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Fractional calculus on parabolic Bergman spaces

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Abstract. The parabolic Bergman space is the set of all L^p-solutions of the parabolic operator $L^{(\alpha)}$. In this paper, we study fractional calculus on parabolic Bergman spaces. In particular, we investigate properties of fractional derivatives of the fundamental solution of the parabolic operator. We show the reproducing property of fractional derivatives of the fundamental solution.

1. Introduction

Let H be the upper half-space of the $(n+1)$-dimensional Euclidean space $\mathbf{R}^{n+1} (n \geq 1)$, that is, $H = \{(x,t) \in \mathbf{R}^{n+1}; x \in \mathbf{R}^n, t > 0\}$. For $1 \leq p < \infty$ and $\lambda > -1$, $L^p(\lambda)$ is the Banach space of Lebesgue measurable functions u on H with

$$\|u\|_{L^p(\lambda)} = \left(\int_H |u(x,t)|^p t^\lambda \, dV(x,t) \right)^{1/p} < \infty,$$

where dV is the Lebesgue volume measure on H. For $0 < \alpha \leq 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)} = \partial_t + (-\Delta_x)^\alpha,$$

where $\partial_t = \partial/\partial t$ and Δ_x is the Laplacian with respect to x. A continuous function u on H is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)} u = 0$ in the sense of distributions (for details, see section 3). The parabolic Bergman space $\boldsymbol{b}_\alpha^p(\lambda)$ is the set of all $L^{(\alpha)}$-harmonic functions u on H which belong to $L^p(\lambda)$. We remark that $\boldsymbol{b}_{1/2}^p(\lambda)$ coincides with the harmonic Bergman space of Koo, Nam, and Yi [3]. Therefore, usual harmonic Bergman spaces are the classes of $L^p(\lambda)$-solutions of the parabolic operator $L^{(1/2)}$.

Our aim of this paper is the analysis of fractional integrals and derivatives of parabolic Bergman functions. The fundamental solution of the parabolic operator $L^{(\alpha)}$ plays an important role for the analysis of parabolic Bergman

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spaces. Therefore, in this paper, we investigate the fractional integrals and derivatives of the fundamental solution. Consequently, we show that the reproducing kernel for $\boldsymbol{b}_\alpha^2(\lambda)$ is given by a fractional derivative of the fundamental solution. Furthermore, our results in this paper can also be applied to study conjugate functions of parabolic Bergman functions, whose application will be described elsewhere.

To state our results, we present some definitions. Let $\mathbf{N}_0$ be the set of all non-negative integers. For a multi-index $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{N}_0^n$, let $\partial_x^\beta = \partial^{|\beta|}/\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}$. For a real number κ, a fractional differential operator $\mathscr{D}_t^\kappa$ is defined by $\mathscr{D}_t^\kappa = (-\partial_t)^\kappa$ (the explicit definition will be described in section 2). And let $W^{(\alpha)}$ be the fundamental solution of the parabolic operator $L^{(\alpha)}$ (for details, see section 3). The following theorems are main results in this paper. In Theorem 1, we give properties of fractional integrals and derivatives of the fundamental solution. Theorem 2 establishes the reproducing property of fractional derivatives of the fundamental solution (our result is more general, see Theorem 5.2).

THEOREM 1. *Let $0 < \alpha \le 1$, $\beta \in \mathbf{N}_0^n$, and $\kappa > -\frac{n}{2\alpha}$ be a real number. Then, the following statements hold.*

(1) *The derivative $\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x,t) = \mathscr{D}_t^\kappa \partial_x^\beta W^{(\alpha)}(x,t)$ is well-defined. Moreover, there exists a constant $C > 0$ such that*

$$|\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x,t)| \le C(t + |x|^{2\alpha})^{-(n+|\beta|)/2\alpha - \kappa}$$

for all $(x,t) \in H$.

(2) *If $0 < q < \infty$ and $\theta > -1$ satisfy the condition $\frac{n}{2\alpha} + \theta + 1 - \left(\frac{n+|\beta|}{2\alpha} + \kappa\right)q < 0$, then there exists a constant $C > 0$ such that*

$$\int_H |\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x-y, t+s)|^q s^\theta \, dV(y,s) \le C t^{n/2\alpha + \theta + 1 - ((n+|\beta|)/2\alpha + \kappa)q}$$

for all $(x,t) \in H$.

(3) *$\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H.*

THEOREM 2. *Let $0 < \alpha \le 1$, $1 \le p < \infty$, and $\lambda > -1$. Then, the reproducing property*

$$u(x,t) = C_{\lambda+1} \int_H u(y,s) \mathscr{D}_t^{\lambda+1} W^{(\alpha)}(x-y, t+s) s^\lambda \, dV(y,s)$$

holds for all $u \in \boldsymbol{b}_\alpha^p(\lambda)$ and $(x,t) \in H$, where $C_\lambda = 2^\lambda/\Gamma(\lambda)$.

By Theorems 1 and 2, we obtain the following corollary, which shows that the reproducing kernel for $\boldsymbol{b}_\alpha^2(\lambda)$ is given by a fractional derivative of the fundamental solution.

Corollary. *Let* $0 < \alpha \le 1$ *and* $\lambda > -1$. *We define the function* R_α^λ *on* $H \times H$ *such that*

$$R_\alpha^\lambda(x, t; y, s) = C_{\lambda+1} \mathscr{D}_t^{\lambda+1} W^{(\alpha)}(x - y, t + s).$$

Then, $R_\alpha^\lambda(x, t; \cdot, \cdot)$ *is the reproducing kernel for* $\boldsymbol{b}_\alpha^2(\lambda)$.

Throughout this paper, we will denote by C a positive constant whose value may not necessarily be the same at each occurrence.

2. Fractional differential operators

We introduce fractional differential operators which are main tools for our study. Let $C(\mathbf{R}_+)$ be the set of all continuous functions on $\mathbf{R}_+ = (0, \infty)$. For a positive real number κ, let $\mathscr{FC}^{-\kappa}$ be the set of all functions $\varphi \in C(\mathbf{R}_+)$ such that there exist constants ε, $C > 0$ with $|\varphi(t)| \le Ct^{-\kappa-\varepsilon}$ for all $t \in \mathbf{R}_+$. We remark that $\mathscr{FC}^{-\nu} \subset \mathscr{FC}^{-\kappa}$ if $0 < \kappa \le \nu$. For $\varphi \in \mathscr{FC}^{-\kappa}$, we can define the fractional integral of φ with order κ by

$$\mathscr{D}^{-\kappa}\varphi(t) = \frac{1}{\Gamma(\kappa)} \int_0^\infty \tau^{\kappa-1}\varphi(t + \tau)d\tau = \frac{1}{\Gamma(\kappa)} \int_t^\infty (\tau - t)^{\kappa-1}\varphi(\tau)d\tau, \qquad t \in \mathbf{R}_+, \quad (2.1)$$

where $\Gamma(\cdot)$ is the gamma function. Furthermore, for a positive real number κ, let $\mathscr{FC}^{\kappa}$ be the set of all functions $\varphi \in C(\mathbf{R}_+)$ such that $d_t^{\lceil\kappa\rceil}\varphi \in \mathscr{FC}^{-(\lceil\kappa\rceil-\kappa)}$, where $d_t = d/dt$ and $\lceil\kappa\rceil$ is the smallest integer greater than or equal to κ. In particular, we will write $\mathscr{FC}^0 = C(\mathbf{R}_+)$. For $\varphi \in \mathscr{FC}^{\kappa}$, we can also define the fractional derivative of φ with order κ by

$$\mathscr{D}^{\kappa}\varphi(t) = \mathscr{D}^{-(\lceil\kappa\rceil-\kappa)}(-d_t)^{\lceil\kappa\rceil}\varphi(t), \qquad t \in \mathbf{R}_+. \quad (2.2)$$

Also, we define $\mathscr{D}^0\varphi = \varphi$. For a real number κ, we may often call both (2.1) and (2.2) *the fractional derivative of* φ *with order* κ. Moreover, we call $\mathscr{D}^\kappa$ *the fractional differential operator with order* κ. The following proposition shows that fractional differential operators enjoy the commutative and exponential laws under some conditions.

Proposition 2.1. *Let* κ *and* ν *be positive real numbers. Then, the following statements hold.*

(1) *If* $\varphi \in \mathscr{FC}^{-\kappa}$, *then* $\mathscr{D}^{-\kappa}\varphi \in C(\mathbf{R}_+)$.

(2) *If* $\varphi \in \mathscr{FC}^{-\kappa-\nu}$, *then* $\mathscr{D}^{-\kappa}\mathscr{D}^{-\nu}\varphi = \mathscr{D}^{-\kappa-\nu}\varphi$.

(3) *If* $d_t^k\varphi \in \mathscr{FC}^{-\nu}$ *for all integers* $0 \le k \le \lceil\kappa\rceil - 1$ *and* $d_t^{\lceil\kappa\rceil}\varphi \in \mathscr{FC}^{-(\lceil\kappa\rceil-\kappa)-\nu}$, *then* $\mathscr{D}^{\kappa}\mathscr{D}^{-\nu}\varphi = \mathscr{D}^{-\nu}\mathscr{D}^{\kappa}\varphi = \mathscr{D}^{\kappa-\nu}\varphi$.

(4) *If* $d_t^{k+\lceil\nu\rceil}\varphi \in \mathscr{FC}^{-(\lceil\nu\rceil-\nu)}$ *for all integers* $0 \le k \le \lceil\kappa\rceil - 1$, $d_t^{\lceil\kappa\rceil+\ell}\varphi \in \mathscr{FC}^{-(\lceil\kappa\rceil-\kappa)}$ *for all integers* $0 \le \ell \le \lceil\nu\rceil - 1$, *and* $d_t^{\lceil\kappa\rceil+\lceil\nu\rceil}\varphi \in \mathscr{FC}^{-(\lceil\kappa\rceil-\kappa)-(\lceil\nu\rceil-\nu)}$, *then* $\mathscr{D}^{\kappa}\mathscr{D}^{\nu}\varphi = \mathscr{D}^{\kappa+\nu}\varphi$.

Proof. (1) Let $\varphi \in \mathscr{F}\mathscr{C}^{-\kappa}$ and $f_t(\tau) = \tau^{\kappa-1}\varphi(t+\tau)$, and let $t_0 \in \mathbf{R}_+$ be fixed. By the continuity of φ, $f_t(\tau) \to f_{t_0}(\tau)$ as $t \to t_0$ for all $\tau \in \mathbf{R}_+$. Since $\varphi \in \mathscr{F}\mathscr{C}^{-\kappa}$, we can easily find a function $g \in L^1(\mathbf{R}_+)$ and a constant $\delta > 0$ such that $|f_t(\tau)| \le g(\tau)$ for all $\tau \in \mathbf{R}_+$ and $t \in [t_0 - \delta, t_0 + \delta]$. Therefore, the Lebesgue dominated convergence theorem implies that $\mathscr{D}^{-\kappa}\varphi(t) \to \mathscr{D}^{-\kappa}\varphi(t_0)$ as $t \to t_0$.

(2) By using (2.1) and the change of variables $\frac{\tau_2 - t}{\tau_1 - t} \mapsto \tau$, we have

$$\begin{aligned}
\mathscr{D}^{-\kappa}\mathscr{D}^{-\nu}\varphi(t) &= \frac{1}{\Gamma(\kappa)}\frac{1}{\Gamma(\nu)}\int_t^\infty \int_t^{\tau_1} (\tau_2 - t)^{\kappa-1}(\tau_1 - \tau_2)^{\nu-1} d\tau_2 \varphi(\tau_1) d\tau_1 \\
&= \frac{1}{\Gamma(\kappa)}\frac{1}{\Gamma(\nu)}\int_t^\infty \int_0^1 \tau^{\kappa-1}(1-\tau)^{\nu-1} d\tau (\tau_1 - t)^{\kappa+\nu-1}\varphi(\tau_1) d\tau_1 \\
&= \frac{B(\kappa,\nu)}{\Gamma(\kappa)\Gamma(\nu)}\int_t^\infty (\tau_1 - t)^{\kappa+\nu-1}\varphi(\tau_1) d\tau_1 = \mathscr{D}^{-\kappa-\nu}\varphi(t),
\end{aligned}$$

where $B(\cdot\,,\cdot)$ stands for the beta function.

(3) By using (2) and differentiating under the integral sign, we can easily obtain the first equality of (3). We show the second equality. First, we prove the second equality in case of $\nu = \kappa$ by the induction, that is,

$$\mathscr{D}^{-\kappa}\mathscr{D}^{\kappa}\varphi = \varphi. \tag{2.3}$$

If $\kappa = 1$, then

$$\mathscr{D}^{-1}\mathscr{D}^{1}\varphi(t) = -\int_0^\infty d_t\varphi(t+\tau) d\tau = -\lim_{h\to\infty}(\varphi(t+h) - \varphi(t)) = \varphi(t) \tag{2.4}$$

for all $t \in \mathbf{R}_+$. If (2.3) holds for some $\kappa \in \mathbf{N}$, then by (2) and (2.4), we have

$$\mathscr{D}^{-\kappa-1}\mathscr{D}^{\kappa+1}\varphi = \mathscr{D}^{-\kappa}\mathscr{D}^{-1}(-d_t)(-d_t)^{\kappa}\varphi(t) = \mathscr{D}^{-\kappa}(-d_t)^{\kappa}\varphi(t) = \varphi(t).$$

Thus, (2.3) holds for all $\kappa \in \mathbf{N}$. Moreover, for a positive real number κ, (2) shows that

$$\mathscr{D}^{-\kappa}\mathscr{D}^{\kappa}\varphi = \mathscr{D}^{-\lceil\kappa\rceil}(-d_t)^{\lceil\kappa\rceil}\varphi = \varphi,$$

because $\lceil\kappa\rceil \in \mathbf{N}$. Thus, we obtain (2.3) holds for all positive real numbers κ.

Next, we show the second equality in case of $\kappa \ge \nu$. By (2), we have

$$\mathscr{D}^{-\nu}\mathscr{D}^{\kappa}\varphi = \mathscr{D}^{-\nu-(\lceil\kappa\rceil-\kappa)}(-d_t)^{\lceil\kappa\rceil}\varphi. \tag{2.5}$$

Put $\eta = \nu + \lceil\kappa\rceil - \kappa > 0$. Then, (2) and (2.3) show that (2.5) is equal to

$$\mathscr{D}^{-\eta}(-d_t)^{\lceil\kappa\rceil}\varphi = \mathscr{D}^{-\eta+[\eta]}\mathscr{D}^{-[\eta]}(-d_t)^{[\eta]}(-d_t)^{\lceil\kappa\rceil-[\eta]}\varphi = \mathscr{D}^{-\eta+[\eta]}(-d_t)^{\lceil\kappa\rceil-[\eta]}\varphi,$$

where $[\cdot]$ is the Gauss symbol. Since $\lceil\kappa\rceil - [\eta] = \lceil\kappa - \nu\rceil$ and $\eta - [\eta] = \lceil\kappa - \nu\rceil - \kappa + \nu$, (2.5) is equal to $\mathscr{D}^{\kappa-\nu}\varphi$. Thus, we obtain the second equality in case of $\kappa \geq \nu$.

Finally, we show the second equality in case of $\kappa < \nu$. Since $\kappa - \nu < 0$ and $-\kappa < 0$, (2) and (2.3) show that

$$\mathscr{D}^{-\nu}\mathscr{D}^{\kappa}\varphi = \mathscr{D}^{\kappa-\nu}\mathscr{D}^{-\kappa}\mathscr{D}^{\kappa}\varphi = \mathscr{D}^{\kappa-\nu}\varphi.$$

(4) By using (2) and differentiating under the integral sign, we have

$$\mathscr{D}^{\kappa}\mathscr{D}^{\nu}\varphi = \mathscr{D}^{-(\lceil\kappa\rceil+\lceil\nu\rceil)+\kappa+\nu}(-d_t)^{\lceil\kappa\rceil+\lceil\nu\rceil}\varphi.$$

If $\lceil\kappa\rceil + \lceil\nu\rceil = \lceil\kappa+\nu\rceil$, then we can directly obtain the equality of (4). If $\lceil\kappa\rceil + \lceil\nu\rceil = \lceil\kappa+\nu\rceil + 1$, then (2) and (2.3) show that

$$\mathscr{D}^{-(\lceil\kappa\rceil+\lceil\nu\rceil)+\kappa+\nu}(-d_t)^{\lceil\kappa\rceil+\lceil\nu\rceil}\varphi = \mathscr{D}^{-\lceil\kappa+\nu\rceil+\kappa+\nu}\mathscr{D}^{-1}(-d_t)(-d_t)^{\lceil\kappa+\nu\rceil}\varphi = \mathscr{D}^{\kappa+\nu}\varphi.$$

This completes the proof.

We give some examples of fractional derivatives of elementary functions.

Example 2.2. *Let $\kappa > 0$ and ν be real numbers. Then, we have the following.*

(1) $\mathscr{D}^{\nu}e^{-\kappa t} = \kappa^{\nu}e^{-\kappa t}$.

(2) *Moreover, if $-\kappa < \nu$, then* $\mathscr{D}^{\nu}t^{-\kappa} = \dfrac{\Gamma(\kappa+\nu)}{\Gamma(\kappa)}t^{-\kappa-\nu}$.

3. Fractional derivatives of the fundamental solution

In this section, we study fractional derivatives of the fundamental solution of the parabolic operator $L^{(\alpha)}$ and give the proof of Theorem 1. First, we begin with recalling the definition of $L^{(\alpha)}$-harmonic functions. We explain about the operator $(-\Delta_x)^{\alpha}$. Since the case $\alpha = 1$ is trivial, we only describe the case $0 < \alpha < 1$. Let $C_c^{\infty}(H)$ be the set of all infinitely differentiable functions on H with compact support. For $0 < \alpha < 1$, $(-\Delta_x)^{\alpha}$ is the convolution operator defined by

$$(-\Delta_x)^{\alpha}\psi(x,t) = -c_{n,\alpha}\lim_{\delta\to 0^+}\int_{|x-y|>\delta}(\psi(y,t) - \psi(x,t))|x-y|^{-n-2\alpha}dy \tag{3.1}$$

for all $\psi \in C_c^{\infty}(H)$ and $(x,t) \in H$, where $c_{n,\alpha} = -4^{\alpha}\pi^{-n/2}\Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$. A continuous function u on H is said to be $L^{(\alpha)}$-harmonic on H if u satisfies the following condition: for every $\psi \in C_c^{\infty}(H)$,

$$\int_H |u\cdot\tilde{L}^{(\alpha)}\psi|dV < \infty \qquad \text{and} \qquad \int_H u\cdot\tilde{L}^{(\alpha)}\psi\, dV = 0, \tag{3.2}$$

where $\tilde{L}^{(\alpha)} = -\partial_t + (-\Delta_x)^{\alpha}$ is the adjoint operator of $L^{(\alpha)}$. By (3.1) and the compactness of $\mathrm{supp}(\psi)$ (the support of ψ), there exist $0 < t_1 < t_2 < \infty$ and a constant $C > 0$ such that $\mathrm{supp}(\tilde{L}^{(\alpha)}\psi) \subset S = \mathbf{R}^n \times [t_1, t_2]$ and $|\tilde{L}^{(\alpha)}\psi(x,t)| \le C(1+|x|)^{-n-2\alpha}$ for all $(x,t) \in S$. Thus, the integrability condition $\int_H |u \cdot \tilde{L}^{(\alpha)}\psi| dV < \infty$ is equivalent to the following: for any $0 < t_1 < t_2 < \infty$,

$$\int_{t_1}^{t_2} \int_{\mathbf{R}^n} |u(x,t)|(1+|x|)^{-n-2\alpha} dV(x,t) < \infty. \tag{3.3}$$

Next, we introduce the fundamental solution of $L^{(\alpha)}$. For $x \in \mathbf{R}^n$, the fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ is defined by

$$W^{(\alpha)}(x,t) = \begin{cases} \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbf{R}^n} \exp(-t|\xi|^{2\alpha} + \sqrt{-1}x \cdot \xi) d\xi & t > 0 \\ 0 & t \le 0, \end{cases} \tag{3.4}$$

where $x \cdot \xi$ denotes the inner product on $\mathbf{R}^n$. It is known that $W^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H and $W^{(\alpha)} \in C^\infty(H)$, where $C^\infty(H)$ is the set of all infinitely differentiable functions on H. Furthermore,

$$W^{(\alpha)}(x,t) \ge 0 \text{ for all } (x,t) \in H \qquad \text{and} \qquad \int_{\mathbf{R}^n} W^{(\alpha)}(x,t) dx = 1 \text{ for all } t \in \mathbf{R}_+. \tag{3.5}$$

Let $0 < \alpha \le 1$, $\beta \in \mathbf{N}_0^n$, and $k \in \mathbf{N}_0$. Lemma 1 of [5] says that there exists a constant $C > 0$ such that

$$|\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t)| \le C(t + |x|^{2\alpha})^{-(n+|\beta|)/2\alpha - k} \tag{3.6}$$

for all $(x,t) \in H$. Moreover, $W^{(\alpha)}$ satisfies the homogeneous property (see section 3 of [4]), that is,

$$\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t) = t^{-(n+|\beta|)/2\alpha - k} (\partial_x^\beta \partial_t^k W^{(\alpha)})(t^{-1/2\alpha}x, 1)$$

for all $(x,t) \in H$.

Now, we describe properties of fractional derivatives of the fundamental solution. Here, we define a function $\omega(\kappa)$, which is frequently used throughout this paper. For a real number κ, let

$$\omega(\kappa) = \begin{cases} \lceil \kappa \rceil & \kappa \ge 0 \\ 0 & \kappa < 0. \end{cases}$$

Basic properties of fractional derivatives of $W^{(\alpha)}$ are given in the following theorem. The assertions (1) and (3) of Theorem 3.1 are (1) and (3) of Theorem 1, respectively.

THEOREM 3.1. *Let* $0 < \alpha \le 1$, $\beta \in \mathbf{N}_0^n$, *and* $\kappa > -\frac{n}{2\alpha}$ *be a real number. Then, the following statements hold.*

(1) *The derivative* $\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x,t) = \mathscr{D}_t^\kappa \partial_x^\beta W^{(\alpha)}(x,t)$ *is well-defined, and there exists a constant* $C > 0$ *such that*

$$|\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x,t)| \le C(t + |x|^{2\alpha})^{-(n+|\beta|)/2\alpha-\kappa}$$

for all $(x,t) \in H$.

(2) *Let* ν *be a real number such that* $\kappa + \nu > -\frac{n}{2\alpha}$. *Then,*

$$\mathscr{D}_t^\nu \partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x,t) = \partial_x^\beta \mathscr{D}_t^{\kappa+\nu} W^{(\alpha)}(x,t)$$

for all $(x,t) \in H$.

(3) $\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}$ *is* $L^{(\alpha)}$*-harmonic on* H.

(4) $\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}$ *satisfies the homogeneous property, that is,*

$$\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x,t) = t^{-(n+|\beta|)/2\alpha-\kappa}(\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)})(t^{-1/2\alpha}x, 1)$$

for all $(x,t) \in H$.

PROOF. (1) Let $\beta \in \mathbf{N}_0^n$ and $\kappa > -\frac{n}{2\alpha}$ be a real number. By (2.1), (2.2), and (3.6), it is easy to see that the derivative $\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x,t) = \mathscr{D}_t^\kappa \partial_x^\beta W^{(\alpha)}(x,t)$ is well-defined. By (3.6) and (2) of Example 2.2, there exists a constant $C > 0$ such that

$$\begin{aligned}
|\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x,t)| &\le C \int_0^\infty \tau^{\omega(\kappa)-\kappa-1} |\partial_x^\beta \mathscr{D}_t^{\omega(\kappa)} W^{(\alpha)}(x, t+\tau)| d\tau \\
&\le C \int_0^\infty \tau^{\omega(\kappa)-\kappa-1} (t + \tau + |x|^{2\alpha})^{-(n+|\beta|)/2\alpha-\omega(\kappa)} d\tau \\
&= C\mathscr{D}_t^{-(\omega(\kappa)-\kappa)}(t + |x|^{2\alpha})^{-(n+|\beta|)/2\alpha-\omega(\kappa)} = C(t + |x|^{2\alpha})^{-(n+|\beta|)/2\alpha-\kappa}
\end{aligned}$$

for all $(x,t) \in H$.

(2) By (1) of Theorem 3.1 and (4) of Proposition 2.1, we can easily show the statement.

(3) For any $0 < t_1 < t_2 < \infty$, we can easily show that

$$\int_{t_1}^{t_2} \int_{\mathbf{R}^n} (t + |x|^{2\alpha})^{-(n+|\beta|)/2\alpha-\kappa} (1 + |x|)^{-n-2\alpha} dx dt < \infty. \tag{3.7}$$

Therefore, by (3.3) and (1) of Theorem 3.1, we obtain the integrability condition of (3.2) for $\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}$. Next, we show that $\int_H \partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)} \cdot \tilde{L}^{(\alpha)} \psi \, dV = 0$ for all $\psi \in C_c^\infty(H)$. Since the case $\kappa \in \mathbf{N}_0$ is trivial, we only show this for $\kappa \in \mathbf{R} \backslash \mathbf{N}_0$. Let $\psi \in C_c^\infty(H)$. Thanks to (2) of Theorem 3.1 and (3.7), the Fubini theorem implies that

$$\int_H \partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x,t)\tilde{L}^{(\alpha)}\psi(x,t)dV(x,t)$$

$$= \frac{1}{\Gamma(\omega(\kappa)-\kappa)}\int_0^\infty \tau^{\omega(\kappa)-\kappa-1}\int_H \partial_x^\beta \mathscr{D}_t^{\omega(\kappa)} W^{(\alpha)}(x,t+\tau)\tilde{L}^{(\alpha)}\psi(x,t)dV(x,t)d\tau$$

$$= 0.$$

(4) By differentiating under the integral sign in (3.4), the Fubini theorem implies that

$$\begin{aligned}\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x,t) &= \frac{1}{\Gamma(\omega(\kappa)-\kappa)}\int_0^\infty \tau^{\omega(\kappa)-\kappa-1}\int_{\mathbf{R}^n} \mathscr{D}_t^{\omega(\kappa)} e^{-(t+\tau)|\xi|^{2\alpha}}\partial_x^\beta e^{ix\cdot\xi}\, d\xi d\tau \\ &= \int_{\mathbf{R}^n} |\xi|^{2\alpha\omega(\kappa)} e^{-t|\xi|^{2\alpha}} i^{|\beta|}\prod_{j=1}^n \xi_j{}^{\beta_j} e^{ix\cdot\xi} \\ &\quad\times \frac{1}{\Gamma(\omega(\kappa)-\kappa)}\int_0^\infty \tau^{\omega(\kappa)-\kappa-1} e^{-\tau|\xi|^{2\alpha}}\, d\tau d\xi \\ &= \int_{\mathbf{R}^n} |\xi|^{2\alpha\kappa} e^{-t|\xi|^{2\alpha}} i^{|\beta|}\prod_{j=1}^n \xi_j{}^{\beta_j} e^{ix\cdot\xi}\, d\xi.\end{aligned}$$

By making the change of variables $\xi_j \mapsto t^{-1/2\alpha}\xi_j'$ for all integers $1 \le j \le n$, we have

$$\begin{aligned}\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x,t) &= \int_{\mathbf{R}^n} t^{-\kappa}|\xi'|^{2\alpha\kappa} e^{-|\xi'|^{2\alpha}} t^{-|\beta|/2\alpha} i^{|\beta|}\prod_{j=1}^n \xi_j'^{\beta_j} e^{it^{-1/2\alpha}x\cdot\xi'} t^{-n/2\alpha}\, d\xi' \\ &= t^{-(n+|\beta|)/2\alpha-\kappa}(\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)})(t^{-1/2\alpha}x, 1),\end{aligned}$$

where $\xi' = (\xi_1', \dots, \xi_n')$. This completes the proof.

In (1) of Theorem 3.1, we give upper estimates of fractional derivatives of $W^{(\alpha)}$. By (4) of Theorem 3.1, we can also give lower estimates of fractional derivatives of $W^{(\alpha)}$. In section 4, we will show that the reproducing kernel for the parabolic Bergman space coincides with a fractional derivative of $W^{(\alpha)}$ with suitable order. Usually, lower estimates of the reproducing kernel for the harmonic Bergman space are given on the pseudo-hyperbolic balls in H. In our case, lower estimates of fractional derivatives of $W^{(\alpha)}$ are given on the following parabolic Carleson boxes, which are defined in [6]. For $(y,s) \in H$, the parabolic Carleson Box $Q^{(\alpha)}(y,s)$ is defined by

$$Q^{(\alpha)}(y,s) = \{(x,t) \in H; |x_j - y_j| < 2^{-1}s^{1/2\alpha}(1 \le j \le n), s \le t \le 2s\}.$$

The proofs of (1) and (2) of Proposition 3.2 are similar to that of (2) of Proposition 1 and Corollary 1 of [6], respectively. Hence, we omit the proofs.

Proposition 3.2. *Let* $0 < \alpha \leq 1$, $\beta \in \mathbf{N}_0^n$, *and* $\kappa > -\frac{n}{2\alpha}$ *be a real number. Then, the following statements hold.*

(1) *If each* β_j *is even, then there exist constants* σ, $C > 0$ *such that*

$$\inf\{|\partial_x^\beta \mathcal{D}_t^\kappa W^{(\alpha)}(x,t)|; |x| \leq \sigma t^{1/2\alpha}\} \geq C t^{-(n+|\beta|)/2\alpha-\kappa},$$

where σ *and* C *depend on* n, α, β *and* κ. *Otherwise,*

$$\inf\{|\partial_x^\beta \mathcal{D}_t^\kappa W^{(\alpha)}(x,t)|; |x| \leq \sigma t^{1/2\alpha}\} = 0$$

for all $\sigma > 0$.

(2) *If each* β_j *is even, then there exist constants* ρ, $C > 0$ *such that*

$$C^{-1} s^{-(n+|\beta|)/2\alpha-\kappa} \leq |\partial_x^\beta \mathcal{D}_t^\kappa W^{(\alpha)}(x-y, t+s)| \leq C s^{-(n+|\beta|)/2\alpha-\kappa}.$$

for all $(x,t) \in Q^{(\alpha)}(y, \rho s)$, *where* ρ *and* C *depend on* n, α, β *and* κ.

Finally, we show $L^p(\lambda)$-norm estimates of fractional derivatives of the fundamental solution in (2) of Theorem 1. By (1) of Theorem 3.1 and Lemma 3.3 below, we can directly obtain (2) of Theorem 1. This completes the proof of Theorem 1.

Lemma 3.3 (Lemma 5 of [5]). *Let* θ *and* c *be real numbers such that* $\theta > -1$ *and* $\frac{n}{2\alpha} + \theta + 1 - c < 0$. *Then, there exists a constant* $C > 0$ *such that*

$$\int_H \frac{s^\theta}{(t+s+|x-y|^{2\alpha})^c} \, dV(y,s) \leq C t^{n/2\alpha+\theta+1-c}$$

for all $(x,t) \in H$.

4. Fractional calculus and reproducing properties on parabolic Bergman spaces

In this section, we study properties of fractional derivatives of parabolic Bergman functions and give the proof of Theorem 2. First, we present the estimates of ordinary derivatives of parabolic Bergman functions. Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, $\beta \in \mathbf{N}_0^n$, and $k \in \mathbf{N}_0$. The following estimate is established in Lemma 3.4 of [9]: if $u \in \boldsymbol{b}_\alpha^p(\lambda)$, then $u \in C^\infty(H)$ and there exists a constant $C > 0$ such that

$$|\partial_x^\beta \partial_t^k u(x,t)| \leq C t^{-|\beta|/2\alpha-k-(n/2\alpha+\lambda+1)(1/p)} \|u\|_{L^p(\lambda)} \tag{4.1}$$

for all $(x,t) \in H$. Basic properties of fractional derivatives of parabolic Bergman functions are given in the following proposition.

Proposition 4.1. *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, $\lambda > -1$, $\beta \in \mathbf{N}_0^n$, *and* $\kappa > -\left(\frac{n}{2\alpha} + \lambda + 1\right)\frac{1}{p}$ *be a real number. If* $u \in \boldsymbol{b}_\alpha^p(\lambda)$, *then the following statements hold.*

(1) *The derivative* $\partial_x^\beta \mathscr{D}_t^\kappa u(x,t) = \mathscr{D}_t^\kappa \partial_x^\beta u(x,t)$ *is well-defined, and there exists a constant* $C > 0$ *such that*

$$|\partial_x^\beta \mathscr{D}_t^\kappa u(x,t)| \le C t^{-|\beta|/2\alpha - \kappa - (n/2\alpha + \lambda + 1)(1/p)} \|u\|_{L^p(\lambda)}$$

for all $(x,t) \in H$.

(2) *Let* ν *be a real number such that* $\kappa + \nu > -\left(\frac{n}{2\alpha} + \lambda + 1\right)\frac{1}{p}$. *Then,*

$$\mathscr{D}_t^\nu \partial_x^\beta \mathscr{D}_t^\kappa u(x,t) = \partial_x^\beta \mathscr{D}_t^{\kappa+\nu} u(x,t)$$

for all $(x,t) \in H$.

(3) $\partial_x^\beta \mathscr{D}_t^\kappa u$ *is* $L^{(\alpha)}$*-harmonic on* H.

Proof. (1) Let $\beta \in \mathbf{N}_0^n$ and $\kappa > -\left(\frac{n}{2\alpha} + \lambda + 1\right)\frac{1}{p}$ be a real number. And let $u \in \boldsymbol{b}_\alpha^p(\lambda)$. By (2.1), (2.2), and (4.1), it is easy to see that the derivative $\partial_x^\beta \mathscr{D}_t^\kappa u(x,t) = \mathscr{D}_t^\kappa \partial_x^\beta u(x,t)$ is well-defined. By (4.1) and (2) of Example 2.2, there exists a constant $C > 0$ such that

$$\begin{aligned}
|\partial_x^\beta \mathscr{D}_t^\kappa u(x,t)| &\le \frac{1}{\Gamma(\omega(\kappa) - \kappa)} \int_0^\infty \tau^{\omega(\kappa)-\kappa-1} |\partial_x^\beta \mathscr{D}_t^{\omega(\kappa)} u(x, t+\tau)| d\tau \\
&\le C(\mathscr{D}_t^{-(\omega(\kappa)-\kappa)} t^{-|\beta|/2\alpha - \omega(\kappa) - (n/2\alpha+\lambda+1)(1/p)}) \|u\|_{L^p(\lambda)} \\
&\le C t^{-|\beta|/2\alpha - \kappa - (n/2\alpha+\lambda+1)(1/p)} \|u\|_{L^p(\lambda)}.
\end{aligned}$$

The proofs of (2) and (3) are similar to those (2) and (3) of Theorem 3.1, respectively. This completes the proof of Proposition 4.1.

For $\delta > 0$ and a function u on H, we define an auxiliary function u_δ of u by $u_\delta(x,t) = u(x, t+\delta)$. First, we show the reproducing property for fractional derivatives of u_δ in Proposition 4.5 below. In order to prove Proposition 4.5, we need the following lemmas. We note that the Huygens property described in Lemma 4.2 is important for our study. Furthermore, Lemma 4.3 is Remark 3.2 of [9].

Lemma 4.2 (Lemma 3.1 of [9]). *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, *and* $\lambda > -1$. *If* $u \in \boldsymbol{b}_\alpha^p(\lambda)$, *then* u *satisfies the Huygens property, that is,*

$$u(x,t) = \int_{\mathbf{R}^n} u(x-y, t-s) W^{(\alpha)}(y,s) dy$$

holds for all $x \in \mathbf{R}^n$ *and* $0 < s < t < \infty$.

Lemma 4.3 (Remark 3.2 of [9]). *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, *and* $\lambda > -1$. *For* $u \in \boldsymbol{b}_\alpha^p(\lambda)$, *the function* $t \mapsto \int_{\mathbf{R}^n} |u(x,t)|^p dx$ *is decreasing on* $(0, \infty)$.

Lemma 4.4. *Let* $m \in \mathbf{N}_0$ *and* κ, ν *be real numbers such that* $m - \nu > 0$ *and* $\kappa + \nu > 0$. *Then,*

$$\int_0^\infty \tau^{m-\nu-1}(c+\tau)^{-m-\kappa} d\tau = c^{-\kappa-\nu} B(m-\nu, \kappa+\nu)$$

for all $c > 0$.

Proposition 4.5. *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, $\lambda > -1$, *and* $\delta > 0$. *And let* ν, κ *be real numbers such that* $\nu > -\left(\frac{n}{2\alpha} + \lambda + 1\right)\frac{1}{p}$, $\kappa \ge 0$, *and* $\nu + \kappa > 0$. *Then,*

$$u_\delta(x,t) = C_{\nu+\kappa} \int_H \mathscr{D}_t^\nu u_\delta(y,s) \mathscr{D}_t^\kappa W^{(\alpha)}(x-y, t+s) s^{\nu+\kappa-1}\, dV(y,s) \tag{4.2}$$

holds for all $u \in \boldsymbol{b}_\alpha^p(\lambda)$ *and* $(x,t) \in H$, *where* $C_{\nu+\kappa}$ *is the constant defined in Theorem* 2.

Proof. Let $u \in \boldsymbol{b}_\alpha^p(\lambda)$ and $\delta > 0$. And let $m, k \in \mathbf{N}_0$ such that $m + k > 0$. First, we show that

$$u_\delta(x,t) = \frac{(c_1+c_2)^{m+k}}{\Gamma(m+k)} \int_H \mathscr{D}_t^m u_\delta(y, c_1 s) \mathscr{D}_t^k W^{(\alpha)}(x-y, t+c_2 s) s^{m+k-1}\, dV(y,s) \tag{4.3}$$

for all $(x,t) \in H$ and real numbers $c_1, c_2 > 0$. We prove the equality (4.3) with $m \in \mathbf{N}$ and $k = 0$. Since (3) of Proposition 4.1, Lemmas 4.2 and 4.3 imply that $\mathscr{D}_t^m u_\delta$ satisfies the Huygens property, we have

$$\begin{aligned} &\int_0^\infty \int_{\mathbf{R}^n} \mathscr{D}_t^m u_\delta(y, c_1 s) W^{(\alpha)}(x-y, t+c_2 s) s^{m-1}\, dy ds \\ &\qquad = \int_0^\infty \mathscr{D}_t^m u_\delta(x, t+(c_1+c_2)s) s^{m-1}\, ds, \end{aligned} \tag{4.4}$$

where (3.5) and (1) of Proposition 4.1 guarantee that the integrand in (4.4) belongs to $L^1(H, dV)$. Moreover, integrating by parts $m-1$ times, (1) of Proposition 4.1 implies that the right-hand side of (4.4) is equal to

$$\begin{aligned} &\frac{-1}{c_1+c_2}\left[\mathscr{D}_t^{m-1} u_\delta(x, t+(c_1+c_2)s) s^{m-1}\right]_0^\infty \\ &\qquad + \frac{m-1}{c_1+c_2} \int_0^\infty \mathscr{D}_t^{m-1} u_\delta(x, t+(c_1+c_2)s) s^{m-2}\, ds \\ &= \frac{\Gamma(m)}{(c_1+c_2)^{m-1}} \int_0^\infty \mathscr{D}_t u_\delta(x, t+(c_1+c_2)s) ds = \frac{\Gamma(m)}{(c_1+c_2)^m} u_\delta(x,t). \end{aligned}$$

Therefore, we obtain (4.3) with $m \in \mathbf{N}$ and $k = 0$. We show (4.3) with $m \in \mathbf{N}_0$ and $k \in \mathbf{N}$ by induction on k. Let $k = 1$. If $m = 0$, then the Huygens property and (4.4) imply that

$$\begin{aligned}
&\int_H u_\delta(y, c_1 s) \mathscr{D}_t W^{(\alpha)}(x - y, t + c_2 s) dyds \\
&\quad = -\frac{1}{c_2}\left[\int_{\mathbf{R}^n} u_\delta(y, c_1 s) W^{(\alpha)}(x - y, t + c_2 s) dy\right]_0^\infty \\
&\qquad - \frac{c_1}{c_2}\int_H \mathscr{D}_t u_\delta(y, c_1 s) W^{(\alpha)}(x - y, t + c_2 s) dyds \\
&\quad = \frac{1}{c_2} u_\delta(x, t) - \frac{c_1}{c_2(c_1 + c_2)} u_\delta(x, t) = \frac{1}{c_1 + c_2} u_\delta(x, t).
\end{aligned}$$

If $m \geq 1$, then the Huygens property, (4.1), and (4.4) imply that

$$\begin{aligned}
&\int_H \mathscr{D}_t^m u_\delta(y, c_1 s) \mathscr{D}_t W^{(\alpha)}(x - y, t + c_2 s) s^m \, dyds \\
&\quad = -\frac{1}{c_2}\left[\int_{\mathbf{R}^n} \mathscr{D}_t^m u_\delta(y, c_1 s) W^{(\alpha)}(x - y, t + c_2 s) dy s^m\right]_0^\infty \\
&\qquad - \frac{c_1}{c_2}\int_H \mathscr{D}_t^{m+1} u_\delta(y, c_1 s) W^{(\alpha)}(x - y, t + c_2 s) s^m \, dyds \\
&\qquad + \frac{m}{c_2}\int_H \mathscr{D}_t^m u_\delta(y, c_1 s) W^{(\alpha)}(x - y, t + c_2 s) s^{m-1} \, dyds \\
&\quad = -\frac{c_1 \Gamma(m+1)}{c_2(c_1 + c_2)^{m+1}} u_\delta(x, t) + \frac{\Gamma(m+1)}{c_2(c_1 + c_2)^m} u_\delta(x, t) \\
&\quad = \frac{\Gamma(m+1)}{(c_1 + c_2)^{m+1}} u_\delta(x, t).
\end{aligned}$$

Therefore, (4.3) holds for all $m \in \mathbf{N}_0$ whenever $k = 1$. Let $k \geq 1$ and suppose that (4.3) holds for all $m \in \mathbf{N}_0$. Then, (3.6), (4.1), and the assumption of the induction imply that

$$\begin{aligned}
&\int_H \mathscr{D}_t^m u_\delta(y, c_1 s) \mathscr{D}_t^{k+1} W^{(\alpha)}(x - y, t + c_2 s) s^{m+k} \, dyds \\
&\quad = -\frac{1}{c_2}\left[\int_{\mathbf{R}^n} \mathscr{D}_t^m u_\delta(y, c_1 s) \mathscr{D}_t^k W^{(\alpha)}(x - y, t + c_2 s) dy s^{m+k}\right]_0^\infty \\
&\qquad - \frac{c_1}{c_2}\int_H \mathscr{D}_t^{m+1} u_\delta(y, c_1 s) \mathscr{D}_t^k W^{(\alpha)}(x - y, t + c_2 s) s^{m+k} \, dyds
\end{aligned}$$

$$+\frac{m+k}{c_2}\int_H \mathscr{D}_t^m u_\delta(y,c_1 s)\mathscr{D}_t^k W^{(\alpha)}(x-y,t+c_2 s)s^{m+k-1}\,dyds$$
$$=-\frac{c_1\Gamma(m+k+1)}{c_2(c_1+c_2)^{m+k+1}}u_\delta(x,t)+\frac{\Gamma(m+k+1)}{c_2(c_1+c_2)^{m+k}}u_\delta(x,t)$$
$$=\frac{\Gamma(m+k+1)}{(c_1+c_2)^{m+k+1}}u_\delta(x,t).$$

Therefore, we obtain (4.3) with $m,k\in\mathbf{N}_0$ such that $m+k>0$.

Next, we show the equality (4.2). Let ν and κ be real numbers such that $\nu>-\left(\frac{n}{2\alpha}+\lambda+1\right)\frac{1}{p}$, $\kappa\geq 0$, and $\nu+\kappa>0$. Then, by (2.1) and (2.2), we have

$$\int_H \mathscr{D}_t^\nu u_\delta(y,s)\mathscr{D}_t^\kappa W^{(\alpha)}(x-y,t+s)s^{\nu+\kappa-1}\,dV(y,s)$$
$$=\int_H \frac{1}{\Gamma(\omega(\nu)-\nu)}\int_0^\infty \tau_1^{\omega(\nu)-\nu-1}\mathscr{D}_t^{\omega(\nu)}u_\delta(y,s+\tau_1)d\tau_1$$
$$\times\frac{1}{\Gamma(\omega(\kappa)-\kappa)}\int_0^\infty \tau_2^{\omega(\kappa)-\kappa-1}\mathscr{D}_t^{\omega(\kappa)}W^{(\alpha)}(x-y,t+s+\tau_2)d\tau_2$$
$$\times s^{\nu+\kappa-1}\,dV(y,s). \tag{4.5}$$

We assert that the integrand on the right-hand side of (4.5) belongs to $L^1(H,dV)$. In fact, when $\kappa=0$, (4.1) and (3.5) guarantee our assertion by the condition $\nu=\nu+\kappa>0$. Also, when $\kappa>0$, (4.1), (3.6), and Lemma 3.3 guarantee our assertion by the condition $\nu>-\left(\frac{n}{2\alpha}+\lambda+1\right)\frac{1}{p}$ and $\nu+\kappa>0$. Thus, by (4.3), the Fubini theorem implies that

$$\int_H \mathscr{D}_t^\nu u_\delta(y,s)\mathscr{D}_t^\kappa W^{(\alpha)}(x-y,t+s)s^{\nu+\kappa-1}\,dV(y,s)$$
$$=\frac{1}{\Gamma(\omega(\nu)-\nu)}\frac{1}{\Gamma(\omega(\kappa)-\kappa)}\int_0^\infty \tau_1^{\omega(\nu)-\nu-1}\int_0^\infty \tau_2^{\omega(\kappa)-\kappa-1}$$
$$\times\int_H \mathscr{D}_t^{\omega(\nu)}u_\delta(y,(1+\tau_1)s)\mathscr{D}_t^{\omega(\kappa)}W^{(\alpha)}(x-y,t+(1+\tau_2)s)$$
$$\times s^{\omega(\nu)+\omega(\kappa)-1}\,dV(y,s)d\tau_2 d\tau_1$$
$$=u_\delta(x,t)\frac{\Gamma(\omega(\nu)+\omega(\kappa))}{\Gamma(\omega(\nu)-\nu)\Gamma(\omega(\kappa)-\kappa)}$$
$$\times\int_0^\infty \tau_1^{\omega(\nu)-\nu-1}\int_0^\infty \frac{\tau_2^{\omega(\kappa)-\kappa-1}}{(\tau_1+\tau_2+2)^{\omega(\nu)+\omega(\kappa)}}\,d\tau_2 d\tau_1.$$

Furthermore, by Lemma 4.4, we have

$$\frac{\Gamma(\omega(\nu)+\omega(\kappa))}{\Gamma(\omega(\nu)-\nu)\Gamma(\omega(\kappa)-\kappa)}\int_0^\infty \tau_1^{\omega(\nu)-\nu-1}\int_0^\infty \frac{\tau_2^{\omega(\kappa)-\kappa-1}}{(\tau_1+\tau_2+2)^{\omega(\nu)+\omega(\kappa)}}\,d\tau_2 d\tau_1$$
$$=\frac{\Gamma(\nu+\kappa)}{2^{\nu+\kappa}}.$$

Therefore, the proof of Proposition 4.5 is completed.

The following lemma shows that the auxiality function u_δ of $u \in \boldsymbol{b}_\alpha^p(\lambda)$ converges to u in $L^p(\lambda)$ as $\delta \to 0^+$.

Lemma 4.6. *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, $\lambda > -1$, $\delta > 0$, *and* $u \in \boldsymbol{b}_\alpha^p(\lambda)$. *Then,*

$$\lim_{\delta\to 0^+} \|u_\delta - u\|_{L^p(\lambda)} = 0.$$

Proof. Let $u \in \boldsymbol{b}_\alpha^p(\lambda)$. Then, by Lemma 4.3, we have

$$\int_H |u_\delta(x,t)|^p t^\lambda \, dV(x,t) \le \int_H |u(x,t)|^p t^\lambda \, dV(x,t).$$

Moreover, the Fatou lemma shows that

$$\int_H |u(x,t)|^p t^\lambda \, dV(x,t) \le \liminf_{\delta\to 0^+} \int_H |u_\delta(x,t)|^p t^\lambda \, dV(x,t).$$

Therefore, we obtain $\lim_{\delta\to 0^+} \|u_\delta\|_{L^p(\lambda)} = \|u\|_{L^p(\lambda)}$. Hence, the desired result is obtained by the Egoroff theorem.

Now, we give the reproducing properties of fractional derivatives of the fundamental solution. The reproducing property in Theorem 2 is obtained by Theorem 4.7 with $\kappa = \lambda + 1$.

Theorem 4.7. *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, *and* $\lambda > -1$. *And let* κ *be a real number such that* $\kappa > \frac{\lambda+1}{p}$. *Then, the reproducing property*

$$u(x,t) = C_\kappa \int_H u(y,s) \mathscr{D}_t^\kappa W^{(\alpha)}(x-y, t+s) s^{\kappa-1} \, dV(y,s) \tag{4.6}$$

holds for all $u \in \boldsymbol{b}_\alpha^p(\lambda)$ *and* $(x,t) \in H$, *where* C_κ *is the constant defined in Theorem 2. Moreover,* (4.6) *also holds whenever* $p = 1$ *and* $\kappa = \lambda + 1$.

PROOF. Let $u \in \boldsymbol{b}_\alpha^p(\lambda)$ and $(x,t) \in H$. Then, Proposition 4.5 implies that

$$\begin{aligned}
&\left| u(x,t) - C_\kappa \int_H u(y,s) \mathscr{D}_t^\kappa W^{(\alpha)}(x-y,t+s) s^{\kappa-1}\, dV(y,s) \right| \\
&\quad \le |u(x,t) - u_\delta(x,t)| \\
&\qquad + \left| u_\delta(x,t) - C_\kappa \int_H u(y,s) \mathscr{D}_t^\kappa W^{(\alpha)}(x-y,t+s) s^{\kappa-1}\, dV(y,s) \right| \\
&\quad \le |u(x,t) - u_\delta(x,t)| + C_\kappa \int_H |u_\delta(y,s) - u(y,s)| s^{\lambda/p} \\
&\qquad \times |\mathscr{D}_t^\kappa W^{(\alpha)}(x-y,t+s)| s^{\kappa-1-\lambda/p}\, dV(y,s) \qquad (4.7)
\end{aligned}$$

for all $\delta > 0$. If $p = 1$ and $\kappa \ge \lambda + 1$, then $|\mathscr{D}_t^\kappa W^{(\alpha)}(x-y,t+s)| s^{\kappa-1-\lambda}$ is bounded. Therefore, Lemma 4.6 implies that (4.7) tends to 0 as $\delta \to 0^+$. If $p > 1$ and $\kappa > \frac{\lambda+1}{p}$, then by the Hölder inequality, (4.7) is dominated by

$$\begin{aligned}
&|u(x,t) - u_\delta(x,t)| \\
&\quad + C\|u_\delta - u\|_{L^p(\lambda)} \left(\int_H |\mathscr{D}_t^\kappa W^{(\alpha)}(x-y,t+s)|^q s^{(\kappa-1-\lambda/p)q}\, dV(y,s) \right)^{1/q},
\end{aligned}$$

where q is the exponent conjugate to p. Therefore, Lemma 4.6 and (2) of Theorem 1 imply that (4.7) tends to 0 as $\delta \to 0^+$. This completes the proof of Theorem 4.7.

5. Generalization of reproducing properties

In this section, we give generalization of reproducing properties in Theorem 4.7. The following proposition gives upper estimates of derivative norms of $u \in \boldsymbol{b}_\alpha^p(\lambda)$.

PROPOSITION 5.1. *Let $0 < \alpha \le 1$, $1 \le p < \infty$, and $\lambda > -1$. If a real number ν satisfies the condition $\nu > -\frac{\lambda+1}{p}$, then there exists a constant $C > 0$ such that $\|t^\nu \mathscr{D}_t^\nu u\|_{L^p(\lambda)} \le C\|u\|_{L^p(\lambda)}$ for all $u \in \boldsymbol{b}_\alpha^p(\lambda)$.*

PROOF. Let $u \in \boldsymbol{b}_\alpha^p(\lambda)$. Then, by Theorem 4.7 with $\kappa = \lambda + 2$, we have

$$u(x,t) = C_{\lambda+2} \int_H u(y,s) \mathscr{D}_t^{\lambda+2} W^{(\alpha)}(x-y,t+s) s^{\lambda+1}\, dV(y,s). \qquad (5.1)$$

By (1) and (2) of Theorem 1, we can differentiate under the integral sign on the right-hand side of (5.1). Therefore, we have

$$\mathscr{D}_t^{\omega(\nu)} u(x,t) = C_{\lambda+2} \int_H u(y,s) \mathscr{D}_t^{\omega(\nu)+\lambda+2} W^{(\alpha)}(x-y,t+s) s^{\lambda+1}\, dV(y,s).$$

Furthermore, by (2) of Theorem 1 and the condition $\nu > -\frac{\lambda+1}{p}$, the Fubini theorem implies that

$$\mathscr{D}_t^{\nu} u(x,t) = C_{\lambda+2} \int_H u(y,s) \mathscr{D}_t^{\nu+\lambda+2} W^{(\alpha)}(x-y,t+s) s^{\lambda+1}\, dV(y,s). \quad (5.2)$$

Suppose $p > 1$ and let q be the exponent conjugate to p. Then, by the condition $\nu > -\frac{\lambda+1}{p}$, we can choose a real number θ such that $-(\nu+\lambda+1) < \frac{\theta}{q} < 1$ and $-(\lambda+2) < \frac{\theta}{p} < \nu$. Thus, by the Hölder inequality and (2) of Theorem 1, there exists a constant $C > 0$ such that

$$\begin{aligned} &|\mathscr{D}_t^{\nu} u(x,t)| \\ &\quad \le C_{\lambda+2} \int_H |u(y,s)|\, |\mathscr{D}_t^{\nu+\lambda+2} W^{(\alpha)}(x-y,t+s)| s^{\theta/pq} s^{-\theta/pq} s^{\lambda+1}\, dV(y,s) \\ &\quad \le C_{\lambda+2} \left(\int_H |u(y,s)|^p |\mathscr{D}_t^{\nu+\lambda+2} W^{(\alpha)}(x-y,t+s)| s^{-\theta/q+\lambda+1}\, dV(y,s) \right)^{1/p} \\ &\qquad \times \left(\int_H |\mathscr{D}_t^{\nu+\lambda+2} W^{(\alpha)}(x-y,t+s)| s^{\theta/p+\lambda+1}\, dV(y,s) \right)^{1/q} \\ &\quad \le C \left(\int_H |u(y,s)|^p s^{-\frac{\theta}{q}+\lambda+1} |\mathscr{D}_t^{\nu+\lambda+2} W^{(\alpha)}(x-y,t+s)| dV(y,s) \right)^{1/p} t^{(1/q)(\theta/p-\nu)}. \end{aligned}$$

Therefore, by the Fubini theorem and (2) of Theorem 1, we have

$$\begin{aligned} \|t^{\nu} \mathscr{D}_t^{\nu} u\|_{L^p(\lambda)} &\le C \int_H |u(y,s)|^p s^{-\theta/q+\lambda+1} \\ &\qquad \times \int_H t^{(p/q)(\theta/p-\nu)+p\nu+\lambda} |\mathscr{D}_t^{\nu+\lambda+2} W^{(\alpha)}(x-y,t+s)| dV(x,t) dV(y,s) \\ &\le C \int_H |u(y,s)|^p s^{-\theta/q+\lambda+1} s^{\theta/q-1}\, dV(y,s) = C\|u\|_{L^p(\lambda)}. \end{aligned}$$

Suppose $p = 1$. Then, by (5.2) and the Fubini theorem, (2) of Theorem 1 implies that

$$\begin{aligned} &\|t^{\nu} \mathscr{D}_t^{\nu} u\|_{L^1(\lambda)} \\ &\quad \le C \int_H |u(y,s)| s^{\lambda+1} \int_H |\mathscr{D}_t^{\nu+\lambda+2} W^{(\alpha)}(x-y,t+s)| t^{\nu+\lambda}\, dV(x,t) dV(y,s) \\ &\quad \le C \int_H |u(y,s)| s^{\lambda+1} s^{-1}\, dV(y,s) = C\|u\|_{L^1(\lambda)}. \end{aligned}$$

This completes the proof.

Now, we give a generalization of reproducing properties in Theorem 4.7.

THEOREM 5.2. *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, *and* $\lambda > -1$. *And let* ν *and* κ *be real numbers such that* $\nu > -\frac{\lambda+1}{p}$ *and* $\kappa > \frac{\lambda+1}{p}$. *Then, the reproducing property*

$$u(x,t) = C_{\nu+\kappa} \int_H \mathscr{D}_t^{\nu} u(y,s) \mathscr{D}_t^{\kappa} W^{(\alpha)}(x-y, t+s) s^{\nu+\kappa-1}\, dV(y,s) \tag{5.3}$$

holds for all $u \in \boldsymbol{b}_{\alpha}^{p}(\lambda)$ *and* $(x,t) \in H$, *where* C_{ν} *is the constant defined in Theorem 2. Moreover,* (5.3) *also holds whenever* $p = 1$ *and* $\kappa = \lambda + 1$.

PROOF. Let $u \in \boldsymbol{b}_{\alpha}^{p}(\lambda)$ and $(x,t) \in H$. Then, Proposition 4.5 implies that

$$\begin{aligned} &\left| u(x,t) - C_{\nu+\kappa} \int_H \mathscr{D}_t^{\nu} u(y,s) \mathscr{D}_t^{\kappa} W^{(\alpha)}(x-y, t+s) s^{\nu+\kappa-1}\, dV(y,s) \right| \\ &\quad \le |u(x,t) - u_{\delta}(x,t)| + C_{\nu+\kappa} \int_H |\mathscr{D}_t^{\nu} u_{\delta}(y,s) - \mathscr{D}_t^{\nu} u(y,s)| s^{\nu+\lambda/p} \\ &\quad\quad \times |\mathscr{D}_t^{\kappa} W^{(\alpha)}(x-y, t+s)| s^{\kappa-1-\lambda/p}\, dV(y,s) \end{aligned} \tag{5.4}$$

for all $\delta > 0$. If $p = 1$ and $\kappa \ge \lambda + 1$, then $|\mathscr{D}_t^{\kappa} W^{(\alpha)}(x-y, t+s)| s^{\kappa-1-\lambda}$ is bounded. Therefore, Proposition 5.1 and Lemma 4.6 imply that (5.4) tends to 0 as $\delta \to 0^+$. If $p > 1$ and $\kappa > \frac{\lambda+1}{p}$, then by the Hölder inequality and Proposition 5.1, (5.4) is dominated by

$$\begin{aligned} &|u(x,t) - u_{\delta}(x,t)| \\ &\quad + C \|u_{\delta} - u\|_{L^p(\lambda)} \left(\int_H |\mathscr{D}_t^{\kappa} W^{(\alpha)}(x-y, t+s)|^q s^{(\kappa-1-\lambda/p)q}\, dV(y,s) \right)^{1/q}, \end{aligned}$$

where q is the exponent conjugate to p. Therefore, Lemma 4.6 and (2) of Theorem 1 imply that (5.4) tends to 0 as $\delta \to 0^+$. This completes the proof of Theorem 5.2.

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