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# Eigenvalues of generalized Laplacians for generalized Poisson-Cauchy transforms on classical domains

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**Abstract.** We develop a group-theoretic method to generalize the Laplace-Beltrami operators on the classical domains. In [11], inspired by Helgason's paper [3], we defined the "Poisson transforms" for homogeneous vector bundles over symmetric spaces. In [13], we defined the generalized Poisson-Cauchy transforms for homogeneous holomorphic line bundles over hermitian symmetric spaces and computed explicitly the kernel functions for each type of the classical domains. In [7], making use of the Casimir operator, we defined the "generalized Laplacians" on homogeneous holomorphic line bundles over hermitian symmetric spaces and showed that the generalized Poisson-Cauchy transforms give rise to eigenfunctions of the "generalized Laplacians". In this paper, using the canonical coordinates for each type of the classical domains, we carry out the direct computation to obtain the explicit formulas of (line bundle valued) invariant differential operators which we call the generalized Laplacians and compute their eigenvalues evaluated at the generalized Poisson-Cauchy kernel functions.

## 1. Introduction

We denote by $\boldsymbol{D}_I$ (resp. $\boldsymbol{D}_{II}$, $\boldsymbol{D}_{III}$, $\boldsymbol{D}_{IV}$) the classical domain of type $I$ (resp. $II$, $III$, $IV$).

In [6], Hua gave the explicit formula of the Laplace-Beltrami operator for $\boldsymbol{D}_I$. In [13], we generalized some results by Hua so that we followed his classification about the classical domains. In this paper, we use the classification given in [15], which means that we exchange the definition of the type II and that of the type III.

Following Hua's idea, it is straightforward to compute the Laplace-Beltrami operators for $\boldsymbol{D}_{II}$ and $\boldsymbol{D}_{III}$. In the case of the type IV, however, it is quite difficult to pursue Hua's method.

In this paper, we give another method to obtain the explicit formulas of the Laplace-Beltrami operators which can be carried out also for $\boldsymbol{D}_{IV}$. We use

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the canonical coordinate system defined by the Harish-Chandra decomposition. This leads us to a certain "canonical riemannian metric" (see Section 3). It turns out that Hua's differential operator is the Laplace-Beltrami operator defined by this metric.

Looking closely at the explicit formula of the Laplace-Beltrami operator, we found a way to modify the Laplace-Beltrami operator in order to obtain the line bundle valued invariant differential operator which we call the generalized Laplacian. (See Section 3 for the precise definition of generalized Laplacians).

In [13], we gave the explicit formulas of the generalized Poisson-Cauchy kernel functions. We compute explicitly the value of the generalized Laplacian evaluated at the generalized Poisson-Cauchy kernel function at the origin, which, owing to the invariance by the group action, gives us the eigenvalue of the generalized Laplacian.

The "generalized Laplacians" defined in [7] coincide, up to a constant factor and scalar operators, with the generalized Laplacians defined in this paper. Making use of the Casimir operator, we give another method to compute the above mentioned eigenvalues.

It is easy to see that Theorem 3 in this paper still holds for (line bundle valued) Sato's hyperfunctions on the Shilov boundary. The image of the generalized Poisson-Cauchy transform of Sato's hyperfunctions on the Shilov boundary do not exhaust all eigenfunctions of the generalized Laplacian (see [8]). It is an interesting problem to give the characterization of this image. Another interesting problem is to generalize the results of this paper to the case of vector bundles (see [14]).

In [13], we followed [15] to obtain the explicit formula of the action of the group $SO_0(n,2)$ on $\boldsymbol{D}_{IV}$. In this paper we use the Harish-Chandra decomposition to get the explicit formula of the group action.

We gave the proof of each theorem (Theorem 1, 2, 4 and 5) in the separate subsections for each type of classical domains. We showed each step of the proof in detail but omitted the elementary calculus of matrices. (For details see [16].)

There are some misprints in [13]. We correct them in this paper.

## 2. Definitions and notation

Let $G$ be a connected non-compact Lie group admitting a finite dimensional faithful representation. Let $K$ be a maximal compact subgroup of $G$. We assume that $G/K$ is an irreducible hermitian symmetric space. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$, respectively. Let $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$. We put

$$\mathfrak{p} = \{Y \in \mathfrak{g}; B(X, Y) = 0 \text{ for all } X \in \mathfrak{k}\},$$

where $B$ denotes the Killing form of the Lie algebra $\mathfrak{g}_c$. Then we have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \qquad \mathfrak{k} \cap \mathfrak{p} = \{0\}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}.$$

For any subset $\mathfrak{s}$ of $\mathfrak{g}$ we denote by $\mathfrak{s}_c$ the complex subspace of $\mathfrak{g}_c$ spanned by $\mathfrak{s}$. Since $G/K$ is hermitian symmetric, there exist abelian subalgebras $\mathfrak{p}_+$ and $\mathfrak{p}_-$ of $\mathfrak{g}_c$ such that

$$\mathfrak{p}_c = \mathfrak{p}_+ + \mathfrak{p}_-, \quad \mathfrak{p}_+ \cap \mathfrak{p}_- = \{0\}, \quad \mathfrak{p}_+ = \bar{\mathfrak{p}}_-, \quad [\mathfrak{k}_c, \mathfrak{p}_+] \subset \mathfrak{p}_+, \quad [\mathfrak{k}_c, \mathfrak{p}_-] \subset \mathfrak{p}_-.$$

Let $G_c$ be the complexification of $G$ with Lie algebra $\mathfrak{g}_c$. We denote by $K_c$ (resp. $P_+$, $P_-$) the complex analytic subgroup of $G_c$ corresponding to $\mathfrak{k}_c$ (resp. $\mathfrak{p}_+$, $\mathfrak{p}_-$). Then $P_+K_cP_-$ is an open subset of $G_c$, and any element $w \in P_+K_cP_-$ is uniquely expressed as $w = p_+k_cp_-$ $(p_+ \in P_+, k_c \in K_c, p_- \in P_-)$. This is called the Harish-Chandra decomposition (see [1]). Put $U = K_cP_-$. Then $U$ is a complex analytic subgroup of $G_c$ and $P_-$ a normal subgroup of $U$. Consider the complex homogeneous space $G_c/U$. Then $G/K$ can be canonically identified with an open submanifold $GU/U$ of $G_c/U$ which is the $G$-orbit of the point $U$ in $G_c/U$.

We introduce an inner product $(\cdot\,,\cdot)$ on the complex vector space $\mathfrak{p}_+$ by

$$(z_1, z_2) = B(z_1, \overline{z_2}) \qquad (z_1, z_2 \in \mathfrak{p}_+).$$

In [1], it was proved by Harish-Chandra that $GU \subset P_+U$ and that there exists a unique bounded domain $\boldsymbol{D}$ in $\mathfrak{p}_+$ such that $GU = (\exp \boldsymbol{D})U$. For any $w$ in $P_+U$, we denote by $z(w)$ (resp. $u(w)$) the unique element of $\mathfrak{p}_+$ (resp. $U$) such that $w = (\exp z(w))u(w)$. For any $g \in G$ and $z \in \boldsymbol{D}$, we denote by $g[z]$ the unique element of $\boldsymbol{D}$ such that $g(\exp z)U = (\exp g[z])U$. Then $G$ acts on $G/K \cong GU/U \cong \boldsymbol{D}$ by the commutative diagram: for any $g_1 \in G$,

$$\begin{array}{ccccc} G/K & \cong & GU/U & \cong & \boldsymbol{D} \\ \Cup & & \Cup & & \Cup \\ gK & \mapsto & gU & \mapsto & g[0] = z \\ \downarrow & & \downarrow & & \downarrow \\ g_1gK & \mapsto & g_1gU & \mapsto & g_1g[0] = g_1[z] \\ \Cap & & \Cap & & \Cap \\ G/K & \cong & GU/U & \cong & \boldsymbol{D}. \end{array}$$

We fix a point $\mu U \in G_c/U$ such that $\mu U$ belongs to the boundary of $GU/U$ and that the $G$-orbit of $\mu U$ is compact. Then the isotropy subgroup at the point $\mu U$ of $G_c/U$ is a maximal parabolic subgroup of $G$, which we denote by $P$. Put $u_0 = z(\mu)$ and $\mu_0 = \exp u_0$. Then clearly we get $\mu U = \mu_0 U = (\exp u_0)U$ which implies that $G \cap \mu U \mu^{-1} = G \cap \mu_0 U \mu_0^{-1} = P$. Put

$$\check{\boldsymbol{S}} = \{u \in \mathfrak{p}_+ ; (\exp u)U \in G\mu_0 U\}.$$

Then $\check{\boldsymbol{S}}$ is the Shilov boundary of $\boldsymbol{D}$. For any $g \in G$ and $u \in \check{\boldsymbol{S}}$, we denote by $g[u]$ a unique element of $\check{\boldsymbol{S}}$ such that $g(\exp u)U = (\exp g[u])U$. Then $G$ acts on $G/P \cong G\mu_0 U/U \cong \check{\boldsymbol{S}}$ by the commutative diagram: for any $g_1 \in G$,

$$\begin{array}{ccccc} G/P & \cong & G\mu_0 U/U & \cong & \check{\boldsymbol{S}} \\ \Cup & & \Cup & & \Cup \\ gP & \mapsto & g\mu_0 U & \mapsto & g[u_0] = u \\ \downarrow & & \downarrow & & \downarrow \\ g_1 gP & \mapsto & g_1 g\mu_0 U & \mapsto & g_1 g[u_0] = g_1[u] \\ \Cap & & \Cap & & \Cap \\ G/P & \cong & G\mu_0 U/U & \cong & \check{\boldsymbol{S}}. \end{array}$$

For any representation $\pi$ of a Lie group, we denote by $d\pi$ the differential representation of $\pi$.

Since $G/K$ is hermitian symmetric, $\mathfrak{k}$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. For each linear form $\lambda$ on $\mathfrak{h}_c$, we denote by $H_\lambda$ the element of $\mathfrak{h}_c$ such that $B(H_\lambda, H) = \lambda(H)$ for all $H \in \mathfrak{h}_c$. For any two linear forms $\lambda$, $\mu$ we define $\langle \lambda, \mu \rangle = B(H_\lambda, H_\mu)$. Let $\Delta$ be the set of all non-zero roots of the pair $(\mathfrak{g}_c, \mathfrak{h}_c)$. For each $\alpha \in \Delta$ we choose a root vector $X_\alpha$ associated with the root $\alpha$. It is clear that for any $\alpha \in \Delta$ $X_\alpha$ belongs to either $\mathfrak{k}_c$ or $\mathfrak{p}_c$. A root $\alpha \in \Delta$ is called compact or non-compact according to $X_\alpha \in \mathfrak{k}_c$ or $X_\alpha \in \mathfrak{p}_c$, respectively. We can choose a linear order on $\Delta$ such that if $X_\alpha \in \mathfrak{p}_+$ then $\alpha$ is positive. We denote by $\rho$ the half sum of all positive roots. Let $\Lambda$ be an integral form. We assume that $\langle \Lambda, \alpha \rangle = 0$ for all compact roots $\alpha$. Then there exists a unique character $\tau$ of $K$ such that $d\tau(H) = \Lambda(H)$ for all $H \in \mathfrak{h}$. Moreover $\tau$ is uniquely extended to a holomorphic character of $U$ which is trivial on $P_-$.

We regard the complex Lie group $G_c$ as the principal fiber bundle over the complex homogeneous space $G_c/U$. We denote by $\tilde{\boldsymbol{E}}_\tau$ the holomorphic line bundle over $G_c/U$ associated to $\tau$. We denote by $\boldsymbol{E}_\tau$ the restriction of $\tilde{\boldsymbol{E}}_\tau$ to the open submanifold $G/K \cong GU/U$ of $G_c/U$. Then the space of all $\boldsymbol{C}^\infty$-sections of $\boldsymbol{E}_\tau$ is identified with

$$\boldsymbol{C}^\infty(\boldsymbol{E}_\tau) = \{h \in \boldsymbol{C}^\infty(GU); h(wu) = \tau(u)^{-1}h(w) \ \ (w \in GU, u \in U)\}.$$

Let $\eta$ be a $\boldsymbol{C}^\infty$-character of $U$ such that the restriction of $\eta$ to $K$ coincides with $\tau$. We denote by $\tilde{\boldsymbol{L}}_\eta$ the $\boldsymbol{C}^\infty$-line bundle on $G_c/U$ associated to $\eta$. We denote by $\boldsymbol{L}_\eta$ the restriction of $\tilde{\boldsymbol{L}}_\eta$ to the compact submanifold $G/P \cong G\mu_0 U/U$ of $G_c/U$. Then the space of all $\boldsymbol{C}^\infty$-sections of $\boldsymbol{L}_\eta$ is identified with

$$\boldsymbol{C}^\infty(\boldsymbol{L}_\eta) = \{h \in \boldsymbol{C}^\infty(G\mu_0 U); h(wu) = \eta(u)^{-1}h(w) \ \ (w \in G\mu_0 U, u \in U)\}.$$

We define a $\boldsymbol{C}^\infty$-character $\xi$ of $P$ by

$$\xi(p) = \eta(\mu_0^{-1} p \mu_0) \qquad (p \in P).$$

Put

$$\boldsymbol{C}^{\infty}(G)_{\tau} = \{f \in \boldsymbol{C}^{\infty}(G); f(gk) = \tau(k)^{-1} f(g) \ \ (g \in G, k \in K)\},$$

$$\boldsymbol{C}^{\infty}(G)_{\xi} = \{\phi \in \boldsymbol{C}^{\infty}(G); \phi(gp) = \xi(p)^{-1} \phi(g) \ \ (g \in G, p \in P)\}.$$

Then we obtain the following four onto-isomorphisms:

$$\boldsymbol{C}^{\infty}(\boldsymbol{E}_{\tau}) \ni h \mapsto f \in \boldsymbol{C}^{\infty}(G)_{\tau}, \qquad f(g) = h(g) \quad (g \in G),$$

$$\boldsymbol{C}^{\infty}(\boldsymbol{E}_{\tau}) \ni h \mapsto F \in \boldsymbol{C}^{\infty}(\boldsymbol{D}), \qquad F(z) = h(\exp z) \quad (z \in \boldsymbol{D}),$$

$$\boldsymbol{C}^{\infty}(\boldsymbol{L}_{\eta}) \ni \psi \mapsto \phi \in \boldsymbol{C}^{\infty}(G)_{\xi}, \qquad \phi(g) = \psi(g\mu_0) \quad (g \in G),$$

$$\boldsymbol{C}^{\infty}(\boldsymbol{L}_{\eta}) \ni \psi \mapsto \Phi \in \boldsymbol{C}^{\infty}(\check{\boldsymbol{S}}), \qquad \Phi(u) = \psi(\exp u) \quad (u \in \check{\boldsymbol{S}}).$$

For any $g \in G$ and $h \in \boldsymbol{C}^{\infty}(\boldsymbol{E}_{\tau})$, we define

$$(\pi_{\tau}(g)h)(w) = h(g^{-1}w) \qquad (w \in GU).$$

Then $\pi_{\tau}$ is a representation of $G$ on $\boldsymbol{C}^{\infty}(\boldsymbol{E}_{\tau})$.

For any $g \in G$ and $\psi \in \boldsymbol{C}^{\infty}(\boldsymbol{L}_{\eta})$, we define

$$(\pi_{\eta}(g)\psi)(w) = \psi(g^{-1}w) \qquad (w \in G\mu_0 U).$$

Then $\pi_{\eta}$ is a representation of $G$ on $\boldsymbol{C}^{\infty}(\boldsymbol{L}_{\eta})$.

For any $g \in G$, we define $T_{\tau}(g)$ such that the following diagram is commutative.

$$\begin{array}{ccccc} & \boldsymbol{C}^{\infty}(\boldsymbol{E}_{\tau}) & \cong & \boldsymbol{C}^{\infty}(\boldsymbol{D}) & \\ \pi_{\tau}(g) & \downarrow & & \downarrow & T_{\tau}(g) \\ & \boldsymbol{C}^{\infty}(\boldsymbol{E}_{\tau}) & \cong & \boldsymbol{C}^{\infty}(\boldsymbol{D}). & \end{array}$$

Then we have the following lemma.

Lemma 1. *For any* $g \in G$ *and* $F \in \boldsymbol{C}^{\infty}(\boldsymbol{D})$

$$(T_{\tau}(g)F)(z) = \rho_{\tau}(g,z)^{-1} F(g^{-1}[z]) \qquad (z \in \boldsymbol{D}),$$

*where* $\rho_{\tau}(g,z) = \tau(u(g^{-1} \exp z))$.

Suppose now that for each $\tau$, we have a differential operator $D_{\tau}$ on $\boldsymbol{D}$. We denote by $\widehat{D_{\tau}}$ the $\boldsymbol{E}_{\tau}$-valued differential operator such that the following diagram is commutative.

$$\begin{array}{ccccc} & \boldsymbol{C}^{\infty}(\boldsymbol{E}_{\tau}) & \cong & \boldsymbol{C}^{\infty}(\boldsymbol{D}) & \\ \widehat{D_{\tau}} & \downarrow & & \downarrow & D_{\tau} \\ & \boldsymbol{C}^{\infty}(\boldsymbol{E}_{\tau}) & \cong & \boldsymbol{C}^{\infty}(\boldsymbol{D}). & \end{array}$$

If $\widehat{D_{\tau}}$ commutes with $\pi_{\tau}(g)$ for all $g \in G$, we call $\widehat{D_{\tau}}$ an $\boldsymbol{E}_{\tau}$-valued invariant differential operator. It is easy to see that this is the case if and only if $D_{\tau}$

commutes with $T_\tau(g)$ for all $g \in G$. By abuse of the language, we shall call also such a differential operator $D_\tau$ an $\boldsymbol{E}_\tau$-valued invariant differential operator.

For any vector space $V$, we denote by $\boldsymbol{I}_V$ the identity operator on $V$. We use the following notation which are defined in [13].

$\boldsymbol{N}$, $\boldsymbol{Z}$, $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{C}^*$, $\boldsymbol{R}^k$, $\boldsymbol{C}^k$, $M_k(\boldsymbol{R})$, $M_k(\boldsymbol{C})$, $GL(k,\boldsymbol{C})$, $SL(k,\boldsymbol{C})$, $SU(n,m)$, $SO(k)$, $SO(k,\boldsymbol{C})$, $Sp(k,\boldsymbol{C})$, $\mathfrak{sl}(k,\boldsymbol{C})$, $\mathfrak{o}(k,\boldsymbol{C})$, $I_k$, $M_{p,q}(\boldsymbol{R})$, $M_{p,q}(\boldsymbol{C})$, $I_{p,q}$.

**2.1. Type I.** Fix any $n, m \in \boldsymbol{N}$ such that $n \geq m$. The classical domain of type I is defined by

$$\boldsymbol{D}_I = \{z \in M_{n,m}(\boldsymbol{C}); z^*z \ll I_m\}.$$

The Shilov boundary of $\boldsymbol{D}_I$ is given by

$$\check{\boldsymbol{S}}_I = \{u \in M_{n,m}(\boldsymbol{C}); u^*u = I_m\}.$$

We define

$$G_c = SL(m+n, \boldsymbol{C}),$$

$$G = SU(n,m),$$

$$K_c = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in SL(m+n,\boldsymbol{C}); \alpha \in GL(n,\boldsymbol{C}), \delta \in GL(m,\boldsymbol{C}) \right\},$$

$$P_+ = \left\{ \begin{pmatrix} I_n & z \\ 0 & I_m \end{pmatrix}; z \in M_{n,m}(\boldsymbol{C}) \right\},$$

$$P_- = \left\{ \begin{pmatrix} I_n & 0 \\ \zeta & I_m \end{pmatrix}; \zeta \in M_{m,n}(\boldsymbol{C}) \right\},$$

$$U = K_c P_-,$$

$$\mathfrak{g}_c = \mathfrak{sl}(m+n, \boldsymbol{C}),$$

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \in \mathfrak{g}_c; \begin{array}{l} a^* = -a, \\ d^* = -d, \end{array} \begin{array}{l} a \in M_n(\boldsymbol{C}) \\ b \in M_{n,m}(\boldsymbol{C}) \\ d \in M_m(\boldsymbol{C}) \end{array} \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathfrak{g}; \begin{array}{ll} a^* = -a, & a \in M_n(\boldsymbol{C}) \\ d^* = -d, & d \in M_m(\boldsymbol{C}) \end{array} \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}; b \in M_{n,m}(\boldsymbol{C}) \right\},$$

$$\mathfrak{p}_+ = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}; z \in M_{n,m}(\boldsymbol{C}) \right\}.$$

The Killing form $B$ is given by

$$B(X, Y) = 2(n+m)\,\mathrm{Tr}(XY) \qquad (X, Y \in \mathfrak{g}_c).$$

For any $\ell \in \boldsymbol{Z}$ and $s \in \boldsymbol{C}$ we define characters $\tau_\ell$, $\eta_{\ell,s}$ and $\xi_{\ell,s}$ by

$$\tau_\ell : U \ni \begin{pmatrix} \alpha & 0 \\ \zeta & \delta \end{pmatrix} \mapsto (\det(\delta))^\ell \in \boldsymbol{C}^*,$$

$$\eta_{\ell,s} : U \ni \begin{pmatrix} \alpha & 0 \\ \zeta & \delta \end{pmatrix} \mapsto \left(\frac{\det(\delta)}{|\det(\delta)|}\right)^\ell |\det(\delta)|^s \in \boldsymbol{C}^*,$$

$$\xi_{\ell,s} : P \ni p \mapsto \eta_{\ell,s}(\mu_0^{-1} p \mu_0) \in \boldsymbol{C}^*.$$

We identify $\mathfrak{p}_+$ with $M_{n,m}(\boldsymbol{C})$ by:

$$\begin{array}{ccc} \mathfrak{p}_+ & \cong & M_{n,m}(\boldsymbol{C}) \supset \boldsymbol{D}_I \\ \Downarrow & & \Downarrow \\ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} & \mapsto & z. \end{array}$$

For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z \in \boldsymbol{D}_I$, we define $g[z]$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \exp\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} U = \begin{pmatrix} a & az+b \\ c & cz+d \end{pmatrix} U = \exp\begin{pmatrix} 0 & g[z] \\ 0 & 0 \end{pmatrix} U.$$

Notice that

$$(zb^* + a^*)(az+b) = (zd^* + c^*)(cz+d).$$

Then we have the following lemma.

Lemma 2. *For any $g \in G$ and $z \in \boldsymbol{D}_I$, we have*

$$g[z] = (az+b)(cz+d)^{-1} = (zb^* + a^*)^{-1}(zd^* + c^*) \qquad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right),$$

$$(T_{\tau_\ell}(g)F)(z) = (\det(cz+d))^{-\ell} F(g^{-1}[z]) \qquad \left(g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

**2.2. Type II.** Fix any $n \in \boldsymbol{N}$ $(n > 1)$. The classical domain of type II is defined by

$$\boldsymbol{D}_{II} = \{z \in M_n(\boldsymbol{C}); z^*z \ll I_n, {}^t z = -z\}.$$

The Shilov boundary of $\boldsymbol{D}_{II}$ is given as follows.

In case $n$ is even,

$$\check{\boldsymbol{S}}_{II} = \{u \in M_n(\boldsymbol{C}); u^*u = I_n, {}^t u = -u\}.$$

In case $n$ is odd,

$$\check{\boldsymbol{S}}_{II} = \{u \in M_n(\boldsymbol{C}); \mathrm{rank}(I_n - u^*u) = 1, {}^t u = -u\}.$$

We define

$$G_c = \left\{ g \in SL(2n, \boldsymbol{C}); {}^t g \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \right\},$$

$$G = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(n,n); a, b \in M_n(\boldsymbol{C}) \right\},$$

$$K_c = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{pmatrix} \in SL(2n, \boldsymbol{C}); \alpha \in GL(n, \boldsymbol{C}) \right\},$$

$$P_+ = \left\{ \begin{pmatrix} I_n & z \\ 0 & I_n \end{pmatrix}; {}^t z = -z \ \ (z \in M_n(\boldsymbol{C})) \right\},$$

$$P_- = \left\{ \begin{pmatrix} I_n & 0 \\ \zeta & I_n \end{pmatrix}; {}^t \zeta = -\zeta \ \ (\zeta \in M_n(\boldsymbol{C})) \right\},$$

$$U = K_c P_-,$$

$$\mathfrak{g}_c = \left\{ \begin{pmatrix} a & b \\ c & -{}^t a \end{pmatrix}; \begin{matrix} {}^t b = -b, \\ {}^t c = -c \end{matrix} \ \ (a, b, c \in M_n(\boldsymbol{C})) \right\},$$

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathfrak{g}_c; a^* = -a, {}^t b = -b \ \ (a, b \in M_n(\boldsymbol{C})) \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \in \mathfrak{g}; a^* = -a \ \ (a \in M_n(\boldsymbol{C})) \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix}; {}^t b = -b \ \ (b \in M_n(\boldsymbol{C})) \right\},$$

$$\mathfrak{p}_+ = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}; {}^t z = -z \ \ (z \in M_n(\boldsymbol{C})) \right\}.$$

The Killing form $B$ is given by

$$B(X, Y) = 2(n-1)\,\mathrm{Tr}(XY) \qquad (X, Y \in \mathfrak{g}_c).$$

For any $\ell \in \boldsymbol{Z}$ and $s \in \boldsymbol{C}$ we define characters $\tau_\ell$, $\eta_{\ell,s}$ and $\xi_{\ell,s}$ by

$$\tau_\ell : U \ni \begin{pmatrix} {}^t\delta^{-1} & 0 \\ \zeta & \delta \end{pmatrix} \mapsto (\det(\delta))^\ell \in \boldsymbol{C}^*,$$

$$\eta_{\ell,s} : U \ni \begin{pmatrix} {}^t\delta^{-1} & 0 \\ \zeta & \delta \end{pmatrix} \mapsto \left( \frac{\det(\delta)}{|\det(\delta)|} \right)^\ell |\det(\delta)|^s \in \boldsymbol{C}^*,$$

$$\xi_{\ell,s} : P \ni p \mapsto \eta_{\ell,s}(\mu_0^{-1} p \mu_0) \in \boldsymbol{C}^*.$$

We identify $\mathfrak{p}_+$ with $\{z \in M_n(\boldsymbol{C}); {}^t z = -z\}$ by:

$$\begin{array}{ccc} \mathfrak{p}_+ & \cong & \{z \in M_n(\boldsymbol{C}); {}^t z = -z\} \supset \boldsymbol{D}_{II} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} & \mapsto & z. \end{array}$$

For any $g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in G$ and $z \in \boldsymbol{D}_{II}$, we define $g[z]$ by

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \exp\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} U = \begin{pmatrix} a & az+b \\ -\bar{b} & -\bar{b}z+\bar{a} \end{pmatrix} U = \exp\begin{pmatrix} 0 & g[z] \\ 0 & 0 \end{pmatrix} U.$$

Notice that

$$(zb^* + a^*)(az+b) = (z\,{}^t a - {}^t b)(-\bar{b}z + \bar{a}).$$

Then we have the following lemma.

Lemma 3. *For any $g \in G$ and $z \in \boldsymbol{D}_{II}$, we have*

$$g[z] = (az+b)(-\bar{b}z+\bar{a})^{-1} = (zb^* + a^*)^{-1}(z\,{}^t a - {}^t b) \qquad \left(g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\right),$$

$$(T_{\tau_\ell}(g)F)(z) = (\det(-\bar{b}z+\bar{a}))^{-\ell} F(g^{-1}[z]) \qquad \left(g^{-1} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\right).$$

**2.3. Type III.** Fix any $n \in \boldsymbol{N}$. The classical domain of type III is defined by

$$\boldsymbol{D}_{III} = \{z \in M_n(\boldsymbol{C}); z^* z \ll I_n, {}^t z = z\}.$$

The Shilov boundary of $\boldsymbol{D}_{III}$ is given by

$$\check{\boldsymbol{S}}_{III} = \{u \in M_n(\boldsymbol{C}); u^* u = I_n, {}^t u = u\}.$$

We define

$$G_c = Sp(n, \boldsymbol{C}),$$

$$G = \left\{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(n,n); a, b \in M_n(\boldsymbol{C})\right\},$$

$$K_c = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{pmatrix}; \alpha \in GL(n, \boldsymbol{C})\right\},$$

$$P_+ = \left\{\begin{pmatrix} I_n & z \\ 0 & I_n \end{pmatrix}; {}^t z = z \;\; (z \in M_n(\boldsymbol{C}))\right\},$$

$$P_- = \left\{\begin{pmatrix} I_n & 0 \\ \zeta & I_n \end{pmatrix}; {}^t \zeta = \zeta \;\; (\zeta \in M_n(\boldsymbol{C}))\right\},$$

$$U = K_c P_-,$$

$$\mathfrak{g}_c = \mathfrak{sp}(n, \boldsymbol{C}),$$

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathfrak{g}_c; a^* = -a, {}^t b = b \ \ (a, b \in M_n(\boldsymbol{C})) \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \in \mathfrak{g}; a^* = -a \ \ (a \in M_n(\boldsymbol{C})) \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix}; {}^t b = b \ \ (b \in M_n(\boldsymbol{C})) \right\},$$

$$\mathfrak{p}_+ = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}; {}^t z = z \ \ (z \in M_n(\boldsymbol{C})) \right\}.$$

The Killing form $B$ is given by

$$B(X, Y) = 2(n+1)\ \mathrm{Tr}(XY) \qquad (X, Y \in \mathfrak{g}_c).$$

For any $\ell \in \boldsymbol{Z}$ and $s \in \boldsymbol{C}$ we define characters $\tau_\ell$, $\eta_{\ell,s}$ and $\xi_{\ell,s}$ by

$$\tau_\ell : U \ni \begin{pmatrix} {}^t\delta^{-1} & 0 \\ \zeta & \delta \end{pmatrix} \mapsto (\det(\delta))^\ell \in \boldsymbol{C}^*,$$

$$\eta_{\ell,s} : U \ni \begin{pmatrix} {}^t\delta^{-1} & 0 \\ \zeta & \delta \end{pmatrix} \mapsto \left( \frac{\det(\delta)}{|\det(\delta)|} \right)^\ell |\det(\delta)|^s \in \boldsymbol{C}^*,$$

$$\xi_{\ell,s} : P \ni p \mapsto \eta_{\ell,s}(\mu_0^{-1} p \mu_0) \in \boldsymbol{C}^*.$$

We identify $\mathfrak{p}_+$ with $\{z \in M_n(\boldsymbol{C}); {}^t z = z\}$ by:

$$\begin{array}{ccc} \mathfrak{p}_+ & \cong & \{z \in M_n(\boldsymbol{C}); {}^t z = z\} \supset \boldsymbol{D}_{III} \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} & \mapsto & z. \end{array}$$

For any $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in G$ and $z \in \boldsymbol{D}_{III}$, we define $g[z]$ by

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \exp\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} U = \begin{pmatrix} a & az+b \\ \bar{b} & \bar{b}z+\bar{a} \end{pmatrix} U = \exp\begin{pmatrix} 0 & g[z] \\ 0 & 0 \end{pmatrix} U.$$

Notice that

$$(zb^* + a^*)(az+b) = (z\,{}^t a + {}^t b)(\bar{b}z + \bar{a}).$$

Then we have the following lemma.

Lemma 4. *For any* $g \in G$ *and* $z \in \boldsymbol{D}_{III}$, *we have*

$$g[z] = (az+b)(\bar{b}z+\bar{a})^{-1} = (zb^* + a^*)^{-1}(z^t a + {}^t b) \qquad \left(g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}\right),$$

$$(T_{\tau_\ell}(g)F)(z) = (\det(\bar{b}z+\bar{a}))^{-\ell} F(g^{-1}[z]) \qquad \left(g^{-1} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}\right).$$

**2.4. Type IV.** Fix any $n \in \boldsymbol{N}$ such that $n > 2$. The classical domain of type IV is defined by

$$\boldsymbol{D}_{IV} = \left\{ z \in \boldsymbol{C}^n; z^* z < \frac{1}{2}(1 + |{}^t zz|^2) < 1 \right\}.$$

The Shilov boundary of $\boldsymbol{D}_{IV}$ is given by

$$\check{\boldsymbol{S}}_{IV} = \{u = e^{i\theta} x \in \boldsymbol{C}^n; 0 \le \theta < 2\pi, x \in S^{n-1}\},$$

where $S^{n-1} = \{x \in \boldsymbol{R}^n; {}^t xx = 1\}$.

The definitions of $G$, $K$, $G_c$, $\tilde{\mathfrak{p}}_+$ and $\tilde{\mathfrak{p}}_-$ in [13] are not correct. They should be defined as follows.

Define

$$G = SO_0(n,2),$$

$$K = \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}; k_1 \in SO(n), k_2 \in SO(2) \right\}.$$

Then $K$ is a maximal compact subgroup of $G$. The complexification of $G$ is given by

$$G_c = \begin{pmatrix} I_n & 0 \\ 0 & iI_2 \end{pmatrix} SO(n+2, \boldsymbol{C}) \begin{pmatrix} I_n & 0 \\ 0 & iI_2 \end{pmatrix}^{-1}.$$

Put

$$\gamma = \begin{pmatrix} I_n & 0 \\ 0 & \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \end{pmatrix}.$$

We define

$$\tilde{G}_c = \gamma G_c \gamma^{-1},$$

$$\tilde{K}_c = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta^{-1} & 0 \\ 0 & 0 & \delta \end{pmatrix} \in SL(n+2, \boldsymbol{C}); \alpha \in SO(n, \boldsymbol{C}), \delta \in \boldsymbol{C}^* \right\},$$

$$\tilde{P}_+ = \left\{ \begin{pmatrix} I_n & 0 & z \\ 2\,{}^t z & 1 & {}^t zz \\ 0 & 0 & 1 \end{pmatrix}; z \in \boldsymbol{C}^n \right\},$$

$$\tilde{P}_- = \left\{ \begin{pmatrix} I_n & \zeta & 0 \\ 0 & 1 & 0 \\ 2\,{}^t \zeta & {}^t \zeta\zeta & 1 \end{pmatrix}; \zeta \in \boldsymbol{C}^n \right\},$$

$$\tilde{U} = \tilde{K}_c P_-,$$

$$\mathfrak{g}_c = \begin{pmatrix} I_n & 0 \\ 0 & iI_2 \end{pmatrix} \mathfrak{o}(n+2, \boldsymbol{C}) \begin{pmatrix} I_n & 0 \\ 0 & iI_2 \end{pmatrix}^{-1},$$

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ {}^t b & d \end{pmatrix}; \begin{matrix} {}^t a = -a, \\ {}^t d = -d \end{matrix} \quad (a \in M_n(\boldsymbol{R}), b \in M_{n,2}(\boldsymbol{R}), d \in M_2(\boldsymbol{R})) \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}; \begin{matrix} {}^t a = -a, \\ {}^t d = -d \end{matrix} \quad (a \in M_n(\boldsymbol{R}), d \in M_2(\boldsymbol{R})) \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & \eta & \xi \\ {}^t \eta & 0 & 0 \\ {}^t \xi & 0 & 0 \end{pmatrix}; \xi, \eta \in \boldsymbol{R}^n \right\},$$

$$\tilde{\mathfrak{g}}_c = \gamma \mathfrak{g}_c \gamma^{-1},$$

$$\tilde{\mathfrak{p}}_c = \left\{ \begin{pmatrix} 0 & \zeta & z \\ 2\,{}^t z & 0 & 0 \\ 2\,{}^t \zeta & 0 & 0 \end{pmatrix}; z, \zeta \in \boldsymbol{C}^n \right\},$$

$$\tilde{\mathfrak{p}}_+ = \left\{ \begin{pmatrix} 0 & 0 & z \\ 2\,{}^t z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; z \in \boldsymbol{C}^n \right\},$$

$$\tilde{\mathfrak{p}}_- = \left\{ \begin{pmatrix} 0 & \zeta & 0 \\ 0 & 0 & 0 \\ 2\,{}^t \zeta & 0 & 0 \end{pmatrix}; \zeta \in \boldsymbol{C}^n \right\},$$

$$K_c = \gamma^{-1} \tilde{K}_c \gamma,\ P_+ = \gamma^{-1} \tilde{P}_+ \gamma,\ P_- = \gamma^{-1} \tilde{P}_- \gamma,\ U = \gamma^{-1} \tilde{U} \gamma,$$

$$\tilde{\mathfrak{g}} = \gamma \mathfrak{g} \gamma^{-1},\ \tilde{\mathfrak{k}} = \gamma \mathfrak{k} \gamma^{-1},\ \tilde{\mathfrak{p}} = \gamma \mathfrak{p} \gamma^{-1},\ \mathfrak{p}_+ = \gamma^{-1} \tilde{\mathfrak{p}}_+ \gamma,\ \mathfrak{p}_- = \gamma^{-1} \tilde{\mathfrak{p}}_- \gamma.$$

The Killing form $B$ is given by

$$B(X, Y) = n \operatorname{Tr}(XY) \qquad (X, Y \in \mathfrak{g}_c).$$

For any $\ell \in \boldsymbol{Z}$ and $s \in \boldsymbol{C}$ we define characters $\tau_\ell$, $\eta_{\ell,s}$ and $\xi_{\ell,s}$ by

$$\tau_\ell : U = K_c P_- \ni \gamma^{-1}\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta^{-1} & 0 \\ 0 & 0 & \delta \end{pmatrix}\gamma p_- \mapsto (\delta)^\ell \in \boldsymbol{C}^*,$$

$$\eta_{\ell,s} : U = K_c P_- \ni \gamma^{-1}\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta^{-1} & 0 \\ 0 & 0 & \delta \end{pmatrix}\gamma p_- \mapsto \left(\frac{\delta}{|\delta|}\right)^\ell |\delta|^s \in \boldsymbol{C}^*,$$

$$\xi_{\ell,s} : P \ni p \mapsto \eta_{\ell,s}(\mu_0^{-1} p \mu_0) \in \boldsymbol{C}^*.$$

We identify $\mathfrak{p}_+$ with $\boldsymbol{C}^n$ by:

$$\begin{array}{ccc} \mathfrak{p}_+ & \cong & \boldsymbol{C}^n \supset \boldsymbol{D}_{IV} \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \gamma^{-1}\begin{pmatrix} 0 & 0 & z \\ 2\,{}^t z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\gamma & \mapsto & z. \end{array}$$

Notice that

$$\exp\begin{pmatrix} 0 & 0 & z \\ 2\,{}^t z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta^{-1} & 0 \\ 0 & 0 & \delta \end{pmatrix}\begin{pmatrix} I_n & \zeta & 0 \\ 0 & 1 & 0 \\ 2\,{}^t\zeta & {}^t\zeta\zeta & 1 \end{pmatrix} = \begin{pmatrix} * & * & z\delta \\ * & * & {}^t z z\delta \\ * & * & \delta \end{pmatrix}.$$

For any $g \in G$ and $z \in \boldsymbol{D}_{IV}$, we have

$$g\gamma^{-1}\exp\begin{pmatrix} 0 & 0 & z \\ 2\,{}^t z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\gamma = \gamma^{-1}\begin{pmatrix} * & * & az + b\begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix} \\ * & * & (i,1)\left(cz + d\begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix}\right) \\ * & * & (-i,1)\left(cz + d\begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix}\right) \end{pmatrix}\gamma.$$

Notice that

$$(z(-i,1)\,{}^t b + {}^t a)\left(az + b\begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix}\right)$$
$$= (z(-i,1)\,{}^t d + {}^t c)\left(cz + d\begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix}\right).$$

Then we have the following lemma.

Lemma 5. *For any $g \in G$ and $z \in \boldsymbol{D}_{IV}$, we have*

$$g[z] = \left(az + b\begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix}\right)\left((-i, 1)\left(cz + d\begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix}\right)\right)^{-1}$$

$$= (z(-i, 1)^t b + {}^t a)^{-1}(z(-i, 1)^t d + {}^t c)\left(cz + d\begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix}\right)$$

$$\times \left((-i, 1)\left(cz + d\begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix}\right)\right)^{-1},$$

$${}^t g[z]g[z] = (i, 1)\left(cz + d\begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix}\right)\left((-i, 1)\left(cz + d\begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix}\right)\right)^{-1},$$

$$\left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right),$$

$$(T_\tau(g)F)(z) = \left((-i, 1)\left(cz + d\begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix}\right)\right)^{-\ell} F(g^{-1}[z]),$$

$$\left(g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

## 3. Generalized Laplacians

We start with the case where $\tau$ is trivial.

Let $\boldsymbol{D}$ be a classical domain. For any $g \in G$ and $F \in \boldsymbol{C}^\infty(\boldsymbol{D})$, we define

$$T(g)F(z) = F(g^{-1}[z]) \qquad (z \in \boldsymbol{D}).$$

Let $T_0(\boldsymbol{D})$ be the tangent space of $\boldsymbol{D}$ at $z = 0$. Since $K$ is the isotropy subgroup at $z = 0$, we have the linear onto-isomorphism:

$$\begin{array}{ccc} \mathfrak{p} & \cong & T_0(\boldsymbol{D}) \\ \Downarrow & & \Downarrow \\ X & \mapsto & dT(X)|_{z=0}. \end{array}$$

Since the Killing form $B$ is positive definite on $\mathfrak{p}$, this gives an inner product on $T_0(\boldsymbol{D})$. It is clear that this inner product is invariant by the linear isotropy representation of $K$, so that it defines the invariant riemannian metric on $\boldsymbol{D}$. It is well-known (see [2], [5]) that $dT(\Omega)$ coincides with the Laplace-Beltrami operator with respect to the invariant riemannian metric on $\boldsymbol{D}$ defined by the Killing form $B$. Let $N$ be the complex dimension of $\mathfrak{p}_+$. Let $\{z_k\}_{1 \le k \le N}$

denote the canonical coordinates of $\boldsymbol{D}$. We choose a basis $\{Z_k\}_{1 \le k \le N}$ of $\mathfrak{p}_+$ such that

$$\begin{array}{ccc} \mathfrak{p}_+ & \cong & \boldsymbol{C}^N \supset \boldsymbol{D} \\ \Psi & & \Psi \\ \sum_{k=1}^N z_k Z_k & \mapsto & z. \end{array}$$

For any $k$ $(1 \le k \le N)$, we choose $X_k, Y_k \in \mathfrak{p}$ such that

$$Z_k = \frac{1}{2}(X_k - iY_k).$$

Then clearly, $X_k$, $Y_k$ $(1 \le k \le N)$ is a basis of $\mathfrak{p}$. For any $k$ $(1 \le k \le N)$, put $z_k = x_k + iy_k$ $(x_k, y_k \in \boldsymbol{R})$. Then it is straightforward to check

$$dT(X_k)|_{z=0} = -\frac{\partial}{\partial x_k}\bigg|_{z=0}, \qquad dT(Y_k)|_{z=0} = -\frac{\partial}{\partial y_k}\bigg|_{z=0},$$

using the "case by case" method for each type of the classical domains.

On the other hand, by means of matrix calculations, these equations can be proved directly by the following observations. For any $t \in \boldsymbol{R}$, we see

$$\exp(-tX_k) = \exp(-tZ_k + \sigma(-tZ_k)) = \exp(-tZ_k)\exp(\sigma(-tZ_k)) + O(t^2),$$

where $\sigma$ denotes the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$. Since $\exp(\sigma(-tZ_k)) \in P_-$, it is clear that

$$(\exp(tX_k))^{-1}[0] = -tZ_k + O(t^2),$$

from which the first equation follows. In the same way, the second equation is obtained.

In the succeeding subsections, we prove the following proposition.

Proposition 1. *There exist a positive constant $c$ and a number $M$ $(0 \le M \le N)$ such that we can rearrange $X_k$, $Y_k$ $(1 \le k \le N)$ such that*

$$B(X_k, X_k) = B(Y_k, Y_k) = c \qquad (1 \le k \le M),$$

$$B(X_k, X_k) = B(Y_k, Y_k) = 2c \qquad (M+1 \le k \le N).$$

*(In case $M = 0$, one should ignore the part $(1 \le k \le M)$).*

*Moreover, $\frac{1}{\sqrt{c}}X_k$, $\frac{1}{\sqrt{c}}Y_k$ $(1 \le k \le M)$, $\frac{1}{\sqrt{2c}}X_k$, $\frac{1}{\sqrt{2c}}Y_k$ $(M+1 \le k \le N)$ is an orthonormal basis of $\mathfrak{p}$, where $c$ is given as follows:*

$$(Type\ I) \quad c = 4(n+m), \qquad (Type\ II) \quad c = 4(n-1),$$

$$(Type\ III) \quad c = 4(n+1), \qquad (Type\ IV) \quad c = 8n.$$

Consider the invariant riemannian metric on $\boldsymbol{D}$ defined by $\frac{4}{c}$ times the Killing form. We call this the canonical riemannian metric on the classical domain $\boldsymbol{D}$.

Notice that

$$dT(\Omega)|_{z=0} = \frac{4}{c}\left(\sum_{k=1}^{M} \frac{\partial^2}{\partial z_k \partial \bar{z}_k}\bigg|_{z=0} + \frac{1}{2}\sum_{k=M+1}^{N} \frac{\partial^2}{\partial z_k \partial \bar{z}_k}\bigg|_{z=0}\right).$$

We put

$$A = \begin{pmatrix} I_M & 0 \\ 0 & \frac{1}{2} I_{N-M} \end{pmatrix}, \qquad \partial_z = \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_N}\right).$$

Then

$$\partial_z A \partial_z^*|_{z=0} = \sum_{k=1}^{M} \frac{\partial^2}{\partial z_k \partial \bar{z}_k}\bigg|_{z=0} + \frac{1}{2}\sum_{k=M+1}^{N} \frac{\partial^2}{\partial z_k \partial \bar{z}_k}\bigg|_{z=0}.$$

For each type of the classical domains, we seek an invariant differential operator $\Delta$ such that

$$\Delta|_{z=0} = \partial_z A \partial_z^*|_{z=0}.$$

For any $F \in \boldsymbol{D}$, we define

$$\partial F(z) = \sum_{k=1}^{N} \frac{\partial F(z)}{\partial z_k}\, dz_k.$$

Then the operator $\partial$ is independent of the choice of the coordinate system. For any $g \in G$, the action $\boldsymbol{D} \ni z \mapsto w = g[z] \in \boldsymbol{D}$ is a complex analytic isomorphism, so that $w = (w_1, \dots, w_N)$ gives another coordinate system of $\boldsymbol{D}$. We denote by $\frac{\partial(w)}{\partial(z)}$ the Jacobian matrix and put

$$\partial_w = \left(\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_N}\right).$$

Then $\partial_z = \partial_w \frac{\partial(w)}{\partial(z)}$. Fix any $z_0 \in \boldsymbol{D}$. Then there exists $g \in G$ such that $g[0] = z_0$. Putting $\varphi_g(z) = g[z]$ $(z \in \boldsymbol{D})$, we compute explicitly $(\varphi_g)_*(\partial_z A \partial_z^*|_{z=0})$. Then it is clear that there exists a positive definite hermitian matrix $h_g$ which depends on $g$ such that

$$(\varphi_g)_*(\partial_z A \partial_z^*|_{z=0}) = \partial_w h_g \partial_w^*|_{w=z_0}.$$

Since $\partial_z A \partial_z^*|_{z=0}$ $\left(= \frac{c}{4} dT(\Omega)|_{z=0}\right)$ is invariant by the linear isotropy representation of $K$, all coefficients of $h_g$ depend only on $z_0$, so that we write $h_g = h(z_0)$.

Then it is now easy to see that $h(0) = A$ and that $\partial_z \cdot h(z) \cdot \partial_z^*$ is an invariant differential operator, which at the point $z = 0$ coincides with $\partial_z A \partial_z^*|_{z=0}$. (Here the notation "$\cdot$" means that any function between two dots should not be differentiated.)

This means that $\partial_z \cdot h(z) \cdot \partial_z^*$ is the Laplace-Beltrami operator with respect to the canonical riemannian metric.

It is clear that $h(z)$ satisfies the following assumption.

Assumption 1. *For any $g \in G$ and $z \in \boldsymbol{D}$,*

$$h(w) = \frac{\partial(w)}{\partial(z)} h(z) \left(\frac{\partial(w)}{\partial(z)}\right)^* \qquad (w = g[z]).$$

(Remark that the inverse matrix of the Hessian of the Bergman kernel function satisfies this assumption.)

Conversely, if a positive definite hermitian matrix valued $\boldsymbol{C}^\infty$-function $h(z)$ satisfies this assumption, then it is obvious that $\partial_z \cdot h(z) \cdot \partial_z^*$ is an invariant differential operator on $\boldsymbol{D}$. Hua's method is based on this fact.

Next we modify the Laplace-Beltrami operator in order to obtain the line bundle valued invariant differential operator.

Recall (see Lemma 1) that for any $g \in G$ and $F \in \boldsymbol{C}^\infty(\boldsymbol{D})$, we have

$$(T_\tau(g)F)(z) = \rho_\tau(g,z)^{-1} F(g^{-1}[z]) \qquad (z \in \boldsymbol{D}).$$

Suppose that a positive $\boldsymbol{C}^\infty$ function $r_\tau$ on $\boldsymbol{D}$ satisfies the assumption:

Assumption 2. *For any $g \in G$ and $z \in \boldsymbol{D}$,*

$$r_\tau(w) = |\rho_\tau(g,z)|^{-2} r_\tau(z) \qquad (w = g^{-1}[z]).$$

We define

$$\Delta_\tau(z) = r_\tau(z)^{-1} \partial_z r_\tau(z) \cdot h(z) \cdot \partial_z^*.$$

For any $g \in G$, put $w = g^{-1}[z]$. Then we have

$$\begin{aligned}
\Delta_\tau(w) &= r_\tau(w)^{-1} \partial_w r_\tau(w) \cdot h(w) \cdot \partial_w^* \\
&= |\rho_\tau(g,z)|^2 r_\tau(z)^{-1} \partial_z |\rho_\tau(g,z)|^{-2} r_\tau(z) \\
&\quad \times \left(\frac{\partial(w)}{\partial(z)}\right)^{-1} h(w) \left(\left(\frac{\partial(w)}{\partial(z)}\right)^*\right)^{-1} \cdot \partial_z^* \\
&= \rho_\tau(g,z) r_\tau(z)^{-1} \partial_z r_\tau(z) \cdot h(z) \cdot \partial_z^* \rho_\tau(g,z)^{-1}.
\end{aligned}$$

Here, we used the fact that

$$\partial_z \rho_\tau(g,z)^* = 0, \qquad \partial_z^* \rho_\tau(g,z) = 0.$$

Thus for any $g \in G$ and $F \in \boldsymbol{C}^{\infty}(\boldsymbol{D})$, we obtain

$$(T_\tau(g)\Delta_\tau F)(z) = \rho_\tau(g,z)^{-1}\Delta_\tau(w)F(w) = \Delta_\tau(z)T_\tau(g)F(z).$$

This shows that $\Delta_\tau$ defines an $\boldsymbol{E}_\tau$-valued invariant differential operator which we call the generalized Laplacian.

For the notation $\partial_z$ in the following theorem, the reader is referred to the succeeding subsections. For simplicity we denote $\Delta_{\tau_\ell}$ by $\Delta_\ell$.

Theorem 1. *For each type of the classical domains, the generalized Laplacian $\Delta_\ell$ is explicitly given as follows.*

$$(\textit{Type I}) \quad \mathrm{Tr}(\det(I_m - z^*z)^{-\ell}(I_m - z^*z)\partial_z \det(I_m - z^*z)^{\ell} \cdot (I_n - zz^*) \cdot \partial_z^*),$$

$$(\textit{Type II}) \quad \mathrm{Tr}(\det(I_n - z^*z)^{-\ell}(I_n - z^*z)\partial_z \det(I_n - z^*z)^{\ell} \cdot (I_n - zz^*) \cdot \partial_z^*),$$

$$(\textit{Type III}) \quad \mathrm{Tr}(\det(I_n - z^*z)^{-\ell}(I_n - z^*z)\partial_z \det(I_n - z^*z)^{\ell} \cdot (I_n - zz^*) \cdot \partial_z^*),$$

$$(\textit{Type IV}) \quad (1 + |{}^tzz|^2 - 2z^*z)^{-\ell+1}\partial_z(1 + |{}^tzz|^2 - 2z^*z)^{\ell}$$
$$\times \left(I_n - 2zz^* + 2\frac{(I_n - zz^*)\overline{zz^*}(I_n - zz^*)}{(1 + |{}^tzz|^2 - 2z^*z)}\right) \cdot \partial_z^*,$$

*where the function between two dots "$\cdot$" should not be differentiated.*

We prove this theorem in the following subsections for each type of classical domains.

**3.1. Type I.** For any $z = (z_{ij}) \in \boldsymbol{D}_I \subset M_{n,m}(\boldsymbol{C})$, we put $\partial_z = {}^t\left(\frac{\partial}{\partial z_{ij}}\right)$. In [6], Hua gave the following invariant differential operator:

$$\Delta = \mathrm{Tr}((I_m - z^*z)\partial_z \cdot (I_n - zz^*) \cdot \partial_z^*).$$

Here, we modified the coordinate system defined by Hua, so that we have the canonical isomorphism:

$$\begin{array}{ccc} \mathfrak{p}_+ & \cong & M_{n,m}(\boldsymbol{C}) \supset \boldsymbol{D}_I \\ \Downarrow & & \Downarrow \\ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} & \mapsto & z. \end{array}$$

(See Section 2.1).

For any $i$, $j$ $(1 \le i \le n, 1 \le j \le m)$, we put

$$X_{ij} = \begin{pmatrix} 0 & E_{ij} \\ E_{ji} & 0 \end{pmatrix}, \qquad Y_{ij} = \begin{pmatrix} 0 & iE_{ij} \\ -iE_{ji} & 0 \end{pmatrix}.$$

Then we have

$$B(X_{ij}, X_{ij}) = 4(n+m), \qquad B(Y_{ij}, Y_{ij}) = 4(n+m).$$

Moreover, it is easy to check that $\frac{1}{\sqrt{4(n+m)}} X_{ij}, \frac{1}{\sqrt{4(n+m)}} Y_{ij}$ $(1 \le i \le n, 1 \le j \le m)$ is an orthonormal basis of $\mathfrak{p}$. Notice that

$$\begin{aligned}\Delta|_{z=0} = \mathrm{Tr}(\partial_z \partial_z^*)|_{z=0} &= \sum_{1 \le i \le n, 1 \le j \le m} \frac{\partial^2}{\partial z_{ij} \partial \overline{z_{ij}}}\Bigg|_{z=0} \\ &= \frac{1}{4} \sum_{1 \le i \le n, 1 \le j \le m} (X_{ij}^2 + Y_{ij}^2)|_{z=0} = (n+m) dT(\Omega)|_{z=0}.\end{aligned}$$

This shows that $\Delta$ is the Laplace-Beltrami operator with respect to the canonical riemannian metric.

Now we carry out our method to obtain this Laplace-Beltrami operator. From Lemma 2, for any $g \in G$, we have

$$g[0] = bd^{-1}, \qquad \partial_z|_{z=0} = d^{-1} \partial_w|_{w=g[0]} (a - g[0]c) \qquad \left( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$

Moreover, since $g^* I_{n,m} g = I_{n,m}$, we obtain

$$I_m - g[z]^* g[z] = ((cz+d)^*)^{-1} (I_m - z^* z)(cz+d)^{-1},$$

which implies

$$(dd^*)^{-1} = I_m - g[0]^* g[0].$$

Further we observe that

$$\begin{aligned}\mathrm{Tr}(\partial_z \partial_z^*)|_{z=0} &= \mathrm{Tr}(d^{-1} \partial_w (a - g[0]c)(a - g[0]c)^* \partial_w^* (d^*)^{-1})|_{w=g[0]} \\ &= \mathrm{Tr}((dd^*)^{-1} \partial_w (I_n - g[0]g[0]^* - (b - g[0]d)(b - g[0]d)^*) \partial_w^*)|_{w=g[0]} \\ &= \mathrm{Tr}((I_m - g[0]^* g[0]) \partial_w (I_m - g[0]g[0]^*) \partial_w^*)|_{w=g[0]}.\end{aligned}$$

This shows that

$$\mathrm{Tr}((I_m - z^* z) \partial_z \cdot (I_n - zz^*) \cdot \partial_z^*)$$

is the Laplace-Beltrami operator with respect to the canonical riemannian metric.

For any $g \in G$, put $w = g^{-1}[z] = (az+b)(cz+b)^{-1}$ $\left( g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$. From the above formula, it is clear that

$$\det(I_m - w^* w) = |\det(cz+d)|^{-2} \det(I_m - z^* z).$$

In view of Lemma 2, this satisfies Assumption 2, so that

$$\mathrm{Tr}(\det(I_m - z^*z)^{-\ell}(I_m - z^*z)\partial_z \det(I_m - z^*z)^{\ell} \cdot (I_n - zz^*) \cdot \partial_z^*)$$

defines an $\boldsymbol{E}_\tau$-valued invariant differential operator.

**3.2. Type II.** Let us recall the canonical isomorphism:

$$\begin{array}{ccc} \mathfrak{p}_+ & \cong & \{z \in M_n(\boldsymbol{C});\, {}^t z = -z\} \supset \boldsymbol{D}_{II} \\ \Psi & & \Psi \\ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} & \mapsto & z. \end{array}$$

(See Section 2.2).

For any $z = (z_{ij}) \in \boldsymbol{D}_{II} \subset M_n(\boldsymbol{C})$, we see that $z_{ji} = -z_{ij}$. Thus the canonical coordinates of $\boldsymbol{D}_{II}$ is given by $\{z_{ij}\}_{1 \le i < j \le n}$.

For any $i$, $j$ $(1 \le i < j \le n)$, we put

$$X_{ij} = \begin{pmatrix} 0 & E_{ij} - E_{ji} \\ E_{ij} - E_{ji} & 0 \end{pmatrix}, \qquad Y_{ij} = \begin{pmatrix} 0 & i(E_{ij} - E_{ji}) \\ -i(E_{ij} - E_{ji}) & 0 \end{pmatrix}.$$

Then we have

$$B(X_{ij}, X_{ij}) = 8(n-1), \qquad B(Y_{ij}, Y_{ij}) = 8(n-1).$$

Moreover, it is easy to check that $\frac{1}{\sqrt{8(n-1)}} X_{ij}$, $\frac{1}{\sqrt{8(n-1)}} Y_{ij}$ $(1 \le i < j \le n)$ is an orthonormal basis of $\mathfrak{p}$. We define $\partial_z$ by:

$$(i,j) - \mathit{component\ of}\ \ \partial_z = \begin{cases} -\frac{1}{2}\frac{\partial}{\partial z_{ij}} & (i < j), \\ 0 & (i = j), \\ \frac{1}{2}\frac{\partial}{\partial z_{ji}} & (i > j). \end{cases}$$

Notice that

$$\begin{aligned} (n-1)dT(\Omega)|_{z=0} &= \frac{1}{8}\sum_{1 \le i < j \le n} (X_{ij}^2 + Y_{ij}^2)|_{z=0} \\ &= \frac{1}{2}\sum_{1 \le i < j \le n} \frac{\partial^2}{\partial z_{ij} \partial \overline{z_{ij}}}\bigg|_{z=0} = \mathrm{Tr}(\partial_z \partial_z^*)|_{z=0}. \end{aligned}$$

From Lemma 3, for any $g \in G$, we have

$$g[0] = b\bar{a}^{-1}, \qquad \partial_z|_{z=0} = \bar{a}^{-1}\partial_w|_{w=g[0]}(a + g[0]\bar{b}) \qquad \left(g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\right).$$

Moreover, since $g^* I_{n,n} g = I_{n,n}$, we obtain

$$I_n - g[z]^* g[z] = ((-\bar{b}z + \bar{a})^*)^{-1}(I_n - z^*z)(-\bar{b}z + \bar{a})^{-1},$$

which implies

$$(\bar{a}(\bar{a})^*)^{-1} = I_n - g[0]^* g[0].$$

Further we observe that

$$\begin{aligned}\mathrm{Tr}(\partial_z \partial_z^*)|_{z=0} &= \mathrm{Tr}(\bar{a}^{-1}\partial_w (a + g[0]\bar{b})(a + g[0]\bar{b})^* \partial_w^* (\bar{a}^*)^{-1})|_{w=g[0]} \\ &= \mathrm{Tr}((\bar{a}(\bar{a})^*)^{-1}\partial_w (I_n - g[0]g[0]^* - (b - g[0]\bar{a})(b - g[0]\bar{a})^*)\partial_w^*)|_{w=g[0]} \\ &= \mathrm{Tr}((I_n - g[0]^* g[0])\partial_w (I_n - g[0]g[0]^*)\partial_w^*)|_{w=g[0]}.\end{aligned}$$

This shows that

$$\Delta = \mathrm{Tr}((I_n - z^* z)\partial_z \cdot (I_n - zz^*) \cdot \partial_z^*)$$

is the Laplace-Beltrami operator with respect to the canonical riemannian metric.

For any $g \in G$, put $w = g^{-1}[z] = (az + b)(-\bar{b}z + \bar{a})^{-1}$ $\left(g^{-1} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\right)$. From the above formula, it is clear that

$$\det(I_n - w^* w) = |\det(-\bar{b}z + \bar{a})|^{-2} \det(I_n - z^* z).$$

In view of Lemma 3, this satisfies Assumption 2, so that

$$\mathrm{Tr}(\det(I_n - z^* z)^{-\ell}(I_n - z^* z)\partial_z \det(I_n - z^* z)^{\ell} \cdot (I_n - zz^*) \cdot \partial_z^*)$$

defines an $\boldsymbol{E}_\tau$-valued invariant differential operator.

**3.3. Type III.** Let us recall the canonical isomorphism:

$$\begin{array}{ccc} \mathfrak{p}_+ & \cong & \{z \in M_n(\boldsymbol{C});\, {}^t z = z\} \supset \boldsymbol{D}_{III} \\ \Psi & & \Psi \\ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} & \mapsto & z. \end{array}$$

(See Section 2.3).

For any $z = (z_{ij}) \in \boldsymbol{D}_{III} \subset M_n(\boldsymbol{C})$, we see that $z_{ji} = z_{ij}$. Thus the canonical coordinates of $\boldsymbol{D}_{III}$ is given by $\{z_{ij}\}_{1 \le i \le j \le n}$.

For any $i,\ j,\ k$ $(1 \le i < j \le n, 1 \le k \le n)$, we put

$$X_{kk} = \begin{pmatrix} 0 & E_{kk} \\ E_{kk} & 0 \end{pmatrix}, \qquad Y_{kk} = \begin{pmatrix} 0 & iE_{kk} \\ -iE_{kk} & 0 \end{pmatrix},$$

$$X_{ij} = \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ E_{ij} + E_{ji} & 0 \end{pmatrix}, \qquad Y_{ij} = \begin{pmatrix} 0 & i(E_{ij} + E_{ji}) \\ -i(E_{ij} + E_{ji}) & 0 \end{pmatrix}.$$

Then we have

$$B(X_{kk}, X_{kk}) = 4(n+1), \qquad B(Y_{kk}, Y_{kk}) = 4(n+1),$$
$$B(X_{ij}, X_{ij}) = 8(n+1), \qquad B(Y_{ij}, Y_{ij}) = 8(n+1).$$

Moreover, it is easy to check that

$$\frac{1}{\sqrt{4(n+1)}} X_{kk}, \frac{1}{\sqrt{4(n+1)}} Y_{kk} \qquad (1 \le k \le n),$$
$$\frac{1}{\sqrt{8(n+1)}} X_{ij}, \frac{1}{\sqrt{8(n+1)}} Y_{ij} \qquad (1 \le i < j \le n)$$

is an orthonormal basis of $\mathfrak{p}$. We define $\partial_z$ by:

$$(i,j) - component\ \ of\ \ \partial_z = \begin{cases} \frac{1}{2}\frac{\partial}{\partial z_{ij}} & (i<j), \\ \frac{\partial}{\partial z_{ii}} & (i=j), \\ \frac{1}{2}\frac{\partial}{\partial z_{ji}} & (i>j). \end{cases}$$

Notice that

$$\begin{aligned} (n+1)dT(\Omega)|_{z=0} &= \frac{1}{4}\sum_{1\le k\le n}(X_{kk}^2 + Y_{kk}^2)|_{z=0} + \frac{1}{8}\sum_{1\le i<j\le n}(X_{ij}^2 + Y_{ij}^2)|_{z=0} \\ &= \sum_{1\le k\le n}\frac{\partial^2}{\partial z_{kk}\partial\overline{z_{kk}}}\bigg|_{z=0} + \frac{1}{2}\sum_{1\le i<j\le n}\frac{\partial^2}{\partial z_{ij}\partial\overline{z_{ij}}}\bigg|_{z=0} = \mathrm{Tr}(\partial_z\partial_z^*)|_{z=0}. \end{aligned}$$

From Lemma 4, for any $g \in G$, we have

$$g[0] = b\bar{a}^{-1}, \qquad \partial_z|_{z=0} = \bar{a}^{-1}\partial_w|_{w=g[0]}(a - g[0]\bar{b}) \qquad \left(g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}\right).$$

Moreover, since $g^* I_{n,n} g = I_{n,n}$, we obtain

$$I_n - g[z]^* g[z] = ((\bar{b}z + \bar{a})^*)^{-1}(I_n - z^*z)(\bar{b}z + \bar{a})^{-1},$$

which implies

$$(\bar{a}(\bar{a})^*)^{-1} = I_n - g[0]^* g[0].$$

Further we observe that

$$\begin{aligned} \mathrm{Tr}(\partial_z\partial_z^*)|_{z=0} &= \mathrm{Tr}(\bar{a}^{-1}\partial_w(a - g[0]\bar{b})(a - g[0]\bar{b})^*\partial_w^*((\bar{a})^*)^{-1})|_{w=g[0]} \\ &= \mathrm{Tr}((\bar{a}(\bar{a})^*)^{-1}\partial_w(I_n - g[0]g[0]^* - (b - g[0]\bar{a})(b - g[0]\bar{a})^*)\partial_w^*)|_{w=g[0]} \\ &= \mathrm{Tr}((I_n - g[0]^*g[0])\partial_w(I_n - g[0]g[0]^*)\partial_w^*)|_{w=g[0]}. \end{aligned}$$

This shows that

$$\mathrm{Tr}((I_n - z^*z)\partial_z \cdot (I_n - zz^*) \cdot \partial_z^*)$$

is the Laplace-Beltrami operator with respect to the canonical riemannian metric.

For any $g \in G$, put $w = g^{-1}[z] = (az+b)(\bar{b}z+\bar{a})^{-1}$ $\left(g^{-1} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}\right)$. From the above formula, it is clear that

$$\det(I_n - w^*w) = |\det(\bar{b}z+\bar{a})|^{-2} \det(I_n - z^*z).$$

In view of Lemma 4, this satisfies Assumption 2, so that

$$\mathrm{Tr}(\det(I_n - z^*z)^{-\ell}(I_n - z^*z)\partial_z \det(I_n - z^*z)^{\ell} \cdot (I_n - zz^*) \cdot \partial_z^*)$$

defines an $\boldsymbol{E}_\tau$-valued invariant differential operator.

**3.4. Type IV.** Let us recall the canonical isomorphism:

$$\begin{array}{ccc} \mathfrak{p}_+ & \cong & \boldsymbol{C}^n \supset \boldsymbol{D}_{IV} \\ \Psi & & \Psi \\ \gamma^{-1}\begin{pmatrix} 0 & 0 & z \\ 2{}^tz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\gamma & \mapsto & z. \end{array}$$

(See Section 2.4).

We denote by $E_k$ the column vector $\in \boldsymbol{C}^n$ such that the $k$-component is 1 and other components are all zero. For any $k$ $(1 \le k \le n)$, we put

$$X_k = \gamma^{-1}\begin{pmatrix} 0 & E_k & E_k \\ 2{}^tE_k & 0 & 0 \\ 2{}^tE_k & 0 & 0 \end{pmatrix}\gamma, \qquad Y_k = \gamma^{-1}\begin{pmatrix} 0 & -iE_k & iE_k \\ 2i{}^tE_k & 0 & 0 \\ -2i{}^tE_k & 0 & 0 \end{pmatrix}\gamma.$$

Then we have

$$B(X_k, X_k) = 8n, \qquad B(Y_k, Y_k) = 8n.$$

Moreover, it is easy to check that $\frac{1}{\sqrt{8n}}X_k, \frac{1}{\sqrt{8n}}Y_k$ $(1 \le k \le n)$ is an orthonormal basis of $\mathfrak{p}$. We define $\partial_z = \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right)$. Notice that

$$2n\, dT(\Omega)|_{z=0} = \frac{1}{4}\sum_{1 \le k \le n}(X_k^2 + Y_k^2)|_{z=0} = \partial_z\partial_z^*|_{z=0}.$$

For any $g \in G$, we put

$$\delta = \frac{1}{2}(-i, 1)d\begin{pmatrix} i \\ 1 \end{pmatrix} \qquad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

Then from Lemma 5, for any $g \in G$, we have

$$g[0] = \delta^{-1}\frac{1}{2}b\begin{pmatrix} i \\ 1 \end{pmatrix}, \qquad \partial_z|_{z=0} = \partial_w|_{w=g[0]}\delta^{-1}(a - g[0](-i,1)c).$$

Moreover, since ${}^t g I_{n,2} g = I_{n,2}$, we obtain

$$\begin{aligned} &1 + |{}^t g[z]g[z]|^2 - 2g[z]^* g[z] \\ &\quad = \left|\left((-i,1)\left(cz + d\begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix}\right)\right)\right|^{-2}(1 + |{}^t zz|^2 - 2z^* z), \end{aligned}$$

which implies

$$|\delta|^{-2} = 1 + |{}^t g[0]g[0]|^2 - 2g[0]^* g[0].$$

Further we observe that

$$\begin{aligned} \partial_z \partial_z^*|_{z=0} &= (\partial_w |\delta|^{-2}(a - g[0](-i,1)c)(a - g[0](-i,1)c)^* \partial_w^*)|_{w=g[0]} \\ &= (\partial_w |\delta|^{-2}(I_n - 2g[0]g[0]^* + (b - g[0](-i,1)d) \\ &\qquad \times (b - g[0](-i,1)d)^*)\partial_w^*)|_{w=g[0]}. \end{aligned}$$

We notice that

$$(b - g[0](-i,1)d)\begin{pmatrix} i \\ 1 \end{pmatrix} = 2\left(\frac{1}{2}b\begin{pmatrix} i \\ 1 \end{pmatrix} - g[0]\delta\right) = 0.$$

From Lemma 5, we have

$$(i,1)d\begin{pmatrix} i \\ 1 \end{pmatrix}\left((-i,1)d\begin{pmatrix} i \\ 1 \end{pmatrix}\right)^{-1} = {}^t g[0]g[0].$$

Thus we have

$$\begin{aligned} (b - g[0](-i,1)d)\begin{pmatrix} -i \\ 1 \end{pmatrix} &= 2\left(\frac{1}{2}b\begin{pmatrix} -i \\ 1 \end{pmatrix} - g[0](-i,1)\frac{1}{2}d\begin{pmatrix} -i \\ 1 \end{pmatrix}\right) \\ &= 2\bar{\delta}(I_n - g[0]\,{}^t\overline{g[0]})\overline{g[0]}. \end{aligned}$$

Since

$$\begin{pmatrix} i \\ 1 \end{pmatrix}(-i,1) + \begin{pmatrix} -i \\ 1 \end{pmatrix}(i,1) = 2I_2,$$

it follows that

$$\begin{aligned} &(b - g[0](-i,1)d)(b - g[0](-i,1)d)^* \\ &\quad = 2\delta\bar{\delta}(I_n - g[0]\,{}^t\overline{g[0]})\overline{g[0]}((I_n - g[0]\,{}^t\overline{g[0]})\overline{g[0]})^*. \end{aligned}$$

Hence

$$\begin{aligned}&\partial_z \partial_z^*|_{z=0}\\ &\quad = (1 + |{}^t g[0] g[0]|^2 - 2g[0]^* g[0])\partial_w \\ &\qquad \times \left( I_n - 2g[0]g[0]^* + 2\frac{(I_n - g[0]g[0]^*)\overline{g[0]g[0]^*}(I_n - g[0]g[0]^*)}{(1 + |{}^t g[0] g[0]|^2 - 2g[0]^* g[0])} \right) \partial_w^* \Big|_{w=g[0]}.\end{aligned}$$

This shows that

$$(1 + |{}^t zz|^2 - 2z^* z)\partial_z \cdot \left( I_n - 2zz^* + 2\frac{(I_n - zz^*)\overline{zz^*}(I_n - zz^*)}{(1 + |{}^t zz|^2 - 2z^* z)} \right) \cdot \partial_z^*$$

is the Laplace-Beltrami operator with respect to the canonical riemannian metric.

For any $g \in G$, put

$$\begin{aligned} w &= g^{-1}[z] \\ &= \left( az + b \begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix} \right) \left( (-i, 1) \left( cz + d \begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix} \right) \right)^{-1}, \\ &\quad \left( g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \in \boldsymbol{D}_{IV} \right). \end{aligned}$$

From the above formula, it is clear

$$1 + |{}^t ww|^2 - 2w^* w = \left| \left( (-i, 1) \left( cz + d \begin{pmatrix} \frac{i}{2}(1 - {}^t zz) \\ \frac{1}{2}(1 + {}^t zz) \end{pmatrix} \right) \right) \right|^{-2} (1 + |{}^t zz|^2 - 2z^* z).$$

In view of Lemma 5, this satisfies Assumption 2, so that

$$\begin{aligned}(1 + |{}^t zz|^2 - 2z^* z)^{-\ell+1} \partial_z (1 + |{}^t zz|^2 - 2z^* z)^{\ell} \\ \cdot \left( I_n - 2zz^* + 2\frac{(I_n - zz^*)\overline{zz^*}(I_n - zz^*)}{(1 + |{}^t zz|^2 - 2z^* z)} \right) \cdot \partial_z^*\end{aligned}$$

defines an $\boldsymbol{E}_\tau$-valued invariant differential operator.

## 4. Eigenvalues of the generalized Laplacian

Let $\boldsymbol{D}$ be a classical domain, $\check{\boldsymbol{S}}$ its Shilov boundary. Let $\boldsymbol{K}_{\tau,\eta}(z, u)$ ($z \in \boldsymbol{D}, u \in \check{\boldsymbol{S}}$) be the generalized Poisson-Cauchy kernel function given in [13]. Then clearly, $\boldsymbol{K}_{\tau,\eta}(0, u) = 1$. Let $\Delta_\tau$ be the generalized Laplacian defined in the previous section.

The following proposition shows a way to prove the next theorem.

Proposition 2. *Put*

$$c_{\tau,\eta} = \Delta_\tau \boldsymbol{K}_{\tau,\eta}(z,u)|_{z=0}.$$

*If* $c_{\tau,\eta}$ *is independent of* $u \in \check{\boldsymbol{S}}$, *then*

$$\Delta_\tau \boldsymbol{K}_{\tau,\eta}(z,u) = c_{\tau,\eta}\boldsymbol{K}_{\tau,\eta}(z,u) \qquad (z \in \boldsymbol{D}, u \in \check{\boldsymbol{S}}).$$

We prove this proposition, making use of the representation $T_\tau$ as follows. For any $z \in \boldsymbol{D}$, $u \in \check{\boldsymbol{S}}$, we choose $g \in G$ such that $g^{-1}[z] = 0$. Then

$$\begin{aligned} T_\tau(g)\Delta_\tau(z)\boldsymbol{K}_{\tau,\eta}(z,u) &= \rho_\tau(g,z)^{-1}\Delta_\tau(w)\boldsymbol{K}_{\tau,\eta}(w,u) \\ &= c_{\tau,\eta}\rho_\tau(g,z)^{-1}\boldsymbol{K}_{\tau,\eta}(w,u) \\ &= c_{\tau,\eta}T_\tau(g)\boldsymbol{K}_{\tau,\eta}(z,u). \end{aligned}$$

If we apply $T_\tau(g^{-1})$ to the both sides of this equation, the proposition follows at once.

Applying this proposition to each type of the classical domains, we obtain the following theorem, where for simplicity we denote $c_{\tau_\ell,\eta_{\ell,s}}$ by $c_{\ell,s}$.

Theorem 2. *For each type of the classical domains, the eigenvalue* $c_{\ell,s}$ *is given as follows.*

$$\begin{aligned} &(Type\ \ I) && \tfrac{1}{4}m(s-\ell)(s+\ell-2n), \\ &(Type\ \ II,\ n: even) && \tfrac{1}{4}n(s-\ell)(s+\ell-n+1), \\ &(Type\ \ II,\ n: odd) && \tfrac{1}{4}(n-1)(s-\ell)(s+\ell-n), \\ &(Type\ \ III) && \tfrac{1}{4}n(s-\ell)(s+\ell-n-1), \\ &(Type\ \ IV) && (s-\ell)(s+\ell-n). \end{aligned}$$

We prove this theorem in the following subsections for each type of classical domains.

**4.1. Type I.** The generalized Poisson-Cauchy kernel function for Type I is given by

$$\boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u) = \frac{1}{(\det(I_m - u^*z))^\ell}\left(\frac{\det(I_m - z^*z)}{|\det(I_m - u^*z)|^2}\right)^{n-(\ell+s)/2},$$

(Theorem 2 in [13]).

Using Theorem 1, we get

$$\begin{aligned}\Delta_\ell \boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u)|_{z=0} &= \sum_{1\le i\le n,1\le j\le m} \frac{\partial^2}{\partial z_{ij}\partial \overline{z_{ij}}} \boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u)\bigg|_{z=0} \\ &= \sum_{1\le i\le n,1\le j\le m} \left(n-\frac{(\ell+s)}{2}\right)\left(\left(\ell+n-\frac{(\ell+s)}{2}\right)\overline{u_{ij}}u_{ij}-1\right) \\ &= \frac{1}{4}m(s-\ell)(s+\ell-2n).\end{aligned}$$

Here we used the following property of $u \in \check{\boldsymbol{S}}_I$

$$\sum_{1\le i\le n,1\le j\le m} \overline{u_{ij}}u_{ij} = m.$$

## 4.2. Type II.

**4.2.1.** ($n$ : even). The generalized Poisson-Cauchy kernel function for Type II (Type III in [13]) for even $n$ is given by

$$\boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u) = \frac{1}{(\det(I_n+\bar{u}z))^\ell}\left(\frac{\det(I_n+\bar{z}z)}{|\det(I_n+\bar{u}z)|^2}\right)^{(n-1-\ell-s)/2},$$

(Theorem 4 in [13]).

Using Theorem 1, we get

$$\begin{aligned}\Delta_\ell \boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u)|_{z=0} &= \frac{1}{2}\sum_{1\le i<j\le n} \frac{\partial^2}{\partial z_{ij}\partial \overline{z_{ij}}} \boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u)\bigg|_{z=0} \\ &= \sum_{1\le i<j\le n} ((n-1-\ell-s)/2) \\ &\quad \times ((\ell+(n-1-\ell-s)/2)2\overline{u_{ij}}u_{ij}-1) \\ &= \frac{1}{4}n(s-\ell)(s+\ell-n+1).\end{aligned}$$

Here we used the following property of $u \in \check{\boldsymbol{S}}_{II}$

$$\sum_{1\le i<j\le n} \overline{u_{ij}}u_{ij} = \frac{n}{2}.$$

**4.2.2.** ($n$ : odd). The generalized Poisson-Cauchy kernel function for Type II (Type III in [13]) for odd $n$ is given by

$$\boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u) = \frac{1}{(\det(I_n + \bar{u}z))^\ell}\left(\frac{\det(I_n + \bar{z}z)}{|\det(I_n + \bar{u}z)|^2}\right)^{(n-\ell-s)/2},$$

(Theorem 5 in [13]).

Using Theorem 1, we get

$$\begin{aligned}\Delta_\ell \boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u)|_{z=0} &= \frac{1}{2}\sum_{1\le i<j\le n}\frac{\partial^2}{\partial z_{ij}\partial\overline{z_{ij}}}\boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u)\bigg|_{z=0}\\ &= \sum_{1\le i<j\le n}((n-\ell-s)/2)((\ell+(n-\ell-s)/2)2\overline{u_{ij}}u_{ij}-1)\\ &= \frac{1}{4}(n-1)(s-\ell)(s+\ell-n).\end{aligned}$$

Here we used the following property of $u \in \check{\boldsymbol{S}}_{II}$

$$\sum_{1\le i<j\le n}\overline{u_{ij}}u_{ij} = \frac{n-1}{2}.$$

**4.3. Type III.** The generalized Poisson-Cauchy kernel function for Type III (Type II in [13]) is given by

$$\boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u) = \frac{1}{(\det(I_n - \bar{u}z))^\ell}\left(\frac{\det(I_n - \bar{z}z)}{|\det(I_n - \bar{u}z)|^2}\right)^{(n+1-\ell-s)/2},$$

(Theorem 3 in [13]).

Using Theorem 1, we get

$$\begin{aligned}\Delta_\ell \boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u)|_{z=0} &= \left(\sum_{1\le k\le n}\frac{\partial^2}{\partial z_{kk}\partial\overline{z_{kk}}} + \frac{1}{2}\sum_{1\le i<j\le n}\frac{\partial^2}{\partial z_{ij}\partial\overline{z_{ij}}}\right)\boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u)\bigg|_{z=0}\\ &= \sum_{1\le k\le n}((n+1-\ell-s)/2)((\ell+(n+1-\ell-s)/2)\overline{u_{kk}}u_{kk}-1)\\ &\quad + \sum_{1\le i<j\le n}((n+1-\ell-s)/2)\\ &\quad \times((\ell+(n+1-\ell-s)/2)2\overline{u_{ij}}u_{ij}-1)\\ &= \frac{1}{4}n(s-\ell)(s+\ell-n-1).\end{aligned}$$

Here we used the following property of $u \in \check{\boldsymbol{S}}_{III}$

$$\sum_{1\le k\le n}\overline{u_{kk}}u_{kk} + 2\sum_{1\le i<j\le n}\overline{u_{ij}}u_{ij} = n.$$

**4.4. Type IV.** The generalized Poisson-Cauchy kernel function for Type IV is given by

$$\boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u) = \left(\frac{e^{2i\theta}}{{}^t(u-z)(u-z)}\right)^{\ell}\left(\frac{1+|{}^tzz|^2-2z^*z}{|{}^t(u-z)(u-z)|^2}\right)^{(n-\ell-s)/2},$$

(Theorem 6 in [13]).

Using Theorem 1, we get

$$\begin{aligned}\Delta_\ell \boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u)|_{z=0} &= \sum_{1\le i\le n}\frac{\partial^2}{\partial z_i\partial\overline{z_i}}\boldsymbol{K}_{\tau_\ell,\eta_{\ell,s}}(z,u)\Big|_{z=0}\\ &= \sum_{1\le i\le n}((n-\ell-s)/2)((\ell+(n-\ell-s)/2)4\overline{u_i}u_i-2)\\ &= (s-\ell)(s+\ell-n).\end{aligned}$$

Here we used the following property of $u\in\check{\boldsymbol{S}}_{IV}$

$$\sum_{1\le i\le n}\overline{u_i}u_i = 1.$$

## 5. The Casimir operator on various homogeneous line bundles

Let $P=MAN$ be the Langlands decomposition of $P$ and let $\mathfrak{m}$ (resp. $\mathfrak{a}$, $\mathfrak{n}$) be the Lie algebras corresponding to $M$ (resp. $A$, $N$). Then $MA$ is the centralizer of $A$ in $G$ and $N$ a normal subgroup of $P$. Moreover $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{a}$ in the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ with respect to the Killing form. Let $\mathfrak{z}_\mathfrak{m}$ be the center of $\mathfrak{m}$. Then we can show that $\mathfrak{z}_\mathfrak{m}\subset\mathfrak{k}$ (see the following subsections). Since $\mathfrak{m}$ is reductive, we have

$$\mathfrak{m} = \mathfrak{z}_\mathfrak{m} + [\mathfrak{m},\mathfrak{m}] \qquad \text{(direct sum)}.$$

There exists a Cartan subalgebra $\hat{\mathfrak{h}}$ of $\mathfrak{g}$ such that

$$\hat{\mathfrak{h}} = \mathfrak{a} + \mathfrak{z}_\mathfrak{m} + \hat{\mathfrak{h}}\cap[\mathfrak{m},\mathfrak{m}] \qquad \text{(direct sum)}.$$

For each linear form $\lambda$ on $\hat{\mathfrak{h}}_c$, we denote by $\hat{H}_\lambda$ the element of $\hat{\mathfrak{h}}_c$ such that $B(\hat{H}_\lambda,\hat{H})=\lambda(\hat{H})$ for all $\hat{H}\in\hat{\mathfrak{h}}_c$. Let $\hat{\Delta}$ be the set of all non-zero roots of the pair $(\mathfrak{g}_c,\hat{\mathfrak{h}}_c)$. For each $\hat{\alpha}\in\hat{\Delta}$ we choose a root vector $\hat{X}_{\hat{\alpha}}$ associated with the root $\hat{\alpha}$. We can choose a linear order on $\hat{\Delta}$ such that if $\hat{X}_{\hat{\alpha}}\in\mathfrak{n}_c$ then $\hat{\alpha}$ is positive. Let $\theta$ denote the Cartan involution. Then it is easy to see that we can normalize $\hat{X}_{\hat{\alpha}}$ so that $B(\hat{X}_{\hat{\alpha}},\theta\hat{X}_{\hat{\alpha}})=1$. This implies that $[\hat{X}_{\hat{\alpha}},\theta\hat{X}_{\hat{\alpha}}]=\hat{H}_{\hat{\alpha}}$. For each $\hat{\alpha}\in\hat{\Delta}$ we see that

$$\theta(\hat{H}_{\hat{\alpha}}) = \theta([\hat{X}_{\hat{\alpha}},\theta\hat{X}_{\hat{\alpha}}]) = [\theta\hat{X}_{\hat{\alpha}},\hat{X}_{\hat{\alpha}}] = -[\hat{X}_{\hat{\alpha}},\theta\hat{X}_{\hat{\alpha}}] = -\hat{H}_{\hat{\alpha}},$$

which implies $\hat{H}_{\hat{\alpha}} \in \mathfrak{p}$. Since we assumed that $G/K$ is irreducible, we have $\dim \mathfrak{a} = 1$. We can choose $H \in \mathfrak{a}$ such that for any root $\hat{\alpha} \in \hat{\Delta}$, $\hat{\alpha}(H) > 0$ if and only if $\hat{X}_{\hat{\alpha}} \in \mathfrak{n}_c$. We put $\hat{\rho}_+ = \frac{1}{2}\sum_{\hat{\alpha}(H)>0} \hat{\alpha}$. Notice that $\hat{H}_{2\hat{\rho}_+} = \sum_{\hat{\alpha}(H)>0} \hat{H}_{\hat{\alpha}} \in \mathfrak{p}$ and that $\mathfrak{z}_{\mathfrak{m}} \subset \mathfrak{k}$. Then there exist constants $a, b \in \boldsymbol{R}$ and $X \in \hat{\mathfrak{h}} \cap [\mathfrak{m}, \mathfrak{m}]$ such that $\hat{H}_{2\hat{\rho}_+} = aH + bX$. Since $X \in [\mathfrak{m}, \mathfrak{m}]$, $d\xi(\hat{H}_{2\hat{\rho}_+}) = a\, d\xi(H)$. Moreover since $X \in \mathfrak{m}$, we have $2\hat{\rho}_+(H) = B(\hat{H}_{2\hat{\rho}_+}, H) = aB(H, H)$, which implies that

$$d\xi(\hat{H}_{2\hat{\rho}_+}) = \frac{2\hat{\rho}_+(H)}{B(H,H)}\, d\xi(H).$$

We denote by $\Omega$ the Casimir operator (see [5], [17]). It is easy to see that the Killing form $B$ is non-degenerate on $\mathfrak{m}$. We denote by $\Omega_{\mathfrak{m}}$ the Casimir operator of $\mathfrak{m}$ defined by the restriction of $B$ to $\mathfrak{m}$. Put $\hat{\mathfrak{h}}_{\mathfrak{m}} = \mathfrak{z}_{\mathfrak{m}} + \hat{\mathfrak{h}} \cap [\mathfrak{m}, \mathfrak{m}]$. Then it is easy to see that $\mathfrak{n}_c = \sum_{\hat{\alpha}(H)>0} \hat{X}_{\hat{\alpha}}$ and that

$$\mathfrak{m}_c = (\hat{\mathfrak{h}}_{\mathfrak{m}})_c + \sum_{\hat{\alpha}>0,\, \hat{\alpha}(H)=0} (\boldsymbol{C}\hat{X}_{\hat{\alpha}} + \boldsymbol{C}\theta\hat{X}_{\hat{\alpha}}).$$

We put

$$c_\xi = \frac{1}{B(H,H)}(d\xi(H)^2 - 2\hat{\rho}_+(H)d\xi(H)) + d\xi(\Omega_{\mathfrak{m}}).$$

Then it is clear that $c_\xi$ is independent of the choice of $H$. Let $\Omega_\xi$ be the restriction of $\Omega$ on $\boldsymbol{C}^\infty(G)_\xi$. Owing to the definition of $\boldsymbol{C}^\infty(G)_\xi$, for any $X \in \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ and $\phi \in \boldsymbol{C}^\infty(G)_\xi$, we have

$$X\phi = -d\xi(X)\phi.$$

Since $[\mathfrak{a}, \mathfrak{n}] = \mathfrak{n}$, $X\phi = 0$ *for all* $X \in \mathfrak{n}$. Moreover, since $\xi$ is a character, $d\xi([\mathfrak{m}, \mathfrak{m}]) = \{0\}$. Bearing these facts in mind, we see that

$$\begin{aligned}
\Omega_\xi &= \left(\frac{1}{B(H,H)}H^2 + \sum_{\hat{\alpha}(H)>0} (\hat{X}_{\hat{\alpha}}\theta\hat{X}_{\hat{\alpha}} + \theta\hat{X}_{\hat{\alpha}}\hat{X}_{\hat{\alpha}}) + \Omega_{\mathfrak{m}}\right)\Bigg|_{\boldsymbol{C}^\infty(G)_\xi} \\
&= \left(\frac{1}{B(H,H)}H^2 + \sum_{\hat{\alpha}(H)>0} [\hat{X}_{\hat{\alpha}}, \theta\hat{X}_{\hat{\alpha}}] + \Omega_{\mathfrak{m}}\right)\Bigg|_{\boldsymbol{C}^\infty(G)_\xi} \\
&= \left(\frac{1}{B(H,H)}H^2 + \hat{H}_{2\hat{\rho}_+} + \Omega_{\mathfrak{m}}\right)\Bigg|_{\boldsymbol{C}^\infty(G)_\xi} \\
&= \left(\frac{1}{B(H,H)}(d\xi(H)^2 - 2\hat{\rho}_+(H)d\xi(H)) + d\xi(\Omega_{\mathfrak{m}})\right)\boldsymbol{I}_{\boldsymbol{C}^\infty(G)_\xi} \\
&= c_\xi \boldsymbol{I}_{\boldsymbol{C}^\infty(G)_\xi}.
\end{aligned}$$

For any $g \in G$ and $\phi \in \boldsymbol{C}^{\infty}(G)_{\xi}$, we define

$$(\pi_{\xi}(g)\phi)(x) = \phi(g^{-1}x) \qquad (x \in G).$$

Then $\pi_{\xi}$ is a representation of $G$ on $\boldsymbol{C}^{\infty}(G)_{\xi}$. For any $g \in G$, $X \in \mathfrak{g}$ and $\phi \in \boldsymbol{C}^{\infty}(G)_{\xi}$, we have

$$X\phi(g) = \frac{d}{dt}\phi(g \exp tX)|_{t=0} = \frac{d}{dt}\phi(\exp t(Ad(g)X)g)|_{t=0} = d\pi_{\xi}(-Ad(g)X)\phi(g).$$

Notice that the Killing form $B$ is invariant by the adjoint action of $G$. Then it is easy to prove the following proposition, which is crucial in our method.

Proposition 3.

$$d\pi_{\xi}(\Omega) = \Omega_{\xi} = c_{\xi}\boldsymbol{I}_{\boldsymbol{C}^{\infty}(G)_{\xi}}.$$

For any $g \in G$, we define $T_{\eta}(g)$ such that the following diagram is commutative.

$$\begin{array}{ccccccc} & \boldsymbol{C}^{\infty}(G)_{\xi} & \cong & \boldsymbol{C}^{\infty}(\boldsymbol{L}_{\eta}) & \cong & \boldsymbol{C}^{\infty}(\check{\boldsymbol{S}}) & \\ \pi_{\xi}(g) & \downarrow & \pi_{\eta}(g) & \downarrow & & \downarrow & T_{\eta}(g) \\ & \boldsymbol{C}^{\infty}(G)_{\xi} & \cong & \boldsymbol{C}^{\infty}(\boldsymbol{L}_{\eta}) & \cong & \boldsymbol{C}^{\infty}(\check{\boldsymbol{S}}). & \end{array}$$

It follows from Proposition 3 that

$$dT_{\eta}(\Omega) = c_{\xi}\boldsymbol{I}_{\boldsymbol{C}^{\infty}(\check{\boldsymbol{S}})}.$$

The next theorem follows now from the fact that the generalized Poisson-Cauchy transform $\boldsymbol{P}_{\tau,\eta}$ is an intertwining operator between the representations $T_{\eta}$ and $T_{\tau}$.

Theorem 3. *For any $\Phi \in \boldsymbol{C}^{\infty}(\check{\boldsymbol{S}})$, define*

$$F(z) = \int_{\check{\boldsymbol{S}}} \boldsymbol{K}_{\tau,\eta}(z,u)\Phi(u)du \qquad (z \in \boldsymbol{D}).$$

*Then we have*

$$dT_{\tau}(\Omega)F(z) = c_{\xi}F(z) \qquad (z \in \boldsymbol{D}).$$

Before we go into the argument for each type of the classical domains, we would like to correct the statement about $\alpha$ on page 76 and $\rho(H)$ on page 77 in [13]. In general, $\alpha$ is not a root but a linear form on $\mathfrak{a}$. One should define $\rho(H) = \frac{1}{2}\operatorname{Tr}(ad(H)|_{\mathfrak{n}})$ in [13]. However, in this paper, we use the notation $\rho$ for the half sum of positive roots of the pair $(\mathfrak{g}_c, \mathfrak{h}_c)$. Instead of $\rho$ in [13], we use the notation $\hat{\rho}_+$ in this paper, so that $\hat{\rho}_+(H) = \frac{1}{2}\sum_{\hat{\alpha}(H)>0} \hat{\alpha}(H) = \frac{1}{2}\operatorname{Tr}(ad(H)|_{\mathfrak{n}})$.

We remark that if we choose $H$ as in [13] and choose a linear form $\alpha$ on $\mathfrak{a}$ such that $\alpha(H) = 1$, then it is clear that $\hat{\rho}_+(H) = \frac{1}{2}(\dim \mathfrak{n}_1 + 2 \dim \mathfrak{n}_2)$ (see [13], for the definition of $\mathfrak{n}_1$, $\mathfrak{n}_2$ and the value: $\dim \mathfrak{n}_1 + 2 \dim \mathfrak{n}_2$).

In the succeeding subsections, we use the same $H$ as in [13] and the following facts, which were proved in [13].

LEMMA 6. *For each type of the classical domains, $2\hat{\rho}_+(H)$ is given as follows.*

(*Type I*) $2mn$, (*Type II*) $n(n-1)$, (*Type III*) $n(n+1)$, (*Type IV*) $n$.

The explicit formula of the eigenvalue $c_\xi$ is now given by the following theorem.

THEOREM 4. *For each type of the classical domains, $c_{\xi_{\ell,s}}$ is given as follows.*

$$(\textit{Type I}) \qquad \frac{m}{4(n+m)}\left(s^2 - 2ns + \frac{n-m}{n+m}\ell^2\right),$$

$$(\textit{Type II},\ n:\textit{even}) \qquad \frac{ns}{4(n-1)}(s-n+1),$$

$$(\textit{Type II},\ n:\textit{odd}) \qquad \frac{1}{4(n-1)}((n-1)s^2 - n(n-1)s + \ell^2),$$

$$(\textit{Type III}) \qquad \frac{ns}{4(n+1)}(s-n-1),$$

$$(\textit{Type IV}) \qquad \frac{s}{2n}(s-n).$$

We prove this theorem in the following subsections for each type of classical domains. The computations in proof are straightforward and elementary calculus of matrices, so that we omit the details.

**5.1. Type I.** Put

$$H = \begin{pmatrix} 0 & 0 & I_m \\ 0 & 0 & 0 \\ I_m & 0 & 0 \end{pmatrix}, \qquad \mu_0 = \begin{pmatrix} I_m & 0 & I_m \\ 0 & I_{n-m} & 0 \\ 0 & 0 & I_m \end{pmatrix}.$$

Clearly

$$d\xi_{\ell,s}(H) = d\eta_{\ell,s}(\mu_0^{-1} H \mu_0) = s\operatorname{Tr}(I_m) = ms.$$

Thus

$$B(H,H) = 4m(n+m), \qquad d\xi_{\ell,s}(H) = ms, \qquad 2\hat{\rho}_+(H) = 2mn.$$

If $n = m$, then we can prove that $\mathfrak{z}_{\mathfrak{m}} = \{0\}$, $d\xi_{\ell,s}(\Omega_{\mathfrak{m}}) = 0$. If $n > m$, put

$$Z = \begin{pmatrix} i(n-m)I_m & 0 & 0 \\ 0 & -2im\boldsymbol{I}_{n-m} & 0 \\ 0 & 0 & i(n-m)I_m \end{pmatrix}.$$

Then we can prove that $\mathfrak{z}_{\mathfrak{m}} = \boldsymbol{R}Z \subset \mathfrak{k}$.

Clearly

$$d\xi_{\ell,s}(Z) = d\eta_{\ell,s}(\mu_0^{-1}Z\mu_0) = \ell(n-m)i\,\mathrm{Tr}(I_m) = im(n-m)\ell.$$

Thus

$$B(Z,Z) = -4m(n+m)^2(n-m), \qquad d\xi_{\ell,s}(Z) = im(n-m)\ell,$$

$$d\xi_{\ell,s}(\Omega_{\mathfrak{m}}) = \frac{m(n-m)\ell^2}{4(n+m)^2}.$$

Hence

$$c_{\xi_{\ell,s}} = \frac{m}{4(n+m)}\left(s^2 - 2ns + \frac{n-m}{n+m}\ell^2\right).$$

### 5.2. Type II.

**5.2.1.** ($n$ : even, $m = \frac{n}{2}$). Put

$$\sigma = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \qquad \mu_0 = \begin{pmatrix} I_n & \sigma \\ 0 & I_n \end{pmatrix}.$$

Clearly

$$d\xi_{\ell,s}(H) = d\eta_{\ell,s}(\mu_0^{-1}H\mu_0) = s\,\mathrm{Tr}(-\sigma^2) = ns.$$

Thus

$$B(H,H) = 4n(n-1), \qquad d\xi_{\ell,s}(H) = ns, \qquad 2\hat{\rho}_+(H) = n(n-1).$$

We can prove that $\mathfrak{z}_{\mathfrak{m}} = \{0\}$, $d\xi_{\ell,s}(\Omega_{\mathfrak{m}}) = 0$.

Hence

$$c_{\xi_{\ell,s}} = \frac{ns}{4(n-1)}(s-n+1).$$

**5.2.2.** ($n$ : odd, $m = \frac{n-1}{2}$). Put

$$\sigma = \begin{pmatrix} 0 & 0 & I_m \\ 0 & 0 & 0 \\ -I_m & 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}.$$

Clearly

$$d\xi_{\ell,s}(H) = d\eta_{\ell,s}(\mu_0^{-1} H \mu_0) = s\,\mathrm{Tr}(-\sigma^2) = 2m = (n-1)s.$$

Thus

$$B(H,H) = 4(n-1)^2, \qquad d\xi_{\ell,s}(H) = (n-1)s, \qquad 2\hat{\rho}_+(H) = n(n-1).$$

Put

$$Z = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}.$$

Then we can prove that $\mathfrak{z}_{\mathfrak{m}} = \boldsymbol{R}Z \subset \mathfrak{k}$.

Clearly

$$d\xi_{\ell,s}(H) = d\eta_{\ell,s}(\mu_0^{-1} Z \mu_0) = \ell\,\mathrm{Tr}\begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} = i\ell.$$

Thus

$$B(Z,Z) = -4(n-1), \qquad d\xi_{\ell,s}(Z) = i\ell, \qquad d\xi_{\ell,s}(\Omega_{\mathfrak{m}}) = \frac{1}{4(n-1)}\ell^2.$$

Hence

$$c_{\xi_{\ell,s}} = \frac{1}{4(n-1)}((n-1)s^2 - n(n-1)s + \ell^2).$$

**5.3. Type III.** Put

$$H = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \qquad \mu_0 = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}.$$

Clearly

$$d\xi_{\ell,s}(H) = d\eta_{\ell,s}(\mu_0^{-1} H \mu_0) = s\,\mathrm{Tr}(I_n) = ns.$$

Thus

$$B(H,H) = 4n(n+1), \qquad d\xi_{\ell,s}(H) = ns, \qquad 2\hat{\rho}_+(H) = n(n+1).$$

We can prove that $\mathfrak{z}_{\mathfrak{m}} = \{0\}$, $d\xi_{\ell,s}(\Omega_{\mathfrak{m}}) = 0$.

Hence

$$c_{\xi_{\ell,s}} = \frac{ns}{4(n+1)}(s-n-1).$$

**5.4. Type IV.** Put

$$\gamma = \begin{pmatrix} I_n & 0 \\ 0 & \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \end{pmatrix}, \qquad \mu_0 = \gamma^{-1} \begin{pmatrix} I_n & 0 & u_0 \\ 2\,{}^t u_0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \gamma,$$

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in M_{n+2}(R).$$

Clearly

$$d\xi_{\ell,s}(H) = d\eta_{\ell,s}(\mu_0^{-1} H \mu_0) = d\eta_{\ell,s}\left(\gamma^{-1} \begin{pmatrix} 0 & \frac{1}{2}u_0 & 0 \\ 0 & -1 & 0 \\ {}^t u_0 & 0 & 1 \end{pmatrix} \gamma\right) = s.$$

Thus

$$B(H,H) = 2n, \qquad d\xi_{\ell,s}(H) = s, \qquad 2\hat{\rho}_+(H) = n.$$

We can prove that $\mathfrak{z}_{\mathfrak{m}} = \{0\}$, $d\xi_{\ell,s}(\Omega_{\mathfrak{m}}) = 0$.

Hence

$$c_{\xi_{\ell,s}} = \frac{s}{2n}(s-n).$$

## 6. The Casimir operator and generalized Laplacians

We keep the notation of the previous sections. Since we assumed that $G/K$ is irreducible, it is easy to show that there exist constants $a$ and $b$ such that

$$dT_\tau(\Omega) = a\Delta_\tau + b\boldsymbol{I}_{\boldsymbol{C}^\infty(\boldsymbol{D})}.$$

From Theorem 4.1 in [12], considering the case where $q = 0$ (namely, differential forms of degree zero), we obtain the following lemma.

Lemma 7. *Let* $\square^0$ *denote the* $\boldsymbol{E}_\tau$*-valued complex Laplacian defined on* $\boldsymbol{C}^\infty(\boldsymbol{E}_\tau)$ *(see [12]). Then*

$$d\pi_\tau(\Omega) = -2\square^0 + \langle \Lambda + 2\rho, \Lambda \rangle \boldsymbol{I}_{\boldsymbol{C}^\infty(\boldsymbol{E}_\tau)}.$$

Recall that the representations $\pi_\tau$ and $T_\tau$ satisfy the following commutative diagram.

$$\begin{array}{ccccc} & \boldsymbol{C}^\infty(\boldsymbol{E}_\tau) & \cong & \boldsymbol{C}^\infty(\boldsymbol{D}) & \\ \pi_\tau(g) & \downarrow & & \downarrow & T_\tau(g) \\ & \boldsymbol{C}^\infty(\boldsymbol{E}_\tau) & \cong & \boldsymbol{C}^\infty(\boldsymbol{D}). & \end{array}$$

We define $\tilde{\square}^0$ such that the following diagram is commutative.

$$\begin{array}{ccccc} & \boldsymbol{C}^\infty(\boldsymbol{E}_\tau) & \cong & \boldsymbol{C}^\infty(\boldsymbol{D}) & \\ \square^0 & \downarrow & & \downarrow & \tilde{\square}^0 \\ & \boldsymbol{C}^\infty(\boldsymbol{E}_\tau) & \cong & \boldsymbol{C}^\infty(\boldsymbol{D}). & \end{array}$$

Then clearly

$$dT_\tau(\Omega) = -2\tilde{\square}^0 + \langle \Lambda + 2\rho, \Lambda \rangle \boldsymbol{I}_{\boldsymbol{C}^\infty(\boldsymbol{D})}.$$

Notice that

$$\tilde{\square}^0 = \vartheta\bar{\partial},$$

where $\vartheta$ is the adjoint operator of $\bar{\partial}$. Remark that for any $F \in \boldsymbol{C}^\infty(\boldsymbol{D})$, by definition, $\bar{\partial}F(z) = \sum_{1 \le k \le N} \frac{\partial F(z)}{\partial \bar{z}} d\bar{z}$ $(z \in \boldsymbol{D})$. Applying $dT_\tau(\Omega)$ to the constant function 1 on $\boldsymbol{D}$, from the above two formulas we have

$$b = \langle \Lambda + 2\rho, \Lambda \rangle.$$

Thus we obtain the following Proposition.

Proposition 4. *There exists a constant a such that*

$$dT_\tau(\Omega) = a\Delta_\tau + \langle \Lambda + 2\rho, \Lambda \rangle \boldsymbol{I}_{\boldsymbol{C}^\infty(\boldsymbol{D})}.$$

It is obvious that the value of $a$ in this proposition can be obtained from the value $c$ in Proposition 1, namely $a = \frac{4}{c}$.

This gives the explicit formulas of $a$ in the following theorem.

Theorem 5. *For each type of the classical domains, a and $\langle \Lambda + 2\rho, \Lambda \rangle$ are given as follows.*

$$(Type\ I) \quad a = \frac{1}{n+m}, \qquad \langle \Lambda + 2\rho, \Lambda \rangle = \frac{mn\ell(\ell - n - m)}{2(n+m)^2},$$

$$(Type\ II) \quad a = \frac{1}{n-1}, \qquad \langle \Lambda + 2\rho, \Lambda \rangle = \frac{n\ell(\ell - n + 1)}{4(n-1)},$$

$$(Type\ III) \quad a = \frac{1}{n+1}, \qquad \langle \Lambda + 2\rho, \Lambda \rangle = \frac{n\ell(\ell - n - 1)}{4(n+1)},$$

$$(Type\ IV) \quad a = \frac{1}{2n}, \qquad \langle \Lambda + 2\rho, \Lambda \rangle = \frac{\ell(\ell - n)}{2n}.$$

We prove this theorem in the following subsections for each type of classical domains.

Remark. It is clear that $c_{\xi_{\ell,s}} = ac_{\ell,s} + \langle \Lambda + 2\rho, \Lambda \rangle$. This shows that Theorem 2 follows from Theorem 4 and Theorem 5, namely the eigenvalue $c_{\ell,s}$ is determined by $\frac{1}{a}(c_{\xi_{\ell,s}} - \langle \Lambda + 2\rho, \Lambda \rangle)$.

In the following subsections, we use the list of root systems given at Appendix on page 20 in [17].

**6.1. Type I.** The type of the Lie algebra $\mathfrak{g}$ is $(\mathbf{A}_{\boldsymbol{l}})$, where $\boldsymbol{l} = n + m - 1$.
It is easy to see

$$\Lambda = -\ell(e_1 + \cdots + e_n), \qquad H_\Lambda = \frac{\ell}{2(n+m)^2}\begin{pmatrix} -mI_n & 0 \\ 0 & nI_m \end{pmatrix}.$$

Thus

$$\langle \Lambda + 2\rho, \Lambda \rangle = \frac{mn\ell(\ell - n - m)}{2(n+m)^2}.$$

**6.2. Type II.** The type of the Lie algebra $\mathfrak{g}$ is $(\mathbf{D}_{\boldsymbol{l}})$, where $\boldsymbol{l} = n$.
It is easy to see

$$\Lambda = -\ell(e_1 + \cdots + e_n), \qquad H_\Lambda = \frac{-\ell}{4(n-1)}\begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

Thus

$$\langle \Lambda + 2\rho, \Lambda \rangle = \frac{n\ell(\ell - n + 1)}{4(n-1)}.$$

**6.3. Type III.** The type of the Lie algebra $\mathfrak{g}$ is $(\mathbf{C}_{\boldsymbol{l}})$, where $\boldsymbol{l} = n$.
It is easy to see

$$\Lambda = -\ell(e_1 + \cdots + e_n), \qquad H_\Lambda = \frac{-\ell}{4(n+1)}\begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

Thus

$$\langle \Lambda + 2\rho, \Lambda \rangle = \frac{n\ell(\ell - n - 1)}{4(n+1)}.$$

**6.4. Type IV.** If $n$ is even, the type of the Lie algebra $\mathfrak{g}$ is $(\mathbf{D}_{\boldsymbol{l}})$, where $\boldsymbol{l} = \frac{n+2}{2}$. If $n$ is odd, the type of the Lie algebra $\mathfrak{g}$ is $(\mathbf{B}_{\boldsymbol{l}})$, where $\boldsymbol{l} = \frac{n+1}{2}$.

It is easy to see

$$\Lambda = -\ell e_1, \qquad H_\Lambda = \frac{-\ell}{2n}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right).$$

Thus

$$\langle \Lambda + 2\rho, \Lambda \rangle = \frac{\ell(\ell - n)}{2n}.$$

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