}, \qquad \lambda_2 = \alpha^2; \qquad t = \beta = m.$$

Note that in Corollary 3.3.3

$$\lambda_1 = \alpha(\alpha - 1) + \frac{\alpha(\alpha - 1)}{m - 1}$$

which causes some restriction on the values of $\alpha(\geq 2)$ for given m in $v = mn$.

As a method of construction of an SRGD design belonging to Corollary 3.3.3, Kageyama and Mohan [30; Corollary 2.1] show that when v^* is a prime, the existence of a symmetric BIB$(v^* = b^*, r^* = k^*, \lambda^*)$ implies the existence of an affine α-resolvable symmetric SRGD design with parameters $v = b = (v^*)^2$, $r = k = v^*k^*$, $\lambda_1 = \lambda^*v^*$, $\lambda_2 = (k^*)^2$, $q_1 = \lambda^*v^*$, $q_2 = (k^*)^2$, $\alpha = r^*$, $t = \beta = v^*$ for $m = n = v^*$. By use of this result, for example, the following can be given. (i) Since a symmetric BIB$(3, 3, 2, 2, 1)$ exists, we get a design of No. 6 of Table 3.4, i.e., SR23. (ii) Since a symmetric BIB$(5, 5, 4, 4, 3)$ exists, we get an affine 4-resolvable SRGD design with parameters $v = b = 25$, $r = k = 20$, $\lambda_1 = 15$, $\lambda_2 = 16$, $t = \beta = 5$; $m = n = 5$, whose complement is, by Lemma 2.3, an affine resolvable SRGD design with parameters $v = b = 25$, $r = k = 5$, $\lambda_1 = 0$, $\lambda_2 = 1$; $m = n = 5$, i.e., a design of No. 13, which may be different from SR60. (iii) Since a symmetric BIB$(7, 7, 3, 3, 1)$ exists (cf. [43]), we get an affine 3-resolvable SRGD design with parameters $v = b = 49$, $r = k = 21$, $\lambda_1 = 7$, $\lambda_2 = 9$, $t = \beta = 7$ for $m = n = 7$.

For the next section the case of $\alpha = 1$ will be investigated in detail. For an affine resolvable SRGD design, $t = r$ and then Theorem 3.3.1 with $q_2 = k^2/v$ shows the expression of design parameters as

$$v = mn, \qquad b = \frac{\beta m(n - 1)}{\beta - 1}, \qquad r = \frac{m(n - 1)}{\beta - 1}, \qquad k = \frac{mn}{\beta}, \qquad \lambda_1 = \frac{m(n - \beta)}{\beta(\beta - 1)},$$

$$\lambda_2 = \frac{m(n - 1)}{\beta(\beta - 1)}, \qquad q_1 = 0, \qquad q_2 = \frac{mn}{\beta^2}, \qquad \frac{k}{m} = \frac{n}{\beta}.$$

Then it holds that $\lambda_2 - \lambda_1 = m/\beta$. Therefore, there exist positive integers x and y such that

$$m = x\beta \qquad \text{and} \qquad n = y\beta.$$

These x and y can be used to express the required parameters as

$$(3.3.1)\quad v = xy\beta^2, \qquad b = \frac{x\beta^2(y\beta - 1)}{\beta - 1}, \qquad r = \frac{x\beta(y\beta - 1)}{\beta - 1}, \qquad k = xy\beta,$$

$$(3.3.2)\quad \lambda_1 = \frac{x\beta(y - 1)}{\beta - 1}, \quad \lambda_2 = \frac{x(y\beta - 1)}{\beta - 1}, \quad q_1 = 0, \quad q_2 = xy, \quad \frac{k}{m} = y.$$

In this case $\lambda_2 - \lambda_1 = x$ and $\lambda_1 = \beta(\lambda_2 - xy)$ (≥ 0). Note that $\lambda_1 = 0$ if and only if $y = 1$, i.e., the design is symmetric.

Now a way of presentation of the design parameters is made according to four patterns on the values of positive integers x and y.

Case 1: $x = y = 1$, i.e., $m = n = \beta$. Then the design parameters are shown as

$$v = b = \beta^2, \qquad r = k = \beta, \qquad \lambda_1 = 0, \qquad \lambda_2 = 1, \qquad q_2 = 1, \qquad \frac{k}{m} = 1,$$

which is symmetric. In fact, the existing SR1, SR23, SR44, SR60, SR87, SR97 and SR105 in Table VI of Clatworthy [12] belong to this class. By Lemma 2.3, note that the complement of the design of Case 1 is an affine $(\beta - 1)$-resolvable symmetric SRGD design with parameters $v^* = b^* = \beta^2$, $r^* = k^* = \beta(\beta - 1)$, $\lambda_1^* = \beta(\beta - 2)$, $\lambda_2^* = \beta(\beta - 2) + 1$, $q_1^* = \beta(\beta - 2)$, $q_2^* = \beta(\beta - 2) + 1$, and vice versa. For the present case a construction result can be provided.

Theorem 3.3.2. *When β is a prime or a prime power, there exists an affine resolvable symmetric SRGD design with parameters*

$$v = b = \beta^2, \quad r = k = \beta, \quad \lambda_1 = 0, \quad \lambda_2 = 1, \quad q_1 = 0, \quad q_2 = 1; \quad m = n = \beta.$$

Proof. It is well known (cf. [10; Chapter 6]) that when β is a prime or a prime power, an affine resolvable BIB$(v^* = \beta^2, b^* = \beta(\beta + 1), r^* = \beta + 1, k^* = \beta, \lambda^* = 1)$ can be constructed by use of an affine plane. The dual of this design can yield an SRGD design with parameters $v = \beta(\beta + 1)$, $b = \beta^2$, $r = \beta$, $k = \beta + 1$, $\lambda_1 = 0$, $\lambda_2 = 1$. In this design by deleting a group of β treatments corresponding to a partition for the affine resolvability of the original BIB design, we can obtain an SRGD design with parameters $v = b = \beta^2$, $r = k = \beta$, $\lambda_1 = 0$, $\lambda_2 = 1$. The remaining problem is to introduce the affine resolvability for the present design. It can be shown that this affine resolvability is naturally given when the incidence structure corresponding to β treatments of the group deleted in the dual design is

$$I_\beta \otimes \mathbf{1}'_\beta,$$

where $A \otimes B$ denotes the Kronecker product of matrices A and B, and I_β is the identity matrix of order β. ■

Remark 3.3.1. From the combinatorial structure on incidence in the construction process given in the proof of Theorem 3.3.2, it is obvious that the existence of an "affine resolvable" SRGD design as in Theorem 3.3.2 is equivalent to the existence of an affine plane of order β.

Case 2: $y = 1$, i.e., $n = \beta$. The the design parameters can be shown as

$$v = b = x\beta^2, \qquad r = k = x\beta, \qquad \lambda_1 = 0, \qquad \lambda_2 = x, \qquad q_2 = x, \qquad \frac{k}{m} = 1,$$

which is symmetric. When $x = 1$, this case coincides with Case 1 and then $x > 1$ is mainly considered. In fact, the existing SR36, SR72, SR92, SR95 and SR102 in Table VI of Clatworthy [12] belong to this class for $x = 2, 2, 4, 2$ and 3, respectively. By Lemma 2.3 note that the complement of the design of Case 2 is an affine $(\beta - 1)$-resolvable symmetric SRGD design with parameters $v^* = b^* = x\beta^2$, $r^* = k^* = x\beta(\beta - 1)$, $\lambda_1^* = x\beta(\beta - 2)$, $\lambda_2^* = x[\beta(\beta - 2) + 1]$, $q_1^* = x\beta(\beta - 2)$, $q_2^* = x[\beta(\beta - 2) + 1]$.

As a method of construction of a design for Case 2, Bose, Shrikhande and Bhattacharya [8] show that when s is a prime or a prime power, there exists an affine resolvable symmetric SRGD design with parameters $v = b = s^3$, $r = k = s^2$, $\lambda_1 = 0$, $\lambda_2 = s$, $q_2 = s$; $m = s^2$, $n = s$. Here $x = s$ and $y = 1$. When $s = 2$ and 3, we have designs of Nos. 8 and 23 in Table 3.4, respectively. When $s = 4$, we can obtain a solution of an affine resolvable SRGD design of No. 37 with parameters $v = b = 64$, $r = k = 16$, $\lambda_1 = 0$, $\lambda_2 = 4$, $q_2 = 4$; $m = 16$, $n = 4$.

Furthermore, to construct affine resolvable symmetric SRGD designs of Case 2, a special type of a difference scheme (cf. [17]) will be utilized.

An $m \times m$ matrix A with entries from a set $S = \{0, 1, \ldots, s - 1\}$ for $s \geq 2$ is here called a difference scheme, denoted by $DS(m, s; x)$, if on a vector difference in any two columns of A every entry of S occurs x times.

Remark 3.3.2. The same concept as the difference scheme has been discussed under other names of a difference matrix $D(m, m, s)$ or a generalized Hadamard matrix $GH(s, x)$ by interchanging roles of rows and columns (see [3, 13]).

It is easily seen that (i) all entries in the first row and first column of a $DS(m, s; x)$ can be set 0, and (ii) in each of columns except for the first, every entry of S occurs x times. The property (ii) implies that $m = xs$ in a $DS(m, s; x)$.

Furthermore, the following properties can be derived (see [3; pp. 532–534, especially, Remark 3.9(a)], or [17; p. 115]).

(iii) In each of rows except for the first one of a $DS(m, s; x)$, every entry of S occurs x times.

(iv) On a vector difference in any two rows of a $DS(m, s; x)$, every entry of S occurs x times.

Now the following construction result can be shown.

THEOREM 3.3.3. *The existence of a $DS(m, s; x)$ implies the existence of an affine resolvable symmetric SRGD design with parameters*

$$v = b = xs^2, \ \ r = k = xs, \ \ \lambda_1 = 0, \ \ \lambda_2 = x, \ \ q_1 = 0, \ \ q_2 = x; \ \ m = xs, \ \ n = s$$

for $s \geq 2$.

PROOF. Replace the entries $0, 1, \ldots, s-1$ in an $m \times m$ matrix as a $DS(m, s; x)$ by $s \times s$ matrices $\pi^i I_s$, $i = 0, 1, \ldots, s-1$, respectively, where π is a row permutation such that $\pi R_\ell = R_{\ell+1}$ and R_ℓ is the ℓth row of I_s. Then from $m = xs$ such replacement can show the required design with a GD association scheme on an $xs \times s$ array. In fact, under the property (ii), parameters $v = b = xs^2$, $k = xs$, $\lambda_1 = 0$, $m = xs$ and $n = s$ are obvious. The property (iii) with $m = xs$ implies $r = xs$. It is also clear that the replacement of $s \times s$ $(0, 1)$-matrices shows the resolvability consisting of m resolution sets of s blocks each, and then $q_1 = 0$. Furthermore, the properties (i) and (ii) of the $DS(m, s; x)$ with properties (iii) and (iv) can yield $\lambda_2 = x$ and $q_2 = x$ (affine resolvability). ■

When $s = 5$ and $x = 2$ in Theorem 3.3.3, it is illustrated by use of a $DS(10, 5; 2)$ given as follows (see Table 6.35 in [17]).

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 1 & 2 & 1 & 0 & 4 & 2 & 3 \\ 0 & 3 & 1 & 2 & 4 & 4 & 2 & 0 & 1 & 3 \\ 0 & 1 & 2 & 4 & 3 & 1 & 2 & 3 & 0 & 4 \\ 0 & 2 & 4 & 3 & 1 & 4 & 1 & 3 & 2 & 0 \\ 0 & 2 & 3 & 2 & 3 & 0 & 4 & 1 & 4 & 1 \\ 0 & 1 & 1 & 3 & 0 & 2 & 4 & 4 & 3 & 2 \\ 0 & 0 & 4 & 4 & 2 & 3 & 3 & 1 & 1 & 2 \\ 0 & 3 & 0 & 1 & 1 & 2 & 3 & 2 & 4 & 4 \\ 0 & 4 & 2 & 0 & 4 & 3 & 1 & 2 & 3 & 1 \end{bmatrix}$$

which obviously satisfies the above properties (i) to (iv).

EXAMPLE 3.3.1. There exists an affine resolvable symmetric SRGD design with parameters $v = b = 50$, $r = k = 10$, $\lambda_1 = 0$, $\lambda_2 = 2$, $q_2 = 2$; $m = 10$, $n = 5$, whose GD association scheme of 50 treatments is

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \\ 26 & 27 & 28 & 29 & 30 \\ 31 & 32 & 33 & 34 & 35 \\ 36 & 37 & 38 & 39 & 40 \\ 41 & 42 & 43 & 44 & 45 \\ 46 & 47 & 48 & 49 & 50 \end{bmatrix}.$$

Now, replace 0, 1, 2, 3, 4 in the above $DS(10,5;2)$ by the following five matrices of order 5:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

respectively. Then the 50 blocks of 10 resolution sets (i.e., each resolution set showing a bracket [ ] below) of 5 blocks each are given by

[(1, 6, 11, 16, 21, 26, 31, 36, 41, 46), (2, 7, 12, 17, 22, 27, 32, 37, 42, 47), (3, 8, 13, 18, 23, 28, 33, 38, 43, 48), (4, 9, 14, 19, 24, 29, 34, 39, 44, 49), (5, 10, 15, 20, 25, 30, 35, 40, 45, 50)],

[(1, 10, 14, 17, 23, 28, 32, 36, 44, 50), (2, 6, 15, 18, 24, 29, 33, 37, 45, 46), (3, 7, 11, 19, 25, 30, 34, 38, 41, 47), (4, 8, 12, 20, 21, 26, 35, 39, 42, 48), (5, 9, 13, 16, 22, 27, 31, 40, 43, 49)],

[(1, 9, 12, 18, 25, 29, 32, 40, 41, 48), (2, 10, 13, 19, 21, 30, 33, 36, 42, 49), (3, 6, 14, 20, 22, 26, 34, 37, 43, 50), (4, 7, 15, 16, 23, 27, 35, 38, 44, 46), (5, 8, 11, 17, 24, 28, 31, 39, 45, 47)],

[(1, 7, 13, 20, 24, 28, 34, 40, 42, 46), (2, 8, 14, 16, 25, 29, 35, 36, 43, 47), (3, 9, 15, 17, 21, 30, 31, 37, 44, 48), (4, 10, 11, 18, 22, 26, 32, 38, 45, 49), (5, 6, 12, 19, 23, 27, 33, 39, 41, 50)],

[(1, 8, 15, 19, 22, 29, 31, 38, 42, 50), (2, 9, 11, 20, 23, 30, 32, 39, 43, 46), (3, 10, 12, 16, 24, 26, 33, 40, 44, 47), (4, 6, 13, 17, 25, 27, 34, 36, 45, 48), (5, 7, 14, 18, 21, 28, 35, 37, 41, 49)],

[(1, 7, 15, 17, 25, 26, 33, 39, 43, 49), (2, 8, 11, 18, 21, 27, 34, 40, 44, 50), (3, 9, 12, 19, 22, 28, 35, 36, 45, 46), (4, 10, 13, 20, 23, 29, 31, 37, 41, 47), (5, 6, 14, 16, 24, 30, 32, 38, 42, 48)],

[(1, 6, 13, 18, 22, 30, 35, 39, 44, 47), (2, 7, 14, 19, 23, 26, 31, 40, 45, 48), (3, 8, 15, 20, 24, 27, 32, 36, 41, 49), (4, 9, 11, 16, 25, 28, 33, 37, 42, 50), (5, 10, 12, 17, 21, 29, 34, 38, 43, 46)],

[(1, 10, 11, 19, 24, 27, 35, 37, 43, 48), (2, 6, 12, 20, 25, 28, 31, 38, 44, 49), (3, 7, 13, 16, 21, 29, 32, 39, 45, 50), (4, 8, 14, 17, 22, 30, 33, 40, 41, 46), (5, 9, 15, 18, 23, 26, 34, 36, 42, 47)],

[(1, 8, 12, 16, 23, 30, 34, 37, 45, 49), (2, 9, 13, 17, 24, 26, 35, 38, 41, 50), (3, 10, 14, 18, 25, 27, 31, 39, 42, 46), (4, 6, 15, 19, 21, 28, 32, 40, 43, 47), (5, 7, 11, 20, 22, 29, 33, 36, 44, 48)],

[(1, 9, 14, 20, 21, 27, 33, 38, 45, 47), (2, 10, 15, 16, 22, 28, 34, 39, 41, 48), (3, 6, 11, 17, 23, 29, 35, 40, 42, 49), (4, 7, 12, 18, 24, 30, 31, 36, 43, 50), (5, 8, 13, 19, 25, 26, 32, 37, 44, 46)].

Six designs of Nos. 23, 29, 30, 33, 39 and 42 in Table 3.4 are also constructed by use of Theorem 3.3.3 with $DS(9,3;3)$, $DS(12,3;4)$, $DS(12,2^2;3)$, $DS(14,7;2)$, $DS(18,3;6)$ and $DS(20,5;4)$, respectively. Many useful information on the existence of a difference scheme can be found in [3, 13] and [17; Chapter 6].

Another characterization for Case 2 is provided. It is clear (see, for example, [17; Theorem 7.6]) that a $DS(2x,2;x)$ exists iff a Hadamard matrix of order $2x$ exists. Here Theorem 3.3.3 with $s=2$ can be especially expressed as an equivalence existence.

Theorem 3.3.4. *The existence of a Hadamard matrix of order* $2x$ *is equivalent to the existence of an affine resolvable symmetric SRGD design with parameters*

$$v=b=4x,\ \ r=k=2x,\quad \lambda_1=0,\ \ \lambda_2=x,\ \ q_1=0,\ \ q_2=x;\ \ m=2x,\ \ n=2.$$

Proof. (Necessity) In a Hadamard matrix H of order $2x$, replace $+1$ and -1 by I_2 and $\mathbf{1}_2\mathbf{1}_2'-I_2$ respectively. Then the relation $HH'=2xI_{2x}=H'H$ can yield that $\lambda_1=0$ and $\lambda_2=x$ with the affine resolvability. Thus the required design can be obtained. Or apply Theorem 3.3.3.

(Sufficiency) Since $v=2k$, from the properties of the GD association scheme on a $2x\times 2$ array, the resolvability and $\lambda_1=0$, it follows that the $4x\times 4x$ incidence matrix is partitioned into $(2x)^2$ submatrices of order 2, whose pattern is either I_2 or $\mathbf{1}_2\mathbf{1}_2'-I_2$. Now replace I_2 and $\mathbf{1}_2\mathbf{1}_2'-I_2$ by $+1$ and -1 respectively. Then we get a $2x\times 2x$ matrix H whose elements are $+1$

or -1. In the original incidence matrix of the design, each of four rows (consisting of two columns each) corresponding to the replacement, which follows the above partition of the incidence matrix, has one of four patterns as $(I_2, I_2)'$, $(I_2, \mathbf{1}_2\mathbf{1}_2' - I_2)'$, $(\mathbf{1}_2\mathbf{1}_2' - I_2, \mathbf{1}_2\mathbf{1}_2' - I_2)'$, $(\mathbf{1}_2\mathbf{1}_2' - I_2, I_2)'$. Hence, on account of $\lambda_2 = x$, it can be shown that $HH' = 2xI_{2x}$. ■

In Theorem 3.3.4, when $x = 6$, by use of a Hadamard matrix H_{12} of order 12 as

$$\begin{bmatrix}
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1
\end{bmatrix},$$

an affine resolvable symmetric SRGD design of No. 28 in Table 3.4 can be obtained. This will be given in Example 3.3.2.

EXAMPLE 3.3.2. There exists an affine resolvable symmetric SRGD design with parameters $v = b = 24$, $r = k = 12$, $\lambda_1 = 0$, $\lambda_2 = 6$, $q_2 = 6$; $m = 12$, $n = 2$, whose GD association scheme of 24 treatments is given by the usual 12×2 array. If the entries $+1$ and -1 in H_{12} are replaced by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

respectively, then the 24 blocks of 12 resolution sets of 2 blocks each are given by

$[(1,3,5,7,9,11,13,15,17,19,21,23),\ (2,4,6,8,10,12,14,16,18,20,22,24)]$,
$[(2,3,5,8,9,11,13,16,18,20,21,24),\ (1,4,6,7,10,12,14,15,17,19,22,23)]$,
$[(2,4,5,7,10,11,13,15,18,20,22,23),\ (1,3,6,8,9,12,14,16,17,19,21,24)]$,
$[(2,3,6,7,9,12,13,15,17,20,22,24),\ (1,4,5,8,10,11,14,16,18,19,21,23)]$,
$[(2,4,5,8,9,11,14,15,17,19,22,24),\ (1,3,6,7,10,12,13,16,18,20,21,23)]$,
$[(2,4,6,7,10,11,13,16,17,19,21,24),\ (1,3,5,8,9,12,14,15,18,20,22,23)]$,
$[(2,4,6,8,9,12,13,15,18,19,21,23),\ (1,3,5,7,10,11,14,16,17,20,22,24)]$,
$[(2,3,6,8,10,11,14,15,17,20,21,23),\ (1,4,5,7,9,12,13,16,18,19,22,24)]$,

$[(2,3,5,8,10,12,13,16,17,19,22,23),\ (1,4,6,7,9,11,14,15,18,20,21,24)]$,
$[(2,3,5,7,10,12,14,15,18,19,21,24),\ (1,4,6,8,9,11,13,16,17,20,22,23)]$,
$[(2,4,5,7,9,12,14,16,17,20,21,23),\ (1,3,6,8,10,11,13,15,18,19,22,24)]$,
$[(2,3,6,7,9,11,14,16,18,19,22,23),\ (1,4,5,8,10,12,13,15,17,20,21,24)]$.

It is well known that a necessary condition for the existence of a Hadamard matrix is that the order is either 2 or a multiple of 4. Then Theorem 3.3.4 can produce the following.

Corollary 3.3.4. *When x is odd (≥ 3), there does not exist an affine resolvable symmetric SRGD design with parameters $v = b = 4x$, $r = k = 2x$, $\lambda_1 = 0$, $\lambda_2 = x$, $q_1 = 0$, $q_2 = x$; $m = 2x$, $n = 2$.*

Remark 3.3.3. The existence of a Hadamard matrix of order $2x$ is known for all $2x \leq 664$ (i.e., the smallest order in which a Hadamard matrix is undecided is 668) ([32]). Hence an affine resolvable symmetric SRGD design of Theorem 3.3.4 exists for all even $x \leq 332$. In fact, it is conjectured that a Hadamard matrix always exists for any order ($\equiv 0 \mod 4$) (see [16]).

Remark 3.3.4. By Theorem 3.3.3, Theorem 3.3.4 and Corollary 3.3.4, the nonexistence information on designs of Nos. 14, 17, 25, 27, 32, 34, 35 and 38 in Source 1 of Table 3.4 for $y = 1$ implies the nonexistence of difference schemes $DS(m, s; x)$ in $DS(6,2;3)$, $DS(6,6;1)$, $DS(10,2;5)$, $DS(10,10;1)$, $DS(14,2;7)$, $DS(15,3;5)$, $DS(15,5;3)$ and $DS(18,2;9)$, respectively. Since the existence of $DS(12,6;2)$ and $DS(20,2^2;5)$ is unknown, designs of Nos. 31 and 41 may not be constructed through Theorem 3.3.3. In general, it also follows from Theorem 3.3.4 and Corollary 3.3.4 that there does not exist a difference scheme $DS(2x,2;x)$ for any odd $x \geq 3$.

Case 3: $x = 1$, i.e., $m = \beta$. This case shows the design parameters as

$$v = y\beta^2, \qquad b = \frac{\beta^2(y\beta - 1)}{\beta - 1}, \qquad r = \frac{\beta(y\beta - 1)}{\beta - 1}, \qquad k = y\beta,$$

$$\lambda_1 = \frac{\beta(y - 1)}{\beta - 1}, \qquad \lambda_2 = \frac{y\beta - 1}{\beta - 1}, \qquad q_2 = y, \qquad \frac{k}{m} = y.$$

When $y = 1$, this case coincides with Case 1 and then $y > 1$ is mainly considered. In fact, the existing SR38 and SR71 in Table VI of Clatworthy [12] belong to this class for $y = 2$ and 3, respectively. In this case, all the existing designs satisfy $v = 2k$ (self-complementary). However, note that the parameters of an unknown design of No. 12 do not satisfy $v = 2k$.

As a method of construction of a design belonging to Case 3, Kageyama, Banerjee and Verma [28] show that the existence of an affine resolvable

$\mathrm{BIB}(v^* = 2k^*, b^* = 2r^*, r^* = 2k^* - 1, k^*, \lambda^* = k^* - 1)$ implies the existence of an affine resolvable SRGD design with parameters $v = 4k^*$, $b = 4(2k^* - 1)$, $r = 2(2k^* - 1)$, $k = 2k^*$, $\lambda_1 = 2(k^* - 1)$, $\lambda_2 = 2k^* - 1$; $m = 2$, $n = 2k^*$. Here $x = 1$ and $y = k^*$. Note that this design has only possibility of existence when k^* is even. When $k^* = 2$ we have a design of No. 2 in Table 3.4, i.e., SR38. When $k^* = 4$, a design of No. 4 in Table 3.4 is newly constructed as will be constructed in Example 3.3.3.

Example 3.3.3. There exists an affine resolvable SRGD design with parameters $v = 16$, $b = 28$, $r = 14$, $k = 8$, $\lambda_1 = 6$, $\lambda_2 = 7$, $q_2 = 4$; $m = 2$, $n = 8$ whose GD association scheme of 16 treatments is

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{bmatrix}.$$

The 28 blocks of 14 resolution sets of 2 blocks each are given by

$$\begin{aligned}
&[(1,2,3,5,9,10,11,13),\ (4,6,7,8,12,14,15,16)],\\
&[(4,6,7,8,9,10,11,13),\ (1,2,3,5,12,14,15,16)],\\
&[(2,3,4,6,10,11,12,14),\ (1,5,7,8,9,13,15,16)],\\
&[(1,5,7,8,10,11,12,14),\ (2,3,4,6,9,13,15,16)],\\
&[(3,4,5,7,11,12,13,15),\ (1,2,6,8,9,10,14,16)],\\
&[(1,2,6,8,11,12,13,15),\ (3,4,5,7,9,10,14,16)],\\
&[(1,4,5,6,9,12,13,14),\ (2,3,7,8,10,11,15,16)],\\
&[(2,3,7,8,9,12,13,14),\ (1,4,5,6,10,11,15,16)],\\
&[(2,5,6,7,10,13,14,15),\ (1,3,4,8,9,11,12,16)],\\
&[(1,3,4,8,10,13,14,15),\ (2,5,6,7,9,11,12,16)],\\
&[(1,3,6,7,9,11,14,15),\ (2,4,5,8,10,12,13,16)],\\
&[(2,4,5,8,9,11,14,15),\ (1,3,6,7,10,12,13,16)],\\
&[(1,2,4,7,9,10,12,15),\ (3,5,6,8,11,13,14,16)],\\
&[(3,5,6,8,9,10,12,15),\ (1,2,4,7,11,13,14,16)].
\end{aligned}$$

This is constructed by use of Theorem 1 and Corollary 2 of Kageyama, Banerjee and Verma [28] with an affine resolvable solution, $[(0,1,2,4),(3,5,6,\infty)]$ mod 7, of a $\mathrm{BIB}(8,14,7,4,3)$ (cf. [21]), having the incidence matrix N, i.e., the constructed design has

$$N \otimes \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + (\mathbf{1}_8 \mathbf{1}'_{14} - N) \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

with some renumbering of 16 new treatments to suit the present GD association scheme from the original scheme

$$\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 \end{bmatrix}.$$

Case 4: $x > 1$ and $y > 1$. In this case we have the design parameters as in (3.3.1) and (3.3.2). In general, since

$$\lambda_1 = x(y-1) + \frac{x(y-1)}{\beta - 1},$$

for given x and y there are a finite number of values of β since λ_1 is an integer. Thus all parameters of an affine resolvable SRGD design are systematically expressed in terms of parameters x, y and β.

Some special cases are taken below.

Case 4.1: $x = 2$ and $y = 2$. Then $\lambda_1 = 2 + 2/(\beta - 1)$ which implies $\beta = 2, 3$. When $\beta = 2$, we have $v = 16$, $b = 24$, $r = 12$, $k = 8$, $\lambda_1 = 4$, $\lambda_2 = 6$; $m = n = 4$. This is a design of No. 10 in Table 3.4 and will be constructed as in Example 3.3.4. This is the only existing affine resolvable SRGD design for $x > 1$ and $y > 1$ as far as the authors are aware of. When $\beta = 3$, we have $v = 36$, $b = 45$, $r = 15$, $k = 12$, $\lambda_1 = 3$, $\lambda_2 = 5$; $m = n = 6$ a design of which is shown to be nonexistent by Theorem 12.6.2 in Raghavarao [39].

Example 3.3.4. There exists an affine resolvable SRGD design with parameters $v = 16$, $b = 24$, $r = 12$, $k = 8$, $\lambda_1 = 4$, $\lambda_2 = 6$, $q_2 = 4$; $m = n = 4$ whose GD association scheme of 16 treatments is

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}.$$

The 24 blocks of 12 resolution sets of 2 blocks each are given by

$$\begin{aligned}
&[(1,2,5,6,9,10,13,14),\ (3,4,7,8,11,12,15,16)],\\
&[(1,2,5,6,11,12,15,16),\ (3,4,7,8,9,10,13,14)],\\
&[(1,2,7,8,9,10,15,16),\ (3,4,5,6,11,12,13,14)],\\
&[(1,2,7,8,11,12,13,14),\ (3,4,5,6,9,10,15,16)],\\
&[(1,3,5,7,9,11,13,15),\ (2,4,6,8,10,12,14,16)],\\
&[(1,3,5,7,10,12,14,16),\ (2,4,6,8,9,11,13,15)],\\
&[(1,3,6,8,9,11,14,16),\ (2,4,5,7,10,12,13,15)],\\
&[(1,3,6,8,10,12,13,15),\ (2,4,5,7,9,11,14,16)],\\
&[(1,4,5,8,9,12,13,16),\ (2,3,6,7,10,11,14,15)],\\
&[(1,4,5,8,10,11,14,15),\ (2,3,6,7,9,12,13,16)],\\
&[(1,4,6,7,9,12,14,15),\ (2,3,5,8,10,11,13,16)],\\
&[(1,4,6,7,10,11,13,16),\ (2,3,5,8,9,12,14,15)].
\end{aligned}$$

This is constructed by trial and error under some manner.

Case 4.2: $x = 2$ and $y = 3$. Then $\lambda_1 = 4 + 4/(\beta - 1)$ which implies $\beta = 2, 3, 5$. When $\beta = 2$, we have $v = 24$, $b = 40$, $r = 20$, $k = 12$, $\lambda_1 = 8$, $\lambda_2 = 10$; $m = 4$, $n = 6$ whose affine resolvable solution as a design of No. 11 in Table 3.4 is unknown. A 5-resolvable solution under the usual 4×6 GD association scheme of 24 treatments can be constructed by trial and error. However, it is not affine 5-resolvable.

When $\beta = 3$, we get $v = 54$, $b = 72$, $r = 24$ (> 20), $k = 18$, $\lambda_1 = 6$, $\lambda_2 = 8$; $m = 6$, $n = 9$ whose solution as a design is unknown. When $\beta = 5$, we obtain $v = 150$, $b = 175$, $r = 35$, $k = 30$, $\lambda_1 = 5$, $\lambda_2 = 7$; $m = 10$, $n = 15$ a design of which is shown to be nonexistent by Theorem 12.6.2 in Raghavarao [39].

Case 4.3: $x = 3$ and $y = 2$. Then $\lambda_1 = 3 + 3/(\beta - 1)$ which implies $\beta = 2, 4$. When $\beta = 2$, we have $v = 24$, $b = 36$, $r = 18$, $k = 12$, $\lambda_1 = 6$, $\lambda_2 = 9$; $m = 6$, $n = 4$ a design of which is shown to be nonexistent by Theorem 12.6.2 in Raghavarao [39]. When $\beta = 4$, we get $v = 96$, $b = 112$, $r = 28$ (> 20), $k = 24$ (> 20), $\lambda_1 = 4$, $\lambda_2 = 7$; $m = 12$, $n = 8$ whose solution as a design is unknown.

Case 4.4: $x = 3$ and $y = 3$. Then $\lambda_1 = 6 + 6/(\beta - 1)$ which implies $\beta = 2, 3, 4, 7$. When $\beta = 2$, we have $v = 36$, $b = 60$, $r = 30$, $k = 18$, $\lambda_1 = 12$, $\lambda_2 = 15$; $m = n = 6$ a design of which is shown to be nonexistent by Theorem 12.6.2 in Raghavarao [39]. When $\beta = 3$, we get $v = 81$, $b = 108$, $r = 36$, $k = 27$, $\lambda_1 = 9$, $\lambda_2 = 12$; $m = n = 9$ whose solution is unknown as a design. When $\beta = 4$, we obtain $v = 144$, $b = 176$, $r = 44$, $k = 36$, $\lambda_1 = 8$, $\lambda_2 = 11$; $m = n = 12$ whose solution is unknown as a design. When $\beta = 7$, we obtain $v = 441$, $b = 490$, $r = 70$, $k = 63$, $\lambda_1 = 7$, $\lambda_2 = 10$; $m = n = 21$ a design of which is shown to be nonexistent by Theorem 12.6.2 in Raghavarao [39]. All designs of Case 4.4 have r or $k > 20$ which are beyond the scope in Table 3.4.

Other cases may have r and/or $k > 20$.

The above-mentioned information will be summarized in Table 3.4.

3.4. Table of affine resolvable SRGD designs with $v \leq 100$ and $r, k \leq 20$

According to the values of positive integers x and y as expressed in (3.3.1) and (3.3.2), we now systematically search affine resolvable SRGD designs with admissible parameters within the scope of $v \leq 100$ and $r, k \leq 20$. (Note that in Clatworthy [12] $r, k \leq 10$.) In fact, there are 42 parameters' combinations, among of which 26 designs are existent, 11 designs do not exist, while other 5 cases are unknown for the existence.

In Table 3.4, the admissible parameters of the affine resolvable SRGD designs are listed along with existence information. The designs are numbered

Table 3.4. Affine resolvable SRGD designs

No.	m	n	v	b	r	k	λ_1	λ_2	q_2	Source 1	Source 2	x	y
1	2	2	4	4	2	2	0	1	1	SR1		1	1
2	2	4	8	12	6	4	2	3	2	SR38		1	2
3	2	6	12	20	10	6	4	5	3	SR71		1	3
4	2	8	16	28	14	8	6	7	4	Exist		1	4
5	2	10	20	36	18	10	8	9	5	?	?	1	5
6	3	3	9	9	3	3	0	1	1	SR23		1	1
7	3	9	27	36	12	9	3	4	3	Non-E	?	1	3
8	4	2	8	8	4	4	0	2	2	SR36		2	1
9	4	4	16	16	4	4	0	1	1	SR44		1	1
10	4	4	16	24	12	8	4	6	4	Exist		2	2
11	4	6	24	40	20	12	8	10	6	?	Exist	2	3
12	4	16	64	80	20	16	4	5	4	?	?	1	4
13	5	5	25	25	5	5	0	1	1	SR60		1	1
14	6	2	12	12	6	6	0	3	3	Non-E	SR67	3	1
15	6	3	18	18	6	6	0	2	2	SR72		2	1
16	6	4	24	36	18	12	6	9	6	Non-E	?	3	2
17	6	6	36	36	6	6	0	1	1	Non-E	?	1	1
18	6	6	36	45	15	12	3	5	4	Non-E	?	2	2
19	7	7	49	49	7	7	0	1	1	SR87		1	1
20	8	2	16	16	8	8	0	4	4	SR92		4	1
21	8	4	32	32	8	8	0	2	2	SR95		2	1
22	8	8	64	64	8	8	0	1	1	SR97		1	1
23	9	3	27	27	9	9	0	3	3	SR102		3	1
24	9	9	81	81	9	9	0	1	1	SR105		1	1
25	10	2	20	20	10	10	0	5	5	Non-E	SR108	5	1
26	10	5	50	50	10	10	0	2	2	Exist		2	1
27	10	10	100	100	10	10	0	1	1	Non-E	?	1	1
28	12	2	24	24	12	12	0	6	6	Exist		6	1
29	12	3	36	36	12	12	0	4	4	Exist		4	1
30	12	4	48	48	12	12	0	3	3	Exist		3	1
31	12	6	72	72	12	12	0	2	2	?	?	2	1
32	14	2	28	28	14	14	0	7	7	Non-E	?	7	1
33	14	7	98	98	14	14	0	2	2	Exist		2	1
34	15	3	45	45	15	15	0	5	5	Non-E	?	5	1
35	15	5	75	75	15	15	0	3	3	Non-E	?	3	1
36	16	2	32	32	16	16	0	8	8	Exist		8	1
37	16	4	64	64	16	16	0	4	4	Exist		4	1
38	18	2	36	36	18	18	0	9	9	Non-E	?	9	1
39	18	3	54	54	18	18	0	6	6	Exist		6	1
40	20	2	40	40	20	20	0	10	10	Exist		10	1
41	20	4	80	80	20	20	0	5	5	?	?	5	1
42	20	5	100	100	20	20	0	4	4	Exist		4	1

in the ascending order of m and for the same m in the order of n. Since $q_1 = 0$, the parameter is not listed. "Non-E" means the nonexistence of the design. Source 1 has some information on the existence of the corresponding affine resolvable SRGD design, while Source 2 shows some information on the existence of the corresponding SRGD design when the affine resolvable solution does not exist or is unknown. The symbol ? means that the existence or nonexistence of the corresponding design is unknown. Half of the existence is confirmed in Table VI of Clatworthy [12], for example, SR1, etc. By Theorem 12.6.2 of Raghavarao [39], it can be seen that affine resolvable designs of Nos. 7, 14, 16, 17, 18, 32, 34 and 35 do not exist. The nonexistence of designs of Nos. 17 and 27 also follows from Remark 3.3.1 since an affine plane of order 6 or 10 does not exist (cf. [34]). The nonexistence of designs of Nos. 25 and 38 follows from Corollary 3.3.4.

3.5. Affine α-resolvable L_2 designs

For the description of an L_2 design with $v = s^2$ treatments, having the incidence matrix N, see Definition 2.5. Note (cf. [39]) that NN' has eigenvalues $r + (s-2)\lambda_1 - (s-1)\lambda_2$ $(= \theta_1$, say) and $r - 2\lambda_1 + \lambda_2$ $(= \theta_2$, say) other than simple rk with respective multiplicities $2(s-1)$ and $(s-1)^2$.

Now we consider an affine α-resolvable L_2 design with parameters $v = s^2$, $b = \beta t$, $r = \alpha t$, k, λ_1, λ_2, $q_1 = k(\alpha-1)/(\beta-1)$ and $q_2 = k^2/v$.

By Lemma 3.1, we have the following.

Theorem 3.5.1. *If $r + (s-2)\lambda_1 - (s-1)\lambda_2 > 0$ and $r - 2\lambda_1 + \lambda_2 > 0$, then there does not exist an affine α-resolvable L_2 design for any $\alpha \geq 1$.*

Therefore, by Remark 3.2, other two cases are considered to investigate L_2 designs with the affine α-resolvability.

Case 3.5.1. Affine α-resolvable L_2 designs with $\theta_1 = r + (s-2)\lambda_1 - (s-1)\lambda_2 = 0$ and $\theta_2 = r - 2\lambda_1 + \lambda_2 > 0$

In this case, it is clear that $\lambda_2 > \lambda_1$.

At first, an integral expression of q_1 is derived as in Corollaries 3.1.1 and 3.3.1 for affine α-resolvable GD designs.

Corollary 3.5.1. *In an affine α-resolvable L_2 design of Case 3.5.1, $q_1 = k(\alpha-1)/(\beta-1) = k - r + 2\lambda_1 - \lambda_2$ holds.*

Proof. Since $\theta_1 = r + (s-2)\lambda_1 - (s-1)\lambda_2 = 0$ and $\theta_2 = r - 2\lambda_1 + \lambda_2 > 0$, Theorem 3.3 implies that $q_1 = k - r + 2\lambda_1 - \lambda_2$. ∎

Furthermore, a useful result is remarked.

Lemma 3.5.1 ([31]). *In an L_2 design of Case* 3.5.1, *k is divisible by s.*

Hence the following can be shown.

Theorem 3.5.2. *The parameters of an affine α-resolvable L_2 design of Case* 3.5.1 *are given by*

$$v = s^2, \qquad b = \frac{\beta(s-1)^2}{\beta - 1}, \qquad r = \frac{\alpha(s-1)^2}{\beta - 1}, \qquad k = \frac{\alpha s^2}{\beta},$$

$$\lambda_1 = \frac{\alpha(s-1)(\alpha s - \beta)}{\beta(\beta - 1)}, \qquad \lambda_2 = \frac{\alpha(\alpha s^2 + \beta - 2\alpha s)}{\beta(\beta - 1)}; \qquad t = \frac{(s-1)^2}{\beta - 1},$$

where $\alpha s/\beta$ is an integer.

Proof. Since eigenvalues of NN' are $r + (s-2)\lambda_1 - (s-1)\lambda_2 = 0$ and $r - 2\lambda_1 + \lambda_2 > 0$ with respective multiplicities $2(s-1)$ and $(s-1)^2$, by Theorem 3.3 it holds that $b - t = (s-1)^2$, i.e., $b = v + t - 2s + 1$ which also implies that $t = (s-1)^2/(\beta - 1)$. Then it follows that $v = s^2$, $b = \beta t = \beta(s-1)^2/(\beta - 1)$, $r = \alpha t = \alpha(s-1)^2/(\beta - 1)$, $k = vr/b = \alpha s^2/\beta$. Furthermore, from relations $r(k-1) = n_1\lambda_1 + n_2\lambda_2$ and $r + (s-2)\lambda_1 - (s-1)\lambda_2 = 0$, we get $\lambda_1 = \alpha(s-1)(s\alpha - \beta)/[\beta(\beta-1)]$ and $\lambda_2 = \alpha(s^2\alpha + \beta - 2s\alpha)/[\beta(\beta - 1)]$. Also by Lemma 3.5.1, $k/s = \alpha s/\beta$ must be an integer. ■

Thus, all parameters of an affine α-resolvable L_2 design of Case 3.5.1 can be expressed in terms of s, α and β.

Note that

$$q_1 = \frac{s^2\alpha(\alpha - 1)}{\beta(\beta - 1)} \qquad \text{and} \qquad q_2 = \left(\frac{s\alpha}{\beta}\right)^2.$$

Next the case of $\alpha = 1$ will be investigated in detail. For an affine resolvable L_2 design of Case 3.5.1, $t = r$ and then Theorem 3.5.2 shows the expression of design parameters as

$$v = s^2, \qquad b = \frac{\beta(s-1)^2}{\beta - 1}, \qquad r = \frac{(s-1)^2}{\beta - 1}, \qquad k = \frac{s^2}{\beta}, \qquad \lambda_1 = \frac{(s-1)(s-\beta)}{\beta(\beta - 1)},$$

$$\lambda_2 = \frac{s^2 + \beta - 2s}{\beta(\beta - 1)}, \qquad q_1 = 0, \qquad q_2 = \frac{s^2}{\beta^2}, \qquad \frac{k}{s} = \frac{s}{\beta}.$$

Then there exists a positive integer ℓ such that

$$s = \ell\beta$$

which implies that

$$(3.5.1)\quad v = (\ell\beta)^2, \qquad b = \frac{\beta(\ell\beta - 1)^2}{\beta - 1}, \qquad r = \frac{(\ell\beta - 1)^2}{\beta - 1}, \qquad k = \ell^2\beta,$$

$$(3.5.2)\quad \lambda_1 = \ell(\ell - 1) + \frac{(\ell - 1)^2}{\beta - 1}, \qquad \lambda_2 = \frac{\ell^2\beta + 1 - 2\ell}{\beta - 1}, \qquad q_2 = \ell^2.$$

Thus all parameters of an affine resolvable L_2 design of Case 3.5.1 are expressed in terms of ℓ and β. In particular, the above expression of λ_1 means that for given ℓ, we have a finite number of β since λ_1 in (3.5.2) is an integer. For example, some ℓ are investigated.

(i) $\ell = 1$: $\lambda_1 = 0$ and then we have the design parameters as $v = \beta^2$, $b = \beta(\beta - 1)$, $r = \beta - 1$, $k = \beta$, $\lambda_1 = 0$, $\lambda_2 = 1$. The existing LS36 and LS61 in Table XII of Clatworthy [12] belong to this case. Note ([39; Theorem 8.10.1]) that there exists an L_2 design, whose solution may not be affine resolvable, with the above parameters for any β of a prime or a prime power. However, the following can be further obtained.

THEOREM 3.5.3. *The existence of an affine resolvable symmetric SRGD design with parameters*

$$v = b = n^2, \quad r = k = n, \quad \lambda_1 = 0, \quad \lambda_2 = 1, \quad q_1 = 0, \quad q_2 = 1; \quad m = n$$

is equivalent to the existence of an affine resolvable L_2 design of Case 3.5.1 *with parameters*

$$v^* = n^2,\ b^* = n(n-1),\ r^* = n-1,\ k^* = n,\ \lambda_1^* = 0,\ \lambda_2^* = 1,\ q_1^* = 0,\ q_2^* = 1.$$

PROOF. In the first resolution set of the given affine resolvable SRGD design, without loss of generality, we can put the incidence structure, by suitable permutations on rows for each of n groups of n treatments, as follows:

$$\mathbf{1}_n \otimes I_n,$$

where the GD association scheme is

$$(3.5.3)\qquad \begin{bmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & \cdots & \vdots \\ (n-1)n+1 & (n-1)n+2 & \cdots & n^2 \end{bmatrix}.$$

Now, by deleting the first resolution set $\mathbf{1}_n \otimes I_n$ of n blocks from the original affine resolvable SRGD design, it can be seen that the remaining structure forms an affine resolvable L_2 design of Case 3.5.1 with parameters $v^* = v = n^2$, $b^* = b - n = n(n-1)$, $r^* = r - 1 = n - 1$, $k^* = k = n$, $\lambda_1^* = \lambda_1$ or $\lambda_2 - 1$, $\lambda_2^* = \lambda_2$, whose association scheme is the same as in (3.5.3) by following Definition 2.5. The converse process is obvious. ∎

We should know the existence of the SRGD design in Theorem 3.5.3 as described in Theorem 3.3.2 and Remark 3.3.1. Four designs of Nos. 1, 6, 7 and 8 in Table 3.6.1 are provided by Theorem 3.5.3 with $n = 3, 7, 8$ and 9, respectively. When $n = 4$ and 5, the designs are available as LS36 and LS61.

(ii) $\ell = 2$: $\lambda_1 = 2 + 1/(\beta - 1)$ which yields $\beta = 2$. Hence we have $v = 16$, $b = 18$, $r = 9$, $k = 8$, $\lambda_1 = 3$, $\lambda_2 = 5$ whose solution is known as LS100 in Table XII of Clatworthy [12].

(iii) $\ell = 3$: $\lambda_1 = 6 + 4/(\beta - 1)$ which yields $\beta = 2, 3, 5$. When $\beta = 2$, we have $v = 36$, $b = 50$, $r = 25$, $k = 18$, $\lambda_1 = 10$, $\lambda_2 = 13$. When $\beta = 3$, we have $v = 81$, $b = 96$, $r = 32$, $k = 27$, $\lambda_1 = 8$, $\lambda_2 = 22$. When $\beta = 5$, we have $v = 225$, $b = 245$, $r = 49$, $k = 45$, $\lambda_1 = 7$, $\lambda_2 = 10$. All have r and/or $k > 20$ which are beyond the scope in Table 3.6.1.

(iv) $\ell \geq 4$: Since $r, k > 30$, the parameters are not described here.

Case 3.5.2. Affine α-resolvable L_2 designs with $\theta_1 = r + (s-2)\lambda_1 - (s-1)\lambda_2 > 0$ and $\theta_2 = r - 2\lambda_1 + \lambda_2 = 0$

In this case, it is clear that $\lambda_1 > \lambda_2$.

At first, an integral expression of q_1 is derived as in Corollary 3.5.1 for an affine α-resolvable L_2 design of Case 3.5.1.

Corollary 3.5.2. *In an affine α-resolvable L_2 design of Case* 3.5.2, $q_1 = k - r - (s-2)\lambda_1 + (s-1)\lambda_2$ *holds.*

Proof. Since $\theta_1 = r + (s-2)\lambda_1 - (s-1)\lambda_2 > 0$ and $\theta_2 = r - 2\lambda_1 + \lambda_2 = 0$, Theorem 3.3 implies the required expression. ■

Furthermore, the same result as in Corollary 3.5.1 is remarked in this case as follows.

Lemma 3.5.2 ([31]). *In an L_2 design of Case* 3.5.2, *k is divisible by s.*

In this case the following is also seen.

Theorem 3.5.4. *The parameters of an affine α-resolvable L_2 design of Case* 3.5.2 *are given by*

$$v = s^2, \qquad b = \frac{2\beta(s-1)}{\beta - 1}, \qquad r = \frac{2\alpha(s-1)}{\beta - 1}, \qquad k = \frac{\alpha s^2}{\beta},$$

$$\lambda_1 = \frac{\alpha(\alpha s + \beta s - 2\beta)}{\beta(\beta - 1)}, \qquad \lambda_2 = \frac{2\alpha(\alpha s - \beta)}{\beta(\beta - 1)}; \qquad t = \frac{2(s-1)}{\beta - 1},$$

where $\alpha s/\beta$ is an integer.

Proof. Since eigenvalues of NN' are $r + (s-2)\lambda_1 - (s-1)\lambda_2 > 0$ and $r - 2\lambda_1 + \lambda_2 = 0$ with respective multiplicities $2(s-1)$ and $(s-1)^2$, by Theorem

3.3 it holds that $b - t = 2(s-1)$, i.e., $t = 2(s-1)/(\beta-1)$. Then it follows that $v = s^2$, $b = \beta t = 2\beta(s-1)/(\beta-1)$, $r = \alpha t = 2\alpha(s-1)/(\beta-1)$, $k = vr/b = \alpha s^2/\beta$. Furthermore, from relations $r(k-1) = n_1\lambda_1 + n_2\lambda_2$ and $r - 2\lambda_1 + \lambda_2 = 0$, we obtain $\lambda_1 = \alpha(s\alpha + s\beta - 2\beta)/[\beta(\beta-1)]$ and $\lambda_2 = 2\alpha(s\alpha - \beta)/[\beta(\beta-1)]$. Also by Lemma 3.5.2, $k/s = \alpha s/\beta$ must be an integer. ■

Thus, all parameters of an affine α-resolvable L_2 design of Case 3.5.2 can be expressed in terms of s, α and β.

Note that

$$q_1 = \frac{s^2\alpha(\alpha-1)}{\beta(\beta-1)} \qquad \text{and} \qquad q_2 = \left(\frac{s\alpha}{\beta}\right)^2.$$

Next the case of $\alpha = 1$ will be investigated in detail. For an affine resolvable L_2 design of Case 3.5.2, $t = r$ and then Theorem 3.5.4 shows the design parameters as

$$v = s^2, \qquad b = \frac{2\beta(s-1)}{\beta-1}, \qquad r = \frac{2(s-1)}{\beta-1}, \qquad k = \frac{s^2}{\beta}, \qquad \lambda_1 = \frac{(s-2)\beta+s}{\beta(\beta-1)},$$

$$\lambda_2 = \frac{2(s-\beta)}{\beta(\beta-1)}, \qquad q_1 = 0, \qquad q_2 = \frac{s^2}{\beta^2}, \qquad \frac{k}{s} = \frac{s}{\beta}.$$

Then there exists a positive integer ℓ such that

$$s = \ell\beta$$

which implies that

$$(3.5.4) \quad v = (\ell\beta)^2, \qquad b = \frac{2\beta(\ell\beta-1)}{\beta-1}, \qquad r = \frac{2(\ell\beta-1)}{\beta-1}, \qquad k = \ell^2\beta,$$

$$(3.5.5) \qquad \lambda_1 = \ell + \frac{2(\ell-1)}{\beta-1}, \qquad \lambda_2 = \frac{2(\ell-1)}{\beta-1}, \qquad q_2 = \ell^2.$$

Thus all parameters of affine resolvable L_2 design of Case 3.5.2 are expressed in terms of ℓ and β. In particular, the above expression of λ_1 or λ_2 in (3.5.5) means that for given ℓ, we have a finite number of β. For example, some ℓ are investigated.

(i) $\ell = 1$: The design of this case always exists for any β as the following shows.

THEOREM 3.5.5. *There exists an affine resolvable L_2 design of Case* 3.5.2 *with parameters*

$$v = \beta^2, \quad b = 2\beta, \quad r = 2, \quad k = \beta, \quad \lambda_1 = 1, \quad \lambda_2 = 0, \quad q_1 = 0, \quad q_2 = 1.$$

PROOF. It follows that the present design can be provided by the incidence matrix as

$$[I_\beta \otimes \mathbf{1}_\beta : \mathbf{1}_\beta \otimes I_\beta].$$

Here the association scheme is given by the $\beta \times \beta$ array as

$$\begin{bmatrix} 1 & 2 & \cdots & \beta \\ \beta+1 & \beta+2 & \cdots & 2\beta \\ \vdots & \vdots & \cdots & \vdots \\ (\beta-1)\beta+1 & (\beta-1)\beta+2 & \cdots & \beta^2 \end{bmatrix}$$

which is the same structure as in (3.5.3). ■

When $\beta = 2$, a design of No. 10 in Table 3.6.2 is provided. The existing LS7, LS28, LS51, LS74, LS84, LS102, LS119 and LS137 in Table XII of Clatworthy [12] belong to this case.

(ii) $\ell = 2$: $\lambda_1 = 2 + 2/(\beta - 1)$ which yields $\beta = 2, 3$. When $\beta = 2$, we have $v = 16$, $b = 12$, $r = 6$, $k = 8$, $\lambda_1 = 4$, $\lambda_2 = 2$ a design of which exists as LS98 in Table XII of Clatworthy [12]. When $\beta = 3$, we have $v = 36$, $b = 15$, $r = 5$, $k = 12$, $\lambda_1 = 3$, $\lambda_2 = 1$ a design of which does not exist by Theorem 12.6.6 of Raghavarao [39].

(iii) $\ell = 3$: $\lambda_1 = 3 + 4/(\beta - 1)$ which yields $\beta = 2, 3, 5$. When $\beta = 2$, we have $v = 36$, $b = 20$, $r = 10$, $k = 18$, $\lambda_1 = 7$, $\lambda_2 = 4$ whose solution is unknown. When $\beta = 3$, we have $v = 81$, $b = 24$, $r = 8$, $k = 27$, $\lambda_1 = 5$, $\lambda_2 = 2$. When $\beta = 5$, we have $v = 225$, $b = 35$, $r = 7$, $k = 45$, $\lambda_1 = 4$, $\lambda_2 = 1$. The last two designs have $k > 20$ which are beyond the scope of Table 3.6.2.

(iv) $\ell \geq 4$: Since r and/or $k > 30$, the parameters are not described here.

3.6. Tables of affine resolvable L_2 designs with $v \leq 100$ and $r, k \leq 20$

According to the values of positive integers ℓ in (3.5.1), (3.5.2), (3.5.4) and (3.5.5), we now systematically search affine resolvable L_2 designs, of two cases, with admissible parameters within the scope of $v \leq 100$ and $r, k \leq 20$. (Note that in Clatworthy [12] $r, k \leq 10$.) In fact, there are 21 parameters' combinations, among of which 17 designs are existent, 3 designs do not exist, while only one case is unknown for the existence.

In Tables 3.6.1 and 3.6.2, the admissible parameters of the affine resolvable L_2 designs are listed along with existence information. The designs are numbered in the ascending order of v and for the same v in the order of b. Since $q_1 = 0$, the parameter is not listed. "Non-E" means the nonexistence of the design. Most of the existence is confirmed in Table VII of Clatworthy [12], for example, LS36, etc. Source has some information on the existence of the

Table 3.6.1. Affine resolvable L_2 designs with $r+(s-2)\lambda_1-(s-1)\lambda_2=0$

No.	v	b	r	k	λ_1	λ_2	q_2	Source	Comment
1	9	6	2	3	0	1	1	Exist	Theorem 3.5.3
2	16	12	3	4	0	1	1	LS36	Theorem 3.5.3
3	16	18	9	8	3	5	4	LS100	
4	25	20	4	5	0	1	1	LS61	Theorem 3.5.3
5	36	30	5	6	0	1	1	Non-E	
6	49	42	6	7	0	1	1	Exist	Theorem 3.5.3
7	64	56	7	8	0	1	1	Exist	Theorem 3.5.3
8	81	72	8	9	0	1	1	Exist	Theorem 3.5.3
9	100	90	9	10	0	1	1	Non-E	

Table 3.6.2. Affine resolvable L_2 designs with $r-2\lambda_1+\lambda_2=0$

No.	v	b	r	k	λ_1	λ_2	q_2	Source	Comment
10	4	4	2	2	1	0	1	Exist	Theorem 3.5.5
11	9	6	2	3	1	0	1	LS7	Theorem 3.5.5
12	16	8	2	4	1	0	1	LS28	Theorem 3.5.5
13	16	12	6	8	4	2	4	LS98	
14	25	10	2	5	1	0	1	LS51	Theorem 3.5.5
15	36	12	2	6	1	0	1	LS74	Theorem 3.5.5
16	36	15	5	12	3	1	4	Non-E	
17	36	20	10	18	7	4	9	?	
18	49	14	2	7	1	0	1	LS84	Theorem 3.5.5
19	64	16	2	8	1	0	1	LS102	Theorem 3.5.5
20	81	18	2	9	1	0	1	LS119	Theorem 3.5.5
21	100	20	2	10	1	0	1	LS137	Theorem 3.5.5

corresponding affine resolvable L_2 design (cf. [12]). Comment shows theorems on the construction. The existence of designs of Nos. 1, 6, 7 and 8 is newly shown by Theorem 3.5.3. It is also shown by Theorems 12.6.5 and 12.6.6 of Raghavarao [39] that two designs of Nos. 5 and 16 do not exist. The nonexistence of a design of No. 9 is shown by Theorem 3.5.3 with Remark 3.3.1. A design of No. 10 is newly listed by Theorem 3.5.5.

4. Bounds in affine resolvable PBIB designs

A simple comparison between the number of treatments v and the number of blocks b will be made. As mentioned in Section 2, Fisher's inequality $b \geq v$ holds for a BIB design, but it is not always valid in a PBIB design.

The following results are well known (cf. [39]): (i) In a regular GD design $b \geq v$ holds. (ii) In an SGD design with $v = mn$, $b \geq m$ holds. (iii) In an

SRGD design with $v = mn$, $b \geq v - (m-1)$ holds. (iv) In an L_2 design with $v = s^2$, $\theta_1 = r + (s-2)\lambda_1 - (s-1)\lambda_2$ and $\theta_2 = r - 2\lambda_1 + \lambda_2$, (iv-1) when $\theta_1 > 0$ and $\theta_2 > 0$, $b \geq v$ holds, (iv-2) when $\theta_1 > 0$ and $\theta_2 = 0$, $b \geq v - (s-1)^2$ holds, and (iv-3) when $\theta_1 = 0$ and $\theta_2 > 0$, $b \geq v - 2(s-1)$ holds. Thus, for the incidence matrix N of a block design, if one of eigenvalues of NN' is zero, then an inequality $b \geq v$ may not hold in general. Through the property of affine resolvability, this inequality will be examined as in Theorem 4.1.

By Theorem 3.3, we can see some relations on v and b through other parameters in an affine α-resolvable 2-associate PBIB design. Even so, a property of the affine resolvability shows the following as a simple comparison between v and b only.

Theorem 4.1. *In affine resolvable PBIB designs, it holds that*
(1) *for an SGD design, $b < v$;*
(2) *for an SRGD design, $b \geq v$;*
(3) *for an L_2 design with $\theta_1 > 0$ and $\theta_2 = 0$, $b < v$.*

Proof. (1) It follows that $b = r + m - 1 = (m-1)/(\beta - 1) + m - 1 = [1 + 1/(\beta - 1)](m-1) < 2(m-1) \leq n(m-1) < nm = v$. (2) Since $\lambda_1 = r - k \geq 0$, $r \geq k$ and hence $b = \beta r \geq \beta k = v$. (3) Since $b = v - 1 - (s-1)^2 + r$, $v - b = s^2 - 2s + 2 - 2(s-1)/(\beta - 1) = s^2 - 2(s-1)[1 + 1/(\beta - 1)] \geq s^2 - 4(s-1) = (s-2)^2 \geq 0$. ■

Note that (2) in Theorem 4.1 is interesting in the sense that one of eigenvalues is zero and further Fisher inequality holds. Also note that in an affine resolvable L_2 design with $\theta_1 = 0$ and $\theta_2 > 0$, two cases $b < v$ or $b > v$ hold. Both such examples exist. For example, see the existing LS51, LS61 and LS100 in Table XII of Clatworthy [12].

The argument made in this section is motivated by the discussion given in Kadowaki and Kageyama [18, 19, 20].

5. Conclusions

We show the usefulness of number-theoretic approach to investigate combinatorial structure of affine α-resolvable PBIB designs. Usually, this kind of approach may not yield much results in design theory. However, as far as a property of the affine α-resolvability is concerned, the approach is powerful. Of course this does not solve the problem completely. We may require other combinatorial consideration. If we restrict ourselves to $\alpha = 1$, i.e., affine resolvability, then we could get more concise results on existence. Within the practical range of parameters, it reveals that there are not many such designs as in tables given in Sections 3.2, 3.4 and 3.6. In fact, we cannot find many new series of such PBIB designs other than ones in Theorems 3.3.2, 3.3.3, 3.3.4 and 3.5.5, except for designs constructed by use of the result that the complement of an affine resolvable block design is an affine α-resolvable block design for some α. Theorems 3.3.4 and 3.5.3 have some potential to produce many affine resolvable designs.

As a practical investigation (i.e., $v \leq 100$ and $r, k \leq 20$) of affine resolvable SGD, SRGD, L_2, triangular and cyclic designs, only six designs are left unknown on existence (i.e., five SRGD designs, one L_2 design).

References

[1] R. A. Bailey, H. Monod and J. P. Morgan, Construction and optimality of affine-resolvable designs, Biometrika **82** (1995), 187–200.

[2] S. Banerjee and S. Kageyama, Existence of α-resolvable nested incomplete block designs, Utilitas Math. **38** (1990), 237–243.

[3] T. Beth, D. Jungnickel and H. Lenz, Design Theory: Volume 1, 2nd ed., Cambridge Univ. Press, UK, 1999.

[4] R. C. Bose, A note on the resolvability of balanced incomplete block designs, Sankhyā A **6** (1942), 105–110.

[5] R. C. Bose, Symmetric group divisible designs with the dual property, J. Statist. Plann. Inference **1** (1977), 87–101.

[6] R. C. Bose and W. S. Connor, Combinatorial properties of group divisible incomplete block designs, Ann. Math. Statist. **23** (1952), 367–383.

[7] R. C. Bose and T. Shimamoto, Classification and analysis of partially balanced incomplete block designs with two associate classes, J. Amer. Statist. Assoc. **47** (1952), 151–184.

[8] R. C. Bose, S. S. Shrikhande and K. N. Bhattacharya, On the construction of group divisible incomplete block designs, Ann. Math. Statist. **24** (1953), 167–195.

[9] T. Caliński and S. Kageyama, Block Designs: A Randomization Approach, Vol. I: Analysis, Lecture Notes in Statistics 150, Springer, New York, 2000.

[10] T. Caliński and S. Kageyama, Block Designs: A Randomization Approach, Vol. II: Design, Lecture Notes in Statistics 170, Springer, New York, 2003.

[11] T. Caliński and S. Kageyama, On the analysis of experiments in affine resolvable designs, J. Statist. Plann. Inference **138** (2008), 3350–3356.

[12] W. H. Clatworthy, Tables of Two-Associate-Class Partially Balanced Designs, NBS Applied Mathematics Series 63, U.S. Department of Commerce, National Bureau of Standards, Washington, D.C., 1973.

[13] C. J. Colbourn and J. H. Dinitz (editors), Handbook of Combinatorial Designs, Second Edition, Chapman & Hall/CRC, New York, 2007.

[14] S. Furino, Y. Miao and J. Yin, Frames and Resolvable Designs, CRC Press, Boca Raton FL, 1996.

[15] D. K. Ghosh, G. C. Bhimani and S. Kageyama, Resolvable semi-regular group divisible designs, J. Japan Statist. Soc. **19** (1989), 163–165.

[16] M. Hall, Jr., Combinatorial Theory, 2nd ed., John Wiley, 1986.

[17] A. S. Hedayat, N. J. A. Sloane and J. Stufken, Orthogonal Arrays: Theory and Applications, Springer, New York, 1999.

[18] S. Kadowaki and S. Kageyama, An inequality on α-resolvable balanced incomplete block designs, Bulletin of the ICA. **37** (2003), 51–57.

[19] S. Kadowaki and S. Kageyama, A 2-resolvable BIBD$(10, 15, 6, 4, 2)$ does not exist, Bulletin of the ICA. **53** (2008), 87–98.

[20] S. Kadowaki and S. Kageyama, On bounds of the number of blocks in α-resolvable BIB designs, J. Stat. & Appl. **3** (2008), 23–31.

[21] S. Kageyama, A survey of resolvable solutions of balanced incomplete block designs, Internat. Statist. Rev. **40** (1972), 269–273.

[22] S. Kageyama, On μ-resolvable and affine μ-resolvable balanced incomplete block designs, Ann. Statist. **1** (1973), 195–203.

[23] S. Kageyama, Resolvability of block designs, Ann. Statist. **4** (1976), 655–661.

[24] S. Kageyama, Conditions for α-resolvability and affine α-resolvability of incomplete block designs, J. Japan Statist. Soc. **7** (1977), 19–25.

[25] S. Kageyama, On non-existence of affine α-resolvable triangular designs, J. Statist. Theor. Practice **1** (2007), 291–298.

[26] S. Kageyama, Non-validity of affine α-resolvability in regular group divisible designs, J. Statist. Plann. Inference **138** (2008), 3295–3298.

[27] S. Kageyama, Non-existence of affine α-resolvable triangular designs under $1 \leq \alpha \leq 10$, J. Indian Soc. Agri. Statist. **62** (2008), 132–137.

[28] S. Kageyama, S. Banerjee and A. Verma, A construction of self complementary semi-regular group divisible designs, Sankhyā B **51** (1989), 335–338.

[29] S. Kageyama and Y. Miao, A construction for resolvable designs and its generalizations, Graph and Combin. **14** (1998), 11–24.

[30] S. Kageyama and R. N. Mohan, Construction of α-resolvable PBIB designs, Calcutta Statist. Assoc. Bull. **34** (1985), 221–224.

[31] S. Kageyama and T. Tsuji, Characterization of certain incomplete block designs, J. Statist. Plann. Inference **1** (1977), 151–161.

[32] H. Kharaghani and B. Tayfeh-Rezaie, A Hadamard matrix of order 428, J. Combin. Designs **13** (2005), 435–440.

[33] T. P. Kirkman, Note on an unanswered prize question, Cambridge Dublin Math. J. **5** (1950), 191–204.

[34] C. W. H. Lam, L. Thiel and S. Swiercz, The non-existence of finite projective planes of order 10, Can. J. Math. **XLI** (1989), 1117–1123.

[35] S. Lang, Introduction to Linear Algebra, 2nd ed., Springer, New York, 1986.

[36] S. L. Ma, Partial difference sets, Discrete Math. **52** (1984), 75–89.

[37] R. N. Mohan and S. Kageyama, Two constructions of α-resolvable PBIB designs, Utilitas Math. **36** (1989), 115–119.

[38] Y. Qian, Z. Meng and B. Du, α-Resolvable group divisible designs with block size four and group size three, Utilitas Math. **77** (2008), 201–224.

[39] D. Raghavarao, Constructions and Combinatorial Problems in Design of Experiments, Dover, New York, 1988.

[40] S. S. Shrikhande, Affine resolvable balanced incomplete block designs: a survey, Aequationae Math. **14** (1976), 251–269.

[41] S. S. Shrikhande and D. Raghavarao, A method of construction of incomplete block designs, Sankhyā A **25** (1963), 399–402.

[42] S. S. Shrikhande and D. Raghavarao, Affine α-resolvable incomplete block designs, Contributions to Statistics, Volume presented to Professor P. C. Mahalanobis on his 75th birthday, Pergamon Press, Oxford and Statistical Publishing Society, Calcutta, 1964, 471–480.

[43] K. Takeuchi, A table of difference sets generating balanced incomplete block designs, Rev. Inst. Internat. Statist. **30:3** (1962), 361–366.

[44] T. M. J. Vasiga, S. Furino and A. C. H. Ling, The spectrum of α-resolvable designs with block size four, J. Combin. Designs **9** (2001), 1–16.

[45] F. Yates, The recovery of inter-block information in variety trials arranged in three-dimensional lattices, Ann. Eugen. **9** (1939), 136–156.

[46] F. Yates, The recovery of inter-block information in balanced incomplete block designs, Ann. Eugen. **10** (1940), 317–325.

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