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Non-invertible knots having toroidal Dehn surgery of hitting number four

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ABSTRACT. We show that there exist infinitely many non-invertible, hyperbolic knots that admit toroidal Dehn surgery of hitting number four. The resulting toroidal manifold contains a unique incompressible torus meeting the core of the attached solid torus in four points, but no incompressible torus meeting it less than four points.

1. Introduction

For a hyperbolic knot in the 3-sphere S^3, at most finitely many Dehn surgeries yield non-hyperbolic 3-manifolds by Thurston's hyperbolic Dehn surgery theorem. Such Dehn surgeries are called *exceptional Dehn surgeries*. A typical one is Dehn surgery creating an incompressible torus, called *toroidal Dehn surgery*. By Gordon-Luecke [7, 8], the surgery slope of toroidal Dehn surgery is integral or half-integral, and the latter happens only for Eudave-Muñoz knots [3]. Thus the study of integral toroidal Dehn surgery is the next challenging task.

We now introduce the notion of hitting number for toroidal Dehn surgery. Let K be a hyperbolic knot in S^3. Suppose that the resulting 3-manifold by r-Dehn surgery, denoted by $K(r)$, is toroidal. For an incompressible torus T contained in $K(r)$, let $|K^* \cap T|$ denote the number of points in $K^* \cap T$, where K^* is the core of the attached solid torus of $K(r)$. For a pair (K, r), we call

$$\min\{|K^* \cap T| : T \text{ is an incompressible torus in } K(r)\}$$

the *hitting number* of (K, r). Since K is hyperbolic, a hitting number is positive. This is a natural measure of the complexity of toroidal Dehn surgery. We should mention that the only possible odd hitting number is one.

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For, if (K, r) has odd hitting number, then $K(r)$ contains a non-separating torus, so $r = 0$ by homological reason. Thus the 0-surgered manifold $K(0)$ contains a non-separating torus, which implies that K has genus one by Gabai [5]. Then a minimal genus Seifert surface extends to an incompressible torus. This means that the hitting number is one. Also, Osoinach [16] showed that hitting number can be arbitrarily large. His construction was studied more precisely in [19], where we showed that any even integer can be realized as hitting number.

Recall that a knot K is *invertible* if there exists an orientation-preserving homeomorphism $h : S^3 \to S^3$ such that its restriction $h|_K$ is an orientation-reversing homeomorphism of K onto itself. Furthermore, if h is chosen to be an involution whose fixed point set is an unknotted circle C such that C meets K in two points, then K is said to be *strongly-invertible*. We remark that an invertible hyperbolic knot is strongly-invertible ([10]).

For most toroidal Dehn surgeries, including half-integral ones, their hitting numbers are at most two. This led to a natural question, as in [11] for example, asking whether it is always the case. However, Eudave-Muñoz [4] first gave an infinite family of hyperbolic knots, each of which admits toroidal Dehn surgery of hitting number four. We note that his knots are strongly-invertible, and that the simplest one seems to have genus 37. In [18], we gave a new infinite family of hyperbolic knots admitting toroidal Dehn surgery of hitting number four. The simplest one has genus 9, but all knots are still strongly-invertible.

Any genus one hyperbolic knot admits toroidal Dehn surgery of hitting number one as explained before, so the existence of such toroidal Dehn surgery does not imply the invertibility, in general. It seems unknown that the existence of toroidal Dehn surgery of hitting number two implies the invertibility. The knots constructed in [19], which realize hitting number greater than two, seem to be non-invertible, as suggested by a computer experiment via SnapPea [22], but we could not prove it.

The purpose of this paper is to give the first infinite family of non-invertible hyperbolic knots, each of which admits toroidal Dehn surgery of hitting number four. Our starting point is the following fact. As stated in [23], 1-surgery on the $(-3, 3, 7)$-pretzel knot is toroidal, and it has hitting number four in our language. This pretzel knot is known to be non-invertible (see [1, section 12D]).

Let K be the $(-3, 3, 7)$-pretzel knot. Consider an unknotted circle C as shown in Figure 1. For an integer n, let K_n be the knot obtained from K by twisting n times along C. (For convenience, set $K = K_0$.) That is, K_n is the image of K under $-1/n$-surgery along C. Since C is unknotted, the ambient space remains the 3-sphere.

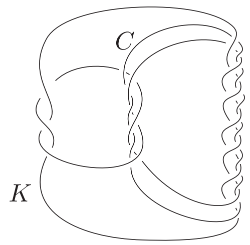

Fig. 1. The $(-3, 3, 7)$-pretzel knot K and C

Theorem 1. *The knots K_n defined as above satisfy the following properties.*
(1) *K_n is a non-invertible hyperbolic knot.*
(2) *K_n has genus one, tunnel number two, cyclic period two.*
(3) 1-*surgery on K_n is toroidal Dehn surgery of hitting number four.*
(4) *If $m \neq n$, then K_m and K_n are not equivalent.*

In general, it is a hard task to show that a knot is non-invertible. We will accomplish this by analyzing the mapping class group of the resulting toroidal manifold, which is a graph manifold.

Our construction is inspired by [12], where non-invertible hyperbolic knots that admit exceptional Dehn surgeries yielding Seifert fibered spaces are constructed by twisting the $(-3, 3, 5)$-pretzel knot along an unknotted circle.

In this paper, two knots are said to be *equivalent* if there exists an ambient isotopy of S^3 sending one knot to the other. Also, a Seifert fibered space over the disk with two exceptional fibers of indices p, q is denoted by $D^2(p, q)$.

2. Toroidal Dehn surgery

In this section, we prove that 1-surgery on K_n yields a toroidal manifold which contains the unique separating incompressible torus by using the Montesinos trick ([13]).

Lemma 1. *For any n,* 1-*surgery on K_n yields a toroidal manifold. More precisely, it is a graph manifold which contains the unique separating incompressible torus splitting it into two Seifert fibered spaces over the disk with two exceptional fibers. Their regular fibers meet in one point on the torus. The core of the attached solid torus of $K_n(1)$ meets the torus in four points minimally.*

Proof. Since the linking number between K and C are zero, 1-surgery on K_n is equivalent to 1-surgery on K and $-1/n$-surgery on C. As shown in Figure 2, there is an orientation-preserving involution $f : S^3 \to S^3$ such that $f(K) = K$ and $f(C) = C$ with an axis A.

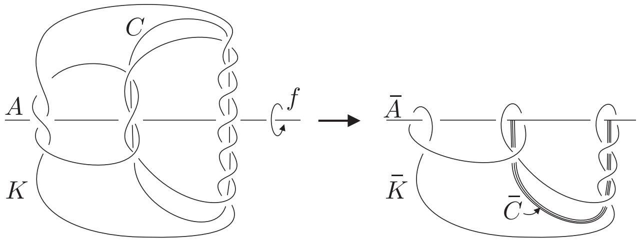

Fig. 2. The involution f for $K \cup C$

Take the quotient of S^3 by f. Let $\bar{K}$, $\bar{C}$, $\bar{A}$ denote the image of K, C and A, respectively. We note that the factor knot $\bar{K}$ is unknotted in $S^3/f \cong S^3$, and that $\bar{C}$ is an arc whose endpoints lie on $\bar{A}$ (see Figure 2).

Now, 1-surgery on K corresponds to $1/2$-surgery on $\bar{K}$, and $-1/n$-surgery on C corresponds to the tangle surgery which replaces the $1/0$-tangle with the $1/n$-tangle. The right one of Figure 2 is deformed to Figure 3. To keep track of the framing on C, $\bar{C}$ is accompanied by a band whose center is $\bar{C}$. Then (-2)-twisting along $\bar{K}$ changes $\bar{A}$ to $\bar{A}'$ as shown in Figure 4. Finally, perform the tangle surgery along the arc $\bar{C}$ to yield a knot $\bar{A}''$.

Then we can see that there exists a Conway sphere S which meets $\bar{A}''$ in four points (Figure 4). Both sides of S are Montesinos tangles. Hence the

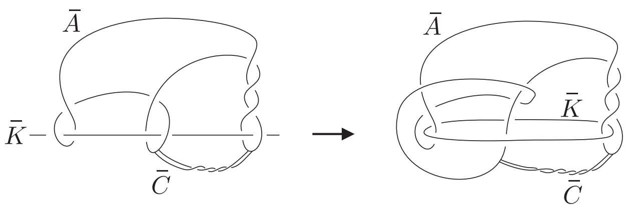

Fig. 3

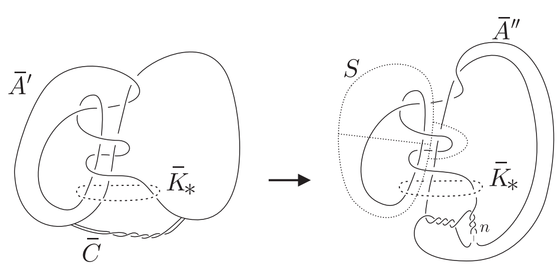

Fig. 4

double branched covering space of S^3 branched over $\bar{A}''$, which is $K_n(1)$, is a toroidal manifold, where the lift T of S gives a separating incompressible torus. More precisely, it is a (non-Seifert) graph manifold which is the union of two Seifert fibered spaces $D^2(2,3)$ and $D^2(2,|14n+3|)$. Their Seifert fibers intersect once on T as seen from Figure 4. It is well known that such a manifold contains the unique incompressible torus.

As shown in Figure 4, the circle $\overline{K}_*$ lifts to the core of the attached solid torus in $K_n(1)$. Thus the core meets the unique incompressible torus T in four times. The minimality follows from the fact that in each side of the Conway sphere S, $\overline{K}_*$ lies on a disk decomposing the Montesinos tangles into two rational tangles [4, Example 1.4].

3. Symmetry

Lemma 2. *K_n has cyclic period two.*

Proof. The argument is the same as the proof of [12, Claim 2.1]. There is an involution $f : S^3 \to S^3$ such that $f(K) = K$ and $f(C) = C$, as used in the proof of Lemma 1 (Figure 2). Then $f|_{S^3 - \mathrm{Int}\, N(C)}$ naturally extends to an involution $\tilde{f}$ of $C(-1/n) \cong S^3$ with an axis $\tilde{C}$ such that $\tilde{f}(K_n) = K_n$ and $\tilde{C} \cap K_n = \emptyset$. This shows that K_n has cyclic period two.

For a pair of manifolds (M,N) (possibly, $N = \emptyset$), let $\mathrm{Diff}(M,N)$ be the group of pairwise diffeomorphisms of (M,N), and $\mathrm{Diff}^*(M,N)$ the subgroup of $\mathrm{Diff}(M,N)$ consisting diffeomorphisms preserving the orientation of M.

Lemma 3. *K_n is not strongly-invertible.*

Proof. It suffices to show for $n \neq 0$, because $K = K_0$ is known to be non-invertible (see [1]). Suppose that K_n is strongly-invertible for contradiction. Then there is a finite subgroup G of $\mathrm{Diff}^*(S^3, K_n)$ generated by an involution f realizing the cyclic period two and a strong inversion g. As in [15, 6.1], G is isomorphic to the dihedral group of order four. For simplicity, put $M = K_n(1)$. Then, the action of G on S^3 induces an action of $\overline{G}$ ($\cong G$) on M, where $\overline{G}$ is generated by the extensions of f and g.

Recall that M contains the unique separating incompressible torus T up to isotopy, which splits M into two Seifert fibered spaces M_1 and M_2 by Lemma 1. For any $n \neq 0$, M_1 and M_2 are not diffeomorphic. Thus any diffeomorphism of M can be isotoped so that it preserves M_i for $i = 1, 2$. Let Δ be the subgroup of $\pi_0\, \mathrm{Diff}(M_1, T) \times \pi_0\, \mathrm{Diff}(M_2, T)$ consisting of all elements $([f_1], [f_2])$ such that $f_1|_T$ is isotopic to $f_2|_T$. Then we have the following exact sequence [17, Lemma 1.2]

$$1 \to \mathscr{D} \to \pi_0\, \mathrm{Diff}(M) \to \Delta \to 1,$$

for $\mathscr{D} \cong \pi_1(T)/(Z(\pi_1(M_1)) + Z(\pi_1(M_2)))$, where $Z(\pi_1(M_i))$ denotes the center of $\pi_1(M_i)$.

Since M_i is a Seifert fibered space over the disk with two exceptional fibers, which is not $D^2(2,2)$, its fibration is unique, and any regular fiber lies in the center of $\pi_1(M_i)$. By Lemma 1, the regular fibers of M_1 and M_2 meet once on T. Hence $\mathscr{D} \cong \{1\}$, so that $\pi_0 \operatorname{Diff}(M) \cong \Delta$. On the other hand, $\pi_0 \operatorname{Diff}(M_i) \cong \mathbf{Z}_2$ ([9, Proposition 25.3]), which is generated by an involution of the orbit surface (disk) fixing the exceptional points. Thus $\Delta \cong \mathbf{Z}_2$, giving $\pi_0 \operatorname{Diff}(M) \cong \mathbf{Z}_2$.

However, M is Haken, so aspherical. Since M is not Seifert fibered, $\pi_1(M)$ has trivial center by [20]. Then Borel's theorem (cf. [2]) implies that there is a monomorphism from $\bar{G}$ to $\operatorname{Out}(\pi_1(M))$, which is isomorphic to $\pi_0 \operatorname{Diff}(M)$ by [21]. This contradicts the fact that $\bar{G}$ is the dihedral group of order four. □

4. Other properties

LEMMA 4. *K_n has genus one.*

PROOF. The unknotted circle C lies on the standard genus one Seifert surface F of K, which is obtained as a checkerboard surface. By pushing C off from F, we can see that F remains a genus one Seifert surface of K_n for any n. Since K_n admits toroidal surgery by Lemma 1, K_n is non-trivial. Thus K_n has genus one. □

LEMMA 5. *K_n is hyperbolic.*

PROOF. Since 0-surgery on the trefoil is the only toroidal Dehn surgery for torus knots ([14]), K_n is not a torus knot.

Suppose that K_n is a satellite knot. Then the knot exterior contains an essential torus F, which bounds a solid torus V in S^3 containing K_n. After 1-surgery on K_n, F is compressible, because it is disjoint from the core of the attached solid torus in $K_n(1)$. Since $K_n(1)$ is a homology sphere, F still bounds a solid torus after the surgery [6]. This implies that K_n is either a 0- or 1-bridge braid in V. In particular, its winding number in V is at least two. Thus K_n has genus at least two by [1, Proposition 2.10], contradicting Lemma 4. □

LEMMA 6. *K_n has tunnel number two.*

PROOF. The argument is the same as the proof of [12, Claim 2.5]. Consider a genus two handlebody H which is obtained by thickening the obvious genus one Seifert surface of K. Its boundary gives a genus two Heegaard splitting of S^3. Since C is a core of H, H remains a genus two handlebody by surgery on C. Thus K_n lies on a genus two Heegaard surface of S^3. This implies that the tunnel number of K_n is at most two. However, a tunnel number one knot is strongly-invertible. By Lemma 3, K_n has tunnel number two.

Lemma 7. *If $m \neq n$, then K_m and K_n are not equivalent.*

Proof. This follows from the fact that $K_m(1)$ and $K_n(1)$ are not diffeomorphic.

Proof of Theorem 1. This follows from Lemmas proved in Sections 2, 3 and 4.

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