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An elementary proof of Sobolev embeddings for Riesz potentials of functions in Morrey spaces $L^{1,\nu,\beta}(G)$

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ABSTRACT. Our aim is to give an elementary proof of Sobolev embeddings for Riesz potentials of functions in Morrey spaces $L^{1,\nu,\beta}(G)$, as an extension of Serrin [13]. We are mainly concerned with Trudinger's type exponential integrability for Riesz potentials.

1. Introduction

Let G be a bounded open set in $\mathbf{R}^n$. For $0 < \alpha < n$, we define the Riesz potential of order α for an integrable function f on G by

$$I_\alpha f(x) = \int_G |x - y|^{\alpha - n} f(y) dy.$$

In what follows we assume that $f = 0$ outside G.

For an integrable function u on a measurable set $E \subset \mathbf{R}^n$ of positive measure, we define the integral mean over E by

$$⨍_E u(x) dx = \frac{1}{|E|} \int_E u(x) dx,$$

where $|E|$ denotes the Lebesgue measure of E.

In the present paper, f is assumed to satisfy the Morrey condition: if $0 \le \nu \le n$ and β are real numbers, then

$$⨍_{B(x,r)} |f(y)| dy \le r^{-\nu} (\log(2 + r^{-1}))^{-\beta} \tag{1.1}$$

for all $x \in G$ and $0 < r < d_G$, where $B(x,r)$ denotes the open ball centered at x of radius $r > 0$ and d_G denotes the diameter of G. It is worth pointing out that (1.1) is essentially equivalent to

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$$\fint_{B(x,r)} |f(y)|dy \le r^{-\nu}(\log(r^{-1}))^{-\beta} \tag{1.2}$$

for all $x \in G$ and $0 < r < \min\{2^{-1}, d_G\}$. We denote by $L^{1,\nu,\beta}(G)$ the family of all measurable functions f on G satisfying condition (1.1) or (1.2); for Morrey spaces, we refer to [8] and [12].

The famous Trudinger's inequality ([15]) insists that Sobolev functions in $W^{1,n}$ satisfy finite exponential integrability (see also [2], [4] and [16]). Recently Serrin [13] gave an elementary proof of Trudinger's inequality ([15]), which relies on Hölder's inequality and integration by parts.

Our first aim in this note is to give a local Morrey version of Trudinger's type exponential integrability and continuity for Riesz potentials of functions satisfying (1.1), as an extension of [13] and [15].

Theorem 1.1. *Let f be a nonnegative measurable function on G satisfying (1.1) with $\nu = \alpha$ and a real number β. If $\alpha/2 \le \varepsilon < \alpha$, then there exist constants $c_1, c_2 > 0$ such that*
(1) *in case $\beta < 1$,*

$$\fint_{B(z,r)} \exp\left(\frac{(I_\alpha f(x))^{1/(1-\beta)}}{c_1}\right) dx \le c_2 r^{-\alpha+\varepsilon}(\log(2+r^{-1}))^{-\beta} \tag{1.3}$$

for all $z \in G$ and $0 < r < d_G$;
(2) *in case $\beta = 1$,*

$$\fint_{B(z,r)} \exp\left(\frac{1}{c_1} \exp\left(\frac{I_\alpha f(x)}{C}\right)\right) dx \le c_2 r^{-\alpha+\varepsilon}(\log(2+r^{-1}))^{-1} \tag{1.4}$$

for all $z \in G$ and $0 < r < d_G$;
(3) *in case $\beta > 1$,*

$$|I_\alpha f(x) - I_\alpha f(z)| \le C(\log(2+|x-z|^{-1}))^{-\beta+1} \tag{1.5}$$

for all $x, z \in G$. Here we can take

$$c_1 = C(\alpha - \varepsilon)^{-1}$$

and

$$c_2 = C(\alpha - \varepsilon)^{-\beta_+ - 1},$$

where $\beta_+ = \max\{\beta, 0\}$ and $C = C(n, \alpha, \beta, d_G)$ denotes a various constant depending on n, α, β and d_G.

We also give the following Morrey version of Sobolev's type inequality for Riesz potentials of functions satisfying (1.1), as an extension of [13].

Theorem 1.2. *Let f be a nonnegative measurable function on G satisfying* (1.1) *with $\alpha < \nu \le n$ and a real number β. If $p = \nu/(\nu - \alpha)$ and $\gamma > 1$, then there exists a constant $C = C(n, \alpha, \nu, \beta, \gamma, d_G) > 0$ such that*

$$\left(\fint_{B(z,r)} (I_\alpha f(x))^p (\log(2 + I_\alpha f(x)))^{-\gamma+\alpha\beta p/\nu} dx \right)^{1/p} \le C r^{\alpha-\nu} (\log(2 + r^{-1}))^{(1-\gamma-\beta)/p} \tag{1.6}$$

for all $z \in G$ and $0 < r < d_G$.

For related results, we also refer to Adams [1], Chiarenza-Frasca [3] and the authors [5, 6, 7, 10, 11].

2. Proof of Theorem 1.1

Throughout this paper, let C denote various positive constants independent of the variables in question and $C(a, b, \dots)$ be a constant which may depend on $a, b, \dots$.

When $\gamma > 0$, note that

$$\int |x - y|^{-\gamma} f(y) dy = \int_0^\infty \left(\int_{B(x,t)} f(y) dy \right) d(-t^{-\gamma}) \tag{2.1}$$

and

$$\int_0^r \left(\log \frac{1}{t} \right)^{-\gamma-1} \frac{dt}{t} = \frac{1}{\gamma} \left(\log \frac{1}{r} \right)^{-\gamma} \tag{2.2}$$

for $0 < r < 1$.

Lemma 2.1. *Suppose $0 < a \le R_0$ and $0 < b \le R_0$. Then there exists a constant $C(R_0) > 0$ such that*

$$\int_\delta^{1/2} t^{-a} (\log(1/t))^{-b} \frac{dt}{t} \le C(R_0) a^{-b-1} \delta^{-a} (\log(1/\delta))^{-b}$$

for all $0 < \delta < 1/2$.

Proof. Note that $u_a(s) = s^{-a}(\log(1/s))^{-b}$ attains a minimum value of $e^b b^{-b} a^b$ at $s = e^{-b/a}$ for $0 < s < 1$. If $1/2 \le e^{-b/a}$, then u_a is decreasing on $(0, 1/2]$. Hence

$$u_a(t) \le u_a(\delta) \qquad \text{for } 0 < \delta \le t < 1/2.$$

If $e^{-b/a} < 1/2$, then u_a is decreasing on $(0, e^{-b/a}]$ and increasing on $[e^{-b/a}, 1/2]$. Hence, in the case $e^{-b/a} \le \delta$ we have

$$u_a(t) \le \frac{u_a(1/2)}{u_a(e^{-b/a})} u_a(\delta) = \frac{2^a(\log 2)^{-b}}{e^b b^{-b} a^b} u_a(\delta) \qquad \text{for } 0 < \delta \le t < 1/2,$$

and, in the case $0 < \delta < e^{-b/a}$ we have

$$u_a(t) \le u_a(\delta) \qquad \text{for } 0 < \delta \le t < e^{-b/a},$$

$$u_a(t) \le \frac{2^a(\log 2)^{-b}}{e^b b^{-b} a^b} u_a(\delta) \qquad \text{for } e^{-b/a} \le t < 1/2.$$

Therefore, we obtain

$$u_a(t) \le C(R_0) a^{-b} u_a(\delta) \qquad \text{for } 0 < \delta \le t < 1/2, \tag{2.3}$$

so that

$$\int_\delta^{1/2} t^{-a}(\log(1/t))^{-b} \frac{dt}{t} \le C(R_0)(a/2)^{-b} u_{a/2}(\delta) \int_\delta^{1/2} t^{-a/2} \frac{dt}{t}$$
$$\le C(R_0) 2^{b+1} a^{-b-1} \delta^{-a} (\log(1/\delta))^{-b}$$

for all $0 < \delta < 1/2$, as required.

LEMMA 2.2. *Let $\alpha/2 \le \varepsilon \le \alpha$. Let f be a nonnegative measurable function on G satisfying* (1.1) *with $\nu = \alpha$.*

(1) *If $\alpha/2 \le \varepsilon < \alpha$, then*

$$\int_{G \setminus B(x,\delta)} |x - y|^{\varepsilon - n} f(y) dy \le C(\alpha - \varepsilon)^{-\beta_+ - 1} \delta^{\varepsilon - \alpha} (\log(2 + \delta^{-1}))^{-\beta};$$

(2) *if $\varepsilon = \alpha$ and $\beta < 1$, then*

$$\int_{G \setminus B(x,\delta)} |x - y|^{\varepsilon - n} f(y) dy \le C(\log(2 + \delta^{-1}))^{-\beta + 1};$$

(3) *if $\varepsilon = \alpha$ and $\beta = 1$, then*

$$\int_{G \setminus B(x,\delta)} |x - y|^{\varepsilon - n} f(y) dy \le C \log(2 + (\log(2 + \delta^{-1})))$$

for $x \in G$ and $\delta > 0$, where $C = C(n, \beta, d_G)$.

PROOF. If $\alpha/2 \le \varepsilon < \alpha$, then we have by (2.1) and (1.1)

$$\int_{G \setminus B(x,\delta)} |x - y|^{\varepsilon - n} f(y) dy \le \int_\delta^{2d_G} \left(\int_{B(x,r)} f(y) dy \right) d(-r^{\varepsilon - n})$$
$$\le C \int_\delta^\infty r^{\varepsilon - \alpha} (\log(2 + r^{-1}))^{-\beta} \frac{dr}{r}.$$

When $\beta > 0$, Lemma 2.1 gives

$$\int_\delta^\infty r^{\varepsilon-\alpha}(\log(2+r^{-1}))^{-\beta}\frac{dr}{r} \le C(\beta)(\alpha-\varepsilon)^{-\beta-1}\delta^{\varepsilon-\alpha}(\log(2+\delta^{-1}))^{-\beta}$$

and when $\beta \le 0$,

$$\begin{aligned}\int_\delta^\infty r^{\varepsilon-\alpha}(\log(2+r^{-1}))^{-\beta}\frac{dr}{r} &\le (\log(2+\delta^{-1}))^{-\beta}\int_\delta^\infty r^{\varepsilon-\alpha}\frac{dr}{r}\\ &\le (\alpha-\varepsilon)^{-1}\delta^{\varepsilon-\alpha}(\log(2+\delta^{-1}))^{-\beta}.\end{aligned}$$

Thus it follows that

$$\int_{G\setminus B(x,\delta)} |x-y|^{\varepsilon-n}f(y)dy \le C(\alpha-\varepsilon)^{-\beta_+-1}\delta^{\varepsilon-\alpha}(\log(2+\delta^{-1}))^{-\beta},$$

where C is a positive constant depending on β.

The remaining cases can be proved similarly.

Lemma 2.3. *Let $\alpha < \nu \le n$. Let f be a nonnegative measurable function on G satisfying* (1.1). *Then*

$$\int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n}f(y)dy \le C\delta^{\alpha-\nu}(\log(2+\delta^{-1}))^{-\beta}$$

for $x \in G$ and $\delta > 0$, where $C = C(n,\alpha,\nu,\beta)$.

Lemma 2.4. *Let $\alpha/2 \le \varepsilon < \alpha$ and $\alpha \le \nu$. Let f be a nonnegative measurable function on G satisfying* (1.1). *Then*

$$\int_{B(z,\delta)} I_\varepsilon f(x)dx \le C(\nu-\varepsilon)^{-\beta_+-1}\delta^{\varepsilon-\nu+n}(\log(2+\delta^{-1}))^{-\beta}$$

for $z \in G$ and $\delta > 0$, where $C = C(n,\alpha,\beta)$.

Proof. Write

$$I_\varepsilon f(x) = \int_{B(z,2\delta)} |x-y|^{\varepsilon-n}f(y)dy + \int_{G\setminus B(z,2\delta)} |x-y|^{\varepsilon-n}f(y)dy = I_1(x) + I_2(x).$$

By Fubini's theorem, we have by (1.1) and the fact that $\int_{B(z,\delta)}|x-y|^{\varepsilon-n}dx$ attains its maximum at $y = z$

$$\begin{aligned}\int_{B(z,\delta)} I_1(x)dx &\le \int_{B(z,2\delta)}\left(\int_{B(z,\delta)}|x-y|^{\varepsilon-n}dx\right)f(y)dy\\ &\le C\delta^\varepsilon\int_{B(z,2\delta)} f(y)dy \le C\delta^{\varepsilon-\nu+n}(\log(2+\delta^{-1}))^{-\beta}.\end{aligned}$$

For I_2, note that

$$I_2(x) \le C \int_{G \setminus B(z,2\delta)} |z-y|^{\varepsilon-n} f(y)dy$$

for $x \in B(z,\delta)$. Hence the proof of Lemma 2.2 gives

$$\int_{B(z,\delta)} I_2(x)dx \le C(v-\varepsilon)^{-\beta_+-1}\delta^{n+\varepsilon-v}(\log(2+\delta^{-1}))^{-\beta}$$

since $\alpha/2 \le \varepsilon < \alpha$. Thus this lemma is proved.

PROOF OF THEOREM 1.1. Let f be a nonnegative measurable function on G satisfying (1.1).

First suppose $\beta < 1$. For $\alpha/2 \le \varepsilon < \alpha$, by Lemma 2.2, we have

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ &\le \delta^{\alpha-\varepsilon} \int_{B(x,\delta)} |x-y|^{\varepsilon-n} f(y)dy + C(\log(2+\delta^{-1}))^{-\beta+1} \\ &\le \delta^{\alpha-\varepsilon} I_\varepsilon f(x) + C(\log(2+\delta^{-1}))^{-\beta+1} \end{aligned}$$

for $\delta > 0$. Considering $\delta = (I_\varepsilon f(x))^{-1/(\alpha-\varepsilon)}(\log(2+I_\varepsilon f(x)))^{(1-\beta)/(\alpha-\varepsilon)}$ when $I_\varepsilon f(x)$ is large enough, we see that

$$I_\alpha f(x) \le C(\alpha-\varepsilon)^{\beta-1}(\log(2+I_\varepsilon f(x)))^{-\beta+1},$$

so that

$$⨍_{B(z,r)} \exp\left(\frac{(I_\alpha f(x))^{1/(1-\beta)}}{c_1}\right)dx \le ⨍_{B(z,r)} \{2+I_\varepsilon f(x)\}dx$$

for $z \in G$ and $0 < r < d_G$, where $c_1 = C(\alpha-\varepsilon)^{-1}$. Hence Lemma 2.4 with $v = \alpha$ gives

$$⨍_{B(z,r)} \exp\left(\frac{(I_\alpha f(x))^{1/(1-\beta)}}{c_1}\right)dx \le C(\alpha-\varepsilon)^{-\beta_+-1} r^{-\alpha+\varepsilon}(\log(2+r^{-1}))^{-\beta}$$

for such z and r, which implies (1.3).

Next suppose $\beta = 1$. For $\alpha/2 \le \varepsilon < \alpha$, by Lemma 2.2, we have

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy + \int_{G\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ &\le \delta^{\alpha-\varepsilon} I_\varepsilon f(x) + C \log(2+\log(2+\delta^{-1})) \end{aligned}$$

for $\delta > 0$. Considering $\delta = (I_\varepsilon f(x))^{-1/(\alpha-\varepsilon)}(\log(2 + \log(2 + I_\varepsilon f(x))))^{1/(\alpha-\varepsilon)}$ when $I_\varepsilon f(x)$ is large enough, we see that

$$I_\alpha f(x) \le C \log\left(2 + \frac{\log(2 + I_\varepsilon f(x))}{\alpha - \varepsilon}\right),$$

so that

$$⨍_{B(z,r)} \exp\left(\frac{1}{c_1} \exp\left(\frac{I_\alpha f(x)}{C}\right)\right) dx \le ⨍_{B(z,r)} \{2 + I_\varepsilon f(x)\} dx$$

for $z \in G$ and $0 < r < d_G$, where $c_1 = C(\alpha - \varepsilon)^{-1}$. Hence Lemma 2.4 with $v = \alpha$ gives

$$⨍_{B(z,r)} \exp\left(\frac{1}{c_1} \exp\left(\frac{I_\alpha f(x)}{C}\right)\right) dx \le c_2 r^{-\alpha+\varepsilon}(\log(2 + r^{-1}))^{-1}$$

with $c_2 = C(\alpha - \varepsilon)^{-2}$ for such z and r, which implies (1.4).

Finally suppose $\beta > 1$. Write

$$\begin{aligned} I_\alpha f(x) - I_\alpha f(z) &= \int_{B(x,2|x-z|)} |x - y|^{\alpha-n} f(y) dy - \int_{B(x,2|x-z|)} |z - y|^{\alpha-n} f(y) dy \\ &\quad + \int_{G \setminus B(x,2|x-z|)} (|x - y|^{\alpha-n} - |z - y|^{\alpha-n}) f(y) dy. \end{aligned}$$

As in the proof of Lemma 2.2, we have

$$\int_{B(x,2|x-z|)} |x - y|^{\alpha-n} f(y) dy \le C(\log(2 + |x - z|^{-1}))^{-\beta+1}$$

and

$$\begin{aligned} \int_{B(x,2|x-z|)} |z - y|^{\alpha-n} f(y) dy &\le \int_{B(z,3|x-z|)} |z - y|^{\alpha-n} f(y) dy \\ &\le C(\log(2 + |x - z|^{-1}))^{-\beta+1} \end{aligned}$$

for $x, z \in G$. On the other hand, by the mean value theorem for analysis, we have by Lemma 2.3

$$\begin{aligned} &\int_{G \setminus B(x,2|x-z|)} \big| |x - y|^{\alpha-n} - |z - y|^{\alpha-n} \big| f(y) dy \\ &\quad \le C|x - z| \int_{G \setminus B(x,2|x-z|)} |x - y|^{\alpha-n-1} f(y) dy \\ &\quad \le C(\log(2 + |x - z|^{-1}))^{-\beta}. \end{aligned}$$

As a consequence we obtain

$$|I_\alpha f(x) - I_\alpha f(z)| \le C(\log(2 + |x - z|^{-1}))^{-\beta+1}$$

for $x, z \in G$, which implies (1.5).

Remark 2.5. Let f be a nonnegative measurable function on G satisfying

$$\left(\fint_{B(x,r)} f(y)^p dy \right)^{1/p} \le r^{-\alpha} (\log(2 + r^{-1}))^{-\beta}$$

for all $x \in G$ and $0 < r < d_G$, where $p > 1$ and a real number β. Then Jensen's inequality yields

$$\fint_{B(x,r)} f(y) dy \le r^{-\alpha} (\log(2 + r^{-1}))^{-\beta}.$$

Hence we can apply Theorem 1.1.

Remark 2.6. In Theorem 1.1 (1), if $\beta = 0$ and $\varepsilon = \alpha/2$, then we can find constants $C_1, C_2 > 0$ depending on n, α and d_G such that

$$\fint_{B(z,r)} \exp\left(\frac{I_\alpha f(x)}{C_1} \right) dx \le C_2 r^{-\alpha/2}$$

for all $z \in G$ and $0 < r < d_G$. If $\alpha/2 < \varepsilon < \alpha$, then Jensen's inequality gives

$$\fint_{B(z,r)} \exp\left(\frac{2(\alpha - \varepsilon) I_\alpha f(x)}{C_1 \alpha} \right) dx \le C_2^{2(\alpha-\varepsilon)/\alpha} r^{-\alpha+\varepsilon},$$

so that

$$\fint_{B(z,r)} \exp\left(\frac{I_\alpha f(x)}{c_1} \right) dx \le c_2 r^{-\alpha+\varepsilon}$$

for all $z \in G$ and $0 < r < d_G$. Here $c_1 = C(\alpha - \varepsilon)^{-1}$ and $c_2 \to 1$ as $\varepsilon \to \alpha$.

Remark 2.7. Theorem 1.1 (3) can also be proved by using Nakai [10, Theorem 3.3] and Spanne [14, p. 601] (see also [9, p. 521]). However our discussions are straightforward.

Remark 2.8. In Theorem 1.1 (1), one can not find positive constants $\tilde{c}_1$ and $\tilde{c}_2$ such that

$$\fint_{B(z,r)} \exp\left(\frac{(I_\alpha f(x))^{1/(1-\beta)}}{\tilde{c}_1} \right) dx \le \tilde{c}_2 (\log(2 + r^{-1}))^{-\beta}$$

holds for all $z \in G$ and $0 < r < d_G$.

To show this, consider

$$f(y) = |y|^{-\alpha}(\log(|y|^{-1}))^{-\beta}$$

for $y \in B(0, 1/2)$ with $\beta < 1$; set $f = 0$ elsewhere. Then

$$\fint_{B(x,r)} |f(y)|dy \le Cr^{-\alpha}(\log(2 + r^{-1}))^{-\beta}$$

for $x \in \mathbf{B} = B(0, 1)$. Further,

$$\begin{aligned} I_\alpha f(x) &\ge \int_{B(0,1/2)\setminus B(0,2|x|)} |x - y|^{\alpha - n} f(y)dy \\ &\ge C \int_{B(0,1/2)\setminus B(0,2|x|)} |y|^{-n}(\log(|y|^{-1}))^{-\beta}dy \\ &\ge C(\log(|x|^{-1}))^{-\beta+1} \end{aligned}$$

for $x \in B(0, 1/8)$. Hence it follows that

$$\fint_{B(0,r)} \exp\left(\frac{I_\alpha f(x)^{1/(1-\beta)}}{C^{1/(1-\beta)}c}\right)dx \ge \fint_{B(0,r)} |x|^{-1/c}dx = C'r^{-1/c}$$

for $0 < r < 1/8$, where $1/c < n$.

3. Proof of Theorem 1.2

For $\gamma > 0$, let

$$\rho_\gamma(r) = r^{-n}(\log(2 + r^{-1}))^{-\gamma}.$$

The following lemma can be proved in the same way as Lemma 2.4.

LEMMA 3.1. *Let $\alpha < \nu \le n$ and $\gamma > 1$. If f is a nonnegative measurable function on G satisfying* (1.1), *then*

$$\int_{B(z,r)} \left(\int_G \rho_\gamma(|x - y|)f(y)dy\right)dx \le Cr^{n-\nu}(\log(2 + r^{-1}))^{-\gamma-\beta+1}$$

whenever $B(z, r) \subset G$, where $C = C(n, \alpha, \nu, \beta, \gamma, d_G)$.

PROOF OF THEOREM 1.2. Let f be a nonnegative measurable function on G satisfying (1.1). Let

$$J_\gamma(x) = \int_G \rho_\gamma(|x - y|)f(y)dy$$

and

$$p = \frac{\nu}{\nu - \alpha}.$$

We find by Lemma 2.3

$$I_\alpha f(x) = \int_{B(x,\delta)} |x - y|^{\alpha - n} f(y) dy + \int_{G \setminus B(x,\delta)} |x - y|^{\alpha - n} f(y) dy$$
$$\leq C\delta^\alpha (\log(2 + \delta^{-1}))^\gamma J_\gamma(x) + C\delta^{\alpha - \nu}(\log(2 + \delta^{-1}))^{-\beta}$$

for $\delta > 0$. Considering $\delta = J_\gamma(x)^{-1/\nu}(\log(2 + J_\gamma(x)))^{-(\gamma+\beta)/\nu}$, we see that

$$I_\alpha f(x) \leq CJ_\gamma(x)^{(\nu-\alpha)/\nu}(\log(2 + J_\gamma(x)))^{\gamma(\nu-\alpha)/\nu - \alpha\beta/\nu}$$
$$= CJ_\gamma(x)^{1/p}(\log(2 + J_\gamma(x)))^{\gamma/p - \alpha\beta/\nu},$$

so that

$$\int_{B(z,r)} \{I_\alpha f(x)(\log(2 + I_\alpha f(x)))^{-\gamma/p + \alpha\beta/\nu}\}^p dx \leq C \int_{B(z,r)} J_\gamma(x) dx$$

whenever $B(z,r) \subset G$. Hence Lemma 3.1 gives

$$⨍_{B(z,r)} \{I_\alpha f(x)(\log(2 + I_\alpha f(x)))^{-\gamma/p + \alpha\beta/\nu}\}^p dx \leq Cr^{-\nu}(\log(2 + r^{-1}))^{-\gamma - \beta + 1}$$

for such z and r, which completes the proof of Theorem 1.2.

REMARK 3.2. The case when $\beta = 0$, $\alpha = 1$ and $1 \leq p \leq 1/\{2(\nu - 1)\}$ was also discussed by Serrin [13] in a different manner.

REMARK 3.3. In general, (1.6) does not hold when $\gamma < 1$.
To show this when $n = 2$, we consider

$$f(y) = f(y_1, y_2) = |y_2|^{-1}(\log(2 + |y_2|^{-1}))^{-\beta - 1}$$

with $\beta > 0$. Then (2.2) gives

$$⨍_{B(x,r)} |f(y)| dy \leq \frac{C}{r} \int_0^r |y_2|^{-1}(\log(2 + |y_2|^{-1}))^{-\beta - 1} dy_2 \leq Cr^{-1}(\log(2 + r^{-1}))^{-\beta}$$

for $x \in \mathbf{B} = B(0,1)$. For $0 < \alpha < 1$, consider the potential

$$I_\alpha f(x) = \int_{\mathbf{B}} |x - y|^{\alpha - 2} f(y) dy.$$

Here we may assume that $x_2 \neq 0$. Setting $Q(x) = \{y = (y_1, y_2) \in \mathbf{B} : |x_1 - y_1| < |x_2|, |y_2| < |x_2|\}$, we note that

$$
\begin{aligned}
I_\alpha f(x) &\geq \int_{Q(x)} |x-y|^{\alpha-2} f(y)dy \\
&\geq C|x_2|^{\alpha-2} \int_{Q(x)} f(y)dy \\
&\geq C|x_2|^{\alpha-1} \int_0^{|x_2|} |y_2|^{-1}(\log(2+|y_2|^{-1}))^{-\beta-1} dy_2 \\
&\geq C|x_2|^{\alpha-1}(\log(2+|x_2|^{-1}))^{-\beta},
\end{aligned}
$$

so that

$$
\begin{aligned}
&\int_{B(0,1)} (I_\alpha f(x))^p (\log(2+I_\alpha f(x))^{-\gamma+\alpha\beta p/\nu} dx \\
&\quad \geq C \int_{B(0,1)} |x_2|^{-1}(\log(2+|x_2|^{-1}))^{-\gamma-\beta} dx = \infty
\end{aligned}
$$

when $p = 1/(1-\alpha), \nu = 1$ and $0 < \beta < 1 - \gamma$.

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