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Singer's formulas for the Steenrod algebra

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Abstract. We prove Singer's formulas and unstable acyclic relations for the mod 2 Steenrod algebra, which are important tools for the computations of products. These are given by analogy of the corresponding dual results for the lambda algebra.

1. Introduction

Singer's formulas [3] are important and basic for the lambda algebra Λ. In fact, by using Singer's formulas, Richter [2] reproved many fundamental results and the author [1] proved some acyclic relations. We refer to [2] for Singer's formulas. On the other hand, the mod 2 Steenrod algebra A, which is the algebra of the cohomology operations, is crucial for the algebrac topology. In this paper, we shall prove Singer's formulas and acyclic relations for A, which are important tools for the computations of products in A.

A monomial $Sq^{a_s}\dots Sq^{a_1}$ is called admissible if $a_{i+1} \geq 2a_i$. Consider the submodule $A^{s,t}[n] \subset A$ $(n \geq 0)$ spanned by the admissible monomials $Sq^{a_s}\dots Sq^{a_1}$ $(t = \sum_{i=1}^{s} a_i)$ with $a_1 > n$, and let $A[n] = \bigcup_{t\geq 1} \bigoplus_{s\geq 1} A^{s,t}[n]$. Our first Singer's formula is the following.

Theorem 1.

$$A[n+t-s]A^{s,t}[n] \subset A[n].$$

Examples

(1) We see $Sq^6 \in A^{1,6}[5] \subset A^{1,6}[0]$, and so $Sq^6 Sq^6 \in A[5]A^{1,6}[0] \subset A[0]$. In fact, $Sq^6 Sq^6 = Sq^{11}Sq^1 + Sq^{10}Sq^2 + Sq^9 Sq^3$.

(2) We see $Sq^8 \in A[7]$, $Sq^6 \in A^{1,6}[2]$, and so $Sq^8 Sq^6 \in A[2]$. In fact, $Sq^8 Sq^6 = Sq^{11}Sq^3 + Sq^{10}Sq^4$.

(3) We see $Sq^8 Sq^4 \in A[3]$, $Sq^4 Sq^1 \in A^{2,5}[0]$, and so $Sq^8 Sq^4 Sq^4 Sq^1 \in A[0]$.

(4) We see $Sq^{15}Sq^6 + Sq^{14}Sq^7 \in A[5]$, $Sq^6 Sq^1 + Sq^5 Sq^2 \in A^{2,7}[0]$, and so

$$(Sq^{15}Sq^6 + Sq^{14}Sq^7)(Sq^6 Sq^1 + Sq^5 Sq^2) \in A[0].$$

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Theorem 1 induces a unstable composition product $\alpha \smile \beta = \alpha\beta \in A[n]$ for $\alpha \in A[n+t-s]$, $\beta \in A^{s,t}[n]$.

Now we define a map (Hopf invariant)

$$H : A^{s,t}[n-1] \to A^{s-1,t-n}[2n-1]$$

by $H(Sq^I Sq^n) = Sq^I$, $H(Sq^I Sq^i) = 0$ for admissible sequences (I,n), (I,i) with $i > n$, where $Sq^I = Sq^{i_{s-1}} \dots Sq^{i_1}$ for $I = (i_{s-1}, \dots, i_1)$, and a map (suspension)

$$E : A^{s,t}[n] \to A^{s,t}[n-1]$$

which is a natural inclusion. We notice that $HE = 0$. Moreover there is a map

$$\theta : A^{s,t}[n] \to A^{s,2t-s}[2n]$$

defined by $\theta(Sq^I) = \theta(Sq^{i_s} \dots Sq^{i_1}) = Sq^{2i_s-1} \dots Sq^{2i_1-1}$ for admissible sequences I. By using these maps, we have the second Singer's formula for the Hopf invariant.

THEOREM 2. *For* $\alpha \in A[n+t-s]$, $\beta \in A^{s,t}[n]$

$$EH(\alpha \smile \beta) = \alpha \smile EH(\beta) + EH(\alpha) \smile \theta(\beta).$$

We notice that

$$H(\alpha \smile \beta) \in A[2n+1], \qquad \alpha \smile EH(\beta), \qquad EH(\alpha) \smile \theta(\beta) \in A[2n].$$

Hence an implication of Theorem 2 is

$$\alpha \smile EH(\beta) + EH(\alpha) \smile \theta(\beta) \in A[2n+1].$$

COROLLARY 1. *Let* $\alpha \in A[m]$, $\beta \in A^{s,t}[n]$.

i) *If* $m > n+t-s$, *then*

$$\alpha \in A[n+t-s], \qquad H(\alpha) = 0, \qquad EH(\alpha \smile \beta) = \alpha \smile EH(\beta).$$

ii) *If* $t-s \le m < n+t-s$, *then*

$$\beta \in A^{s,t}[m-t+s], \qquad H(\beta) = 0, \qquad EH(\alpha \smile \beta) = EH(\alpha) \smile \theta(\beta).$$

Examples

(1) Let $\alpha = Sq^6 \in A[5]$, $\beta = Sq^6 \in A^{1,6}[0]$. Then $\alpha\beta \in A[0]$, $H(\beta) = 0$ and $H(\alpha\beta) = H(\alpha)\theta(\beta) = Sq^{11}$. This implies $\alpha\beta = Sq^{11}Sq^1 + \gamma$ for some $\gamma \in A[1]$. In fact

$$Sq^6 Sq^6 = Sq^{11}Sq^1 + Sq^{10}Sq^2 + Sq^9 Sq^3.$$

(2) We see $\alpha = Sq^8 \in A[7]$, $\beta = Sq^6 \in A^{1,6}[2]$. Then $\alpha\beta \in A[2]$, $H(\beta) = 0$ and $H(\alpha\beta) = H(\alpha)\theta(\beta) = Sq^{11}$. This implies $\alpha\beta = Sq^{11}Sq^3 + \gamma$ for some $\gamma \in A[3]$. In fact, $Sq^8 Sq^6 = Sq^{11}Sq^3 + Sq^{10}Sq^4$.

(3) Let $\alpha = Sq^8Sq^4 \in A[3]$, $\beta = Sq^4Sq^1 \in A^{2,5}[0]$. Then $\alpha\beta \in A[0]$. Moreover

$$H(\alpha\beta) = \alpha H(\beta) + H(\alpha)\theta(\beta) = Sq^8Sq^4Sq^4 + Sq^8Sq^7Sq^1 = Sq^{10}Sq^4Sq^2.$$

Hence $\alpha\beta = Sq^{10}Sq^4Sq^2Sq^1 + \gamma$ for some $\gamma \in A[1]$. In fact $Sq^8Sq^4Sq^4Sq^1 = Sq^{10}Sq^4Sq^2Sq^1$.

(4) Let $\alpha = Sq^{15}Sq^6 + Sq^{14}Sq^7 \in A[5]$, $\beta = Sq^6Sq^1 + Sq^5Sq^2 \in A^{2,7}[0]$. Then $\alpha\beta \in A[0]$. Moreover

$$\begin{aligned} H(\alpha\beta) &= \alpha H(\beta) + H(\alpha)\theta(\beta) \\ &= (Sq^{15}Sq^6 + Sq^{14}Sq^7)Sq^6 + Sq^{15}(Sq^{11}Sq^1 + Sq^9Sq^3) \\ &= Sq^{15}Sq^{10}Sq^2 + Sq^{14}Sq^{11}Sq^2 \\ &= Sq^{19}Sq^6Sq^2 + Sq^{21}Sq^4Sq^2 + Sq^{20}Sq^5Sq^2 + Sq^{18}Sq^7Sq^2. \end{aligned}$$

Hence

$$\begin{aligned} \alpha\beta &= (Sq^{15}Sq^{10}Sq^2 + Sq^{14}Sq^{11}Sq^2)Sq^1 + \gamma \\ &= (Sq^{19}Sq^6 + Sq^{21}Sq^4 + Sq^{20}Sq^5 + Sq^{18}Sq^7)Sq^2Sq^1 + \gamma \end{aligned}$$

for some $\gamma \in A[1]$. In fact

$$\begin{aligned} &(Sq^{15}Sq^6 + Sq^{14}Sq^7)(Sq^6Sq^1 + Sq^5Sq^2) \\ &\quad = Sq^{15}Sq^{10}Sq^2Sq^1 + Sq^{14}Sq^{11}Sq^2Sq^1 \\ &\quad = (Sq^{19}Sq^6 + Sq^{21}Sq^4 + Sq^{20}Sq^5 + Sq^{18}Sq^7)Sq^2Sq^1. \end{aligned}$$

The proof of these theorems is given by analogy to that of Singer's formulas for the lambda algebra. The same approach gives us some acyclic relations.

THEOREM 3. *For $2a - 1 \geq n \geq a \geq 1$, the following is exact:*

$$A[n + 2a - 3] \xrightarrow{\smile Sq^{2a-1}} A[n-1] \xrightarrow{\smile Sq^a} A[n-a].$$

Examples

(1) For $\alpha = Sq^9Sq^4Sq^2 \in A[1]$, we see $\alpha Sq^2 = 0$, and so $\alpha = \beta Sq^3$ for some $\beta \in A[3]$. In fact, for $\beta = Sq^8Sq^4$, we have

$$\beta Sq^3 = Sq^8Sq^4Sq^3 = Sq^9Sq^4Sq^2 = \alpha.$$

(2) For $\alpha = Sq^{33}Sq^{15}Sq^6Sq^3 + Sq^{33}Sq^{14}Sq^7Sq^3 + Sq^{32}Sq^{15}Sq^7Sq^3 \in A[2]$, we see $\alpha Sq^3 = 0$, and so $\alpha = \beta Sq^5$ for some $\beta \in A[6]$. In fact, for $\beta = Sq^{31}Sq^{14}Sq^7 + Sq^{30}Sq^{15}Sq^7$, we have

$$\begin{aligned}\beta Sq^5 &= (Sq^{31}Sq^{14}Sq^7 + Sq^{30}Sq^{15}Sq^7)Sq^5\\ &= Sq^{33}Sq^{15}Sq^6Sq^3 + Sq^{33}Sq^{14}Sq^7Sq^3 + Sq^{32}Sq^{15}Sq^7Sq^3\\ &= \alpha.\end{aligned}$$

Theorem 3 is a special case of the unstable version of the following conjecture: Consider the 2-adic expasion of an integer $a = \sum_{i=1}^{k} 2^{t_i}$ for $t_1 < t_2 < \cdots < t_k$. We write $a^{(j)} = \sum_{i=1}^{j} 2^{t_i}$, and in particular $a^{(k)} = a$.

Conjecture. The following sequence is exact.

$$\bigoplus_{j=1}^{k} A \xrightarrow{f} A \xrightarrow{\cdot Sq^a} A,$$

where $f(\alpha_1, \dots, \alpha_k) = \sum_{j=1}^{k} \alpha_j Sq^{2a^{(j)}-1}$ and $(\cdot Sq^a)(\alpha) = \alpha Sq^a$.

In case $a = 2^t$, this reduces to a conjecture stated in [1]: The following sequence is exact.

$$A \xrightarrow{\cdot Sq^{2^{t+1}-1}} A \xrightarrow{\cdot Sq^{2^t}} A.$$

This paper gives a part of unstable acyclic relations for the Steenrod algebra suggested by W. Richter. I would like to thank him for his invaluable comments about the Steenrod and lambda algebras.

2. Singer's formulas

We refer to [4] for the Steenrod algebra A. The mod 2 Steenrod algebra is generated by Sq^a with Adem relations

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j \qquad a < 2b. \tag{1}$$

If $b - 1 \le a < 2b$, then $b - 1 - j < a - 2j$ for $j < a - b + 1$. Hence, for $a = 2b - 1$, $Sq^a Sq^b = 0$. For $b - 1 \le a < 2b - 1$,

$$Sq^a Sq^b = Sq^{2b-1} Sq^{a-b+1} + R, \tag{2}$$

$$R = \sum_{j=a-b+2}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j \in A^{2,a+b}[a-b+1].$$

Especially, $Sq^{b-1}Sq^b = Sq^{2b-1} + R$. Thus we have

$$Sq^a Sq^b \in A^{2,a+b}[a-b] \qquad \text{for } 1 \le b \le a < 2b-1. \tag{3}$$

By [4, Lemma 2.6], if $a = \sum a_i 2^i$, $b = \sum b_i 2^i$ $(0 \le a_i, b_i < 2)$, then $\binom{b}{a} \equiv \prod \binom{b_i}{a_i}$ (mod 2). By this formula,

$$\binom{n}{m} \equiv \binom{2n+1}{2m+1} \equiv \binom{2n+1}{2m} \equiv \binom{2n}{2m}, \qquad \binom{2n}{2m+1} \equiv 0. \tag{4}$$

If $a < 2b - 1$, we see by the Adem relations that

$$\begin{aligned}\theta(Sq^a)\theta(Sq^b) &= Sq^{2a-1}Sq^{2b-1} \\ &= \sum_{j=0}^{a-1}\binom{2b-2-j}{2a-1-2j}Sq^{2a+2b-2-j}Sq^j \\ &= \sum_{j'=1}^{[a/2]}\binom{b-1-j'}{a-2j'}Sq^{2(a+b-j')-1}Sq^{2j'-1},\end{aligned} \tag{5}$$

since

$$\binom{2b-2-j}{2a-1-2j} \equiv 0 \quad \text{for } j = 2j', \qquad \equiv \binom{b-1-j'}{a-2j'} \quad \text{for } j = 2j'-1$$

by (4). We notice that θ is defined primarily only on admissible elements. By the Adem relations and (5), for $a < 2b$,

$$\begin{aligned}\theta(Sq^aSq^b) &= \binom{b-1}{a}\theta(Sq^{a+b}) + \sum_{j=1}^{[a/2]}\binom{b-1-j}{a-2j}\theta(Sq^{a+b-j}Sq^j) \\ &= \binom{b-1}{a}Sq^{2(a+b)-1} + \sum_{j=1}^{[a/2]}\binom{b-1-j}{a-2j}Sq^{2(a+b-j)-1}Sq^{2j-1} \\ &= \binom{b-1}{a}Sq^{2(a+b)-1} + \theta(Sq^a)\theta(Sq^b).\end{aligned}$$

Hence

$$\theta(Sq^a)\theta(Sq^b) = \theta(Sq^aSq^b) \qquad \text{if } \binom{b-1}{a} \equiv 0 \tag{6}$$

for $a < 2b$. If $b \le a < 2b-1$, then $\binom{b-1}{a} \equiv 0$ in the Adem relations, and so this equation (6) holds for $b \le a < 2b-1$.

Now we prove Theorems 1 and 2 by analogy dual to [2]. The following is a condition for $Sq^I = Sq^{a_s}\dots Sq^{a_1} \in A^s[n]$, where $A^s[n]$ is a submodule spanned by the admissible monomials $Sq^{a_s}\dots Sq^{a_1}$ with $a_1 > n$.

Definition 1. A monomial $Sq^{a_s}\dots Sq^{a_1}$ is called n-pseudo-admissible if $a_i > n - i + 1 + \sum_{1 \le j < i} a_j$ for all $1 \le i \le s$.

Lemma 1. *If Sq^I is admissible and $a_1 > n$, then it is n-pseudo-admissible.*

Proof. We show by induction on i that $a_i > n - i + 1 + \sum_{1 \le j < i} a_j$. This equality obviously holds for $i = 1$. Assuming it is true for i, we have

$$a_{i+1} \ge 2a_i > n - i + 1 + \sum_{1 \le j < i} a_j + a_i > n - i + \sum_{1 \le j < i+1} a_j.$$

Hence Sq^I is n-pseudo-admissible.

Lemma 2. *If Sq^I is n-pseudo-admissible for $n \ge 0$ and $a_1 > 0$, then $Sq^I \in A^s[n]$.*

Proof. We shall show that Adem relations preserve n-pseudo-admissibility. For this purpose, we verify inductively $a_i \ge 1$ and $a_{i+1} > a_i - 1$. In fact,

$$a_{i+1} > n - i + \sum_{1 \le j < i+1} a_j \ge n - i + (i - 1) + a_i \ge a_i - 1 \ge 0.$$

Suppose for some i that $Sq^{a_{i+1}} Sq^{a_i}$ is inadmissible. By using the Adem relations (2) we write $Sq^{a_{i+1}} Sq^{a_i}$ as a sum of admissibles $Sq^a Sq^b$, where each $b \ge a_{i+1} - a_i + 1$. We shall show that all these monomials are n-pseudo-admissible:

$$Sq^J = Sq^{a_s} \dots Sq^{a_{i+2}} Sq^a Sq^b Sq^{a_{i-1}} \dots Sq^{a_1}. \tag{7}$$

Since Sq^I is n-pseudo-admissible,

$$b \ge a_{i+1} - a_i + 1 > n - i + \sum_{1 \le j < i+1} a_j - a_i + 1 = n - i + 1 + \sum_{1 \le j < i} a_j.$$

Now $Sq^a Sq^b$ is admissible, and so

$$a \ge 2b = b + b > n - i + \sum_{1 \le j < i} a_j + b.$$

Since $a + b = a_i + a_{i+1}$, we see that Sq^J is n-pseudo-admissible.

So Adem relations preserve n-pseudo-admissibility. Eventually we shall reach admissible form of which all terms are both admissible and n-pseudo-admissible. Hence all terms belong to $A^s[n]$.

Let $\theta(I) = (2i_s - 1, \dots, 2i_1 - 1)$ for any sequence $I = (i_s, \dots, i_1)$. Then $\theta : A^{s,t}[n] \to A^{s,2t-s}[2n]$ is defined by taking $\theta(Sq^I) = Sq^{\theta(I)}$ for any admissible sequence I. The equation (6) implies $\theta(Sq^a Sq^b) = Sq^{\theta(a,b)}$ for $b \le a$. In the proof above, $Sq^I = \sum Sq^J = \cdots = \sum Sq^K$, where each K is admissible. By definition, $\theta(Sq^K) = Sq^{\theta(K)}$. By induction, we assume $\theta(Sq^J) = Sq^{\theta(J)}$. Since $Sq^{(a_{i+1}, a_i)} = \sum Sq^{(a,b)}$, $Sq^{\theta(a_{i+1}, a_i)} = \sum Sq^{\theta(a,b)}$ by the equation (5) and $a_i \le a_{i+1} < 2a_i - 1$. Hence

$$\theta(Sq^I) = \sum \theta(Sq^J) = \sum Sq^{\theta(J)} = Sq^{\theta(I)}.$$

This implies the folowing:

Lemma 3. *If Sq^I is n-pseudo-admissible, then $\theta(Sq^I) = Sq^{\theta(I)}$.*

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. Take admissible monomials

$$Sq^J = Sq^{b_r} \dots Sq^{b_1} \in A[n+t-s], \qquad Sq^I = Sq^{a_s} \dots Sq^{a_1} \in A^{s,t}[n].$$

By Lemma 1, Sq^J and Sq^I are $(n+t-s)$- and n-pseudo-admissible, respectively. Then

$$b_i > n+t-s-i+1 + \sum_{1 \le j < i} b_j$$
$$= n-(s+i)+1 + \sum_{1 \le j \le s} a_j + \sum_{1 \le j < i} b_j$$

since $t = \sum_{j=1}^{s} a_j$. Hence $Sq^J Sq^I$ is n-pseudo-admissible, and so $Sq^J Sq^I \in A[n]$ by Lemma 2.

By this proof, $Sq^J Sq^I$ is n-pseudo-admissible if Sq^J and Sq^I are $(n+t-s)$- and n-pseudo-admissible, respectively. By Lemma 3,

$$\theta(Sq^J Sq^I) = Sq^{\theta(J)} Sq^{\theta(I)} = \theta(Sq^J)\theta(Sq^I). \tag{8}$$

For $\alpha \in A[n]$, we sometimes will desuspend α without mentioning it to give a smaller n. For instance, in Lemma 4 below, $Sq^t \in A^{1,t}[n]$ and $Sq^{2t-1} \in A^{1,2t-1}[2n+1]$ for $n < t-1$.

We shall prove Theorem 2 after investigating the special case $s = 1$ in Lemma 4.

Lemma 4. *For $\alpha \in A[n+t-1]$,*

$$EH(\alpha \smile Sq^t) = \alpha \smile EH(Sq^t) + EH(\alpha) \smile \theta(Sq^t)$$
$$= \begin{cases} \alpha + EH(\alpha) \smile Sq^{2t-1} & \text{for } n = t-1 \ge 0 \\ EH(\alpha) \smile Sq^{2t-1} & \text{for } 0 \le n < t-1. \end{cases}$$

Proof. Let $\alpha = xSq^{n+t} + y$, $x \in A[2n+2t-1]$, $y \in A[n+t]$. Then $H(\alpha) = x$. For $n = t-1$, $\alpha Sq^t = ySq^t$ is admissible and $H(\alpha Sq^t) = y = \alpha + H(\alpha)Sq^{2t-1}$. For $0 \le n < t-1$, by the equation (2),

$$\alpha Sq^t = xSq^{n+t}Sq^t + ySq^t = xSq^{2t-1}Sq^{n+1} + xR + ySq^t,$$

where $R \in A^{2,2t+n}[n+1]$. Now $xSq^{2t-1} \in A[2n+1]$, $xR \in A[n+1]$, $ySq^t \in A[n+1]$, and so $H(\alpha Sq^t) = xSq^{2t-1} = H(\alpha)Sq^{2t-1}$.

If α and $\beta \in A^{s,t}$ are $(n+t-s)$- and n-pseudo-admissible, respectively, then $\alpha \smile \beta$ is n-pseudo-admissible, and so

$$\theta(\alpha \smile \beta) = \theta(\alpha) \smile \theta(\beta)$$

by the equation (8). Next we can check easily that

$$\alpha \smile (\beta \smile \gamma) = (\alpha \smile \beta) \smile \gamma.$$

Now we prove Theorem 2.

Proof of Theorem 2. We shall prove by induction on s. The $s = 1$ case is Lemma 4.

Let $\beta = \beta' Sq^b$ for $b > n$ and admissible $\beta' \in A^{s-1,t-b}[2b-1] \subset A^{s-1,t-b}[n+b-1]$. Then $\alpha \smile \beta' \in A[n+b-1]$. By induction and Lemma 4,

$$\begin{aligned}
EH(\alpha \smile \beta) &= EH(\alpha \smile (\beta' \smile Sq^b)) \\
&= EH((\alpha \smile \beta') \smile Sq^b) \\
&= (\alpha \smile \beta') \smile EH(Sq^b) + EH(\alpha \smile \beta') \smile \theta(Sq^b) \\
&= \alpha \smile (\beta' \smile EH(Sq^b)) \\
&\quad + \{\alpha \smile EH(\beta') + EH(\alpha) \smile \theta(\beta')\} \smile \theta(Sq^b) \\
&= \alpha \smile \{\beta' \smile EH(Sq^b) + EH(\beta') \smile \theta(Sq^b)\} \\
&\quad + EH(\alpha) \smile (\theta(\beta') \smile \theta(Sq^b)) \\
&= \alpha \smile EH(\beta) + EH(\alpha) \smile \theta(\beta).
\end{aligned}$$

3. Acyclic relations

For the lambda algebra Λ, the acyclic relations are equivalent to the exactness of the following sequence for $p \geq 2n+1$:

$$\Lambda(p+2n+3) \xrightarrow{\lambda_{2n+1}\smile} \Lambda(p+1) \xrightarrow{\lambda_n\smile} \Lambda(p-n).$$

Singer's formulas are crucial to the proof. Now we have Singer's formulas (Theorems 1 and 2) and can prove the acyclic relations for the Steenrod algebra A.

Proof of Theorem 3. By the Adem relation,

$$(\smile Sq^a) \circ (\smile Sq^{2a-1}) = 0.$$

Consider an element $\alpha \in A^{s,t}[n-1]$ with $\alpha \smile Sq^a = 0$. For $n \geq 2a$, αSq^a is admissible, and so $\alpha = 0$.

For $n = 2a - 1$, $\alpha = xSq^{2a-1} + y \in A[2a-2]$, where $x = H(\alpha) \in A[4a-3]$ and $y \in A[2a-1]$. So $\alpha \smile Sq^a = ySq^a$, and so $y = 0$ by the case $n \geq 2a$ above. Thus $\alpha = H(\alpha) \smile Sq^{2a-1}$.

For $n < 2a - 1$, we have the following commutative diagram by Corollary 1:

$$\begin{array}{ccccc} A^{s-1}[n+2a-3] & \xrightarrow{\smile Sq^{2a-1}} & A^s[n-1] & \xrightarrow{\smile Sq^a} & A^{s+1}[n-a] \\ \downarrow H & & \downarrow H & & \downarrow H \\ A^{s-2}[2n+4a-5] & \xrightarrow{\smile Sq^{4a-3}} & A^{s-1}[2n-1] & \xrightarrow{\smile Sq^{2a-1}} & A^s[2n-2a+1] \end{array}$$

Then $0 = H(\alpha \smile Sq^a) = H(\alpha) \smile Sq^{2a-1}$. For $s = 1$ we see $\alpha = 1 \smile Sq^{2a-1}$ if α is non-trivial. When $s = 2$, we may assume that $H(\alpha) = \gamma \smile Sq^{4a-3}$ for some $\gamma \in A[2n+4a-5]$. Since H is surjective, we have an element $f \in A[n+2a-3]$ with $H(f) = \gamma$. Then $H(f \smile Sq^{2a-1}) = H(f) \smile Sq^{4a-3} = \gamma \smile Sq^{4a-3} = H(\alpha)$. Hence $\alpha' = \alpha + f \smile Sq^{2a-1} \in A[n-1]$ has $H(\alpha') = 0$, and so $\alpha' \in A[n]$ and $\alpha' \smile Sq^a = 0$. By reverse-directed induction on n, $\alpha' = \beta' \smile Sq^{2a-1}$ for some $\beta' \in A[n+2a-2]$. Thus $\alpha = \beta \smile Sq^{2a-1}$ for $\beta = f + \beta' \in A[n+2a-3]$.

Here, to prove the exactness of the upper sequence by the exactness of the lower sequence we use the fact that $2n - 1 > n - 1$ which comes from the condition $n \geq a \geq 1$. We proceed by induction on s and we complete the proof of Theorem 3.

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